J Dyn Diff Equat (2012) 24:595–599
DOI 10.1007/s10884-012-9273-2

A Note: An Invariance Theorem for Mappings II

Robert J. Sacker

Received: 11 April 2012 / Published online: 2 August 2012
© Springer Science+Business Media, LLC 2012

Abstract Conditions on a domain D in $\mathbb{R}^n$ are given so that if f is a continuous mapping of $\overline{D}$ into $\mathbb{R}^n$, is an open mapping on the interior of D and maps the boundary of D into the closure of D then f maps the entire set into its closure, i.e. $\overline{D}$ is invariant. This is an improvement over a previous result where f was required to be injective (one-to-one) since a locally injective map on the interior of D is an open map.

Keywords Invariance · Mappings · Difference equation · Discrete dynamics

Mathematics Subject Classification (2010) 39A05 · 37D99

1 Introduction and Statement of Result

For a continuous flow in $\mathbb{R}^n$, e.g. the flow generated by an autonomous ordinary differential equation, it is straightforward to show a domain D is forward invariant by showing that the vector field on the boundary of D is nowhere pointing into the complement of D. For a discrete flow generated by an iterated mapping $x_0 \to x_1 = f(x_0) \to x_2 = f^2(x_0) \to \cdots$ the problem is more difficult.

In this work we will obtain an improvement in the result presented in [4]. In that note we proved the following

Theorem 1.1 *Let $D \subset \mathbb{R}^n$ be a bounded subset and $f : \overline{D} \to \mathbb{R}^n$ continuous. Suppose $f : \mathring{D} \to \mathbb{R}^n$ is injective (one-to-one) and $f(\partial D) \subset \overline{D}$.*

If $\overline{D}^C = \mathbb{R}^n \setminus \overline{D}$ has no bounded components, then $f(\overline{D}) \subset \overline{D}$.

Here we employ the standard notation; $\overline{D}$ denotes the closure of D, $\mathring{D}$ the interior of D, $\partial D = \overline{D} \setminus \mathring{D}$ the boundary of D and $D^C = \mathbb{R}^n \setminus D$, the complement of D.

R. J. Sacker (✉)
Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA
e-mail: rsacker@usc.edu

In [4], counterexamples were given for the theorem whenever any hypothesis is removed. However, the counterexample given for the removal of injectivity also failed to be an open map. An examination of the proof of Theorem 1.1 reveals that the injectivity of f was only used to conclude, via the Brouwer Invariance of Domain Theorem [2], that f is an open map.

The assumption of injectivity is a global assumption and not always easy to verify. Local injectivity, however follows immediately if we know, for example that f has no critical points, see Remark (a). For a general treatment of mappings that are globally injective see Appendix B of [3].

It is easy to construct maps $f : \mathring{D} \to \mathbb{R}^n$ that are locally injective but not injective. But a continuous *local* injection on an open subset of $\mathbb{R}^n$ is locally open by Brouwer's Theorem and therefore an open map, i.e. for each open $\mathcal{V} \subset \mathring{D}$, $f(\mathcal{V})$ is an open subset of $\mathbb{R}^n$. Thus we may state a more general

Theorem 1.2 *Let $D \subset \mathbb{R}^n$ be a bounded subset and $f : \overline{D} \to \mathbb{R}^n$ continuous. Suppose $f : \mathring{D} \to \mathbb{R}^n$ is an open map and $f(\partial D) \subset \overline{D}$.*

If $\overline{D}^C = \mathbb{R}^n \backslash \overline{D}$ has no bounded components, then $f(\overline{D}) \subset \overline{D}$.

Remark

(a) In applications it is sometimes the case that $f \in C^1$ and the following applies. Suppose $\det f'(x) \neq 0$ for all $x \in \mathring{D}$. Then from the Inverse Function Theorem, f is locally injective and from the above discussion, an open map. Thus Theorem 1.2 can be applied.

(b) For $n \geq 2$ the "no bounded components" assumption on $\overline{D}^C$ together with D bounded imply $\overline{D}^C$ is connected and in fact path connected.

It remains to prove Theorem 1.2. In the process we will correct a minor misstep in the choice of the radius of the *ball* $\mathcal{B}$ in the proof of Theorem 1.1 in [4].

Proof If the interior $\mathring{D}$ is empty then $\partial D = D$ and the theorem is trivially true. Thus assume $\mathring{D} \neq \varnothing$. For $n = 1$ the conditions imply D is an interval I and the theorem follows easily from the Intermediate Value Theorem. Under the mapping the left end point moves to the right and the right endpoint to the left. Since f is an open map, f is strictly monotonic and therefore its range lies in $[f(b), f(a)]$ or $[f(a), f(b)]$, where $a \leq b$ are the endpoints of the interval I.

For $n \geq 2$ assume the theorem is not true. Then since f is an open map, $G \doteq f(\mathring{D}) \cap \overline{D}^C \neq \varnothing$ and G is open in $\mathbb{R}^n$. Let $\mathcal{B}$ be an open ball in $\mathbb{R}^n$ so large that $\overline{G} \subset \mathcal{B}$.

Let $y \in G$ and from path connectedness of $\overline{D}^C$, let $\gamma : [0, 1] \to \overline{D}^C$ be a path with $\gamma(0) = y$ and $\gamma(1) \in \partial\mathcal{B}$. Next define $t_0 = \min\{t \in [0, 1] \mid \gamma(t) \in \partial G\}$ and set $\eta = \gamma(t_0) \in \gamma \cap \partial G$, the first point where γ meets the boundary ∂G starting from y.

Now choose a sequence $\eta_j \in G \cap \gamma$ such that $\eta_j \to \eta$ and choose $x_j \in f^{-1}(\eta_j) \cap \mathring{D}$. By compactness of $\overline{D}$ there is a subsequence, that we again call x_j that converges, $x_j \to x$. If $x \in \partial\mathring{D}$ then $\eta = f(x) \in f(\partial\mathring{D}) \subset f(\partial D) \subset \overline{D}$. But $\eta \in \gamma$ and thus $\eta \in \overline{D} \cap \gamma = \varnothing$, a contradiction.

Thus, $x \in \mathring{D}$ and $\eta = f(x) \in f(\mathring{D}) \cap \gamma$. Now $\gamma \subset \overline{D}^C$ so that $\eta \in f(\mathring{D}) \cap \gamma \cap \overline{D}^C = \gamma \cap G$, and G is an open subset of $\mathbb{R}^n$. But $\eta \in \gamma \cap \partial G$, another contradiction and thus the proof is complete. □

2 An Application

In [4] the following mapping $h : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ was considered

$$h_i(x_1, x_2) = \frac{(1+a_i)x_i}{1+b_i(x_1+x_2)}\phi(x_1+x_2), \quad a_i > 0,\ b_i > 0, \quad i = 1, 2, \tag{1}$$

where $\frac{a_1}{b_1} > \frac{a_2}{b_2}$ and ϕ is defined as follows. For *fixed* $\tau > \frac{a_1}{b_1}$,

$$\phi(s) = \begin{cases} 1, & 0 \le s \le \tau \\ e^{-(s-\tau)}, & s > \tau. \end{cases}$$

For $0 \le x_1 + x_2 < \tau$, then (1) is just

$$g_i(x_1, x_2) = \frac{(1+a_i)x_i}{1+b_i(x_1+x_2)}, \quad a_i > 0,\ b_i > 0, \quad i = 1, 2, \tag{2}$$

which is the *pure selection* case of the more general *mutation-selection model* treated in [1],

$$\begin{aligned} x_1(t+1) &= \frac{(\gamma_{11}a_1+1)x_1(t) + \gamma_{12}a_2x_2(t)}{1+b_1(x_1(t)+x_2(t))} \doteq f_1(x_1(t), x_2(t)), \\ x_2(t+1) &= \frac{(\gamma_{22}a_2+1)x_2(t) + \gamma_{21}a_1x_1(t)}{1+b_2(x_1(t)+x_2(t))} \doteq f_2(x_1(t), x_2(t)), \end{aligned} \tag{3}$$

where

$$\gamma_{1j} + \gamma_{2j} = 1, \quad j = 1, 2.$$

Equation (3) express the fact that the total new offspring from x_j consists of growth within the population x_j, i.e. *self growth*, $\gamma_{jj}x_ja_j$ plus the mutation $a_ix_i\gamma_{ji},\ i \ne j$, into x_j. Since mutations are small relative to self growth we express, for small $\varepsilon,\ 0 < \varepsilon < \varepsilon_0$,

$$\gamma_{11} = 1 - \gamma_1\varepsilon, \quad \gamma_{21} = \gamma_1\varepsilon, \quad \gamma_{22} = 1 - \gamma_2\varepsilon, \quad \gamma_{12} = \gamma_2\varepsilon, \tag{4}$$

where $0 < \gamma_i < 1$ and ε_0 is fixed and sufficiently small that all quantities on the left side of the equal signs in (4) are positive. We wish to consider here only the *pure selection* model

$$x(t+1) = g(x(t)), \tag{5}$$

with components given by (2), and its derivative

$$g'(x_1, x_2) = \begin{bmatrix} \frac{(1+a_1)(1+b_1x_2)}{(1+b_1S)^2} & -\frac{(1+a_1)b_1x_1}{(1+b_1S)^2} \\ \frac{(1+a_2)b_2x_2}{(1+b_2S)^2} & \frac{(1+a_2)(1+b_2x_1)}{(1+b_2S)^2} \end{bmatrix}, \tag{6}$$

where $S = x_1 + x_2$. Note that the coordinate axes are invariant for (5) and $P_1 = \left(\frac{a_1}{b_1}, 0\right)$ and $P_2 = \left(0, \frac{a_2}{b_2}\right)$ are fixed points. Recalling that $\frac{a_1}{b_1} > \frac{a_2}{b_2}$, the eigenvalues of $g'(P_1)$ are

$$\lambda_1 = \frac{1}{1+a_1} < 1, \quad \text{and} \quad \lambda_2 = \frac{1+a_2}{1+b_2a_1/b_1} < 1,$$

while those for $g'(P_2)$ are

$$\lambda_1 = \frac{1+a_1}{1+b_1a_2/b_2} > 1, \quad \text{and} \quad \lambda_2 = \frac{1}{1+a_2} < 1.$$

Thus P_1 is asymptotically stable and P_2 is saddle unstable. One wishes to show that P_1 *globally* attracts all orbits of (5) having initial conditions in the *open* first quadrant.

The first step in achieving this, and the *only* step we are concerned with in this note, is to show that all initial conditions in the open first quadrant are ultimately mapped into a compact set D containing P_1 and D is mapped into itself. To this end note that $S(t) = x_1(t) + x_2(t)$ and the following hold for the mapping (5); we prove only the first two.

1. $S(t) > \dfrac{a_1}{b_1} \Longrightarrow S(t+1) < S(t),$
2. $S(t) = \dfrac{a_1}{b_1} \Longrightarrow S(t+1) \le S(t),$ with " = " $\Longleftrightarrow x_2(t) = 0,$
3. $S(t) < \dfrac{a_2}{b_2} \Longrightarrow S(t+1) > S(t),$
4. $S(t) = \dfrac{a_2}{b_2} \Longrightarrow S(t+1) \ge S(t),$ with " = " $\Longleftrightarrow x_1(t) = 0.$

To establish the first inequality note that $b_2 a_1/b_1 > a_2$. Then

$$\begin{aligned} S(t+1) &= \frac{(1+a_1)x_1(t)}{1+b_1 S(t)} + \frac{(1+a_2)x_2(t)}{1+b_2 S(t)} \\ &< \frac{(1+a_1)x_1(t)}{1+a_1} + \frac{(1+a_2)x_2(t)}{1+b_2 a_1/b_1} < S(t). \end{aligned} \tag{7}$$

The second inequality (2) follows by changing the first "<" in (7) to "=" and noting that the second "<" holds if, and only if $x_2 = 0$. The remaining inequalities are proved in a similar manner.

Remark (c): $S = x_1 + x_2$ is a metric in the positive cone K and the inequality (1), for example, says points in K are moved closer to the origin.

Next consider the trapezoidal region D in the first quadrant bounded by the two parallel lines $L_1 : S = \dfrac{a_1}{b_1}$, $L_2 : S = \dfrac{a_2}{b_2}$ and the axes. By (1) and (3) it is easily seen that if ω_P is the ω-limit set of any point $P = (x_1, x_2) \in K$ having $x_1 + x_2 > \frac{a_1}{b_1}$, then $\omega_P \cap K \backslash \overline{D} = \varnothing$, i.e. P is ultimately mapped into D. A similar argument holds for points with $x_1 + x_2 < \frac{a_2}{b_2}$.

By (2) and (4), L_1 and L_2 map into $\overline{D}$, while the bounding segments in the axes are themselves invariant. Thus $g(\partial D) \subset \overline{D}$.

By Remark (a) it only remains to show that f is locally injective on D. This follows immediately since the Jacobian J of the mapping (5) satisfies

$$J = \det g'(x_1, x_2) = \frac{(1+a_1)(1+a_2)(1+b_1 x_2 + b_2 x_1)}{(1+b_1 S)^2 (1+b_2 S)^2}, \tag{8}$$

and all quantities are positive. Thus $J > 0$ so Theorem 1.2 applies and D is invariant.

Remark (d): For the full mapping (3) the Jacobian J_ε is just an ε perturbation of (8) so the local injectivity is immediate. In [4] it was shown that the mapping g is injective, not just locally injective but the proof doesn't generalize easily to handle the full mapping (3). This is in fact what motivated the result of this note.

The choice of domain D and showing $f(\partial D) \subset \overline{D}$ for (3), however, is a much more delicate issue.

Acknowledgments Supported by University of Southern California, Dornsife School of Letters Arts and Sciences Faculty Development Grant.

References

1. Ackleh, A.S., Sacker R.J.: On a discrete selection-mutation model. J. Biol. Dyn. (submitted) (2012)
2. Dugundji, J.: Topology. Allyn and Bacon, Boston (1966)
3. Pötzsche, C.: Theory of Discrete Nonautonomous Dynamical Systems. Vol. 2002 of Lecture Notes in Mathematics. Springer, Berlin (2010)
4. Sacker, R.J.: A note: an invariance theorem for mappings. J. Differ. Equ. Appl. **18**(1), 163–166 (2012)