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ON INVARIANT SURFACES AND BIFURCATION OF PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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The corrections below have been performed on the document to follow. The correction sheet was included with the original publication.

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Corrections to IMM-NYU 333

Page	Line	Reads	Should Read
i	11	multiples	multipliers
7	9	$\alpha(\psi)$	$\alpha(\mu)$
7	9	$\alpha^{'}(\psi_1)$	$\alpha^{'}(\mu_1)$
23	6	mapping	transformation
33	6	$< \hat{c}(r)$	$< 4\hat{c}(r)$
44	-4	p	$\tilde{p}$
44	-5	$P(z,\ldots)$	$P(s,\ldots)$
44	15	$=$	$\neq$
45	4	mapping	transformation
47	11	applying	Applying
55	2	mapping	equation
57	8	$e^{\frac{1}{2}}$	$\varepsilon^{\frac{1}{2}}$
77	3	$c^{r-1}(x,\mu)$	$c^{r-1}(x)$
77	3	$Lip^{r-1}(x,\mu)$	$Lip^{r-1}(x)$
79	7	$\tilde{\lambda}(x,\mu)$	$\tilde{\lambda}(\mu)$
80	7	(all a term)	?
82	7	$\frac{2}{\beta}$	$\frac{\beta}{2}$
82	10	$\frac{1}{\beta}$	$\frac{\beta}{1}$
83	3	$a(x,\ )$	$a(x,\mu)$
84	2	$\|P\|_s$	$\|P\|_s)$
86	-2	brackets	braces
88	2	$[\quad]$	$[\quad]^s$

ON INVARIANT SURFACES AND BIFURCATION OF PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

by Robert John Sacker

Abstract

We consider the ordinary differential equation

$$\frac{dx}{dt} = F(x, \mu)$$

where x is a real n-vector and μ is a real parameter. It is assumed that for all μ in a neighborhood of a critical value μ_1 this equation has a periodic solution $x = \psi(t, \mu)$ which is asymptotically stable for $\mu < \mu_1$ (n-1 Floquet multipliers have modulus < 1). As μ increases through μ_1 we assume a conjugate pair of multipliers $\lambda(\mu)$ and $\overline{\lambda}(\mu)$, leaves the unit disc causing ψ to become unstable. It is shown that a two-dimensional asymptotically stable invariant torus will bifurcate form ψ provided (a) $\lambda(\mu_1)$ is not a second, third, or fourth root of unity, (b) $d/d\mu|\lambda(\mu_1)| > 0$, and (c) a certain nonlinear term obtained from $F(x, \mu)$ has the proper sign (nonlinear damping).

Some cases in which (a) does not hold (resonance cases) are also discussed. In a particular subcase of the case $\lambda^4(\mu_1) = 1$ (but λ and $\lambda^2 \neq 1$), ψ bifurcates into a pair of subharmonic solutions, exactly one of which is stable. When a single multiplier leaves, $\lambda(\mu_1) = 1$, ψ will in general bifurcate into one stable periodic solution. The case $\lambda^3(\mu_1) = 1$ (but $\lambda \neq 1$) is discussed only by means of a an example in which the instability that develops is so violent compared to previous cases that there is no stable manifold or periodic solution in a small neighborhood of ψ.

The existence of the torus is established by two methods. The first is to take an n-1 dimensional hyperplane S normal to the orbit ψ and show that the mapping of S into itself, induced by the vector field near ψ, has an invariant curve. In the second method the torus is obtained as the solution of a certain partial differential equation which is basic in the theory of invariant surfaces.

The second main result is concerned with the theory of perturbation of an invariant surface of a periodic vector field. We consider the differential system

$$\begin{aligned} \dot{x} &= f(x, y, \mu) \\ \dot{y} &= g(x, y, \mu) \end{aligned}$$

where x and y are real vectors, μ is a small real parameter and the functions are periodic in x (e.g., x represents the angular variables on a torus). The problem is to find an invariant manifold $\tau(\mu) : y = \phi(x, \mu)$ whenever $\tau(0)$ is known and has certain properties. One of the properties assumed by other authors is that the flow (vector field) on $\tau(0)$ is parallel, i.e, in appropriate coordinates on the surface, solutions can be obtained from one another by a translation. This restriction is dropped since, in general, parallel flow is lost after a perturbation is made. The essential property is the relative strength of the flow against the unperturbed surface and the flow in the surface.

The function $\phi(x, \mu)$ satisfies the quasilinear partial differential equation

$$\sum_{\nu} f_{\nu}(x, \phi, \mu) \frac{\partial \phi}{\partial x_{\nu}} - g(x, \phi, \mu) = 0$$

which is solved by an iterative scheme. It is linearized and smoothed so that we have a C^{∞} equation $Lu = h$ for the approximate solution u. This is made elliptic by adding the Laplacian, $c\triangle u + Lu = h,\ c =$ constant, so that existence and smoothness of a solution becomes a matter of quoting well known results. The existence of a solution of the quasilinear equation is obtained from "a-priori" estimates for the solution of the elliptic problem and its derivatives. If the flow against the surface is strong compared to the flow in the surface, then the solution $\phi(x, \mu)$ will be very smooth since higher derivatives may be estimated.

TABLE OF CONTENTS

CHAPTER I
INTRODUCTION AND SUMMARY

In this dissertation we are concerned with the ordinary differential equation

$$\frac{dx}{dt} = F(x, \mu) \tag{1.1}$$

where x and F are real n-vectors and μ is a real parameter. We will say that (1.1) has an invariant manifold σ_k, of dimension $k \leq n$, if any solution passing through a point of σ_k remains on σ_k for all time t. The manifold σ_k is said to be asymptotically stable if all solutions of (1.1) sufficiently near σ_k approach σ_k as $t \to +\infty$, and unstable if in any arbitrarily small neighborhood N of σ_k there is at least one solution which leaves N as $t \to +\infty$.

Assume that for $\mu < \mu_0$, σ_k is asymptotically stable, but for $\mu > \mu_0$, it is unstable. One of the simplest examples of this for $n = 2,\ k = 0$, is

$$\begin{aligned} \dot{z} &= (i + \mu)z + \beta z^2 \bar{z} \\ &= i[1 + Im\ \beta|z|^2]z + [\mu + Re\ \beta|z|^2]z \end{aligned} \tag{1.2}$$

where $z = x_1 + ix_2,\ \mu_0 = 0$ and $\sigma_0 = 0$. The characteristic exponents for the singular point $z = 0$ are $\alpha(\mu) = i + \mu$ and $\bar{\alpha}$. If $Re\ \beta < 0$ (nonlinear damping) then for $\mu > 0$ this equation has an asymptotically stable limit cycle

$$|z| = \left(\frac{\mu}{-Re\ \beta}\right)^{\frac{1}{2}}$$

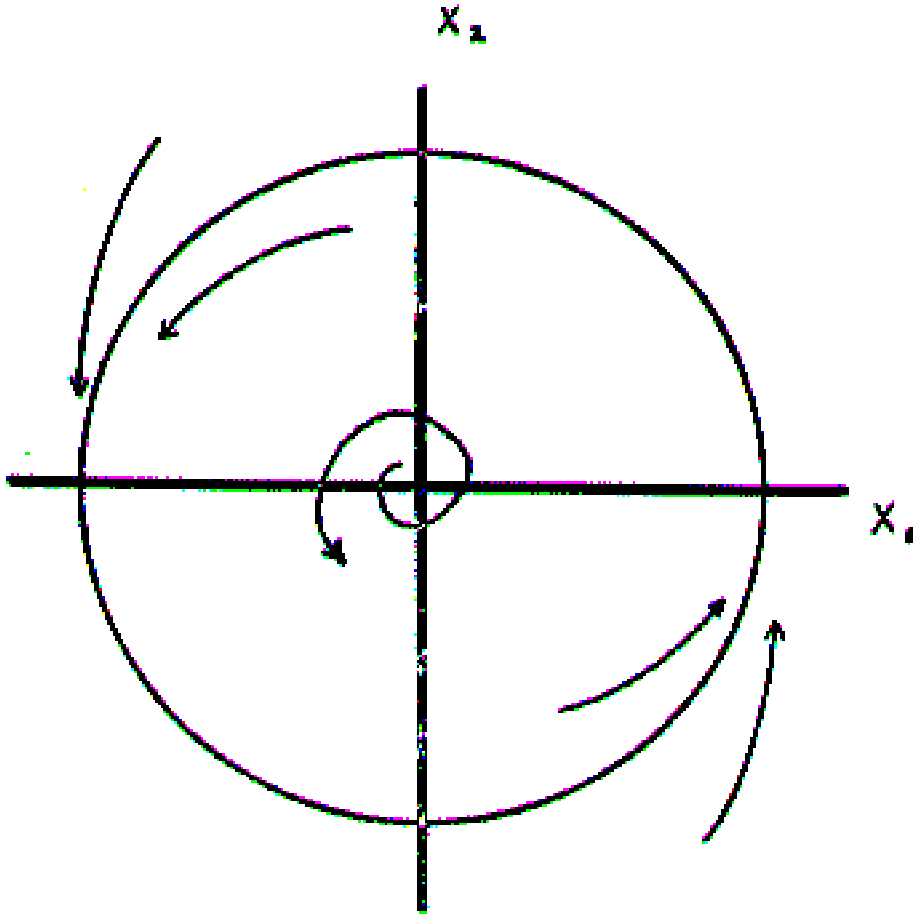
i.e., a stable invariant manifold $\sigma_1(\mu)$ with $\sigma_1(0) = \sigma_0$, which branches off or bifurcates form σ_0 as μ increases through zero (Fig. 1).

<u>Definition</u>: We say that the manifold σ_ℓ bifurcates* from σ_k as μ increases through μ_0 if $\sigma_\ell(\mu_0) = \sigma_k$ but $\sigma_\ell(\mu)$ and σ_k have no points in common for $\mu > \mu_0$.

The above example illustrates a general result obtained by E. Hopf [2] which may be stated in a somewhat weaker form as follows. Assume (1.1) is analytic and has a stationary solution $x = 0$ for $|\mu|$ sufficiently small. Assume that for $\mu < 0$ all the characteristic exponents $a(\mu)$ satisfy $Re\ a(\mu) < 0$. As μ increases through zero assume a conjugate pair of exponents $\alpha(\mu),\ \bar{\alpha}(\mu) \neq 0$ crosses the imaginary axis and enters the right half plane with $Re\ \alpha(0) = 0,\ Re\ \alpha'(0) > 0$, while the remaining exponents satisfy $Re\ \alpha(\mu) \leq a_0$ for some constant $a_0 < 0$. Then if a certain constant depending on terms of degree ≤ 3 in the expansion of F about $x = 0$ has the proper sign (nonlinear damping), then the solution

*See Minorsky [1] for a discussion of bifurcation.

FIG. 1



Radius of circle $= \left(\frac{\mu}{-\mathrm{Re}\ \beta}\right)^{\frac{1}{2}}$, $\mu \geq 0$

$\sigma_0 : x = 0$ bifurcates into an asymptotically stable* periodic solution $\sigma_1(\mu) : x = \psi(t, \mu)$ for $\mu > 0$ sufficiently small. $\psi(t, 0) = 0$ and $\psi(t, \mu) \neq 0$ for $\mu > 0$. If the constant has the opposite sign, then ψ will exist in advance ($\mu < 0$) but will be unstable, i.e, as μ decreases through zero the unstable origin σ_0 becomes stable and bifurcates into an unstable periodic solution. Hopf assumed analyticity of the functions and used power series expansions in his proof- the method of Poincaré.

In this dissertation we consider the case in which $\psi(t, \mu)$ becomes unstable as μ increases further and passes through a second critical point μ_1, and a pair of Floquet exponents $\alpha(\mu)$, $\overline{\alpha(\mu)}$ enters the right half plane $Re\ \varsigma > 0$ with $Re\ \alpha'(\mu_1) > 0$, i.e., a pair of Floquet multipliers $\lambda(\mu), \overline{\lambda(\mu)}$ leaves the unit disc $|\varsigma| \leq 1$ with $d/d\mu|\lambda(\mu_1)| > 0$.

The Floquet exponents are associated with (1.1) linearized along $\psi(t, \mu)$,

$$\dot{y} = F_x(\psi(t, \mu), \mu)y \qquad (1.3)_\mu$$

If the frequency of $\psi_1 \equiv \psi(t, \mu_1)$ is ω_1 then $Im\ \alpha(\mu_1) \equiv \omega_2$ (determined only up to integral multiples of ω_1) is the frequency of oscillation about ψ_1 of solutions of $(1.3)_{\mu_1}$ starting near ψ_1. Floquet theory asserts that $(1.3)_{\mu_1}$ has a family of solutions of the form

$$\phi(t) = A(t)\cos\omega_2 t + B(t)\sin\omega_2 t$$

where A and B have frequency ω_1. If ω_2/ω_1 is irrational (non-resonance case) then $\phi(t)$ is not periodic. If $\omega_2/\omega_1 =$ rational $= p/q$ in lowest form (resonance case) then $\phi(t)$ is periodic and has frequency ω_1/q, i.e., the period $P = qP_1$ ($P_1 = 2\pi/\omega_1$ is the period of ψ_1). If $q > 1$ we call $\phi(t)$ a <u>subharmonic solution</u>. In terms of the Floquet multiplier $\lambda(\mu_1) = e^{i2\pi\frac{\omega_2}{\omega_1}}$, non-resonance is characterized by $\lambda^q(\mu_1) \neq 1$ for any integer q and resonance by $\lambda^q(\mu_1) = 1$ for some integer q.

Throughout this work we will assume that F has $\rho \geq 5$ continuous derivatives with respect to x and μ in some neighborhood of $\psi(t, \mu)$. We will prove the following.

<u>Statement 1</u>: If certain low order resonances are avoided, specifically, if
(a) $\omega_2/\omega_1 \neq p/q$ in lowest form with $q = 1, 2, 3, 4$ i.e., if $\lambda^q(\mu_1) \neq 1$ for those q, and if
(b) $Re\ \alpha'(\mu_1) > 0$ and
(c) a certain constant obtained from the terms of degree ≤ 3 in the expansion of F about $\psi(t, \mu_1)$ has the proper sign, then $\psi(t, \mu)$ bifurcates into a unique asymptotically stable torus $\sigma_2(\mu)$ as μ increases through μ_1. For r an integer, $0 \leq r \leq \rho - 1$, $\sigma_2(\mu)$ exists as an r-times continuously differentiable manifold for $0 < \mu - \mu_1 < \delta_r$ where δ_r depends on r, the terms of

*i.e., $n - 1$ of the Floquet exponents associated with $\psi(t, \mu)$ have negative real parts.

degree ≤ 3 mentioned in (c), and $M_0 = |F|_r{}^*$. In general $\delta_r \to 0$ as $r \to \infty$. If the constant in (c) has the opposite sign, then the torus σ_2 exists in advance ($\mu < \mu_1$) and is unstable.

A model of such bifurcation phenomenon was given by E. Hopf [3]. Hence we show essentially that for the second bifurcation (i.e., periodic solution into torus) Hopf's model is the generic situation.

It is worth noting that even if $F(x, \mu)$ is analytic, this torus may not be analytic but only represented by functions with finitely many derivatives. Thus any method based on power series expansions must fail.

Three resonance cases are discussed and we prove the following.

Statement 2: If
(a) $\omega_2/\omega_1 = 1/4$, i.e., $\lambda^4(\mu_1) = 1$
(b) $Re\ \alpha'(\mu_1) > 0$ and
(c) certain conditions are satisfied by the terms of degree ≤ 3 in the expansion of F about $\psi(t, \mu_1)$, then as μ increases through $\mu_1, \psi(t, \mu)$ bifurcates into a pair of subharmonic solutions**. One solution is asymptotically stable and the other is unstable.

The next result deals with the case in which one real Floquet exponent $\alpha(\mu)$ becomes positive.

Statement 3: If
(a) $\alpha(\mu_1) = 0,\ \alpha(\mu)$ real, (b) $\alpha'(\mu_1) > 0$ and
(c) a certain constant obtained from the terms of degree ≤ 2 in the expansion of F about $\psi(t, \mu_1)$ is not zero, then as μ increases through μ_1, $\psi(t, \mu)$ bifurcates into an asymptotically stable periodic solution having the same period.**

The third resonance case, $\omega_2/\omega_1 = 1/3$, is discussed by means of an example which illustrates the violent instability which develops after bifurcation.

The resonances (Statements 2 and 3 and the example) are treated in Chapter III where the rate of growth of the bifurcating manifolds is also discussed. The proofs of Statements 2 and 3 require the usual construction of periodic solutions by means of the implicit function theorem [4]. In contrast, Statement 1 requires the construction of a 2-dimensional surface and thus different techniques are needed. Statement 1 is proved in Chapter II and again in Chapter III. In Chapter II the problem is treated by working with the mapping induced by

*This norm is defined at the end of the chapter. The differentiation is with respect to x and μ and the supremum is taken over a small neighborhood of $\psi(t, u)$.

** The deviation of period with μ is eliminated by introducing a new parameter s in place of t.

the vector field near ψ, the so-called "method of surfaces of section" of Poincaré and G.D. Birkhoff. The method is described in detail in the introduction to that chapter.

In Chapter III we treat the differential equation directly. The torus is obtained as the solution of a certain partial differential equation which arises in the theory of perturbation of an invariant surface of a periodic vector field. This theory, treated in Chapter IV, is concerned with the problem of finding an invariant manifold $\tau(\mu)$ of

$$\begin{aligned} \dot{x} &= f(x, y, \mu) \\ \dot{y} &= g(x, y, \mu) \end{aligned} \tag{1.4}$$

whenever $\tau(0)$ is known and has certain properties. μ is a small parameter, x and y are vectors, and the functions are periodic in x. If $\tau(0)$ is the surface $\phi_0(x)$, then the problem is to find a solution $\tau(\mu) : y = \phi(x, \mu)$ of the quasilinear partial differential equation

$$\sum_{\nu} f_\nu(x, \phi, \mu) \frac{\partial \phi}{\partial x_\nu} - g(x, \phi, \mu) = 0 \tag{1.5}$$

with $\phi(x, 0) = \phi_0(x)$. The properties of the flow (vector field) in the surface $\tau(0)$ and the flow against* the surface are essential in determining the existence and smoothness of the perturbed surface. We say that the flow in $\tau(0)$ is <u>parallel</u> if $\dot{x}$ = constant on the surface. The flow is parallelizable if it is parallel in appropriate coordinates on the surface. The case of parallel flow has received much attention (Diliberto [5, 6], Bogoliubov and Mitropolski [7, 8], Hale [9, 10], Diliberto and Hufford [11]).

In treating the general problem of perturbation of invariant surfaces the assumption of parallel flow is much too stringent and unnatural since it is generally lost after the perturbation is made. When the unperturbed surface contains lower dimensional invariant manifolds (e.g., stationary points) it is usually impossible to parallelize the flow. The characteristics on the surface may form envelopes and it seems unsatisfactory to use methods such as those of Bogoliubov and Mitropolski [7] in which (1.5) is solved by integrating along characteristics. The characteristics method is associated with initial value problems whereas the problem of solving (1.5) defined on a torus may be formulated more naturally as a boundary value problem in which the boundary condition to be imposed is the periodicity of the solution in x. It is the latter formulation which is considered in Chapter IV. The solution of (1.5) is obtained by an iterative process. At each stage the equation is linearized and smoothed so that we obtain an equation $Lu = h$ where h and the coefficients in L are C^∞ functions. This is then modified to give an elliptic equation

$$c \triangle u + Lu = h$$

*We do not consider the case in which the flow is away from $\tau(0)$.

where $c = c(n)$ is a constant which tends to zero as n, the number of iterations, tends to ∞ and $\triangle$ is the Laplacian. Under the given conditions we obtain existence and C^∞ smoothness of a solution of the above equation. The existence of the solution of the nonlinear equation then follows from the "a priori" estimates for the solution of the linear equation and its derivatives. In deriving these estimates there are two parameters, $\beta > 0$ and $\lambda \leq 0$, which play a fundamental role. β is a measure of the flow against the unperturbed surface and λ is a measure of the flow in the surface. It is shown that if $\beta + r\lambda > 0$, r and integer ≥ 2, and another condition* is satisfied, then for sufficiently small μ the perturbed surface $\tau(\mu)$ has $r - 1$ Lipschitz continuous derivatives. This is the second main result established in this dissertation (Theorem 8). It says essentially that if the flow against the surface is great (β large) compared to the flow in surface, then the perturbed surface will be very smooth. Heuristically we can see the need for such an assumption. For consider (in one x dimension) two characteristics C_1 and C_2 approaching the equilibrium $x = 0$, (see Figs. 2 and 3 where $\tau(0)$ is the line $y \equiv 0$). If as in Fig. 2, the flow against $y \equiv 0$ is small compared to the flow in $y \equiv 0$ (shown by arrows), then a cusp may develop. McCarthy [13] and Kyner [14] discuss an example of such a situation. If the relative strength of these flows is reversed (Fig. 3) then C_1 and C_2 tend to meet in a smooth fashion.

Thus even if f and g are C^∞ functions and $\tau(0)$ is a C^∞ surface it may happen that $\tau(\mu)$ has only finitely many derivatives. In fact, in the example of Kyner the perturbed surface has a cusp for arbitrarily small $\mu > 0$.

The result of Kyner [14] is not restricted to parallel flow. In the case of non-parallelizable flow he postulates the invertibility of a certain linear transformation and obtains a perturbed surface which is Lipschitz continuous. Higher derivatives are not treated.

In the special case in which the flow on $\tau(0)$ is parallel, $\lambda = 0$ and thus $\tau(\mu)$ will have any finite number of continuous derivatives provided μ is sufficiently small. Such is the case if we consider the equation

$$\dot{\xi} = F(\xi) + \mu F^*(\xi, t, \mu) \tag{1.6}$$

(ξ an n-vector) where the functions are C^∞ in all arguments and have period T in t. It is assumed that for $\mu = 0$ there is a limit cycle Γ which is asymptotically stable ($n - 1$ of the characteristic exponents of the linear variational equation have negative real parts). $\tau(0)$ is then the torus $\Gamma \times \{0 \leq t \leq T\}$ with the faces $t = 0$ and $t = T$ identified (see Fig. 4). It is known ([15], [8], [9], [11]) that coordinates can be introduced such that the flow on $\tau(0)$ is parallel. Hale [9] and Diliberto and Hufford [11] treat (1.6) in the degenerate case in which

*This condition (number 3 in Theorem 8) is used to solve the linearized equation and is precisely the "positive" condition of Friedrichs [12].

FIG. 2

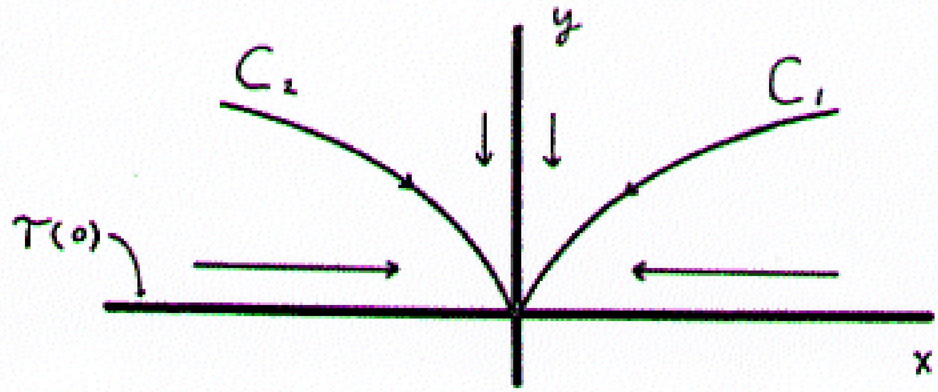
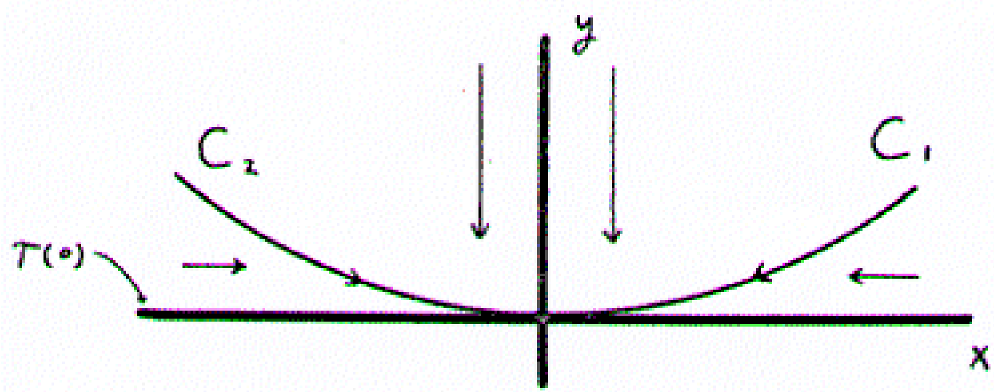


FIG. 3



the flow against $\tau(0)$ vanishes for $\mu = 0$ (some of the characteristic exponents have zero real part for $\mu = 0$). It is this case which arises in the bifurcation problem in Chapter III (see equation 3.24) and thus the results of these authors could have been applied at this stage to prove existence of the torus. The theory of Chapter IV also takes this degeneracy into account and may be considered as a new derivation of the results of the above authors. Its chief advantage, however, lies in its applications to cases of non-parallelizable flow. (see example in Chapter IV).

Notation: We now introduce some notation which will be used throughout the rest of our work. For r a positive integer

$$F(x, y, \mu) \in C^r(x, y) \cap C(\mu)$$

means that F has r derivatives with respect to x and y which are continuous in (x, y, μ).

$$F(x, y, \mu) \in C^r(x, y) \cap Lip^r(x, y) \cap C(\mu)$$

means the same as above and the derivatives are Lipschitz continuous in x and y uniformly in μ.

For a vector $f = (f_i)$ define

$$|f| = \left(\sum_i f_i^2\right)^{\frac{1}{2}} = (f, f)^{\frac{1}{2}}$$

We will consider functions $f(x, \mu)$ defined for x a vector in some domain G and μ a small parameter. For $r \geq 0$ an integer define the norm

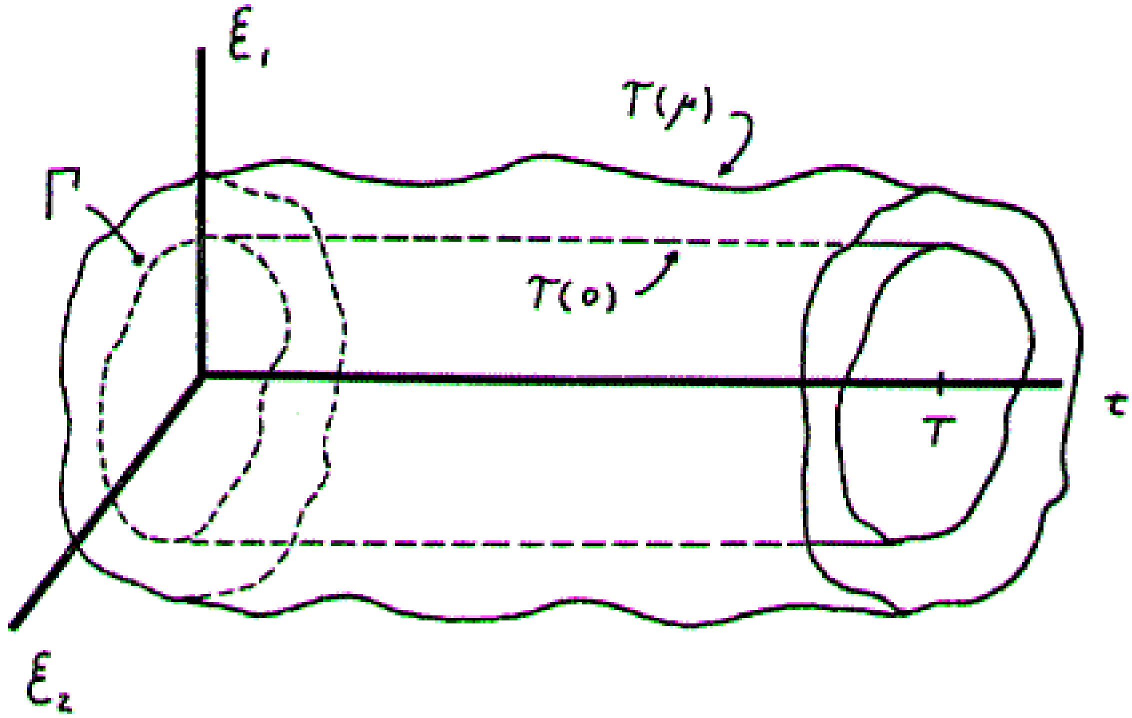
$$|f|_r = \max_{0 \leq \rho \leq r} \sup_{x \in G} |D^\rho f|$$

where D^ρ is any ρ^{th} order derivative of f with respect to the components x_i of x. Unless otherwise stated, differentiation will never be performed with respect to μ.

For two real vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, the symbol (u, v) will denote the usual inner product

$$(u, v) = \sum_{k=1}^{n} u_k v_k.$$

FIG. 4



In this figure Γ is assumed to be in the (ξ_1, ξ_2) plane. The coordinates $(\xi_3, \ldots, \xi_n)$ are not shown.

CHAPTER II
BIFURCATION - MAPPING METHOD

1. Introduction

In this chapter we treat the bifurcation problem by considering the mapping induced by the vector field near the periodic solution: the method of surfaces of section mentioned earlier.

We consider the differential equation

$$\frac{dx}{dt} = F(x, \mu) \tag{2.1}$$

where x and F are real n-vectors and μ is a real parameter. Suppose that for all sufficiently small μ, $|\mu| < \mu_*$, (2.1) has a periodic solution $x = \psi(t, \mu)$ with period 2π. Assume that F has $\rho \geq 5$ continuous derivatives with respect to x and μ in some neighborhood of ψ and that in this neighborhood

$$|F|_\rho \leq M_0$$

where differentiation is with respect to x and μ.

Suppose that ψ is asymptotically stable for $\mu < 0$ in the sense that n-1 of the Floquet multipliers lie inside the unit circle in the complex plane. As μ increases through zero we assume that a conjugate pair of the multipliers

$$\lambda(\mu) = e^{2\pi\alpha(\mu)},\ \overline{\lambda(\mu)} = e^{2\pi\overline{\alpha(\mu)}}$$

leaves the unit circle with

$$Re\ \alpha(0) = 0,\ Re\ \alpha^{'}(0) > 0$$

while the remaining n-3 stay inside. Hence the periodic solution ψ becomes unstable.

We do not require that ψ is obtained by a bifurcation from an equilibrium as discussed in Chapter I but just consider any periodic solution which loses its stability in the above manner as μ passes through a critical value (assumed to be $\mu = 0$).

2. Reduction to a Mapping Problem

We now obtain the mapping described by the flow near the periodic solution ψ. Let C be the curve described by ψ. Choose a point P on C and let S be the n-1 dimensional hyperplane normal to C at P. Let P be the origin of coordinates $x = (x_{(1)}, \ldots, x_{(n-1)})$ describing S. It is known [16; p.50] that any solution of (2.1) starting on S and close enough to P will return to S for the first time after a time lapse of $2\pi + \epsilon(x)$ where $\epsilon \to 0$ as $x \to 0$. In a small neighborhood of P this defines a mapping T of S into itself having P as a fixed point

$$T : x_1 = Dx + \ldots \tag{2.2}$$

It is known [16; p.163] that the eigenvalues of D are just the $n-1$ previously mentioned Floquet multipliers. Thus for $\mu < 0$, P is an asymptotically stable fixed point which becomes unstable as μ increases through zero.

The existence of a torus will be established by showing that P bifurcates into a closed curve $\bar{C}$, lying in S which is invariant under T, i.e., if q is a point of $\bar{C}$, then $Tq = q_1$ is on $\bar{C}$ (see Fig. 5).

3. Normal Form for the Mapping

By appropriately transforming coordinates we will put the mapping in a certain normal form which is needed in the existence theorem. We assume that a real linear transformation of coordinates has been carried out so that D is in the real form

$$D = \begin{bmatrix} Re\ \lambda(\mu) & -Im\ \lambda(\mu) & \\ & & 0 \\ Im\ \lambda(\mu) & Re\ \lambda(\mu) & \\ & & \\ 0 & & S(\mu) \end{bmatrix}$$

where $S(\mu)$ is a matrix whose eigenvalues $\sigma(\mu)$ satisfy

$$|\sigma(0)| < 1 \tag{2.3}$$

and

$$\sup_{|Y|=1} |S(0)Y| = \sigma_0 < 1$$

where $Y = Col(x_{(3)}, \ldots, x_{(n-1)})$ and $|Y|$ is the ordinary Euclidean length of a vector $(Y, Y)^{\frac{1}{2}}$. Let $z = x_{(1)} + ix_{(2)}$. Then in the z, Y coordinates the mapping is

$$T: \quad \begin{aligned} z_1 &= \lambda(\mu)z + U(z, \bar{z}, Y, \mu) \\ Y_1 &= S(\mu)Y + V(z, \bar{z}, Y, \mu) \end{aligned} \tag{2.4}$$

where U is complex and V is real. The functions U, V, λ and S have five continuous derivatives in a neighborhood of $(z, Y, \mu) = 0$ which we assume to be

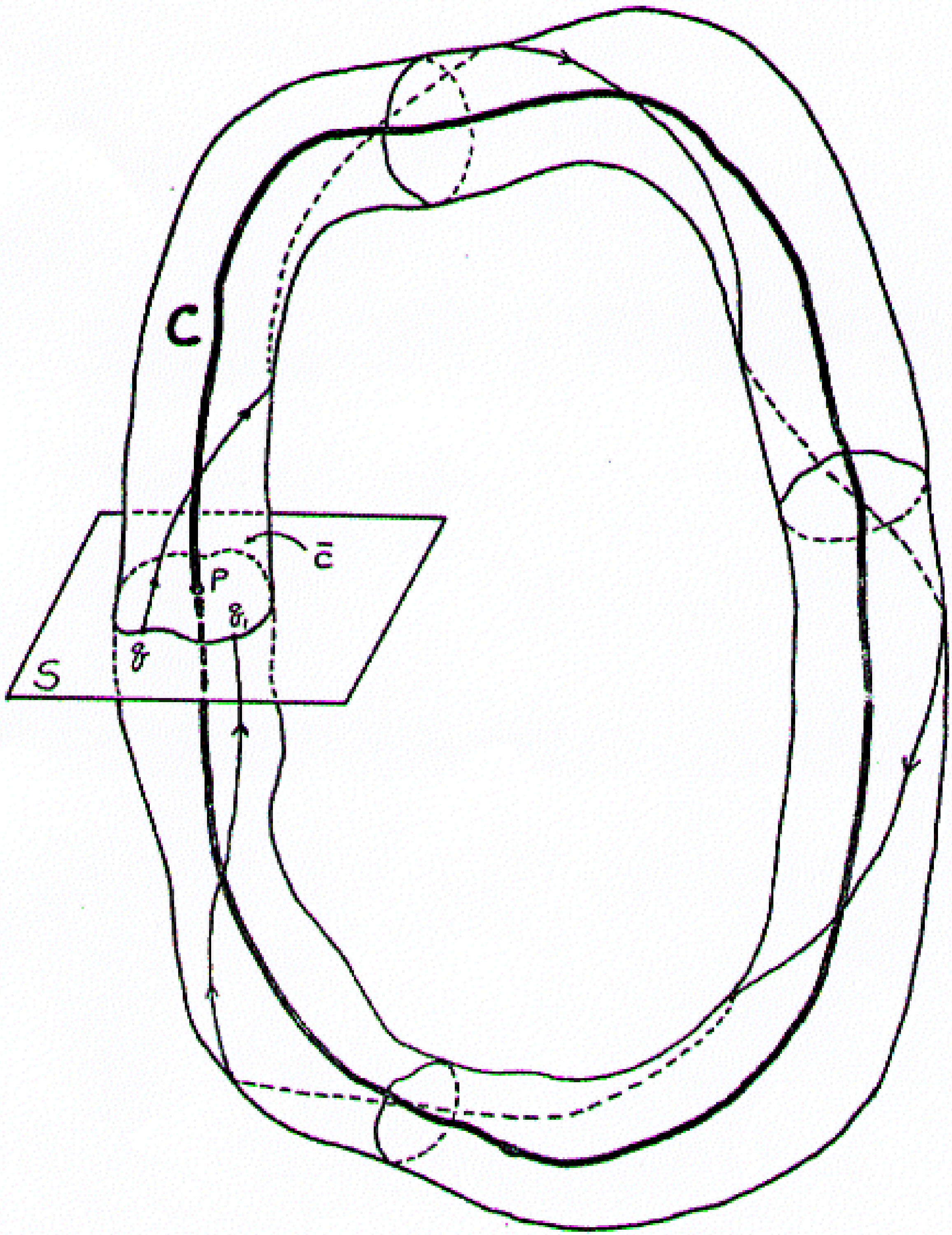
$$|z|^2 + |Y|^2 \leq 2, \qquad |\mu| < \mu_*.$$

We assume U and V to be expanded into Taylor series up to polynomials of degree 4 plus remainder.

In order to motivate the following theorem consider the following example of a mapping

$$\begin{aligned} z_1 &= e^{i+\mu+\beta|z|^2} z \\ Y_1 &= SY + b|z|^2 \end{aligned}$$

FIG. 5



where β is complex, b is a real constant vector and S is a matrix which satisfies (2.3). If $Re\ \beta < 0$, this mapping has the invariant curve

$$|z| = \left(\frac{\mu}{-Re\ \beta}\right)^{\frac{1}{2}}, \quad Y = (S-I)^{-1} b \frac{\mu}{-Re\ \beta}$$

contained in a neighborhood $|z| \leq c\sqrt{\mu},\ |Y| \leq c^2\mu,\ c$ a constant. We now consider the effect of perturbing this example by adding more nonlinear terms. In such an oblate neighborhood Y is small of order μ whereas z is small only to order $\mu^{\frac{1}{2}}$. More precisely consider the neighborhood

$$N : z = a\zeta, \quad Y = a^2 \tilde{Y}$$

where $|\zeta|^2 + |\tilde{Y}|^2 \leq 2$ and a is a small parameter, $0 \leq a \leq 1$. In N the monomial $M_\tau = Y_{(i)}^p z^q \bar{z}^r,\ \tau = 2p + q + r$, satisfies $|M_\tau|_k \leq c(k) a^\tau$ where differentiation is with respect to $\tilde{Y}_{(i)},\ \zeta$, and $\bar{\zeta}$ and $c(k)$ is a constant depending only on k.

Definition. We call τ the weight of M_τ. In the following theorem we will transform the mapping (2.4) into a form similar to the above example in which monomials of weight 2 and 3 will serve to determine the invariant curve in the first approximation while the remaining terms will act as small perturbations. The transformations used are of the type treated by B. Segre [17]. See also A. Kelly [18] for an excellent bibliography.

Theorem 1. If $\lambda^4(0) \neq 1,\ \lambda^3(0) \neq 1$ and (2.3) is satisfied, then there exists a $\mu_0 \leq \mu_*$ such that for $|\mu| < \mu_0$ there exists a transformation

$$\begin{aligned} z &= cw + P(w, \bar{w}, W, c, \mu) \\ Y &= cW + \tilde{P}(w, \bar{w}, W, c, \mu) \end{aligned} \qquad 0 < c \leq 1$$

with P and $\tilde{P}$ polynomials in $w, \bar{w}$, and W, which carries (2.4) into the form

$$\begin{aligned} w_1 &= e^{2\pi\alpha(\mu) + c^2\beta(\mu)|w|^2} w + R_4(w, \bar{w}, W, c, \mu) \\ W_1 &= S(\mu) W + R_3(w, \bar{w}, W, C, \mu) \end{aligned} \qquad (2.5)$$

where R_k are function whose Taylor expansions* about $(w, \bar{w}, W) = 0$ contain no terms of weight $< k$. P and $\tilde{P}$ contain no constant or linear terms. For proper choice of c the transformation is one-to-one in the neighborhood $|w|^2 + |W|^2 \leq 2$.

Proof: In this proof subscripts on functions will carry the same significance as for R_k defined above. We write the first equation of (2.4) as

$$z_1 = \lambda(\mu) z + u(z, \bar{z}, \mu) + v(z, \bar{z}, Y, \mu) + F_4 \qquad (2.6)$$

*By the smoothness assumption made after (2.4), R_k is well defined provided $k \leq 4$.

where u contains quadratic and third order terms in z and $\bar{z}$ and $v = Y'[p(\mu)z + q(\mu)\bar{z}]$ with p, q vectors and prime denoting transpose. We transform (2.6) by

$$z = w + r(\mu)w^k\bar{w}^\ell, \qquad k + \ell = 2, 3 \tag{2.7}$$

to obtain

$$w_1 = \lambda(\mu)w + u(w, \bar{w}, \mu) - g(\mu)w^k\bar{w}^\ell + v(w, \bar{w}, Y, \mu) + \tilde{F}_4$$

where

$$g(\mu) = (\lambda^k\bar{\lambda}^\ell - \lambda)r(\mu), \qquad \lambda = \lambda(\mu)$$

Unless $k = 2,\ l = 1$ we see that $r(\mu)$ may be determined so that (2.7) removes the term $w^k\bar{w}^\ell$ from u. By successive applications of (2.7), we obtain the form

$$w_1 = \lambda(\mu)w + b(\mu)|w|^2w + v(w, \bar{w}, Y, \mu) + G_4. \tag{2.8}$$

The transformation

$$w = \zeta + Y'r(\mu)\zeta^k\bar{\zeta}^\ell, \quad k + \ell = 1 \tag{2.9}$$

with $r(\mu)$ a vector carries (2.8) into

$$\zeta_1 = \lambda(\mu)\zeta + b(\mu)|\zeta|^2\zeta + v(\zeta, \bar{\zeta}, Y, \mu) - Y'g(\mu)\zeta^k\bar{\zeta}^\ell + \tilde{G}_4$$

where

$$g(\mu) = [S'(\mu)\lambda^k\bar{\lambda}^\ell - \lambda I]r(\mu), \quad \lambda = \lambda(\mu)$$

and I is the identity matrix. Since $\lambda(0)$ is on the unit circle and from (2.3) the eigenvalues of $S'(0)$ are inside the unit circle, we see that $r(\mu)$ may be determined for μ sufficiently small such that the term $\zeta^k\bar{\zeta}^\ell$ is removed from v. The mapping now has the form

$$\zeta_1 = \lambda(\mu)\zeta + b(\mu)|\zeta|^2\zeta + H_4. \tag{2.10}$$

The second equation of (2.4) may be written

$$Y_1 = S(\mu)Y + s(\zeta, \bar{\zeta}, \mu) + H_3 \tag{2.11}$$

where s is quadratic in ζ and $\bar{\zeta}$. The transformation

$$Y = X + r(\mu)\zeta^k\bar{\zeta}^\ell, \quad k + \ell = 2$$

reduces (2.11) to

$$X_1 = S(\mu)X + s(\zeta, \bar{\zeta}, \mu) - g(\mu)\zeta^k\bar{\zeta}^\ell + \tilde{H}_3$$

where

$$g(\mu) = [\lambda^k\bar{\lambda}^\ell I - S(\mu)]r(\mu), \quad \lambda = \lambda(\mu).$$

Just as before $r(\mu)$ may be determined for μ sufficiently small. Thus the mapping has the form (2.10) together with

$$X_1 = S(\mu)X + V_3.$$

Using $\lambda(\mu) = e^{2\pi\alpha(\mu)}$, (2.10) may be written

$$\begin{aligned} \zeta_1 &= e^{2\pi\alpha(\mu)}[1 + \beta(\mu)|\zeta|^2]\zeta + H_4 \\ &= e^{2\pi\alpha(\mu)+\beta(\mu)|\zeta|^2}\zeta + \tilde{H}_4 \end{aligned}$$

with $\beta = b/\lambda$. All the previous transformations may be combined into one; $z = \zeta + \ldots, Y = X + \ldots$ where the dots represent polynomials in $\zeta, \bar{\zeta}, X$ without constant and linear terms. It is one-to-one in a neighborhood $|\zeta|^2 + |X|^2 \leq 2c$. The transformation $\zeta = cw$, $X = cW$ then gives the desired results.

4. Existence of an Invariant Curve

Having the mapping in the form (2.5) we may now establish the existence of an invariant curve and hence, according to the discussion of Section 2, an invariant torus of the differential equation (2.1).

Theorem 2. In (2.4) assume

1. $\lambda^4(0) \neq 1,\ \lambda^3(0) \neq 1$ (non-resonance)

2. $\lambda(\mu),\ S(\mu),\ U,\ V \in C^{\ell}(z, \bar{z}, Y, \mu)$ for ℓ an integer ≥ 5, $|\mu| \leq \mu_*$, $|z|^2 + |Y|^2 \leq 2$ and assume $S(\mu)$ satisfies (2.3).

In (2.5) assume

3. $A = 2\pi Re\ \alpha'(0) > 0,\ B = Re\ \beta(0) < 0.$

Let $r,\ 1 \leq r \leq \ell$ be a positive integer. Then there exists $\mu_r \leq \mu_*$ such that for $0 \leq \mu < \mu_r$, (2.4) has an asymptotically stable invariant curve

$$\begin{aligned} z &= a_0\sqrt{\mu}e^{i\theta} + \mu f(\theta,\mu)e^{i\theta} \\ Y &= \mu g(\theta,\mu) \end{aligned} \tag{2.12}$$

where $a_0 = \sqrt{-A/B}$. f and g are defined for all θ and $0 < \mu < \mu_r$, have period 2π in θ,

$$f, g \in C^{r-1}(\theta) \cap Lip^{r-1}(\theta) \cap C(\mu)$$

and

$$|f|_{r-1}, |g|_{r-1} \leq \tilde{c}(r)$$

where $\tilde{c}(r)$ is a constant which depends only on r and differentiation is with respect to θ only. In general $\mu_r \to 0$ as r increases. μ_r depends on the coefficients of terms of degree ≤ 3 in(2.4), i.e., on terms of degree ≤ 3 in the expansion of (2.1) about ψ. μ_r also depends on the constant M_0 defined after (2.1).

Proof. In (2.5) we assume c to be fixed such that the mapping is one-to-one for $|w|^2 + |W|^2 \leq 2$. Note that c depends only on the coefficients of terms of degree ≤ 3 in (2.5).

We restrict attention to a neighborhood $w = az$,

$$W = a^2 Y \tag{2.13}$$

where $|z|^2 + |Y|^2 \leq 2$ and $0 \leq a \leq 1$. Then (2.5) becomes

$$\begin{aligned} z_1 &= e^{2\pi\alpha(\mu)+a^2c^2\beta(\mu)|z|^2} z &+ f_1(z,\bar{z},Y,a,\mu) \\ Y_1 &= S(\mu)Y &+ f_2(z,\bar{z},Y,a,\mu) \end{aligned}$$

and

$$|f_1|_r \leq c(r)a^3, \qquad |f_2|_r \leq c(r)a$$

where differentiation is with respect to z, $\bar{z}$ and Y. Note that the subscripts on the functions are used simply for identification and have nothing to do with "weight" as they did in the previous theorem.

Expanding α, β, and S, we obtain

$$\begin{aligned} z_1 &= \exp[2\pi\alpha(0) + \mu 2\pi\alpha'(0) + a^2c^2\beta(0)|z|^2]z + f_3(z,\bar{z},Y,a,\mu) \\ Y_1 &= S(0)Y + f_4(z,\bar{z},Y,a,\mu) \end{aligned}$$

where

$$|f_3|_r \leq c(r)(\mu^2 + a^2\mu + a^3), \quad |f_4|_r \leq c(r)(a+\mu).$$

Letting

$$\begin{aligned} S(0) &= S, & 2\pi\alpha(0) &= i\tau \\ 2\pi\alpha'(0) &= A + i\tilde{A}, & \beta(0) &= B + i\tilde{B} \end{aligned}$$

and choosing

$$a = \sqrt{\frac{-A\mu}{c^2B}}, \qquad 0 \leq \mu < \mu_0 = \min\left(\mu_*, \frac{c^2B}{-A}\right)$$

we obtain

$$\begin{aligned} z_1 &= \exp[i(\tau + \mu\phi) + \mu A(1 - |z|^2)]z + f_5(z,\bar{z},Y,\mu) \\ Y_1 &= SY + f_6(z,\bar{z},Y,\mu) \end{aligned} \tag{2.14}$$

where

$$\phi = \tilde{A} - \frac{A\tilde{B}}{B}|z|^2$$

and

$$|f_5|_r \leq c(r)\mu^{\frac{3}{2}}, \quad |f_6|_r \leq c(r)\mu^{\frac{1}{2}}.$$

The unperturbed mapping ($f_5, f_6 \equiv 0$) has an invariant curve $|z| = 1,\ Y \equiv 0$. We introduce coordinates in a small toriod (see Fig. 6) surrounding this curve.

$$z = (1+\rho)e^{i\theta},\ |\rho|^2 + |Y|^2 \leq \frac{1}{4}. \tag{2.15}$$

The first equation of (2.14) becomes

$$\rho_1 = (1-2\mu A)\rho - 3\mu A\rho^2 - \mu A\rho^3 + f_7(\rho, \theta, Y, \mu)$$

$$\theta_1 = \theta + \tau + \mu\left[\tilde{A} - \frac{A\tilde{B}}{B}(1+\rho)^2\right] + f_8(\rho, \theta, Y, \mu)$$

where

$$|f_7|_r, \quad |f_8|_r \leq c(r)\mu\frac{3}{2}.$$

Letting $\rho = \mu^{\frac{1}{2}}u,\ Y = \mu^{\frac{1}{2}}v$, we finally obtain (2.16)

$$\begin{aligned} u_1 &= (1-2\mu A)u &&+ \mu[g(\theta) + \tilde{G}(\theta, u, v, \mu)] \\ \theta_1 &= \theta + \tau &&+ \mu[f_0 + \tilde{F}(\theta, u, v, \mu)] \\ v_1 &= Sv &&+ h(\theta) + \tilde{H}(\theta, u, v, \mu) \end{aligned} \tag{2.17}$$

where f_0 is a constant, all functions have period 2π in θ and

$$|g|_r,\ |h|_r \leq \hat{c}(r);\ |\tilde{G}|_r,\ |\tilde{H}|_r,\ |\tilde{F}|_r \leq \tilde{c}(r, K)\mu^{\frac{1}{2}} \tag{2.18}$$

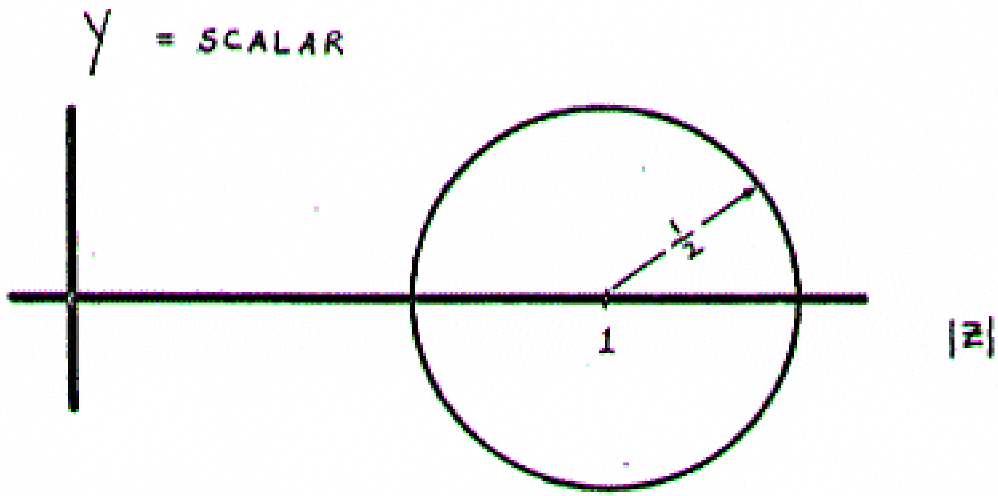
for

$$|u|^2 + |v|^2 \leq K^2, \qquad 0 \leq \mu \leq \frac{1}{4K^2}$$

where differentiation is with respect to all variables except μ. $\hat{c}$ and $\tilde{c}$ are constants which depend only on the arguments shown. See Fig. 7 for a description of the toroidal neighborhood in the $W,\ w$ coordinates. Finding an invariant curve $u = u(\theta, \mu),\ v = v(\theta, \mu)$ of (2.17) is equivalent to solving the functional equation

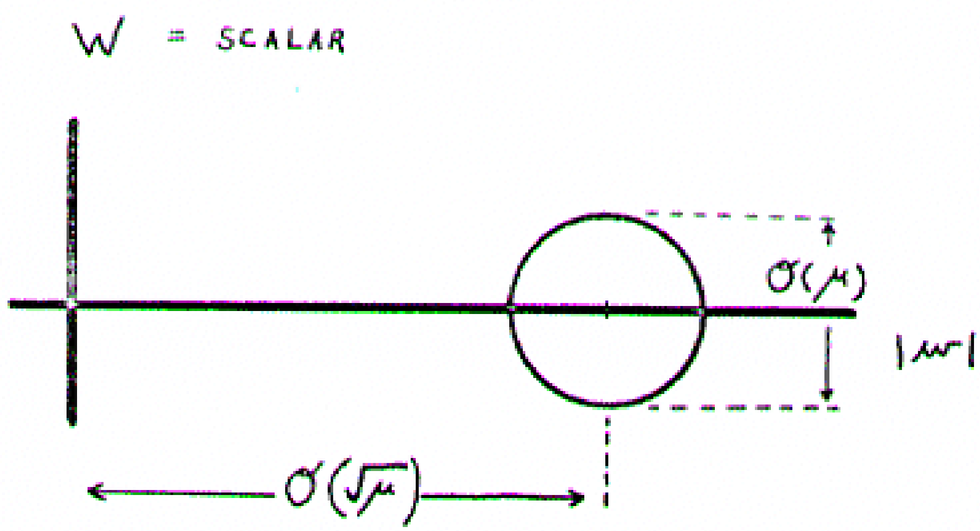
$$\begin{aligned} u(\theta_1) - (1-2\mu A)u(\theta) &= \mu G(\theta, u, v, \mu) \\ v(\theta_1) - Sv(\theta) &= H(\theta, u, v, \mu) \\ \theta_1 &= \theta + \tau + \mu F(\theta, u, v, \mu) \end{aligned} \tag{2.19}$$

FIG. 6



In each figure the toroidal neighborhood is obtained by rotating the circle about the vertical axis.

FIG. 7



where $F = f_0 + \tilde{F},\ f_0$ a constant, $G = g(\theta) + \tilde{G}$ and $H = h(\theta) + \tilde{H}$ satisfy (2.18). Theorem 3 which follows guarantees the existence of a unique solution $u = u(\theta, \mu),\ v = v(\theta, \mu)$ of (2.19) satisfying all our requirements. The form (2.12) is obtained by applying the transformations (2.16), (2.15), (2.13) and finally, the transformation of Theorem 1.

We now prove the stability of the invariant curve. If $u(\theta, \mu),\ v(\theta, \mu)$ is the solution of (2.19) for $0 < \mu < \bar{\mu}_r$, let $u = u(\theta, \mu) + \delta u,\ v = v(\theta, \mu) + \delta v$ in (2.17) to obtain the nonlinear variational mapping

$$\begin{aligned} \delta u_1 &= (1 - 2\mu A)\delta u + \mu(G_u - u' F_u)\delta u + \mu(G_v - u' F_v)\delta v \\ \delta v_1 &= S\delta v + (H_u - \mu v' F_u)\delta u + (H_v - \mu v' F_v)\delta v \end{aligned}$$

where prime is $d/d\theta$ and F_u etc. are evaluated at $u(\theta, \mu) + \epsilon\delta u,\ v(\theta, \mu) + \epsilon\delta v,\ 0 < \epsilon < 1$. From (2.18) we see that the derivatives of F, G, and H with respect to u and v are $\leq$ (constant) $\mu^{\frac{1}{2}}$ uniformly in a small tube $R:\ |\delta u|^2 + |\delta v|^2 \leq$ constant. Hence in R for μ sufficiently small, $0 < \mu < \mu_r \leq \bar{\mu}_r$, the above mapping is a contraction. This completes the proof of the theorem.

5. Solution of Functional Equation

The above stability argument is based on the principle of contraction, i.e., the eigenvalues of the linear part of the mapping are less than unity in modulus. This same principle will now be utilized to prove existence of a solution of equation (2.19).

Theorem 3. In (2.19) assume

1. $A > 0,\ |Sv| \leq \sigma_0 |v|,\ 0 < \sigma_0 < 1\ \sigma_0$ constant
2. $F, G, H \in C^r(\theta, u, v) \cap C(\mu),\ r \geq 1,$

for $0 \leq \mu < \mu_0$ and $|u|^2 + |v|^2 \leq K^2,\ K$ sufficiently large and assume these functions satisfy (2.18) and have period 2π in θ.

Then there exists $\mu_r \leq \mu_0$ such that for $0 < \mu < \mu_r$ (2.19) has a unique solution $u(\theta, \mu),\ v(\theta, \mu)$ having period 2π in θ.

$$u, v \in C^{r-1}(\theta) \cap Lip^{r-1}(\theta) \cap c(\mu)$$

and $|u|_{r-1},\ |v|_{r-1} \leq K$ uniformly in μ where differentiation is with respect to θ only. μ_r depends on A, σ_0, r and the constants $\hat{c}$ and $\tilde{c}$ in (2.18). In general, $\mu_r \to 0$ as r increases.

Proof. Define $b = \min(2A, 1 - \sigma_0)$. Then $0 < b < 1$. Also define

$$K = b^{-r-1} c_0(r)\ \hat{c}(r) 2^{\delta(r+1)+3}$$

where $c_0(r)$ and δ are defined in Lemma 2 and $\hat{c}(r)$ comes from (2.18). We will construct successive approximations u_n and v_n with $u_0 \equiv 0, v_0 \equiv 0$. For convenience let $w_n = \binom{u_n}{v_n}$. As an induction assumption suppose w_{n-1} satisfies

$$w_{n-1} \in C^r(\theta) \cap C(\mu), \qquad 0 < \mu < \mu_0. \tag{2.20}$$

$$|w_{n-1}|_r < K \tag{2.21}$$

Inserting this approximation in (2.19) we obtain

$$\begin{aligned} u(\theta_1) \quad - \quad (1-2\mu A)u(\theta) \quad &= \quad \mu G_n(\theta,\mu) \\ v(\theta_1) \quad - \quad Sv(\theta) \quad &= \quad H_n(\theta,\mu) \end{aligned} \tag{2.22}$$

$$\theta_1 = \theta + \tau + \mu F_n(\theta,\mu)$$

to be solved for $w_n(\theta,\mu)$, where

$$F_n(\theta,\mu) = F(\theta, w_{n-1}(\theta,\mu),\mu) \tag{2.23}$$

and so on. We now verify the conditions of Lemma 1.* We have $F_n, G_n, H_n \in C^r(\theta) \cap C(\mu)$ for $0 < \mu < \mu_0' = \min(\mu_0, 1/4K^2)$. Also if we define $m = \inf_{\theta,\mu} F_{n\theta}$ we have, using (2.18), $b + rm \geq b - cr\mu^{\frac{1}{2}} > b/2$ for μ small, $0 < \mu < \mu' \leq \mu_0'$, where c depends on K and $\tilde{c}(r,K)$. Unless $F \equiv 0$ we see that $\mu' \to 0$ as r increases. By Lemma 1,* (2.22) has a solution $w_n(\theta,\mu)$ which satisfies (2.20) for $0 < \mu < \mu'' \leq \mu'$. To verify (2.21) for w_n define $Q_n = \binom{G_n}{H_n}$ and apply Lemma 2*.

$$|w_n|_r \leq \frac{c_0(r)|Q_n|_r[1 + |F_{n\theta}|_{r-1}]^{\delta(r)}}{(b+rm)^{r+1}}.$$

From (2.18) we have

$$|F_{n\theta}|_{r-1} \leq c_1\mu^{\frac{1}{2}} < 1$$

$$|Q_n|_r \leq 2\hat{c}(r) + c_1\mu^{\frac{1}{2}} < 4\hat{c}(r)$$

for μ small, $0 < \mu < \tilde{\mu} < \mu''$. Thus

$$|w_n|_r < (2/b)^{r+1} 4\hat{c}(r)c_0(r)2^{\delta(r)} = K.$$

Repeating this procedure we generate a sequence $\{w_n\}$ which satisfies (2.20) and (2.21) for $0 < \mu < \tilde{\mu}$.

To show convergence define $\delta u_{n+1}(\theta) = u_{n+1}(\theta,\mu) - u_n(\theta,\mu)$, $\delta G_{n+1} = G_{n+1} - G_n$ and so on. Then from (2.22)

$$\begin{aligned} \delta u_{n+1}(\theta_1) - (1-2\mu A)\delta_{n+1}u(\theta) \quad &= \quad \mu[\delta G_{n+1} - u_n'\delta F_{n+1}] \\ \delta v_{n+1}(\theta_1) - S\delta v_{n+1}(\theta) \quad &= \quad \delta H_{n+1} - v_n'\delta F_{n+1}. \end{aligned}$$

*This lemma is proved later.

$$\theta_1 = \theta + \tau + \mu F_{n+1}(\theta, \mu)$$

Defining Q_n as before, we have from the first estimate of Lemma 2

$$|\delta w_{n+1}|_0 \leq \frac{2}{b}\{|Q_{n+1}|_0 + |w_n|_1 \; |\delta F_{n+1}|_0\}.$$

Using (2.21), (2.23) and (2.18) we have

$$|\delta w_{n+1}|_0 \leq \bar{c}\mu^{\frac{1}{2}}|\delta w_n|_0 \leq \frac{1}{2}|\delta w_n|_0$$

for μ sufficiently small, $0 < \mu < \bar{\mu} \leq \tilde{\mu}$, i.e., uniform convergence of the w_n. For the θ derivative $w_n^{(\lambda)}$ $0 \leq \lambda \leq r-1$ we have from Lemma 3 (with $m = 1,\ x_1 = \theta,\ \ell = r$) and (2.21)

$$|w_{n+p}^{(\lambda)} - w_n^{(\lambda)}|_0 \leq 2K\ c(\lambda, r)|w_{n+p} - w_n|_0^{1-(x/r)}.$$

Hence $\{w_n\}$ converges to a solution w of (2.19) and $w(\theta, \mu) \in C^{r-1}(\theta) \cap C(\mu)$. The Lipschitz condition follows from passing to the limit in

$$|w_n^{(\lambda)}(\theta', \mu) - w_n^{(\lambda)}(\theta, \mu)| \leq |w_n|_{\lambda+1}|\theta' - \theta| \leq K|\theta' - \theta|.$$

Uniqueness follows by applying the convergence argument to the sequence $\{w_1, w_2, w_1, w_2, \ldots\}$ where w_1 and w_2 are two supposed solutions. Finally we note that $|w|_{r-1} \leq K$. Thus the theorem is proved with $\mu_r = \bar{\mu}$.

Lemma 1. (Solution of Linear Functional Equation)
Consider the equation

$$\begin{aligned} u(\theta_1) \quad &- \quad (1 - 2\mu A)u(\theta) &= \quad \mu G(\theta) \\ v(\theta_1) \quad &- \quad Sv(\theta) &= \quad H(\theta) \end{aligned} \tag{2.25}$$

$$\theta_1 = \theta + \tau + \mu F(\theta, \mu) \tag{2.26}$$

where τ is constant, $0 < \mu < \mu',\ A > 0$ and $|Sv| \leq \sigma_0|v|$ for some $\sigma_0,\ 0 < \sigma_0 < 1$. Define $b = \min(2A, 1 - \sigma_0)$ and $m = \inf_{\theta,\mu} F_\theta$. Assume

1. $b + rm > 0$ for some integer $r \geq 0$

2. $F, G, H \in C^r(\theta) \cap C(\mu)$ and have period 2π in θ.

Then for

$$0 < \mu < \mu'' = \min\left[\frac{1}{2A}, \frac{1}{|m|}, \mu', 1\right]$$

(2.25-6) has a solution $u(\theta, \mu), v(\theta, \mu) \in C^r(\theta) \cap c(\mu)$ having period 2π in θ.

Remark. Since F is periodic in θ, m will be ≤ 0 and hence if 1 is true for some integer $r = r_0$ then it is true for any smaller integer $r < r_0$.

Proof. Each equation has the form

$$Y(\theta_1) - D(\mu)Y(\theta) = P(\theta,\mu) \tag{2.27}$$

with

$$|D(\mu)Y| \le (1-\mu b)|Y| \tag{2.28}$$

where $D(\mu)$ represents either $S(\mu)$ or $1-2\mu A$. Let I be any closed subinterval of $(0,\mu'')$. We may solve (2.26) for the negative iterates $\theta_{-1},\theta_{-2},\dots$ where $\theta_{-n+1} = \theta_{-n} + \tau + \mu F(\theta_{-n},\mu)$ since

$$\left|\frac{d\theta_1}{d\theta}\right| \ge 1+\mu m > 0 \text{ for } \mu \in I.$$

We allow D to depend on θ also and write

$$Y(\theta) - D(\theta_{-1},\mu)Y(\theta_{-1}) = P(\theta_{-1},\mu)$$

$$D(\theta_{-1},\mu)Y(\theta_{-1}) - D(\theta_{-1},\mu)D(\theta_{-2},\mu)Y(\theta_{-2}) = D(\theta_{-1},\mu)P(\theta_{-2},\mu)$$
$$\vdots$$

Adding and passing to the limit

$$Y(\theta,\mu) = \sum_{j=1}^{\infty}\prod_{k=1}^{j-1} D(\theta_{-k},\mu)P(\theta_{-j},\mu)$$

where $\prod_{k=1}^{0} \equiv 1$. From (2.28) we see that the series converges uniformly for all θ and for $\mu \in I$. If $r=0$ we are finished. If $r=1$ we differentiate term by term using

$$\frac{d}{d\theta}\theta_{-j} = \frac{d\theta_{-j}}{d\theta_{-j+1}}\cdots\frac{d\theta_{-1}}{d\theta}$$

and

$$\left|\frac{d}{d\theta}\theta_{-j}\right| \le \left(\frac{1}{1+\mu F_\theta}\right)^j.$$

The differentiated series converges uniformly for the same values of θ and μ since for $|y|=1$,

$$\frac{|D(\mu)y|}{1+\mu F_\theta} \le \frac{1-\mu b}{1+\mu m} \le c < 1 \tag{2.29}$$

where c depends only on I, b and m. Hence $Y(\theta,\mu) \in C^1(\theta)\cap C(\mu)$. If $r>1$ assume for $s \le r$ that $Y \in C^{s-1}(\theta)\cap C(\mu)$. Then differentiate (2.27) $s-1$ times with respect to θ

$$Y^{(s-1)}(\theta_1) - \frac{D(\mu)Y^{(s-1)}(\theta)}{(1+\mu F_\theta)^{s-1}} = \tilde{P}(\theta,\mu) \tag{2.30}$$

where $\tilde{P}$ contains derivatives $Y^{(\lambda)}$, $0 \le \lambda \le s-2$ considered as known functions. Thus $\tilde{P} \in C^1(\theta)\cap C(\mu)$. We now consider this as an equation for $Y^{(s-1)}$ and repeat the above procedure with $D(\mu)[1+\mu F_\theta]^{1-s}$ in place of $D(\mu)$. Using the fact that for $x > -1, (1+x)^s \geq 1 + sx$ we see that the corresponding condition (2.29) is

$$\frac{|D(\mu)y|}{[1+\mu F_\theta]^s} \leq \frac{1-\mu b}{1+\mu sm} \leq c_1 < 1$$

hence the solution of (2.30) is in $C^1(\theta) \cap C(\mu)$, i.e.,

$$Y(\theta,\mu) \in C^r(\theta) \cap C(\mu). \qquad \text{Q.E.D.}$$

Lemma 2. "A-priori" Estimates. Consider equation (2.25-6) of Lemma 1 and assume everything in the statement of that lemma. Letting $Q = \binom{G}{H}$ we then have the following estimates for the solution $w = \binom{u}{v}$ of (2.25-6).

$$|w|_0 \leq \frac{2}{b}|Q|_0$$

$$|w|_1 \leq \frac{2|Q|_1}{(b+m)^2}$$

and in general for $1 \leq s \leq r$, s an integer,

$$|w|_s \leq \frac{c_0(s)|Q|_s[1+|F_\theta|_{s-1}]^{\delta(s)}}{(b+sm)^{s+1}}$$

where, as before, $m = \inf_{\theta,\mu} F_\theta$. $c_0(s)$ is a constant which depends only on s. $\delta(1) = 0$ and $\delta(n+1) = \delta(n) + n$.

Proof. We will use the fact that for $x > -1$ and $n \geq 0$ an integer $(1+x)^n \geq 1 + nx$. (2.30)

From the first equation of (2.25)

$$|u(\theta_1)| \leq (1-\mu b)|u|_0 + \mu|G|_0$$

for all θ_1 and in particular at the point where $|u|_0$ is assumed. Thus $|u|_0 \leq b^{-1}|G|_0$.

Similarly for v

$$|v|_0 \leq (1-b)|v|_0 + |H|_0$$

and therefore $|v|_0 \leq b^{-1}|H|_0$. Adding we obtain the first estimate. For the second estimate we differentiate to obtain

$$u^{(1)}(\theta_1)(1+\mu F_\theta) - (1-2\mu A)u^{(1)}(\theta) = \mu G_\theta.$$

Noting the restrictions imposed on μ in Lemma 1,

$$(1+\mu m)|u^{(1)}|_0 \leq (1-\mu b)|u^{(1)}|_0 + \mu|G|_1$$

and similarly for v. Thus

$$|w^{(1)}|_0 \leq 2(b+m)^{-1}|Q|_1.$$

Using $b+m \leq b < 1$ we have $|w|_1 \leq \dfrac{2|Q|_1}{(b+m)^2}$. To prove the general statement for $s \geq 2$ assume it true for $s-1$ and differentiate the first equation in (2.25) s times

$$u^{(s)}(\theta_1)(1+\mu F_\theta)^s - (1-2\mu A)u^{(s)}(\theta) = \mu[G^{(s)} + g_s]$$

where g_s is a linear combination of terms of the form

$$u^{(p)} \prod_{i=1}^{q} F^{(\lambda_i)} \tag{2.31}$$

where $0 \leq p,\ q \leq s-1$ and $1 \leq \lambda_i \leq s$.

At the point θ_1 where $|u^{(s)}(\theta_1)| = |u^{(s)}|_0$ we have, using (2.30), $1+\mu sm \leq (1+\mu F_\theta)^s$ and therefore

$$|u^{(s)}|_0(1+\mu sm) \leq (1-\mu b)|u^{(s)}|_0 + \mu[|G|_s + |g_s|_0]$$

or

$$|u^{(s)}|_0 \leq \frac{|Q|_s + |g_s|_0}{b+sm}.$$

Using the estimate for $s-1$ and the fact that $b+sm \leq \ldots \leq b+m \leq b \leq 1$, we obtain

$$|u^{(s)}|_0 \leq \frac{c(s)|Q|_s[1+|F_\theta|_{s-1}]^{\delta(s)}}{(b+sm)^{s+1}}.$$

In estimating g_s, since the maximum number of terms in the product in (2.31) is $s-1$, we choose $\delta(s) = \delta(s-1)+s-1$. Repeating this for the second equation in (2.25) and adding we obtain the desired result. Q.E.D.

We now establish an inequality which is useful in proving the convergence of the derivatives $g_n^{(\lambda)}$, $0 \leq \lambda < l$, of a sequence whenever the original sequence $\{g_n\}$ converges and the sequence $\{g_n^{(l)}\}$ is uniformly bounded.

Lemma 3. Suppose $f(x) = (f_1, \ldots, f_n) \in C^\ell(x)$ for all $x = (x_1, \ldots, x_m)$. Then for $0 \leq \lambda < \ell$

$$|D^\lambda f|_0 \leq c(\lambda, \ell)|f|_0^{1-(\lambda/\ell)}|f|_\ell^{\lambda/\ell}$$

for positive integers λ and ℓ, where D_λ is any partial derivative of order λ. c depends on m and n but not f.

Proof. We define the norm

$$[f]_r = \sup_{x,i} |D^r f_i|$$

where the supremum is taken over all r^{th} order derivatives. We introduce the smoothing operator (J. Moser [19])

$$T_N f(x) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K_N(x - x')f(x')dx'_1 \dots dx'_m \tag{2.33}$$

valid for any $N > 0$ and all x. The kernel is given by

$$K_N(x) = N^m K(Nx_1) \dots K(Nx_m)$$

where $K(t) \in C^\infty$ for all t and $K(t) \equiv 0$ for $|t| > 1$, and

$$\int_{-\infty}^{\infty} t^k \; K(t)dt = \begin{cases} 1 & \text{for } k = 0 \\ 0, & 0 < k < s \end{cases}$$

with s an integer to be fixed later. K is constructed [19] from a C^∞ function ϕ. We assume that the mode of construction as well as ϕ are fixed. Therefore K depends only on t and s. For (2.33) the following inequalities can be verified for $\sigma, \; s, \; r \geq 0$:

$$[T_N f]_{r+\sigma} \leq c_1(\sigma, s)N^\sigma [f]_r \tag{2.34}$$

$$[f - T_N f]_r \leq c_2(s) \; N^{-s}[f]_{r+s} \tag{2.35}$$

where c_1 and c_2 depend on m but not on N and f. Letting $N = ([f]_\ell/[f]_0)^{1/\ell}$, we use (2.35) with $r = \lambda, \; s = \ell - \lambda$ and (2.34) with $r = 0, \; \sigma = \lambda$ to obtain

$$[f]_\lambda \leq \tilde{c}(\lambda, \ell)[f]_0^{1-(\lambda/\ell)}[f]_\ell^{\lambda/\ell}.$$

The final result follows using

$$c_3(n)|D^p f|_0 \leq [f]_p \leq |f|_p$$

where c_3 is a constant depending only on n. Q.E.D.

CHAPTER III
BIFURCATION - INVARIANT SURFACE METHOD

1. Introduction

In this chapter we treat the bifurcation problem by reducing it to a problem of perturbation of an invariant surface of a vector field.

We consider the differential equation

$$\frac{dx}{dt} = F(x, \mu) \tag{3.1}$$

where x and F are real n-vectors, $n \geq 3$, and μ is a real parameter. Suppose that for all sufficiently small μ, (3.1) has a periodic solution $x = \psi(t, \mu)$ which is asymptotically stable for $\mu < 0$ in the sense that $n-1$ of the Floquet exponents have negative real parts, i.e., $n-1$ of the Floquet multipliers lie inside the unit circle in the complex plane. As μ increases through zero we assume that a conjugate pair of these exponents $\alpha(\mu)$ and $\bar{\alpha}(\mu)$ crosses the imaginary axis and enters the right half plane with

$$Re\ \alpha(0) = 0, \qquad Re\ \alpha'(0) > 0 \tag{3.2}$$

while the remaining $n-3$ stay in the left half plane.

We assume that $F(x, \mu)$ has $\rho \geq 5$ continuous derivatives with respect to x and μ in a neighborhood of ψ and in that neighborhood $|F|_\rho \leq M_0$ where differentiation is with respect to x and μ and M_0 is constant.

2. Reduction to Normal Form

We will now transform equation (3.1) into a form suitable for discussing the behavior of solutions near the trajectory ψ.

Following Nemytskii and Stepanov [20] we introduce the arc length s along ψ as the new independent variable and $y_1, \ldots y_{n-1}$ as new dependent variables. The y_i are coordinates in the hyperplane S_s normal to ψ at the point s. Without loss of generality we assume that the trajectory ψ has arc length 2π, i.e., $\omega_1 = 1$. In these new coordinates (3.1) takes the real form

$$\dot{y} = A(s, \mu)y + \tilde{F}(s, y, \mu) \tag{3.3}$$

where $\tilde{F}(s, 0, \mu) \equiv 0$, $\tilde{F}_y(s, 0, \mu) \equiv 0$. The functions are periodic in s with period 2π and $\cdot$ means d/ds.

Note: If we let S_0 be the hyperplane S discussed at the beginning of Chapter II, then the mapping $T : y_0 \to y$ is just $y(2\pi, y_0, \mu)$ where $y(s, y_0, \mu)$ is the solution of (3.3) which satisfies $y(0, y_0, \mu) = y_0$.

It is known [21] that if $\alpha(0) \neq n(i/2), n =$ odd integer, there exists a real periodic matrix $Q(s, \mu)$ having period 2π such that the change of variables

$$y = Q(s, \mu)u$$

reduces (3.3) to

$$\dot{u} = C(\mu)u + G(s, u, \mu) \tag{3.4}$$

where G has period 2π in s, $G(s, 0, \mu) \equiv 0$, $G_u(s, 0, \mu) \equiv 0$ and $C\mu$ is in the real form

$$C(\mu) = \begin{bmatrix} Re\ \alpha(\mu) & -Im\ \alpha(\mu) & \\ Im\ \alpha(\mu) & Re\ \alpha(\mu) & \\ & & S(\mu) \end{bmatrix}.$$

The roots of $C(\mu)$ are the $n-1$ non-zero Floquet exponents associated with ψ. By assumption the roots $\sigma(0)$ of $S(0)$ satisfy $Re\ \sigma(0) < \sigma_0 < 0$ and $Re\ \alpha(\mu)$ satisfies (3.2). We also assume that a linear transformation has been performed such that $(Y, S, (0)Y) \leq \sigma_0(Y, Y)$ where

$$Y = col(u_3, \ldots, u_{n-1}).$$

If we let

$$z = u_1 + iu_2, \qquad \bar{z} = u_1 - iu_2$$

(3.4) becomes

$$\begin{aligned} \dot{z} &= \alpha(\mu)z + G_1(s, z, \bar{z}, Y, \mu) \\ \dot{Y} &= S(\mu)Y + G_2(s, z, \bar{z}, Y, \mu) \end{aligned} \tag{3.5}$$

where G_1 (complex) and G_2 (real) have period 2π in s, contain no linear terms in $z, \bar{z}$, and Y and $G_i(s, 0, 0, 0, \mu) \equiv 0$. The functions $\alpha(\mu)$, $S(\mu)$, and G_i have $\rho \geq 5$ continuous derivatives in a neighborhood of $(z, y, \mu) = 0$ which we assume to be $|\mu| \leq \mu_*$, $|z|^2 + |Y|^2 \leq 2$. We assume the G_i to be expanded into Taylor series in $z, \bar{z}$, and Y_i up to terms of degree 4 plus remainder. As in Chapter II we will transform (3.5) into a form in which terms of weight ≤ 3 occur in a particular way. We recall that for the term $c(s, \mu)\ z^k \bar{z}^\ell Y_i^m$ the weight τ is defined to be $\tau = k + \ell + 2m$. This problem of removing nonlinear terms from a differential equation by coordinate transformation has been treated by many authors. (See G.D. Birkhoff [22] and the references given in Chapter II prior to Theorem 1.)

Theorem 4. In (3.5) assume that

1. $Re\ \alpha(0) = 0, \quad Re\ \alpha'(0) > 0$

2. the eigenvalues of $S(0)$ satisfy $Re\ \sigma(0) < 0$

3. $\alpha(0) \neq \nu\, i/3,\ \nu\, i/4,\ \nu$ an integer

4. $G_i \in C^4(s, z, \bar{z}, Y, \mu)$

Then there exists a $\mu_0 \leq \mu_*$ depending only on $\alpha(\mu)$ and $S(\mu)$ such that for $|\mu| < \mu_0$, there exists a transformation

$$\begin{aligned} z &= cw + P(s, w, \bar{w}, W, c, \mu) \\ Y &= cW + \tilde{P}(s, w, \bar{w}, W, c, \mu), \end{aligned} \tag{3.6}$$

c constant, $0 < c \leq 1$, which carries (3.5) into the form

$$\begin{aligned} \dot{w} &= \alpha(\mu)w + c^2\beta(\mu)|w|^2 w + R_4(s, w, \bar{w}, W, c, \mu) \\ \dot{W} &= S(\mu)W + R_3(s, w, \bar{w}, W, c, \mu) \end{aligned} \tag{3.7}$$

where P and $\tilde{P}$ are polynomials in $w, \bar{w}$ and W having no terms of degree zero or one, and the R_k are functions whose Taylor expansions about $(w, \bar{w}, W) = 0$ contain no terms having weight* $< k$. The transformation is one-to-one in a neighborhood $|w|^2 + |W|^2 \leq 2$ and all functions of s have period 2π in s. The transformation is unique once c has been fixed. In case 3 is violated, the following is true.

If $\alpha(0) = 0$ then (3.7) should read

$$\begin{aligned} \dot{w} &= \alpha(\mu)w + c[f(\mu)w^2 + g(\mu)w\bar{w} + h(\mu)\bar{w}^2] + R_3 \\ \dot{W} &= S(\mu)W + \tilde{R}_3 \end{aligned} \tag{3.8}$$

where R_3 and $\tilde{R}_3$ are functions of $(s, w, \bar{w}, W, c, \mu)$.**

If $\alpha(0) = i/4$ then (3.7) should read

$$\begin{aligned} \dot{w} &= \alpha(\mu)w + c^2[\beta(\mu)|w|^2 w + \gamma(\mu)e^{is}\bar{w}^3] + R_4 \\ \dot{W} &= S(\mu)W + R_3 \end{aligned} \tag{3.9}$$

where R_3 and R_4 are functions of $(s, w, \bar{w}, W, c, \mu)$.

Proof. Throughout, subscripts will have the same significance as above,i.e., R_k contains no terms of weight $< k$. We write the first equation of (3.5) as

$$\dot{z} = \alpha(\mu)z + U^n(s, z, \bar{z}, \mu) + \dots \tag{3.10}$$

* "Weight" is defined in Chapter II, Section 3.

** For this assertion, hypothesis 4 may be replaced by $G_i \in C^3$.

where U^n contains all the terms of degree $\rho,\ 2 \leq \rho \leq n$, in $z, \bar{z}$. We write U^n displaying a particular n^{th} degree term

$$U^n = \tilde{U}(s, z, \bar{z}, \mu) + b(s, \mu) z^k \bar{z}^\ell,\ k + \ell = n$$

where

$$b(s, \mu) = \sum_{m=-\infty}^{\infty} b^{(m)}(\mu) e^{ims}.$$

Define $d(s, \mu) \equiv 0$ if $(k-1)\alpha(0) + \ell\bar{\alpha}(0) \neq ni,\ n$ an integer, and $d(s, \mu) = b^{(-n)}(\mu)e^{-ins}$ if $(k-1)\alpha(0) + \ell\bar{\alpha}(0) = ni$. Then for sufficiently small μ the equation

$$\dot{\gamma} + [(k-1)\alpha(\mu) + \ell\bar{\alpha}(\mu)]\gamma = b(s, \mu) - d(s, \mu)$$

has a periodic solution $\gamma(s, \mu)$ having period 2π in s. The real part of the bracketed term is $\neq 0$ for $\mu \neq 0$ and therefore $\gamma(s, \mu)$ is uniquely determined. The transformation

$$z = w + \gamma(s, \mu) w^k \bar{w}^\ell \tag{3.11}$$

then carries (3.10) into

$$\dot{w} = \alpha(\mu) w + \tilde{U}(s, w, \bar{w}, \mu) + d(s, \mu) w^k \bar{w}^\ell + \ldots$$

where the dots contain no terms of degree $\leq n$ in w and $\bar{w}$. If $\alpha(0) \neq \nu\ i/3,\ \nu\ i/4$ then successive applications of (3.11) for $n = 2$ and 3 leads to the form $\dot{w} = \alpha(\mu)w + \beta(\mu)|w|^2 w + \ldots$ we write this as

$$\dot{w} = \alpha(\mu) w + \beta(\mu)|w|^2 w + Y \cdot v(s, w, \bar{w}, \mu) + G_4 \tag{3.12}$$

where $v = a(s, \mu)w + b(s, \mu)\bar{w}$, a and b vectors. Consider the vector differential equation

$$\dot{\gamma} + [S'(\mu) + \{(k-1)\alpha(\mu) + \ell\bar{\alpha}(\mu)\}I]\gamma = h(s, \mu)$$

where prime denotes transpose, I is the identity matrix and h is periodic in s. For μ sufficiently small this equation has a unique periodic solution $\gamma(s, \mu)$ and the transformation

$$w = \zeta + Y \cdot \gamma(s, \mu) \zeta^k \bar{\zeta}^\ell \tag{3.13}$$

carries (3.12) into

$$\dot{\zeta} = \alpha(\mu)\zeta + \beta(\mu)|\zeta|^2\zeta + Y \cdot [v(s, \zeta, \bar{\zeta}, \mu) - h(s, \mu)\zeta^k \bar{\zeta}^\ell] + \tilde{G}_4.$$

Applying this for $k = 1,\ \ell = 0,\ h = a(s, \mu)$ and $k = 0,\ \ell = 1,\ h = b(s, \mu)$ we obtain the form

$$\dot{\zeta} = \alpha(\mu)\zeta + \beta(\mu)|\zeta|^2\zeta + H_4.$$

After carrying out the above transformations on the second equation of (3.5) we write that equation as

$$\dot{Y} = S(\mu)Y + v(s, \zeta, \bar{\zeta}, \mu) + \tilde{H}_3 \tag{3.14}$$

where $v = f_1(s,\mu)\zeta^2 + f_2\zeta\bar{\zeta} + f_3\bar{\zeta}^2$. For μ sufficiently small the vector differential equation

$$\dot{\gamma} + [\{k\alpha(\mu) + \ell\bar{\alpha}(\mu)\}I - S(\mu)]\gamma = h(s,\mu)$$

with h periodic in s, has a unique periodic solution $\gamma(s,\mu)$ and the transformation

$$Y = X + \gamma(s,\mu)\zeta^k\bar{\zeta}^\ell$$

carries (3.19) into the form

$$\dot{X} = S(\mu)X + v(s,\zeta,\bar{\zeta},\mu) - h(s,\mu)\zeta^k\bar{\zeta}^\ell + H_3'.$$

Applying this transformation for $k+\ell = 2$ we obtain the form $\dot{X} = S(\mu)X + H_3''$.

The previous transformations may be combined into

$$z = \zeta + Q(s,\zeta,\bar{\zeta},X,\mu)$$

$$Y = X + \tilde{Q}(s,\zeta,\bar{\zeta},X,\mu)$$

which is one-to-one in a small neighborhood $|\zeta|^2 + |X|^2 \le 2c,\ 0 < c \le 1$. The transformation $\zeta = cw,\ X = cW$ then gives the desired result (3.7).

In the case $\alpha(0) = 0$ we apply (3.11) only for $k+\ell = 2$, skip (3.13) and proceed from (3.14) as before. In the case $\alpha(0) = i/4$ the procedure goes as before except that in equation (3.12) there now appears a new term $\gamma(\mu)e^{is}\bar{w}^3$. Q.E.D.

3. Non-resonance Case: $\alpha(0) \ne \nu\, i/3,\ \nu\, i/4,\ \nu$ an integer

In this section we obtain the results of Chapter II by the method of perturbation of an invariant surface. The problem is reduced to solving a certain partial differential equation defined on a torus. The actual solution of this equation is carried out in Chapter IV.

The following theorem states that as μ increases through zero the periodic solution ψ bifurcates into an asymptotically stable torus.

Theorem 5. In (3.5) assume

1. $\alpha(0) \ne \nu\, i/3,\ \nu\, i/4,\ \nu$ an integer

2. $Re\ \alpha(0) = 0,\ A = Re\ \alpha'(0) > 0$ and the matrix $S = S(0)$ satisfies $(Y,SY) \le \sigma_0(Y,Y)$ for $\sigma_0 < 0$ a constant

3. $\alpha(\mu),\ S(\mu),\ G_i \in C^\ell(s,z,\bar{z},Y,\mu), \ell$ and integer ≥ 5

4. In equation (3.7) $B = Re\ \beta(0) < 0$.

Then given any integer $r,\ 2 \le r \le \ell$, there exists $\mu_r \le \mu_*$ such that for $0 < \mu < \mu_r$, (3.5) has an asymptotically stable invariant torus

$$\tau_2(\mu): \quad \begin{aligned} z(s,\theta,\mu) &= a_0\sqrt{\mu}\, e^{i\theta} + \mu f(s,\theta,\mu) \\ Y(s,\theta,\mu) &= \mu g(s,\theta,\mu) \end{aligned} \tag{3.20}$$

where $a_0 = \sqrt{-A/B}$. f and g are defined for $0 < \mu < \mu_r$, have period 2π in θ and s

$$f,\ g \in C^{r-1}(\theta,s) \cap Lip^{r-1}(\theta,s) \cap C(\mu)$$

and $|f|_{r-1},\ |g|_{r-1} \le c(r)$ with $c(r)$ a constant depending only on r. In general $\mu_r \to 0$ as r increases. μ_r depends on M_0 (defined after (3.2)) and the terms of degree ≤ 3 in the expansion of (3.5) about $(z,\bar z,Y) = 0$.

Proof. In (3.6) Assume c to be fixed such that the transformation is one-to-one for $|w|^2 + |W|^2 \le 2$. Note that c depends only on terms of degree ≤ 3 in the expansion of (3.5) about $(z,\bar z,Y) = 0$. We no longer indicate the dependence on c in the functions.

We restrict attention to a neighborhood

$$w = az, \quad W = a^2 Y \tag{3.21}$$

where $|z|^2 + |Y|^2 \le 2$ and $a,\ 0 \le a < 1$, is a small parameter to be fixed later. In the new coordinates (3.7) becomes

$$\dot z = \alpha(\mu)z + a^2c^2\beta(\mu)|z|^2 z + f_1(s,z,\bar z,Y,a,\mu)$$

$$\dot Y = S(\mu)Y + f_2(s,z,\bar z,Y,a,\mu)$$

with $|f_1|_k \le c(k)a^3,\ |f_2|_k \le c(k)a$ where differentiation is with respect to all variables except a and μ. Note that the subscripts on the functions are used simply for identification and have nothing to do with "weight" as they did in the previous theorem.

Expanding $\alpha(\mu),\ S(\mu)$ and $\beta(\mu)$ we obtain

$$\begin{aligned} \dot z = {} & [\alpha(0) + \mu\alpha'(0) + a^2c^2\beta(0)|z|^2]z \\ & + f_3(s,z,\bar z,Y,a,\mu) \\ \dot Y = {} & S(0)Y + f_4(s,z,\bar z,Y,a,\mu) \end{aligned}$$

where

$$|f_3|_k \le c(k)(a^3 + \mu a^2 + \mu^2),\ |f_4|_k \le c(k)(a+\mu).$$

Let $S(0) = S,\ \alpha(0) = i\omega,\ \alpha'(0) = A + i\tilde A,\ \beta(0) = B + i\tilde B$ and choose

$$a = \sqrt{\frac{-A\mu}{c^2B}}, \qquad 0 \le \mu < \mu_0 = \frac{c^2B}{-A}.$$

Then the above equation becomes

$$\dot{z} = i[\omega + \mu\phi]z + \mu A[1 - |z|^2]z + f_5(s, z, \bar{z}, Y, \mu)$$

$$\dot{Y} = SY + f_6(s, z, \bar{z}, Y, \mu)$$

where

$$\phi = \tilde{A} - \frac{\tilde{B}A}{B}|z|^2$$

and

$$|f_5|_k \leq c(k)\mu^{\frac{3}{2}}, \qquad |f_6|_k \leq c(k)\mu^{\frac{1}{2}}.$$

The unperturbed equation ($f_5,\ f_6 \equiv 0$) has an invariant torus $|z| = 1,\ Y = 0$. In order to treat the perturbation terms we introduce coordinates in a neighborhood of this surface by making the transformation

$$z = (1 + \rho)e^{i\theta}, \qquad |\rho|^2 + |Y|^2 \leq \frac{1}{4} \tag{3.22}$$

to obtain

$$\dot{\rho} = -2\mu A\rho + [-\mu A\rho^2 + f_7(s, \theta, \rho, Y, \mu)]$$

$$\dot{\theta} = \omega + [\mu(c_1 + c_2\rho + c_3\rho^2) + f_8(s, \theta, \rho, Y, \mu)]$$

$$\dot{Y} = SY + f_9(s, \theta, \rho, Y, \mu)$$

where the c_i are constants. We now treat f_9 and the expressions in brackets as perturbation and note that the unperturbed system has an invariant torus $\tau_2 :\ \ \rho = 0,\ Y = 0$, which is asymptotically stable for $\mu > 0$. The problem is to find an invariant torus $\tilde{\tau}_2$ for the perturbed system. By making the transformation

$$\rho = \sqrt{\mu}\, u, \qquad Y = \sqrt{\mu}\, v \tag{3.23}$$

we obtain

$$\begin{aligned} \dot{u} &= -2\mu A u + \mu[G_1^0(s, \theta) + \hat{G}_1(s, \theta, u, v, \mu)] \\ \dot{\theta} &= \omega + \mu[F^0 + \hat{F}(s, \theta, u, v, \mu)] \\ \dot{v} &= Sv + G_2^0(s, \theta) + \hat{G}_2(s, \theta, u, v, \mu). \end{aligned} \tag{3.24}$$

The functions on the right have period 2π in θ and s and are defined for $|\mu|^2 + |v|^2 \leq K^2$, $0 \leq \mu < \tilde{\mu}_0 = \min(\mu_0, 1/4K^2)$ where K is an arbitrary constant to be fixed later. These functions satisfy $F^0 \equiv$ constant and

$$|G_i^0|_k \leq c(k); \quad |\hat{G}_i|_k, \quad |\hat{F}|_k \leq c(k)\mu^{\frac{1}{2}}.$$

The problem of finding an invariant surface for (3.24) is equivalent to finding a solution $u = u(s, \theta, \mu)$, $v = v(s, \theta, \mu)$ of the partial differential equation

$$\frac{\partial u}{\partial \theta}[\omega + \mu F] + \frac{\partial u}{\partial s} + 2\mu A u = \mu G_1$$

$$\frac{\partial v}{\partial \theta}[\omega + \mu F] + \frac{\partial v}{\partial s} - S v = G_2$$

where $G_i = G_i^0 + \hat{G}_i$ and $F = F^0 + \hat{F}$. If we define $w = \begin{pmatrix} u \\ v \end{pmatrix}$, $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, $D(w, \theta, s, \mu) = \begin{pmatrix} 1/\mu & 0 \\ 0 & I \end{pmatrix} \left\{ (\omega + \mu F) \frac{\partial}{\partial \theta} + \frac{\partial}{\partial s} \right\}$ and $P = \begin{pmatrix} 2A & 0 \\ 0 & -S \end{pmatrix}$, the equation takes the form

$$D(w, s, \theta, \mu) w + P w = G(w, s, \theta, \mu) \tag{3.25}$$

which is a special case of equation (4.4) of Chapter IV with $m = 2$, $x_1 = \theta$, $x_2 = s$, $A_1^0 \equiv F^0$ and $A_2^0 \equiv 1$. It is shown in that chapter that for a given integer r, $2 \le r \le l$, there exist values K and $\bar{\mu}$ such that for $0 < \mu < \bar{\mu}$, (3.25) has a unique solution $\tilde{\tau}_2 : w = w(\theta, s, \mu)$ having period 2π in θ and s,

$$w \in C^{r-1}(\theta, s) \cap Lip^{r-1}(\theta, s) \cap C(\mu)$$

and $|w|_{r-1} \le K$. The form (3.20) then follows by applying the transformations (3.23), (3.22), (3.21) and finally the transformation (3.6) of the previous theorem.

We now show that $\tilde{\tau}_2$ is asymptotically stable. Let $u^*(s, \mu)$, $v^*(s, \mu)$, $\theta^*(s, \mu)$ be a solution of (3.24) which is sufficiently near $\tilde{\tau}_2$ for some value of s. Write

$$u^*(s, \mu) = u(s, \theta^*(s, \mu), \mu) + \delta u$$

$$v^*(s, \mu) = v(s, \theta^*(s, \mu), \mu) + \delta v$$

which defines the variations δu and δv, where u and v represent $\tilde{\tau}_2$. Putting this into (3.24) we obtain

$$\begin{aligned} \delta \dot{u} + 2\mu A \delta u &= \mu(G_{1_u} - u_\theta F_u)\delta u + \mu(G_{1_v} - u_\theta F_v)\delta v \\ \delta \dot{v} - S \delta v &= (G_{2_u} - \mu v_\theta F_u)\delta u + (G_{2_v} - \mu v_\theta F_v)\delta v \end{aligned} \tag{3.26}$$

where the terms in parentheses are functions obtained by replacing θ by $\theta^*(s, \mu)$ and the derivatives, F_u etc., arise from the mean value theorem, e.g.,

$$F_u = F_u(s, \theta^*, u(\theta^*, s, \mu) + \epsilon \delta u,\ v(\theta^*, s, \mu) + \epsilon \delta v, \mu)$$

where $0 < \epsilon < 1$. From the form of F and G_i given just before (3.25) we see that their derivatives with respect to u and v satisfy

$$|F_u|_0, \ldots \le c_0 \mu^{\frac{1}{2}}, \quad c_0 = \text{ constant}$$

uniformly in a small region $R : |\delta u|^2 + |\delta v|^2 \le M$ containing $\tilde{\tau}_2$. Thus (3.26) has the form

$$\delta \dot{u} = [-2\mu A + P_1]\delta u + P_2 \delta v$$

$$\delta \dot{v} = Q_1 \delta u + [S + Q_2]\delta v$$

where P_i and Q_i are continuous functions of $(\delta u, \delta v, s, \mu)$ which satisfy $|P_i|_0 \le b\mu^{\frac{3}{2}}$, $|Q_i|_0 \le b\mu^{\frac{1}{2}}$, b constant, uniformly in s for δu, δv in R. If we define $U = |\delta u|^2 + \mu|\delta v|^2$ and $m = \min(2A, -\sigma_0)$ then for $0 < \mu < \min[1, (m/8b)^2]$ it can be shown that $\dot{U} \le -\mu(m/2)U$ and therefore $U \to 0$ as $s \to \infty$. Q.E.D.

4. Resonance Case $\alpha(0) = i/4$

We now discuss a case in which a particular low order resonance occurs. Since the frequency of ψ has been normalized ($\omega_1 = 1$), we are in the case in which $\omega_2 = 1/4$, i.e., for solutions of the linearized (Floquet) equation the frequency of motion along the periodic solution ψ is four times as great as the frequency about ψ. In this resonance case we see that the normal form of the equation (Theorem 4) is given by (3.9) in which a new parameter $\gamma(\mu)$ now appears. The relative size of the parameters $\alpha'(0)$, $\beta(0)$, and $\gamma(0)$ determines several cases. We discuss on possible case and show that ψ bifurcates into a pair of subharmonic solutions one of which is asymptotically stable. More precisely

Theorem 6. In (3.5) assume

1. $\alpha(0) = i/4$
2. *Re* $\alpha'(0) > 0$, *Im* $\alpha'(0) = 0$ and $S = S(0)$ satisfies $(Y, SY) \le \sigma_0(Y, Y)$ for some constant $\sigma_0 < 0$
3. $\alpha(\mu)$, $S(\mu)$, $G_i \in C^4(s, z, \bar{z}, Y, \mu)$ and periodic in s with period 2π, and in (3.41) below assume
4. *Re* $\beta(0) < 0$, *Im* $\beta(0) = 0$, and $0 < |\gamma(0)| < |\beta(0)|$

Then there exists a μ_0 depending on M_0 (defined after (3.2)) and the terms of degree ≤ 3 in the expansion of (3.5) about $(z, \bar{z}, Y) = 0$ such that for $0 < \mu < \mu_0$, (3.5) has two subharmonic solutions. Each has the form

$$\begin{aligned} z &= \sqrt{\mu}\, z_0\, e^{i\frac{s}{4}} + \mu f(s, \mu) \\ Y &= \mu g(s, \mu) \end{aligned} \tag{3.40}$$

where z_0 is a constant, f and g have period 8π in s, $f, g \in C^4(s, \mu^{\frac{1}{2}})$, i.e., C^4 a functions of s and $\mu^{\frac{1}{2}}$, and $|f|_4$, $|g|_4 \le$ constant (independent of μ). One solution is asymptotically stable and the other is unstable. For the stable solution $z_0 = \left(\dfrac{-\alpha'(0)}{\beta(0)+|\gamma(0)|}\right)^{\frac{1}{2}}$ and for the unstable solution $z_0 = \left(\dfrac{-\alpha'(0)}{\beta(0)-|\gamma(0)|}\right)^{\frac{1}{2}} e^{i\frac{\pi}{4}}$

Proof. We apply Theorem 4 to obtain the canonical form

$$\begin{aligned} \dot{w} &= \alpha(\mu)w + c^2[\beta(\mu)|w|^2 w + \gamma(\mu)e^{is}\bar{w}^3] + R_4(s, w, \bar{w}, W, \mu) \\ \dot{W} &= S(\mu)W + R_3(s, w, \bar{w}, W, \mu) \end{aligned} \tag{3.41}$$

valid for $|w|^2 + |W|^2 \leq 2$, where R_k contains no terms of weight $< k$. Letting $\alpha'(0) = A,\ c^2\beta(0) = B$ and $c^2\gamma(0) = C$ we have $A > 0,\ B < 0$ and by replacing s by $s+$ constant we may assume $C > 0$. Hence $0 < C < -B$. We then make the transformation

$$w = \sqrt{\mu}\zeta, \quad W = \mu v \tag{3.42}$$

where $|\zeta|^2 + |v|^2 \leq \dfrac{-4A}{B+C}$ and $0 < \mu < \min\left(\dfrac{B+C}{-2A},\ 1\right)$, to obtain

$$\begin{aligned} \dot{\zeta} &= \frac{i}{4}\zeta + \mu h(s, \zeta, \bar{\zeta}) + f_1(s, \zeta, \bar{\zeta}, v, \mu) \\ \dot{v} &= Sv + f_2(s, \zeta, \bar{\zeta}, v, \mu) \end{aligned} \tag{3.43}$$

where

$$h(s, \zeta, \bar{\zeta}) = A\zeta + B|\zeta|^2\zeta + Ce^{is}\bar{\zeta}^3$$

and

$$|f_1|_4 \leq K\mu^{\frac{3}{2}},\ |f_2|_4 \leq K\mu^{\frac{1}{2}},\ K = \text{ constant } > 0 \tag{3.44}$$

In this system we consider the terms f_1 and f_2 as small perturbations and discuss first the system obtained by dropping them. We will show that the unperturbed system has a pair of subharmonic solutions having the stated properties and that this remains true if the perturbation is taken sufficiently small. We thus introduce a new parameter ϵ by writing $f_1 = \mu\epsilon^{\frac{1}{2}}\tilde{f}_1,\ f_2 = \epsilon^{\frac{1}{2}}\tilde{f}_2$ where $|\tilde{f}_j|_4 \leq K$. Then for $\epsilon = \mu$ we have (3.43) and $\epsilon = 0$ gives us the unperturbed equation. Let $\epsilon = 0$ and make the transformation

$$\zeta = \eta e^{i\frac{s}{4}} \tag{3.45}$$

to obtain

$$\begin{aligned} \dot{\eta} &= \mu[A\eta + B|\eta|^2\eta + C\bar{\eta}^3] \\ \dot{v} &= Sv \end{aligned} \tag{3.46}$$

or in polar coordinates, $\eta = re^{i\theta}$

$$\dot{r} = \mu[A + (B + C\cos 4\theta)r^2]r$$

$$\dot{\theta} = -\mu C r^2 \sin 4\theta \tag{3.47}$$

$$\dot{v} = Sv.$$

The stationary points of this equation are (see Fig. 8)

$$P_n = \begin{pmatrix} \eta_n \\ v_n \end{pmatrix}, \qquad Q_n = \begin{pmatrix} \tilde{\eta}_n \\ v_n \end{pmatrix}$$

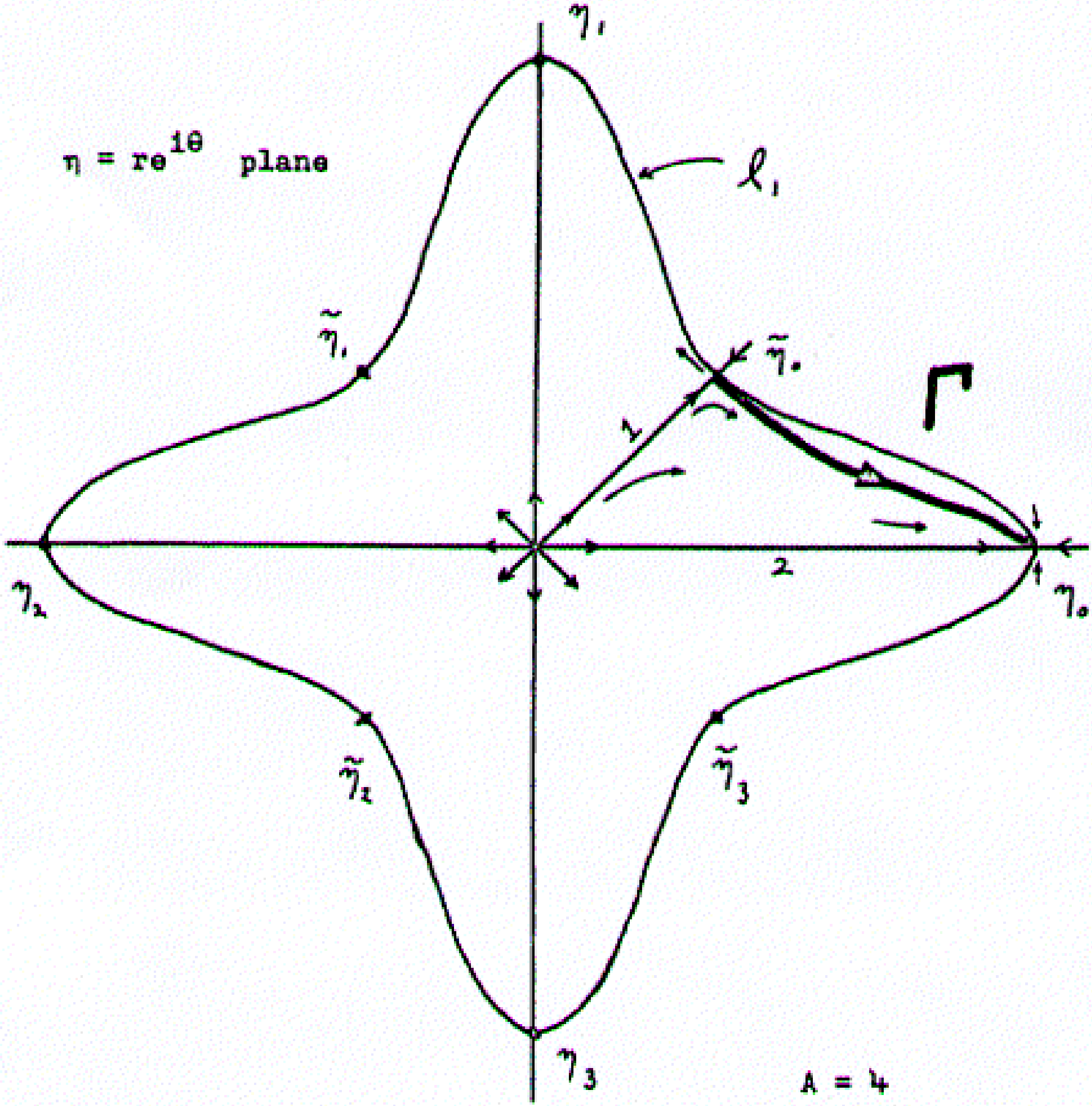
where

$$\eta_n = \sqrt{\left(\frac{-A}{B+C}\right)} e^{in\frac{\pi}{2}}$$

$$\tilde{\eta}_n = \sqrt{\left(\frac{-A}{B-C}\right)} e^{i\left(\frac{\pi}{4} + n\frac{\pi}{2}\right)}$$

$$v_n = 0, \quad n = 0, 1, 2, 3.$$

FIG. 8



$$\ell_1: \quad r^2 = \frac{-A}{B + C \cos 4\theta}$$

$A = 4$

$B = -5/2$

$C = 3/2$

Γ consists of eight such arcs joining the outer singularities. Only one arc is shown.

These stationary points correspond to a pair of periodic solutions of equation (3.43) with $\epsilon = 0$:

$$p_0(s) = \begin{pmatrix} \eta_0 e^{i\dfrac{s}{4}} \\ 0 \end{pmatrix}, \qquad q_0(s) = \begin{pmatrix} \tilde{\eta}_0 e^{i\dfrac{s}{4}} \\ 0 \end{pmatrix}.$$

The linear variational equation obtained from (3.47) has the matrix

$$J = \begin{pmatrix} J_1 & 0 \\ & \\ 0 & S \end{pmatrix}$$

where

$$J_1 = \begin{pmatrix} -2\mu A & 0 \\ & \\ 0 & 4\mu\dfrac{CA}{B+C} \end{pmatrix} \qquad \text{at } P_n$$

and

$$J_1 = \begin{pmatrix} -2\mu A & 0 \\ & \\ 0 & -4\mu\dfrac{CA}{B-C} \end{pmatrix} \qquad \text{at } Q_n.$$

Thus $p_0(s)$ is asymptotically stable and $q_0(s)$ is unstable. It is easily seen that p_0 and q_0 are the only periodic solutions (other than the zero solution) of (3.43) for $\epsilon = 0$.

We now treat the perturbed equation. For $\mu = \epsilon = 0$, (3.43) has a family of periodic solutions $\zeta = \zeta_0 e^{is/4}$, $v = 0$ having period 8π where ζ_0 is an arbitrary constant. According to standard theory, e.g., [4], for sufficiently small μ and ϵ there exist initial conditions $\zeta_0(\mu, \epsilon)$, $v_0(\mu, \epsilon)$ with $\zeta_0(0,0) = \zeta_0$ and $v_0(0,0) = v_0$ and a periodic solution of (3.43) satisfying these initial conditions for $s = 0$ if for $\mu = \epsilon = 0$,

$$(e^{8\pi S} - I)v_0 = 0$$

$$H(\zeta_0, \bar{\zeta}_0) \equiv \int_0^{8\pi} e^{-i\frac{t}{4}} h\left(t, \zeta_0 e^{i\frac{t}{4}}, \bar{\zeta}_0 e^{-i\frac{t}{4}}\right) dt = 0$$

and

$$\tilde{J} = \frac{\partial(H, \bar{H})}{\partial(\zeta_0, \bar{\zeta}_0)} \neq 0.$$

Evaluating the integral, we obtain

$$H(\zeta_0, \bar{\zeta}_0) = 8\pi(A\zeta_0 + B|\zeta_0|^2\zeta_0 + C\bar{\zeta}_0^3).$$

Hence the first two of these conditions are satisfied are satisfied if $\begin{pmatrix} \zeta_0 \\ v_0 \end{pmatrix}$ is one of the points P_n or Q_n. Evaluating $\tilde{J}$, we obtain

$$\begin{aligned}\tilde{J} &= |H_\zeta|^2 - |H_{\bar{\zeta}}|^2 \\ &= \left(\frac{8\pi A}{B+C}\right)^2 [(C-B)^2 - (B+3C)^2]\end{aligned}$$

at P_n and

$$\tilde{J} = \left(\frac{8\pi A}{B-C}\right)^2 [(C+B)^2 - (B-3C)^2]$$

at Q_n. Thus $\tilde{J} \neq 0$ since $0 < C < |B|$ and there exist periodic solutions $p(s, \mu, \epsilon)$ and $q(s, \mu, \epsilon)$ such that

$$p(s, 0, 0) = p(s, \mu, 0) = p_0(s)$$

$$q(s, 0, 0) = q(s, \mu, 0) = q_0(s).$$

These solutions exist for all sufficiently small μ and ϵ and in particular for $\mu = \epsilon$. If we assume the functions in (3.41) are smooth as functions of $(s, w, \bar{w}, W, \mu)$ then (3.43) is smooth as a function of $(s, \zeta, \bar{\zeta}, v, \mu^{\frac{1}{2}})$. Hence $p(s, \mu, \mu) = p_0(s) + \mu^{\frac{1}{2}}\tilde{p}(s, \mu), |\tilde{p}|_4 \leq$ constant, where the constant is independent of μ, and similarly for $q(s, \mu, \mu)$. Using the estimates (3.44), it follows from standard arguments that $p(s, \mu, \mu)$ and $q(s, \mu, \mu)$ have the same stability properties as $p_0(s)$ and $q_0(s)$ respectively. The form (3.40) follows from applying (3.42) and the transformation of Theorem 4.

5. Another Resonance Case

In previous sections we considered cases in which a conjugate pair of Floquet exponents crossed the imaginary axis as μ increased through zero. We now consider the case in which a real exponent $\alpha(\mu)$ enters the right half plane and satisfies

$$\alpha(0) = 0, \qquad \alpha'(0) > 0.$$

Since $\omega_2 = 0$ (notation of Chapter I) there is no tendency for solutions of the linearized equation to oscillate about the periodic solution ψ. Hence in this case we expect that ψ bifurcates into another periodic solution having the same period.

We repeat equation (3.4)

$$\dot{u} = C(\mu)u + G(s, u, \mu) \tag{3.50}$$

where $G(s, 0, \mu) \equiv 0$ and $G_u(s, 0, \mu) \equiv 0$ and assume that $C(\mu)$ is in the real form

$$C(\mu) = \begin{bmatrix} \alpha(\mu) & 0 \\ 0 & S(\mu) \end{bmatrix}$$

where $S(\mu)$ satisfies the same conditions as before. Equation (3.50) is assumed to be valid for $|\mu| \leq \mu_*$ and $|u|^2 \leq 2$ where μ_* is a constant > 0. We introduce the notation $\xi = u_1,\ U = col(u_2, \ldots, u_n)$ and write (3.50) as

$$\begin{aligned} \dot{\xi} &= \alpha(\mu)\xi + G_1(s, \xi, U, \mu) \\ \dot{U} &= S(\mu)U + G_2(s, \xi, U, \mu). \end{aligned} \tag{3.51}$$

We write the scalar function G_1 as

$$G_1(s, \xi, U, \mu) = \beta(s, \mu)\xi^2 + \ldots$$

and define

$$B = \frac{1}{2\pi} \int_0^{2\pi} \beta(s, 0) ds.$$

<u>Theorem 7</u>. Assume in (3.51) that

1. $\alpha(0) = 0,\ \alpha'(0) > 0$
2. $(U, S(0)U) \leq \sigma_0(U, U)$ for some constant $\sigma_0 < 0$
3. $B \neq 0$
4. $G_1,\ G_2 \in C^3(s, \xi, U, \mu)$

Then there exists a $\bar{\mu},\ \bar{\mu} \leq \mu_*$, such that for $0 < \mu < \bar{\mu}$, (3.51) has an asymptotically stable periodic solution

$$\begin{aligned} \xi &= \mu c_0 + \mu^2 f(s, \mu) \\ U &= \mu^2 g(s, \mu) \end{aligned} \tag{3.52}$$

where $c_0 = -(\alpha'(0))/B;\ f, g \in C^3(s, \mu);\ |f|_3,\ |g|_3 \leq$ constant uniformly in μ and f and g have period 2π in s. The constant $\bar{\mu}$ depends on the M_0 defined after (3.2) and the terms of degree ≤ 2 in the expansion of (3.51) about $U = 0,\ \xi = 0$.

<u>Proof</u>. In (3.51) make the substitution

$$\xi = \mu x, \qquad U = \mu Y$$

$$|x|^2 + |Y|^2 \leq \left(\frac{2\alpha'(0)}{|B|}\right)^2, \qquad 0 < \mu < \sqrt{2}\left(\frac{|B|}{2\alpha'(0)}\right)$$

and expand α and S to obtain

$$\begin{aligned} \dot{x} &= \mu F_1(s, x, Y, \mu) \\ \dot{Y} &= SY + \mu F_2(s, x, Y, \mu) \end{aligned} \tag{3.53}$$

where $S = S(0)$ and

$$F_1(s,x,Y,\mu) = \alpha'(0)x + \beta(s,0)x^2 + \mu\tilde{F}(s,x,Y,\mu)$$

where $|\tilde{F}|_3 \leq$ constant uniformly in μ. For $\mu = 0$ (3.53) has a periodic solution $p_0 : x = x_0,\ Y = 0$ where x_0 is an arbitrary constant. It is known that for sufficiently small μ, (3.53) will have a periodic solution $p_\mu(s)$ of period 2π which for $\mu = 0$ reduces to p_0, if x_0 satisfies

$$H(x_0) = \int_0^{2\pi} F_1(s,x_0,0,0)ds = 0$$

and $H'(x_0) \neq 0$. Furthermore the solution will be asymptotically stable if $H'(x_0) < 0$. We see that $H(x_0) = 2\pi[\alpha'(0)x_0 + Bx_0^2]$ and therefore $x_0 = -(\alpha'(0))/B$ and $H'(x_0) = -\alpha'(0)$. From the assumed smoothness of the functions in (3.53) we have $p_\mu(s) = p_0 + \mu\tilde{p}(s,\mu)$ where $|\tilde{p}|_3 \leq$ constant uniformly in μ. Q.E.D.

6. Resonance case $\alpha(0) = i/3$ (An Example)

We now discuss an example of this resonance case for the sake of completeness and to point out the rather violent type of instability which develops as the exponents $\alpha(\mu)$ and $\overline{\alpha(\mu)}$ cross the imaginary axis. We consider the case in which the second equation of (3.5) is absent. By transformations of the type (3.11) we may put the first equation of (3.5) into a standard form similar to those obtained in Theorem 4,

$$\dot{w} = \alpha(\mu)w + c\delta(\mu)e^{is}\bar{w}^2 + R(w,\bar{w},s,\mu).$$

where R is a function whose Taylor expansion about $w = \bar{w} = 0$ contains no quadratic terms and $w = \rho e^{i\phi}$. We consider the case in which $\alpha(\mu) = i/3 + \mu,\ c\delta(\mu) \equiv 1$ and $R = -|w|^2 w$,

$$\dot{w} = \left(\frac{i}{3} + \mu\right)w + e^{is}\bar{w}^2 - |w|^2 w. \tag{3.60}$$

Letting $w = ze^{i\frac{s}{3}}$ we obtain

$$\dot{z} = \mu z + \bar{z}^2 - |z|^2 z. \tag{3.61}$$

In polar coordinates $z = re^{i\theta}$ this equation is

$$\dot{r} = (\mu + r\cos 3\theta - r^2)r$$

$$\dot{\theta} = -r\sin 3\theta.$$

There are seven stationary points,

$$\begin{aligned} r &= 0 \\ P_n : r &= 1 + 0(\mu), &\qquad \theta &= n\tfrac{2\pi}{3} \\ Q_n : r &= \mu + 0(\mu^2), &\qquad \theta &= \tfrac{\pi}{3} + n\tfrac{2\pi}{3} \end{aligned}$$

$n = 0, 1, 2$; and $0(\ )$ is meant as $\mu \to 0$.* For $\mu > 0$, the origin is an unstable node, P_n are stable nodes and Q_n are saddle points (see Fig. 9). The rays $r \geq 0$, $\theta = n(\pi/3)$, $n = 0, 1, \ldots, 5$ are invariant manifolds and the disc $\triangle : r \leq 5/4$ is invariant as $s \to +\infty$. Thus $\triangle$ consists of six invariant subsets

$$\triangle_k : 0 \leq r \leq \frac{5}{4}, \qquad k\frac{\pi}{3} \leq \theta \leq (k+1)\frac{\pi}{3}$$

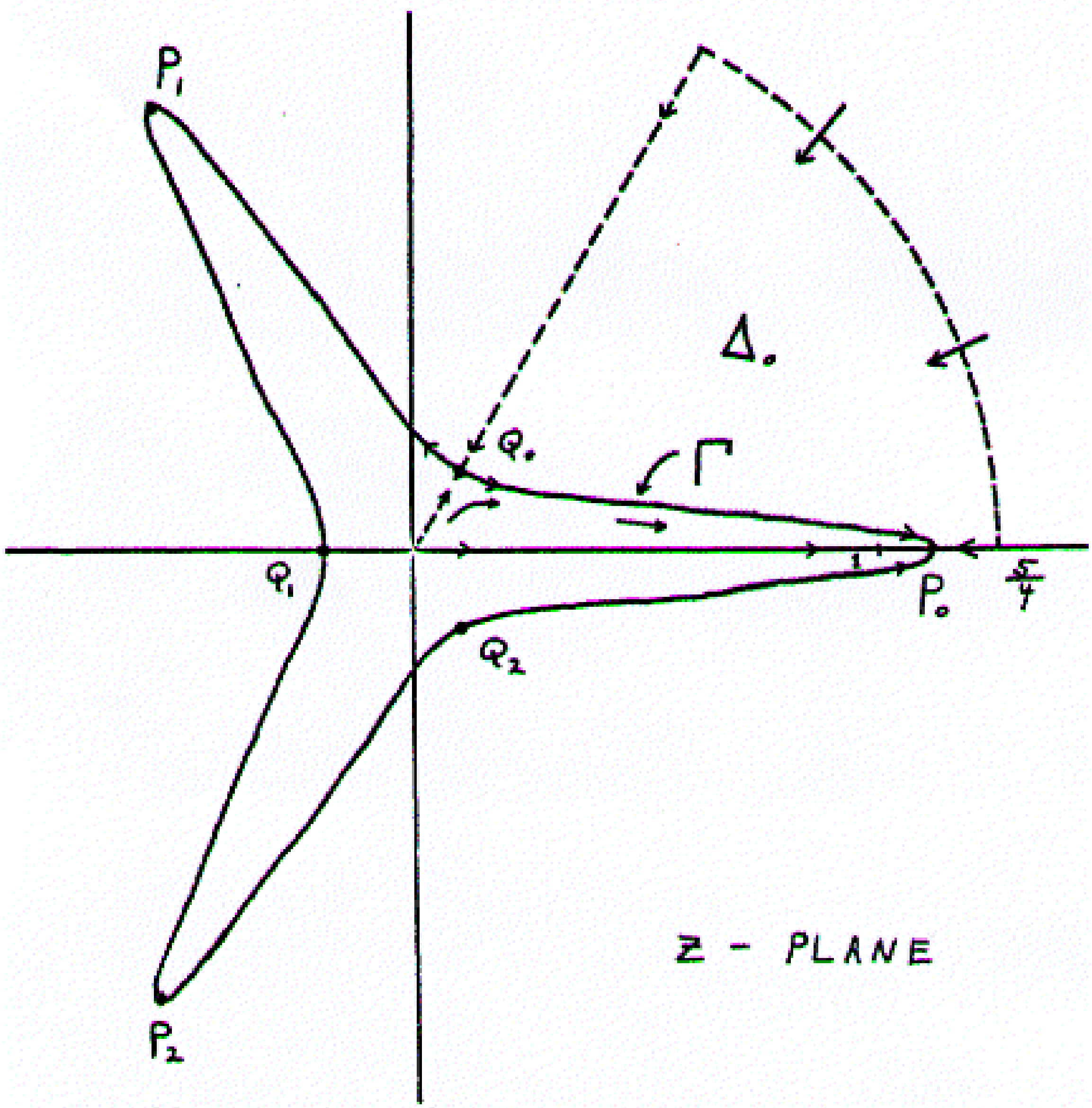
$k = 0, \ldots, 5$. Applying the theorem of Poincaré and Bendixson to the $\triangle_k$ we see that the points P_n and Q_n must be joined by arcs which are solutions of (3.61). Let $\Gamma : r = \gamma(\theta, \mu)$ denote the closed curve formed by the sum of these arcs and the points P_n and Q_n. Then γ has period $2\pi/3$ in θ. In the w, s space the original periodic solution ψ is the line $w = 0$ with points $2n\pi$ identified. The z plane rotates in such a manner that when s increases by 2π, Γ sweeps out a 2-dimensional torus

$$\tau_\mu : w = \gamma\left(\phi - \frac{s}{3}, \mu\right) e^{i\phi}. \tag{3.62}$$

The points P_n sweep out a stable periodic solution $p = p(s, \mu)$ of period 6π obtained by setting $\phi = s/3$ in (3.62). Similarly the Q_n sweep out an unstable periodic solution $q = q(s, \mu)$ obtained by setting $\phi = s/3 + \pi/3$. As $\mu > 0$ tends to zero, $q(s, \mu) \to 0$ and we may say that q bifurcates from ψ as μ increases through zero. However as $\mu \to 0$, p approaches a limiting position $p(s, 0) = e^{i\frac{s}{3}}$. Hence in the z plane Γ has a limiting position which consists of three spikes $0 \leq r \leq 1$, $\theta = n\frac{2\pi}{3}$, $n = 0, 1, 2$ and we see that τ_μ does not bifurcate from ψ in the sense of the definition of Chapter I, i.e., it does not collapse onto ψ as $\mu > 0$ tends to zero. We also see that for arbitrarily small $\mu > 0$ there are solutions starting arbitrarily close to ψ which run out a distance $1 + 0(\mu)$ to the orbit $p(s, \mu)$. This is in contrast to the non-resonance case of Section 3 where nearby solutions may run out a distance of the order $\sqrt{\mu}$ to the torus (3.20). Even is the resonance case $\alpha(0) = i/4$ we see, by the same reasoning above, that the unperturbed equation (3.46) has an invariant curve Γ (Fig. 8). Thus equation (3.43), with f_1 and f_2 omitted, has an invariant torus τ_μ lying at a distance of the order $\sqrt{\mu}$ and therefore solutions starting near ψ can run out only that far. The problem of the existence of τ_μ for the perturbed equation is not treated in this dissertation.

*See Titchmarsh [24] for definition of $0(\ )$.

FIG. 9



7. Rate of Growth of Bifurcating Manifold

We have discussed cases in which the periodic solution ψ becomes unstable and a new stable manifold bifurcates as μ increases through zero. Since the stable oscillations of a system are the only ones observable after a long time, it is of interest to know how these new stable states change as the parameter μ changes. In particular, how far away is the stable motion from the original periodic motion ψ? Such information is given by the quantity $\rho = \rho(\mu)$ which is some appropriate measure of the distance from ψ to the stable bifurcating manifold.* Of course for $\mu < 0$, ψ is stable and $\rho(\mu) \equiv 0$. Let S be the normal hyperplane to the orbit ψ at some point of ψ. In the case in which one real root crosses the origin (Section 5) the bifurcating manifold is a periodic solution which intersects S at some point P. The distance from P to ψ is, in the first approximation, $\rho(\mu) =$ (constant) μ, (Fig. 10). In the non-resonance case of Section 3 the bifurcating manifold is the torus (3.20) whose intersection with S is, in the first approximation, a circle of radius $\rho(\mu) =$ (constant) $\sqrt{\mu}$, (Fig. 11). Also in the resonance case $\alpha(0) = i/4$ the stable bifurcating manifold is the subharmonic solution (3.40) which intersects S at a distance $\rho(\mu) =$ (constant) $\sqrt{\mu}$ in the first approximation.

Thus we see that in the last two cases the bifurcation is more violent in the sense that near the bifurcation point, $\rho'(\mu)$ becomes infinite.

For the example of the last section ($\alpha(0) = i/3$), the subharmonic solution $p(s, \mu)$ intersects S at a distance of 1 in the first approximation. Hence $\rho(\mu)$ is a discontinuous function (Fig. 12)

$$\rho(\mu) = \begin{cases} 0 & \mu < 0 \\ 1 & \mu \geq 0 \end{cases} .$$

*See Minorsky [1] for a discussion of bifurcation and the function ρ.

FIG. 10

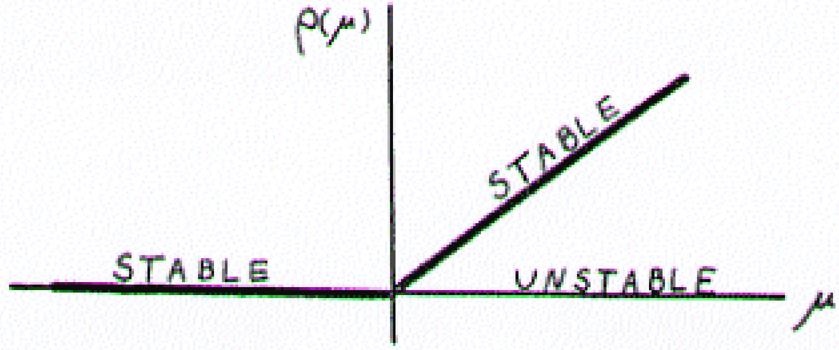


FIG. 11

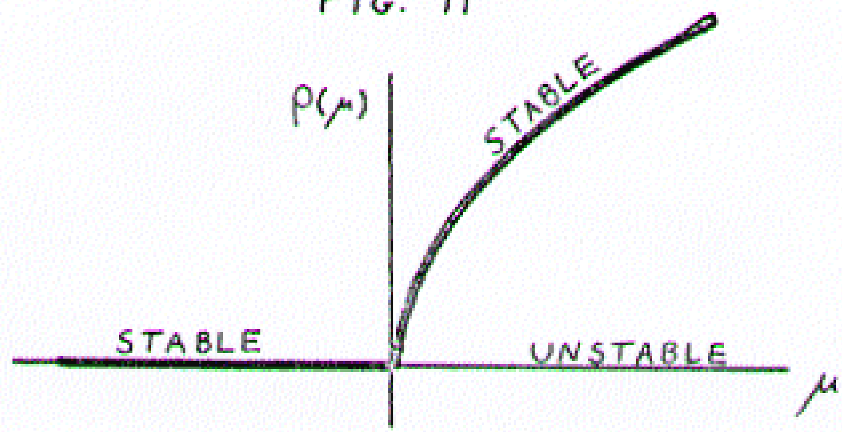
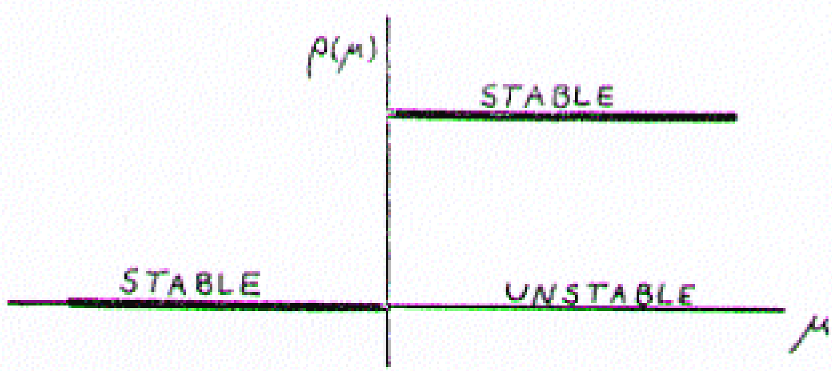


FIG. 12



CHAPTER IV
INVARIANT SURFACES

1. Introduction

In Theorem 5 of Chapter III we saw that a closed invariant surface of a vector field was defined as the solution of a certain partial differential equation of first order involving periodic functions. In this chapter we treat the problem of existence, uniqueness and smoothness of the solution of a slightly more general equation. Thus we consider the real periodic quasilinear system of first order

$$D(w,x,\mu)w + P(x,\mu)w = G(w,x,\mu) \tag{4.0}$$

where $x = (x_1, \ldots, x_m)$, $w = col(w_1, \ldots, w_n) = \binom{u}{v}$ with $u = col(u_1, \ldots, u_{n_1})$ and $v = col(v_1, \ldots, v_{n_2})$.

$$P(x,\mu) = \begin{pmatrix} P_1(x,\mu) & 0 \\ & \\ 0 & P_2(x,\mu) \end{pmatrix}$$

$$D(w,x,\mu) = Cd = \begin{pmatrix} cI_1 & 0 \\ & \\ 0 & I_2 \end{pmatrix} d$$

and

$$G(w,x,\mu) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = G^0(x) + \hat{G}(w,x,\mu)$$

where P_j is an $n_j \times n_j$ matrix, I_j is the $n_j \times n_j$ identity, $c = 1/\mu > 1$ and

$$d = \sum_{\nu=1}^{m} a_\nu(w,x,\mu)\frac{\partial}{\partial x_\nu}$$

with a_ν scalar functions. The functions a_ν, P and G are defined for

$$(w,x,\mu) \in W \times R \times \{0 \le \mu \le \mu_0\}$$

with $W : |w| <$ constant and $R : 0 \le x_i \le R_i$ a period parallelogram.

From the coefficients of d define the vector $a(w,x,\mu) = (a_1, \ldots, a_m)$ and assume that

$$(1)\ a(w,x,\mu) = \omega + \mu A(w,x,\mu) \ \text{ if } \ n_1 > 0$$

$$(2)\ a(w,x,\mu) = A(w,x,\mu) \ \text{ if } \ n_1 = 0$$

where ω is a constant vector and

$$A(w,x,\mu) = A^0(x) + \hat{A}(w,x,\mu). \tag{4.1}$$

The case $n_1 > 0$ is the "degenerate" case mentioned in Chapter I in which the flow against the unperturbed surface vanishes for $\mu = 0$ (see 4.2 below).

Equation (4.0) expresses the fact that $w(x,\mu) = \binom{u}{v}$ should represent an invariant surface for the differential system

$$\begin{aligned} \dot{u} &= \mu P_1(x,\mu)u + \mu G_1(w,x,\mu) \\ \dot{v} &= -P_2(x,\mu)v + G_2(w,x,\mu) \\ \dot{x} &= a(w,x,\mu) \end{aligned} \tag{4.2}$$

In Chapter I we considered the following differential system:

$$\begin{aligned} \dot{x} &= f(x,y,\mu) \\ \dot{y} &= g(x,y,\mu) \end{aligned} \tag{4.3}$$

where x and y are vectors, μ is a small parameter and the functions are periodic in x. We assume that for $\mu = 0$ (4.3) has an invariant surface $y = \phi_0(x)$. The problem is to find an invariant surface $y = \phi(x,\mu)$ for $\mu \geq 0$ such that $\phi(x,0) = \phi_0(x)$. If we assume $\phi_0(x) \equiv 0$ then (4.3) can be put into the form (4.2) with the u equation absent ($n_1 = 0$) by setting $y = \mu v \;\; - g_y(x,0,\mu) = P_2(x,\mu)$ and $a(v,x,\mu) = f(x,y,\mu)$. Then we see that $A^0(x) = f(x,0,0)$.

The basic idea of the method of solution of (4.0) can be illustrated with a single equation in one variable

$$a(u,x)u_x + b(x)u = f(u,x) \tag{4.4}$$

where x is a periodic variable and $b(x) \geq \beta > 0$, β a constant. The equation is first replaced by a similar equation with C^∞ coefficients and right hand side. Then it is linearized by substituting an approximate C^∞ solution into the functions $a(u,x)$ and $f(u,x)$ to obtain the C^∞ equation

$$a(x)u_x + b(x)u = f(x).$$

This equation is made elliptic by adding a higher derivative:

$$-\epsilon u_{xx} + a(x)u_x + b(x)u = f(x) \tag{4.5}$$

with $\epsilon > 0$.

The existence of a C^∞ solution of this equation is then treated from the standpoint of the classical theory (Lemma 5). We then obtain an "a-priori" estimate for the solution by multiplying (4.5) by u to obtain.

$$-\epsilon\left[\frac{1}{2}(u^2)_{xx} - u_x^2\right] + \frac{1}{2}a(x)(u^2)_x + b(x)u^2 = f(x)u.$$

At the maximum of u^2, $(u^2)_{xx} \le 0$ and $(u^2)_x = 0$. Hence we obtain, independently of ϵ,

$$|u|_0 \le \frac{1}{\beta}|f|_0.$$

This estimate and similar ones for higher derivatives are then used to prove convergence of the approximations to a solution of (4.4). The parameter $\epsilon = \epsilon(n)$ goes to zero as n, the number of iterations, goes to ∞.

2. Notation

Let $A(x,\mu) = (A_{j_1}, \dots, _{j_k})$ be a k-dimensional array of functions defined for $x \in R$ and μ in some interval. Define

$$(A,B) = \sum A_{j_1} \dots _{j_k} B_{j_1} \dots _{j_k} \, ,$$

$$|A|_0 = \max_{x \in R} |A(x,\mu)|$$

$$|A| = (A,A)^{\frac{1}{2}}$$

$$(A,B)_2 = \int_R (A,B)dx \quad \text{and} \quad \|A\| = (A,A)_2^{\frac{1}{2}}.$$

For r a positive integer let

$$\partial^r = \frac{\partial}{\partial x_{\ell_1}} \frac{\partial}{\partial x_{\ell_2}} \cdots \frac{\partial}{\partial x_{\ell_r}}$$

$1 \le \ell_i \le m$, denote any r^{th} order partial derivative and let

$$\sum_{[r]} = \sum_{\substack{\ell_1 = 1 \\ i = l, \dots, r}}^{m} \, .$$

The r^{th} derivative of A with respect to x in an $r+k$ dimensional array

$$A^{(r)} = (\partial^r A_{j_1} \dots _{j_k}) \, .$$

For $r = 1$ or 2 we write A_x or A_{xx}. Then

$$|A^{(r)}|^2 = \sum_{[r]} |\partial^r A|^2 = \sum_{[r]} (\partial^r A, \partial^r A).$$

Finally define

$$|A|_r = \max_{0 \le k \le r} |A^{(k)}|_0.$$

Note that the value of this norm is a function of the parameter μ. Any estimate involving $|A|_r$ is understood to be true for each value of μ under consideration.

3. Main Theorem

We will now prove the existence, uniqueness and smoothness of a solution of (4.0). From the vector $A^0(x) = (A_1^0, \ldots, A_m^0)$ define the symmetric matrix

$$\frac{1}{2}\left[\frac{\partial A_i^0}{\partial x_j} + \frac{\partial A_j^0}{\partial x_i}\right].$$

Let $\lambda_k(x)$ denote an eigenvalue of this matrix and define

$$\lambda = \min_{k,x} \lambda_k(x).$$

Since $A^0(x)$ is periodic the functions $\partial A_i^0/\partial x_j$ have zero mean value, i.e.,

$$\int_R \frac{\partial A_i^0}{\partial x_j} dx = 0.$$

In particular, the trace of the above matrix has zero mean value and thus for some k and x, $\lambda_k(x) \leq 0$. Therefore $\lambda \leq 0$

Theorem 8. In (4.0) assume that for all $x \in R$ and μ, $\theta \leq \mu \leq \mu_0$,

1. $(w, P(x,\mu)w) \geq \beta(w,w)$ for some constant $\beta > 0$
2. $\beta + r\lambda > 0$ for some integer $r \geq 2$
3. $(w, P_1 w) - \frac{1}{2} div A^0(x) \geq \beta_0 > 0$ for some constant β_0 and all $|w| = 1$. If $n_1 = 0$, replace P_1 by P_2.
4. $P(x,\mu),\ G(w,x,\mu),\ A(w,x,\mu) \in C^\ell(w,x) \cap C(\mu)$ for ℓ an integer $\geq r$ and $|w| \leq K,\ K$ sufficiently large
5. $$\max\{|\hat{A}|_r,\ |\hat{G}|_r\} = \hat{c}(r,K,\mu)$$

   where $\hat{c} \to 0$ as $\mu \to 0$.

Define $M = \max\{|A^0|_r,\ |G^0|_r,\ \sup_\mu |P|_r\}$. Then there exist constants K and $\bar{\mu},\ \bar{\mu} \leq \mu_0$ such that for $0 < \mu < \bar{\mu}$, (4.0) has a unique solution $w(x,\mu) \in C^{r-1}(x) \cap Lip^{r-1}\ (x) \cap C(\mu)$ and $|w|_{r-1} \leq K$. K depends only on M, $\beta + r\lambda$ and r. $\bar{\mu}$ depends on r, M, $\hat{c}$, β, β_0 and K and in general $\bar{\mu}$ decreases as r increases.

Remark. If $A^0(x) \equiv$ constant as in Theorem 5 of Chapter III, then $\lambda = 0$ and the smoothness of the solution of (4.0) is limited only by the smoothness of the equation and the smallness of μ. Thus if $\ell = \infty$ in (4) above then there exist intervals $0 < \mu < \mu_r$, $\mu_{r+1} \leq \mu_r$ with r arbitrarily large, on which the assertion holds, i.e., w is smoother for smaller values of μ. Reversing this we see that as μ increases,the number of derivatives which the theorem guarantees is diminished.

Proof. Define $\beta' = \beta + r\lambda$ and let K be a constant

$$K \geq \left(\frac{2}{\beta'}\right)^{r+1} 3M \prod^{r}(3M, 2M) + 1$$

where $\prod^{r}$ is the polynomial with positive coefficients defined in Lemma 4 (proved later). We construct successive approximations $w_n(x, \mu)$ with $w_0 \equiv 0$. This is done inductively.

Suppose $w_{n-1}(x, \mu)$ satisfies

$$w_{n-1} \in C^{\infty}(x) \text{ for each fixed } \mu,\ 0 < \mu < \mu' \tag{4.30}$$

$$w_{n-1} \in C^{r-1}(x) \cap C(\mu) \tag{4.31}$$

$$|w_{n-1}|_r < K.$$

On the set $\{|w| \leq K\} \times R \times \{0 \leq \mu \leq \mu_0\}$ approximate the functions $A = A(w, x, \mu)$, $G = G(w, x, \mu)$ and $P = P(x, \mu)$ by functions, $\tilde{A}, \tilde{G}, \tilde{P} \in C^{\infty}(w, x, \mu)$ and consider the linear partial differential equation

$$\tilde{D}_n(x, \mu)w + \tilde{P}(x, \mu)w = \tilde{G}_n(x, \mu)$$

where

$$\tilde{G}_n(x, \mu) = \tilde{G}(w_{n-1}(x, \mu), x, \mu)$$

and (4.33)

$$\tilde{D}_n(x, \mu) = \tilde{D}(w_{n-1}(x, \mu), x, \mu),$$

i.e., in the expression for D, the function A is replaced by

$$\tilde{A}_n(x, \mu) = \tilde{A}(w_{n-1}(x, \mu), x, \mu).$$

We assume the approximation to be so close that

$$|\tilde{A}_n - A_n|_r + |\tilde{G}_n - G_n|_r + |\tilde{P} - P|_r \leq M_n \tag{4.34}$$

where

$$M_n = \frac{1}{n^2} \min\left(\frac{\beta'}{12r}, \frac{\beta_0}{4\sqrt{m}}, \frac{\beta}{4\sqrt{m}}, M\right)$$

and the subscript r denotes differentiation with respect to x. A_n and G_n are defined by (4.33) with the tildes omitted. We modify this equation still further to obtain an elliptic equation for which the existence and smoothness of a solution is well known. Thus consider

$$L_n w \equiv -\epsilon_n \triangle w + \tilde{D}_n(x, \mu)w + \tilde{P}(x, \mu)w = \tilde{G}_n(x, \mu) \tag{4.35}$$

where $\triangle$ is the Laplacian in m variables, $\epsilon_n \to 0$ as $n \to \infty$ and $\epsilon_n > \epsilon_{n+1} > 0$.

Define $\tilde{a}_n = \omega + \mu \tilde{A}_n$ and use (4.34) to obtain

$$\left(w, \left[\tilde{P} - \frac{1}{2} div\ \tilde{a}_n C\right] w\right) \geq \frac{1}{2} \min[\beta_0, \beta](w, w)$$

for μ sufficiently small. Hence by Lemma 5, (4.35) has a solution $w = w_n(x, \mu) \in C^\infty(x)$ for each fixed μ. If we define $p = \beta - \beta'/12$ then using (4.34) we obtain $(w, \tilde{P}w) \geq p(w, w)$. Let $\tilde{\lambda} = \tilde{\lambda}(\mu)$ be obtained from $\tilde{A}_n(x, \mu)$ just as λ was obtained from $A^0(x)$. Then using (4.34) and the fact that $r|\lambda - \tilde{\lambda}| \leq r|A^0 - \tilde{A}_n|_\ell \leq rM_n + 2Kr\hat{c}(l, K, \mu)$ we have $p + r\tilde{\lambda} \geq \beta'/2$ for μ sufficiently small. We see here that for larger r we need smaller μ. Thus applying Lemma 4,

$$|w_n|_r \leq \left(\frac{2}{\beta'}\right)^{r+1} |\tilde{G}_n|_r \prod^r \left(|\tilde{A}_{n_{xx}}|_{r-2},\ |\tilde{P}|_r\right).$$

We have

$$|\tilde{G}_n|_r \leq M_n + |G^0|_r + |\hat{G}|_r \leq 2M + c' \leq 3M$$

$$|\tilde{A}_{n_{xx}}|_{r-2} \leq M_n + |A_0|_r + |\hat{A}|_r \leq 2M + c' \leq 3M$$

and

$$|\tilde{P}|_r \leq M_n + |P|_r \leq 2M$$

for μ sufficiently small. Differentiation is with respect to x only. $\hat{G}$ and $\hat{A}$ are as defined in (4.1) with $w_{n-1}(x, \mu)$ substituted for w. c' depends on r, K and $\hat{c}$ defined in hypothesis 5 and is obtained by applying the chain rule to $\hat{G}$ and $\hat{A}$; $c' \to 0$ as $\mu \to 0$. Hence from the way K was defined, $|W_n|_r < K$. We now verify that w_n satisfies (4.31). For a function $f_n(x, \mu)$ define $\delta f_n = f_{n2} - f_{n1} = f_n(x, \mu_2) - f_n(x, \mu_1)$. From (4.35)

$$L_{n2}\delta w_n = \delta\tilde{G}_n - (\delta\tilde{D}_n)w_{n1} - (\delta\tilde{P})w_{n1}$$

where L_{n2} is L_n with $\mu = \mu_2$. From the estimate of Lemma 4,

$$\frac{p}{2}|\delta w_n|_0 \ \leq\ |\delta\tilde{G}_n|_0 \ +\ |\delta\tilde{A}_n|_0\ |w_n|_1 \ +\ |\delta\tilde{P}|_0\ |w_n|_0.$$

Since $|w_n|_1 < K$ this can be made small for $\mu_2 - \mu_1$ small. From Lemma 3, Chapter II,

$$|\delta w_n^{(j)}|_0 \leq 2Kc(j, r)|\delta w_n|_0^{1-\frac{j}{r}}$$

for $0 \leq j \leq r - 1$. Thus $w_n^{(j)}$ is continuous in μ uniformly in x. Since $w_n \in C^\infty(x)$ for fixed μ, we have $w_n \in C^{r-1}(x) \cap C(\mu)$. Thus for μ' sufficiently small and fixed we have a sequence $\{w_n\}$ satisfying (4.30)-(4.32).

To show convergence define $\delta_{n+1} = w_{n+1} - w_n$. Then from (4.35) $L_{n+1}\delta_{n+1} = H$ where

$$H = \tilde{G}_{n+1} - \tilde{G}_n - \triangle w_n(\epsilon_n - \epsilon_{n+1}) - (\tilde{D}_{n+1} - \tilde{D}_n)w_n.$$

From the estimate of Lemma 4

$$|\delta_{n+1}|_0 \leq \frac{2}{p}|H|_0.$$

We have

$$|(\tilde{D}_{n+1} - \tilde{D}_n)w_n|_0 \leq [2M_n + |A_{n+1} - A_n|_0]K$$

$$\leq 2KM_n + K|A_w|_0\ |\delta_n|_0$$

where $A_n = A(w_{n-1}, x, \mu)$. Similarly

$$|\tilde{G}_{n+1} - \tilde{G}_n|_0 \leq 2M_n + |G_w|_0\ |\delta_n|_0$$

and

$$|\triangle w_n|_0 \leq \sqrt{m}\ |w_n|_2.$$

But $|A_w|_0 = |\hat{A}_w|_0 \leq \hat{c}(1, K, \mu)$ and $|G_w|_0 = |\hat{G}_w|_0 \leq \hat{c}(1, K, \mu)$. Thus for μ sufficiently small

$$|\delta_{n+1}|_0 \leq \frac{1}{2}|\delta_n|_0 + q_n$$

where

$$q_n = 8Kp^{-1}M_n + 2p^{-1}\sqrt{m}(\epsilon_n - \epsilon_{n+1}).$$

Since $\Sigma_1^\infty q_n < \infty$ we see that $\Sigma_0^\infty|\delta_{n+1}|_0 < \infty$ and hence the w_n converge uniformly in x and μ.

Letting $z = w_{n+p} - w_n,\ p > 0$ an integer, we have, using Lemma 3,

$$|z^{(j)}|_0 \leq 2Kc(j, r)|z|_0^{1-\frac{j}{r}}$$

for $0 \leq j \leq r-1$. Thus the sequence $\{w_n\}$ converges to a limit function $w \in C^{r-1}(x) \cap C(\mu)$. The Lipschitz condition follows from

$$\begin{aligned} |w_n^{(r-1)}(x_1, \mu) - w_n^{(r-1)}(x_2, \mu)| &\leq |w_n|_r|x_1 - x_2| \\ &\leq K|x_1 - x_2| \end{aligned}$$

and passing to the limit. For uniqueness assume w_1 and $w_2 \in C^1(x) \cap C(\mu)$ are solutions of (4.4) and satisfy $|w_i|_1 \leq K$. Then $\delta w = w_2 - w_1$ satisfies the linear equation

$$D_2\delta w + P\delta w = \delta G - (\delta D)w_1$$

where

$$D_2 = D(w_2, x, \mu), \delta G = G(w_2, x, \mu) - G(w_1, x, \mu)$$

and

$$\delta D = D(w_2, x, \mu) - D(w_1, x, \mu).$$

Now apply Lemma 4 in the $\epsilon = r = 0$ case to obtain

$$\begin{aligned}\frac{\beta}{2}|\delta w|_0 &\leq |\delta G|_0 + |\delta A|_0 K \\ &= |\delta \hat{G}|_0 + |\delta \hat{A}|_0 K \\ &\leq (|G_w|_0 + |A_w|_0 K)|\delta w|_0 \\ &\leq \frac{\beta}{4}|\delta w|_0\end{aligned}$$

for μ sufficiently small (Hypothesis 5). Thus $|\delta w|_0 = 0$. Q.E.D.

4. "A-priori" Estimates and Solution of Linear Problem

We now derive some estimates appropriate for a solution of the real linear elliptic partial differential equation

$$Lw \equiv -\epsilon \triangle w + D(x,\mu)w + P(x,\mu)w = G(x,\mu) \tag{4.40}$$

where $x = (x_1, \ldots, x_m)$, w and G are column n-vectors with $w = \binom{u}{v}$ $u = col(u_1, \ldots, u_{n_1})$, $v = col(v_1, \ldots, v_{n_2})$,

$$D = Cd = \begin{pmatrix} cI_1 & 0 \\ 0 & I_2 \end{pmatrix} d, \qquad c = \frac{1}{\mu} > 1, \quad I_j = n_j \times n_j \text{ identity}$$

with

$$d = \sum_{\nu=1}^{m} a_\nu(x,\mu)\frac{\partial}{\partial x_\nu}, \quad a_\nu \text{ scalar and}$$

$$P(x,\mu) = \begin{pmatrix} P_1(x,\mu) & 0 \\ 0 & P_2(x,\mu) \end{pmatrix}, \ P_j \text{ an } n_j \times n_j \text{ matrix.}$$

From the coefficients in d we have the vector $a(x,\mu) = (a_1, \ldots, a_m)$ where (1) $a(x,\mu) = \omega + \mu A(x,\mu)$ if $n_1 > 0$ or (2) $a(x,\mu) = A(x,\mu)$ if $n_1 = 0$ where ω is a constant vector. We assume $A = A(x,\mu)$, $P = P(x,\mu)$ and $G = G(x,\mu)$ are defined for $(x,\mu) \in R \times I_{\bar{\mu}}$ where R is a period parallelogram and $I_{\bar{\mu}} : 0 < \mu < \bar{\mu}$.

Consider the symmetric matrix

$$\frac{1}{2}\left[\frac{\partial A_i}{\partial x_j} + \frac{\partial A_j}{\partial x_i}\right]$$

and let $\lambda_k(x,\mu)$, $k = l, \ldots, m$ denote the eigenvalues. Define

$$\lambda' = \inf_{\mu} \min_{k,x} \lambda_k(x,\mu)$$

and note that $\lambda' \leq 0$ due to the periodicity of A and the same argument as given before Theorem 8.

Lemma 4. "A-priori" Estimates

Assume $\epsilon \geq 0$ and for $r \geq 0$ an integer, assume

1. $A, P, G \in C^r(x)$ for each fixed $\mu \in I_{\bar{\mu}}$ and $w \in C^\ell(x)$; $\ell = r+2$ if $\epsilon > 0$, $\ell = r+1$ if $\epsilon = 0$.

2. $(w, Pw) \geq \beta(w, w)$ for $(x, \mu) \in R \times I_{\bar{\mu}}$ where $\beta > 0$ is a constant

3. $\beta + r\lambda' > 0$.

Then

$$|w|_0 \leq \frac{2|G|_0}{\beta}$$

and in general for $0 \leq s \leq r$

$$|w|_s \leq \frac{|G|_s \prod\limits^{s}(|A_{xx}|_{s-2},\ |P|_s)}{(\beta + s\lambda')^{s+1}} \tag{4.41}$$

for all $\mu \in I_{\bar{\mu}}$, where for $k < 0$, $|\ |_k$ is taken to be zero. $\prod\limits^{s}(a, b)$ is a polynomial in a and b of degree s having positive coefficients which depend only on s. The estimates are independent of ϵ.

Proof. Due to the form of P and the fact that $\triangle$ and D are diagonal operators, (4.40) has the form

$$-\epsilon \triangle u + cdu + P_1 u = G_1 \tag{4.42}$$

$$-\epsilon \triangle v + dv + P_2 v = G_2. \tag{4.43}$$

We have $|w|_0 \leq |u|_0 + |v|_0 \leq 2|w|_0$, $|P|_0 \leq |P_1|_0 + |P_2|_0 \leq 2|P|_0$ and $(v, P_j v) \geq \beta(v, v)$ for $j = 1$ and 2. We obtain the estimate for (4.43) and note that the estimate for (4.42) follows by replacing a by ca.

Multiply (4.43) by v (scalar product) and use the identities

$$(v, \triangle v) = \frac{1}{2}\triangle(v, v) - |v_x|^2$$

$$(v, dv) = \frac{1}{2}d(v, v)$$

to obtain

$$-\epsilon\left[\frac{1}{2}\triangle(v, v) - |v_x|^2\right] + \frac{1}{2}d(v, v) + (v, P_2 v) = (v, G_2).$$

Let x_0 be a point at which (v,v) is a maximum, i.e., at x_0, $(v,v)=|v|_0^2$. Since $\triangle(v,v)\le 0$ and $d(v,v)=0$ at x_0 we obtain $(v,P_2v)|_{x=x_0}\le (v,G_2)|_{x=x_0}$ and therefore $\beta|v|_0\le |G_2|_0$.

To obtain the general estimate (4.41) we assume the solution has a sufficient number of derivatives and apply the same argument to the differentiated equation. The rule for applying ∂^s to a product fg of two functions is

$$\partial^s(fg)=f(\partial^s g)+\sum_{j=1}^{s}\frac{\partial f}{\partial x_{\ell_j}}\left(\partial_j^{s-1}g\right)+\dots$$

where

$$\partial_j^{s-1}=\frac{\partial}{\partial x_{\ell_1}}\cdots\frac{\partial}{\partial x_{\ell_{j-1}}}\frac{\partial}{\partial x_{\ell_{j+1}}}\cdots\frac{\partial}{\partial x_{\ell_s}}$$

and the dots (present only if $s\ge 2$) represent a sum which is bilinear in f and g involving derivatives of f of order ≥ 2. Thus

$$\partial^s(dv)=d(\partial^s v)+Q_s+R_s$$

where

$$Q_s=\sum_{j=1}^{s}\sum_{\nu=1}^{m}\frac{\partial a_\nu}{\partial x_{\ell_j}}\frac{\partial}{\partial x_\nu}\left(\partial_j^{s-1}v\right)$$

and

$$R_s=\sum_{\substack{\nu\\ p+q=s+1\\ l\le p\le s-1}}\left[v^{(p)}*a_\nu^{(q)}\right]$$

where $R_s\equiv 0$ for $s\le 1$, $Q_s\equiv 0$ for $s=0$ and the $*$ notation denotes a linear combination of terms $\partial^p v\partial^q a_\nu$. Now apply ∂^s to (4.43), multiply by $\partial^s v$ and sum to obtain

$$\begin{aligned}\sum_{[s]}\Big\{-\epsilon\left[\frac{1}{2}\triangle(\partial^s v,\partial^s v)-|\partial^s v_x|^2\right]+\frac{1}{2}d(\partial^s v,\partial^s v)\\ +(\partial^s v,P_2\partial^s v+Q_s)\Big\}=H\end{aligned}\tag{4.44}$$

where

$$H=\sum_{[s]}(\partial^s v,\partial^s G_2+\tilde{R}_s)$$

with

$$\tilde{R}_s=-R_s+\sum_{\substack{t+p=s\\ 0\le p\le s-1}}\left[P_2^{(t)}*v^{(p)}\right].$$

We now show that

$$\sum_{[s]}(\partial^s v,P_2\partial^s v+Q_s)\ge(\beta+s\lambda')\,|v^{(s)}|^2.\tag{4.45}$$

From the assumption (2),

$$\sum_{[2]}(\partial^s v, P_2 \partial^s v) \;\; \geq \beta \sum_{[s]} (\partial^s v, \partial^s v)$$

$$= \beta \, |v^{(s)}|^2.$$

For the second part, S, we have

$$S \equiv \sum_{[s]} (\partial^s v, Q_s) = \sum_{i=1}^{n_2} \sum_{[s]} \partial^s v_i (Q_s)_i$$

$$= \sum_{i=1}^{n_2} \sum_{j=1}^{s} \sum_{\substack{\ell_p = 1 \\ p=1,\ldots,s \\ p \neq j}}^{m} \left\{ \sum_{\nu, \ell_j = 1}^{m} \frac{\partial a_\nu}{\partial x_{\ell_j}} \left[\frac{\partial}{\partial x_\nu} \left(\partial_j^{s-1} v_i \right) \right] \partial^s v_i \right\}.$$

The term in braces is a quadratic form,

$$K \equiv \sum_{\nu, \ell_j = 1}^{m} \alpha_{\nu, \ell_j} \xi_{\nu e} \xi_{\ell_j e},$$

where

$$\alpha_{\nu, \ell_j} = \frac{\partial a_\nu}{\partial x_{\ell_j}}, \quad \xi_{\nu e} = \frac{\partial}{\partial x_\nu} \left(\partial_j^{s-1} v_i \right)$$

$$\xi_{\ell_j e} = \partial^s v_i = \tfrac{\partial}{\partial x_{\ell_j}} \left(\partial_j^{s-1} v_i \right)$$

and e is the vector of subscripts

$$e = (i, \ell_1, \ldots \ell_{j-1}, \ell_{j+1}, \ldots \ell_s).$$

From the way in which λ' is defined,

$$K \geq \lambda' \sum_{k=1}^{m} \xi_{ke}^2$$

and therefore

$$S \geq \lambda' \sum_{j=1}^{s} \sum_{i=1}^{n_2} \sum_{\substack{k, \ell_p = 1 \\ p=1,\ldots,s \\ p \neq j}}^{m} \left[\frac{\partial}{\partial x_k} \left(\partial_j^{s-1} v_i \right) \right]^2$$

$$= \lambda' \sum_{j=1}^{s} |v^{(s)}|^2 = s\lambda' |v(s)|^2$$

which proves (4.45). We now repeat the argument after (4.43) where x_0 is a point at which

$$|v^{(s)}|^2 \equiv \sum_{[s]} (\partial^s v, \partial^s v)$$

is a maximum to obtain

$$(\beta + s\lambda')|v^{(s)}|_0 \le |H|_0 \le |G^{(s)}|_0 + |\tilde{R}_s|_0.$$

Using the fact that

$$[\beta + s\lambda'] \le [\beta + (s-1)\lambda'] \le \ldots \le \beta + \lambda' \le \beta \le |P|_0 \tag{4.46}$$

we write this as

$$(\beta + s\lambda')^{s+1}|v^{(s)}|_0 \le \beta^s|G|_s + [\beta + (s-1)\lambda']^s|\tilde{R}_s|_0.$$

We now prove (4.41) inductively. Assume it is true with s replaced by $0, 1, \ldots, s-1$. The expression $\tilde{R}_s$ is linear in v and each term contains a factor $\partial^\rho v$ with $\rho \le s-1$. Using the induction assumption and (4.46) we obtain

$$(\beta + s\lambda')^{s+1}|v^{(s)}|_0 \le |G|_s \prod^{s\,\prime}(|A_{xx}|_{s-2}, |P|_s)$$

where $\prod^{s\,\prime}$ is a polynomial of the type described after (4.41). (Note: $|a_x| = |\mu A_x| \le |A_x|$ since $0 < \mu < 1$. In deriving the estimate for u in (4.42) we get $|ca_x| = |A_x|$ and hence the estimates are exactly the same.) From (4.41), with s replaced by $s-1$, we have

$$(\beta + s\lambda')^{s+1}|v|_{s-1} \le |G|_s \prod^{s\,\prime\prime}(|A_{xx}|_{s-2}, |P|_s)$$

where we have again used (4.46) and the fact that $|\ |_p \le |\ |_q$ for $p < q$.

Now define $\prod^{s}$ to be the polynomial whose coefficients are twice the maximum of the corresponding coefficients in $\prod^{s\,\prime}$ and $\prod^{s\,\prime\prime}$. Then (4.41) follows since

$$|v|_s = \max(|v^{(s)}|_0,\ |v|_{s-1}).$$

In regard to the assumption made on ℓ in (1), note that in (4.44) for $s = r$ we need $v \in C^{r+2}$ if $\epsilon > 0$ and $v \in C^{r+1}$ if $\epsilon = 0$. Q.E.D.

Finally comes the problem of existence of a solution of equation (4.40), $Lw = G$. We no longer employ the maximum norm used in the solution of the quasi-linear equation, but introduce the Hilbert space of square integrable functions having period parallelogram R. By obtaining an "a-priori" estimate for a solution of the adjoint equation we prove the existence of a "weak" solution of $Lw = G$. The proof follows that of Friedrichs [12; p. 353]. Certain difficulties, which arise in considering domains with boundaries, disappear due to the periodicity. The C^∞ differentiability of the weak solution then follows from a result of P. D. Lax [23]. The following lemma is a statement of these classical results.

Lemma 5. - (Existence) In (4.40) assume

1. $\epsilon > 0$
2. $(w, [P(x,\mu) - \frac{1}{2}\, div\; a(x,\mu)C]w) \geq p(w,w)$ for $(x,\mu) \in R \times I_{\bar{\mu}}$ and $p > 0$ a constant
3. for each $\mu \in I_{\bar{\mu}},\;\; A, P, G \in C^{\infty}(x)$.

   Then for each fixed $\mu \in I_{\bar{u}}$, (4.40) has a unique solution $w(x,\mu) \in C^{\infty}(x)$.

Proof. For the equation $Lw = G$ we have

$$\int_R (w, Lw) = \int_R (w, G)$$

and integrating by parts

$$\epsilon \int_R |w_x|^2 + \int_R (w, [P - \frac{1}{2}\, div\; a\; C]w) = \int (w, G)$$

where periodicity was used to drop out the boundary terms. From this we obtain

$$\|w\| \leq \frac{1}{p}\|G\|. \tag{4.47}$$

For the adjoint operator $L^*\phi \equiv -\epsilon\triangle\phi - D(x,\mu)\phi + \bar{P}(x,\mu)\phi$ with $\bar{P} = P - (div\; a)C$, we obtain the same estimate

$$\|\phi\| \leq \frac{1}{p}\|L^*\phi\| \tag{4.48}$$

Let $L_2(R)$ denote the Hilbert space of functions $f(x),\; x \in R$, such that $\|f\| < \infty$. Using (4.48), for each fixed μ, we prove the existence of a "weak" solution of $Lw = G$, i.e., a function $w(x,\mu) \in L_2(R)$ such that $(w, L^*\phi)_2 = (G,\phi)_2$ for all ϕ in a dense subset of $L_2(R)$. For define

$$Q = \{q : q = L^*\phi, \quad \phi \in C^{\infty}(x), \quad x \in R, \quad \phi$$

$$\text{periodic with period parallelogram } R\}$$

On Q define the functional $\psi(q) = (G,\phi)_2$. The estimate (4.48) implies uniqueness for the adjoint equation and hence ψ is well defined, i.e., given $q \in Q$, there is exactly one $\phi \in C^{\infty}(x)$ such that $q = L^*\phi$. ψ is bounded and linear due to (4.48) and the linearity of L^*. Thus by Riesz's representation theorem $\psi(q) = (w,q)_2$ for some $w(x,\mu) \in L_2(R)$, i.e., $(G,\phi)_2 = (w, L^*\phi)_2$.

In the previous reference to P. D. Lax, it is shown that for such a weak solution $w(x,\mu)$, if G and the coefficients of L have derivatives in x up to order t which are in $L_2(R)$ then

$w(x,\mu)$ has derivatives in x up to order $t+2$ which are in $L_2(R)$. This implies, by Sobolev's lemma, that $w(x,\mu) \in C^r(x)$ where $r = t - [m/2] + 1$. Hence in our case $w(x,\mu) \in C^\infty(x)$ for each fixed μ.

Uniqueness follows from the estimate (4.47).

Q.E.D.

5. An Example of Non-parallelizable Flow

The following is an example of the situation described in Chapter I in which the flow on the unperturbed surface $\tau(0)$ is not parallelizable. Here $\tau(0)$ is a 2-dimensional torus having 1-dimensional invariant submanifolds which intersect at stationary points (Fig. 13).

$$\dot{\theta}_1 = -k_1 \sin\theta_1 + \mu F_1(y,\theta_1,\theta_2,\mu)$$

$$\dot{\theta}_2 = -k_2 \sin\theta_2 + \mu F_2(y,\theta_1,\theta_2,\mu)$$

$$\dot{y} = -y + \mu G(y,\theta_1,\theta_2,\mu)$$

where $k_i > 0$ are constants, y is distance measured along the normal to the surface and F_i and $G \in C^\infty$ in all arguments.

In this example $\beta = 1$ and $\lambda = -\max(k_1,k_2)$. Applying Theorem 8 we see that if $-\lambda < 1/r$, r an integer ≥ 2, then for μ sufficiently small there exists an invariant manifold $y = \psi(\theta_1,\theta_2,\mu)$ where ψ has $r-1$ Lipschitz continuous derivatives with respect to θ_1 and θ_2 and is continuous in μ with $\psi(\theta_1,\theta_2,0) \equiv 0$.

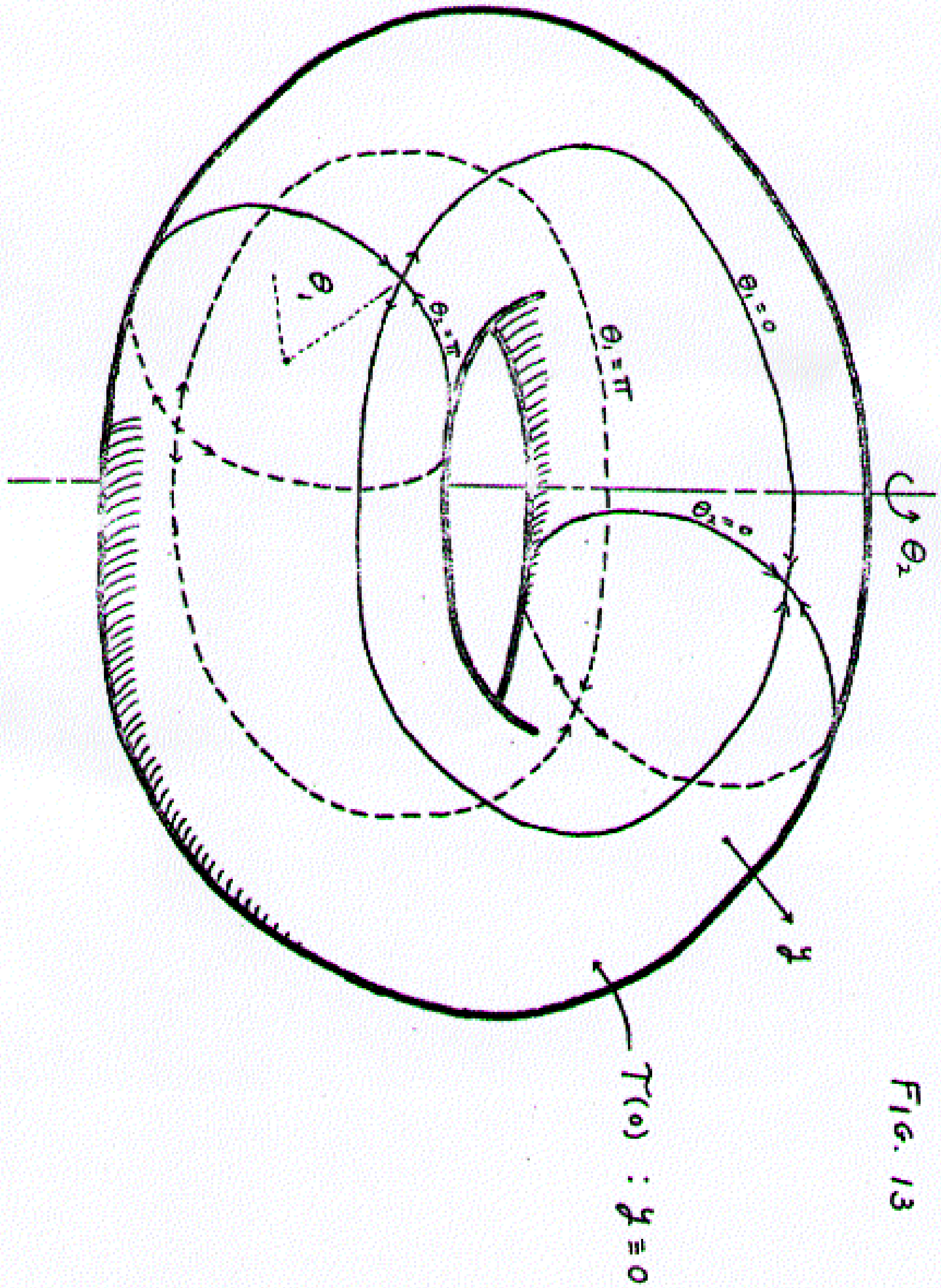


FIG. 13

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