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RESEARCH ARTICLE

Introduction to the 2008 Re-Publication of the "Neimark-Sacker Bifurcation Theorem"

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We give a brief introduction to re-publication of Chapter 2 of the author's 1964 Dissertation. The result has come to be known as the Neimark-Sacker bifurcation theorem. Some of the applications of the theorem are cited and a brief overview of the reduction of the mapping to normal form is given in 3-dimensions. By introducing weighted monomials the reduction method is carried out without the use of the Center Manifold theorem which was not known to the author and was published the same year as the dissertation. The first detailed proof of the theorem is carried out in the above cited Chapter 2 and is contained in the companion article to follow this one.

Keywords: Neimark-Sacker bifurcation theorem, normal form, weighted monomials

AMS Subject Classification: 37C05

1. Introduction

The article which follows this one is a re-publication of Chapter 2 of the author's 1964 Thesis and bears the same title as the Thesis. The result has come to be known, due in great measure to the book of Yuri Kuznetsov [16], as the Neimark-Sacker bifurcation theorem.

The Theorem has found its way into many branches of Science, e.g. Fluids Mechanics [18, 21, 30], Mathematical Biology [15, 36], Cycling in Genetics [28], Epidemiology [4], Neural Networks [9, 33, 38], Economics [1, 6, 7], Computational Methods [20, 35], Mechanical Engineering [29, 32], Chaos and Time Series Analysis [31], Habit Formation and Addiction [5], Decoding Algorithms [34] to name a few.

Along the lines of an earlier result of Andronov [2] for ordinary differential equations, Neimark [22] conjectured that an invariant curve would bifurcate from a fixed point of a mapping provided a certain non-linear term has the correct sign. Later the author [25] independently stated and proved the bifurcation theorem for maps for the first time and discovered that certain low order "resonances", referred to as strong resonances by Iooss & Joseph [13, p.254] had to be avoided in order to transform the mapping into normal form, i.e. the two complex eigenvalues of the linearized map at the fixed point must avoid 3rd and 4th roots of unity as they leave the unit circle. An additional condition is needed and, not unlike that required by [2] and Hopf [11] says that initial states far from the fixed point are mapped toward the fixed point. The result was later restated by [24]. Also see

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[3, 8, 10, 17, 19, 37]. Strictly speaking, Hopf did not "require" nonlinear damping but rather stated an alternative [11, p. 5]; generically either the family of periodic orbits exists and is asymptotically stable as the parameter μ *increases* through zero or the family exists and is unstable as μ *decreases* through zero.

Once the mapping is in normal form the problem reduces to one of *perturbation of invariant manifolds*. In the following article that problem is solved for the case of an invariant circle. For ordinary differential equations it was extended in [26] to invariant tori and in [27], it was extended to imbedded Riemann manifolds and the Stable Manifold Theorem was established as a corollary.

The normal form is used in the *proof* of the theorem and is achieved by applying near-identity transformations to remove all quadratic terms and all but one of the cubic terms whose coefficient determines the nonlinear damping. As many authors above observed, the nonlinear damping coefficient remains unchanged when removing the other cubic terms. Thus in applying the theorem, removal of the quadratic terms in the two variables in the plane spanned by the eigenvectors of the eigenvalues causing the bifurcation, is all that is needed in order to *verify* the nonlinear damping.

All the proofs known to the author invoke the Pliss Reduction Principle [23] (aka the Center Manifold Theorem, Kelley [14]) and then proceed to attain the normal form in $\mathbb{R}^2$ with the usual near-identity transformations. In an application, this involves two separate operations which may not be the most efficient way to proceed. In the original work, the Center Manifold Theorem was not known to the author so the entire reduction was done using near-identity transformations with "weighted monomials". There is, however, a formula for this nonlinear damping term in Kuznetsov [16, p. 187] that combines these two steps. It was originally derived by Iooss, *et.al.* [12]. For similar derivations see [8, 10, 19]. The procedure is illustrated in the following brief overview (excerpt taken from [28]).

2. Brief Overview (in $\mathbb{R}^3$)

We start with a C^4 mapping $x^* = F(x, \mu), \quad \mu \in (-1, 1)$ and assume there is a smooth family of fixed points $\hat{x}(\mu)$ depending on the parameter μ. Assume further that the spectrum of the linearized equation along the family lies inside the unit circle S in the complex plane for $\mu \in (-1, 0)$ and a complex pair of eigenvalues $\{\lambda(\mu), \bar{\lambda}(\mu)\}$ crosses S transversally as μ increases through 0. The remaining eigenvalue $\sigma(\mu)$ remains inside S. We can transform the family of fixed points to the origin in $\mathbb{R}^3$ and by linear transformation put the mapping into the form

$$y^* = \begin{bmatrix} \Re\lambda(\mu) & -\Im\lambda(\mu) & 0 \\ \Im\lambda(\mu) & \Re\lambda(\mu) & 0 \\ 0 & 0 & \sigma(\mu) \end{bmatrix} y + f(y, \mu), \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{1}$$

Using the procedure introduced in [25] we define $z = y_1 + iy_2$ and $W = y_3$. Then the mapping takes the form

$$\begin{aligned} z^* &= \lambda(\mu)z + P(z, \bar{z}, W, \mu) \\ W^* &= \sigma(\mu)W + Q(z, \bar{z}, W, \mu), \end{aligned} \tag{2}$$

where P and Q are functions in $z, \bar{z}$ and W whose series expansions begin with quadratic terms whereas $P(0, 0, 0, \mu) = Q(0, 0, 0, \mu) = 0$ identically for μ in some small neighborhood of zero. Since all the conditions for bifurcation are verified at $\mu = 0$ we shall suppress the μ dependence until it is actually needed. Thus we consider

$$\begin{aligned} z^* &= \lambda z + P(z, \bar{z}, W) \\ W^* &= \sigma W + Q(z, \bar{z}, W), \end{aligned} \tag{3}$$

and proceed to put this into the desired canonical form where the bifurcation conditions can be verified.

The first step is to "remove", from the first equation of (3), all the quadratic terms of the form $z^k \bar{z}^l,\ k + l = 2$ by means of the transformation

$$z = \zeta + a\zeta^2 + b\zeta\bar{\zeta} + c\bar{\zeta}^2\ ,$$

in a sufficiently small neighborhood of the origin. If $\lambda^3 \neq 1,\ a, b$ and c can be uniquely chosen so that in the new mapping

$$\begin{aligned} \zeta^* &= \lambda\zeta + \ldots \\ W^* &= \sigma W + \ldots, \end{aligned} \tag{4}$$

the *first* equation is devoid of these *specific* quadratic terms.

The next step is to "remove" the same type quadratic terms from the second equation of (4) by means of the transformation

$$W = Y + \alpha\zeta^2 + \beta\zeta\bar{\zeta} + \gamma\bar{\zeta}^2.$$

The fact that $|\sigma| < 1$ guarantees that α,β and γ can be uniquely chosen so that in the new mapping

$$\begin{aligned} \zeta^* &= \lambda\zeta + \ldots \\ Y^* &= \sigma Y + \ldots, \end{aligned} \tag{5}$$

the second equation is devoid of these *specific* quadratic terms. The others can stay.

In the *proof* of the bifurcation theorem it is assumed that $\lambda^\nu \neq 1\ for\ \nu = 1, 2, 3, 4$. The exclusion of the fourth roots of unity (together with $|\sigma| < 1$) is needed to remove terms of the form

$$\zeta^k\bar{\zeta}^l, \quad k + l = 3, k \neq 2; \qquad \zeta Y\ and\ \bar{\zeta}Y \tag{6}$$

from the first equation of (5) to obtain the canonical form

$$\begin{aligned} \zeta^* &= \lambda\zeta + \delta\zeta^2\bar{\zeta} + \mathcal{F}_4(\zeta, \bar{\zeta}, Y) \\ Y^* &= \sigma Y + \mathcal{F}_3(\zeta, \bar{\zeta}, Y), \end{aligned} \tag{7}$$

where the Taylor expansion of $\mathcal{F}_j$ has no monomials of "weight" $< j$. The *weight* τ of the monomial $Y^s\zeta^t\bar{\zeta}^u$ is defined to be $\tau = 2s + t + u$. It is this unbalanced scaling of the variables that allows us to avoid the center manifold theorem and treat the problem by straightforward perturbation techniques.

Upon close inspection of the transformations needed to eliminate the terms (6), e.g,

$$\zeta = w + \alpha w^k \bar{w}^l, \quad \zeta = w + \gamma w Y \quad \textit{and} \quad \zeta = w + \gamma \bar{w} Y,$$

it is easy to see that the coefficient δ in (7) is the same as the coefficient of the same monomial in (5). Thus in the *verification* that a bifurcation happens (as opposed to the *proof* of the bifurcation theorem) one may stop at (5) and read off the coefficient δ in (7).

Continuing with (7) we rewrite

$$\begin{aligned}
\zeta^* &= \lambda\zeta + b\zeta^2\bar{\zeta} + \mathcal{F}_4(\zeta, \bar{\zeta}, Y) && (8)\\
&= \lambda(1 + \beta|\zeta|^2)\zeta + \mathcal{F}_4 \\
&= e^{\alpha} e^{\beta|\zeta|^2}\zeta + \hat{\mathcal{F}}_4 \\
&= e^{\alpha(\mu)+\beta(\mu)|\zeta|^2}\zeta + \hat{\mathcal{F}}_4(\zeta, \bar{\zeta}, Y, \mu) \\
&= e^{\alpha(0)+\alpha'(0)\mu+\beta(0)|\zeta|^2}\zeta + \mu^2\hat{\mathcal{F}}_2 + \mu\hat{\mathcal{F}}_3 + \hat{\mathcal{F}}_4 && (9)\\
Y^* &= \sigma(\mu)Y + \mathcal{F}_3(\zeta, \bar{\zeta}, Y, \mu),
\end{aligned}$$

where

$$\begin{aligned}
&\alpha(\mu) = \ln\lambda(\mu) = \alpha(0) + \alpha'(0)\mu + \ldots \quad \text{and} \\
&\beta(0) = b(0)/\lambda(0).
\end{aligned}$$

In the last three equalities in (8) we have returned the μ dependence. Note, however, that the μ dependence in the $\hat{\mathcal{F}}_j$ comes about only through *coefficients* of monomials of weight j and higher.

In the first approximation we drop the $\mathcal{F}$ terms and the mapping takes the form

$$\begin{aligned}
\zeta^* &= \zeta \exp\left[\Re\alpha'(0)\mu + \Re\beta(0)|\zeta|^2\right] \exp i[\Im\alpha(0) + \Im\beta(0)|\zeta|^2] && (10)\\
Y^* &= \sigma Y.
\end{aligned}$$

It follows that this mapping has an invariant circle

$$\begin{aligned}
\zeta &= a_0\sqrt{\mu}\exp(i\theta), \qquad 0 \le \theta \le 2\pi && (11)\\
Y &= 0,
\end{aligned}$$

provided $\Re\alpha'(0) > 0$ and $\Re\beta(0) < 0$ with $a_0 = \sqrt{-\Re\alpha'(0)/\Re\beta(0)}$.

The task remaining is to prove the main theorem which is esentially a *perturbation* theorem.

Theorem 2.1: *Assume that in the C^4 system (1),*

(1) $|\lambda(0)| = 1, \quad \lambda^3(0) \neq 1,$ *and* $\lambda^4(0) \neq 1$ *(non resonance)*
(2) $\Re\alpha'(0) > 0$ *where* $\alpha(\mu) = \log\lambda(\mu)$ *(loss of stability)*
(3) $|\sigma(0)| < 1$ *(stability in "normal" direction) and*
(4) In the normal form (9), $\Re\beta(0) < 0$ *(nonlinear damping)*

Then in a sufficiently small μ neighborhood, $0 < \mu < \mu_*$, (9) has an asymptotically *stable invariant curve parametrized by* $\theta,\ 0 \le \theta \le 2\pi$

$$\begin{aligned} \zeta &= a_0\sqrt{\mu}\exp{(i\theta)} + \mu g(\theta,\mu) \\ Y &= \mu h(\theta,\mu), \end{aligned} \tag{12}$$

where $a_0 = \sqrt{-\Re\alpha'(0)/\Re\beta(0)}$.

The proof of this theorem will be carried out in the next article where the significance of the "weighted monomials" will become apparent.

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