QUENCHED LIMITING ENTRY DISTRIBUTIONS FOR RANDOM EXPANDING MAPS WITH NULL TARGETS

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Abstract. We describe an approach that allows us to deduce the limiting return times distribution for arbitrary sets to be compound Poisson distributed. We establish a relation between the limiting return times distribution and the probability of the cluster sizes, where clusters consist of the portion of points that have finite return times in the limit where random return times go to infinity. In the special case of periodic points we recover the known Pólya-Aeppli distribution which is associated with geometrically distributed cluster sizes. We apply this method to several examples the most important of which is synchronisation of coupled map lattices. For the invariant absolutely continuous measure we establish that the returns to the diagonal is compound Poisson distributed where the coefficients are given by certain integrals along the diagonal.

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Date: August 30, 2023.

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1. Introduction

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2. Assumptions and main results

2.1. **General setup.** Let M be a Polish space, and Ω, the so-called driving space, be a Polish space equipped with a measurably-invertible probability preserving system (θ, ν).

Consider maps $T_\omega : M \to M$ $(\omega \in \Omega)$ which combine to make the a measurable skew product $S : \Omega \times M \to \Omega \times M$, $(\omega, x) \mapsto (\theta\omega, T_\omega x)$. As usual, for higher-order iterates we denote $S^n(\omega, x) = (\theta^n \omega, T^n_\omega(x))$ where $T^n_\omega = T_{\theta^{n-1}\omega} \circ \cdots \circ T_{\theta\omega} \circ T_\omega$.

Consider probability measures $\mu^\omega \in \mathcal{P}(M)$ $(\omega \in \Omega)$ which combine to form i) a random measure over M in the sense that $\omega \mapsto \mu^\omega(B)$ is measurable $(\forall B \in \mathscr{B}_M)$, and ii) a quasi invariant family in the sense that $\mu^{\theta\omega} = T_{\omega *}\mu^\omega$ $(\forall \omega \in \Omega)$.

Consider $\Gamma \subset \Omega \times M$ a measurable subset whose ω-sections $\Gamma(\omega) \subset M$ have null μ^ω-measure. The set Γ is the so-called random target.

The objects considered above comprise what we call a 'targeted random dynamical system', or simply 'system', to be denoted by the tuple $(\theta, \nu, T_\omega, \mu^\omega, \Gamma)$. Now we introduce some other objects, derived from the previous ones.

Define the stationary measure $\check{\mu} = \int_\Omega \mu^\omega \, d\nu(\omega) \in \mathcal{P}(M)$ and S-invariant measure $\hat{\mu} \in \mathcal{P}(\Omega \times M)$ given by $d\hat{\mu}(\omega, x) = d\mu^\omega(x) d\nu(\omega)$. Notice that $\check{\mu} = \pi_{\Omega *}\hat{\mu}$.

Define $\Gamma_\rho(\omega) = B_\rho(\Gamma(\omega))$ $(\rho > 0)$ and the corresponding ω-collection by Γ_ρ. Moreover, for $U \subset \Omega \times M$ whose ω-sections $U(\omega) \subset M$ are sets of positive μ^ω-measure, put $I^\omega_i = \mathbb{1}_{U(\theta^i\omega)} \circ T^i_\omega$ and define the counting functions

$$Z^{\omega,L}_U = \sum_{i=0}^{L} I^\omega_i \text{ and } Z^{\omega,L}_{*U} = \sum_{i=1}^{L} I^\omega_i \ (L \in \mathbb{N}_{\geqslant 0}). \tag{1}$$

Finally, define first hitting times by

$$\tau^\omega_U(x) = \inf\{j \geqslant 1 : T^j_\omega(x) \in U(\theta^j\omega)\}.$$

and their higher order counterparts by putting $\tau^{\omega,1}_U = \tau^\omega_U$ and recursively

$$\tau^{\omega,\ell}_U(x) = \tau^{\omega,\ell-1}_U + \tau^{\omega'}_U(T^{\tau^{\omega,\ell-1}_U}_\omega(x)),$$

where $\omega' = \theta^{\tau^{\omega,\ell-1}_U(x)}\omega$. Notice that $\{Z^{\omega,L}_{*U} \geqslant \ell\} = \{\tau^{\omega,\ell}_U \leqslant L\}$ and $\{Z^{\omega,L}_{*U} = \ell\} = \{\tau^{\omega,\ell}_U \leqslant L < \tau^{\omega,\ell+1}_U\}$.

Notation: A $\mathbb{R}$-valued function defined on the product space, $f(\omega, x)$, is often rewritten with the random seed in the sup/subscript, like $f^\omega(x)$ or $f_\omega(x)$, which can be seen as an Ω-family of functions defined on M. And vice versa. When integrating a function, we may simply omit the dummy variable of integration, even if it is a sup/subscript. We leave it for the reader to infer what variables and parameters are being integrated and were omitted. Some examples: i) $\hat{\mu}(f) = \int_{\Omega \times M} f(\omega, x) d\hat{\mu}(\omega, x) = \int_{\Omega \times M} f_\omega(x) d\hat{\mu}(\omega, x) = \int_\Omega \mu^\omega(f_\omega) d\nu(\omega)$, with $\mu^\omega(f_\omega) = \int_M f_\omega(x) d\mu^\omega(x) = \int_M f(\omega, x) d\mu^\omega(x)$; ii) $\hat{\mu}(Z^L_{\Gamma_\rho}) = \int_{\Omega \times M} Z^{\omega,L}_{\Gamma_\rho}(x) d\hat{\mu}(\omega, x) = \int_\Omega \mu^\omega(Z^{\omega,L}_{\Gamma_\rho}) d\nu(\omega)$, with $\mu^\omega(Z^{\omega,L}_{\Gamma_\rho}) = \int_M Z^{\omega,L}_{\Gamma_\rho}(x) d\mu^\omega(x)$. If the aforementioned f is $\{0, 1\}$-valued we identify it with the set $F = f^{-1}(\{1\})$, since $f = \mathbb{1}_F$, whereas its partials f_ω

are identified with the ω-sections of F, denoted F_ω, since $f_\omega = \mathbb{1}_{F_\omega}$. And vice versa. So, instead of i), we could write i') $\hat{\mu}(F) = \int_{\Omega\times M} \mathbb{1}_F(\omega, x)d\hat{\mu}(\omega, x) = \int_{\Omega\times M} \mathbb{1}_{F_\omega}(x)d\hat{\mu}(\omega, x) = \int_M \mu^\omega(F_\omega)d\nu(\omega)$, with $\mu^\omega(F_\omega) = \int_M \mathbb{1}_{F_\omega}(x)d\mu^\omega(x) = \int_M \mathbb{1}_F(\omega, x)d\mu^\omega(x)$.

2.2. **Working setup.** Now we upgrade the general setup of section 2.1. To optimize for generality, we present in abstract terms the conditions which are required from the systems we'll work with. In concrete examples, these conditions need to be verified, but one should keep in mind that they are conceived to accommodate non-uniformly expanding behavior and random targets which don't overlap very badly with the regions where uniformity breaks.

Convention: We'll write $\lim_{L\to\infty} \overline{\underline{\lim}}_{\rho\to 0} a(L, \rho)$ to refer to the coinciding value of $\lim_{L\to\infty} \overline{\lim}_{\rho\to 0} a(L, \rho)$ and $\lim_{L\to\infty} \underline{\lim}_{\rho\to 0} a(L, \rho)$, when they do exist and coincide.

We start introducing new objects and notation. Before items identified with **H** we are not introducing new hypotheses, in particular, the following objects are not a priori assumed to exist – if they do, then the notation to be proposed stands.

Whenever the following limits exist (and the appropriate ones coincide), denote

I)

$$\lambda_\ell^\omega = \lim_{L\to\infty} \overline{\underline{\lim}}_{\rho\to 0} \lambda_\ell^\omega(L, \rho) = \lim_{L\to\infty} \underbrace{\overline{\lim}_{\rho\to 0} \lambda_\ell^\omega(L, \rho)}_{=\overset{+}{\lambda}{}_\ell^\omega(L)} = \lim_{L\to\infty} \underbrace{\underline{\lim}_{\rho\to 0} \lambda_\ell^\omega(L, \rho)}_{=\overset{-}{\lambda}{}_\ell^\omega(L)},$$

where

$$\lambda_\ell^\omega(L, \rho) = \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} = \ell | Z_{\Gamma_\rho}^{\omega,L} > 0). \tag{2}$$

II)

$$\lambda_\ell = \lim_{L\to\infty} \overline{\underline{\lim}}_{\rho\to 0} \lambda_\ell(L, \rho) = \lim_{L\to\infty} \underbrace{\overline{\lim}_{\rho\to 0} \lambda_\ell(L, \rho)}_{=\overset{+}{\lambda}_\ell(L)} = \lim_{L\to\infty} \underbrace{\underline{\lim}_{\rho\to 0} \lambda_\ell(L, \rho)}_{=\overset{-}{\lambda}_\ell(L)},$$

where

$$\begin{aligned} \lambda_\ell(L, \rho) = \hat{\mu}(Z_{\Gamma_\rho}^L = \ell | Z_{\Gamma_\rho}^L > 0) &= \frac{\int_\Omega \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} = \ell)d\nu(\omega)}{\int_\Omega \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} > 0)d\nu(\omega)} \\ = \int_\Omega \lambda_\ell^\omega(L, \rho) \frac{\mu^\omega(Z_{\Gamma_\rho}^{\omega,L} > 0)}{\int_\Omega \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} > 0)d\nu(\omega)} d\nu(\omega) &= \int_\Omega \lambda_\ell^\omega(L, \rho)d\nu_{L,\rho}(\omega), \end{aligned} \tag{3}$$

with

$$d\nu_{L,\rho}(\omega) := \frac{\mu^\omega(Z_{\Gamma_\rho}^{\omega,L} > 0)}{\int_\Omega \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} > 0)d\nu(\omega)} d\nu(\omega).$$

III)

$$\hat{\alpha}_\ell^\omega = \lim_{L\to\infty} \overline{\underline{\lim}}_{\rho\to 0} \hat{\alpha}_\ell^\omega(L, \rho) = \lim_{L\to\infty} \underbrace{\overline{\lim}_{\rho\to 0} \hat{\alpha}_\ell^\omega(L, \rho)}_{=\overset{+}{\hat{\alpha}}{}_\ell^\omega(L)} = \lim_{L\to\infty} \underbrace{\underline{\lim}_{\rho\to 0} \hat{\alpha}_\ell^\omega(L, \rho)}_{=\overset{-}{\hat{\alpha}}{}_\ell^\omega(L)},$$

where

$$\hat{\alpha}_\ell^\omega(L,\rho) = \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} \geqslant \ell | I_0^\omega = 1) = \mu^\omega(Z_{*\Gamma_\rho}^{\omega,L} \geqslant \ell - 1 | I_0^\omega = 1) = \mu^\omega(\tau_{\Gamma_\rho}^{\omega,\ell-1} \leqslant L | I_0^\omega = 1). \quad (4)$$

Notice that, by L-monotonicity, the outer limits always exist provided that the inner ones do.

IV)

$$\alpha_\ell^\omega = \lim_{L\to\infty} \overline{\underline{\lim}}_{\rho\to 0} \alpha_\ell^\omega(L,\rho) = \lim_{L\to\infty} \underbrace{\overline{\lim}_{\rho\to 0} \alpha_\ell^\omega(L,\rho)}_{=\overset{+}{\alpha}_\ell^\omega(L)} = \lim_{L\to\infty} \underbrace{\underline{\lim}_{\rho\to 0} \alpha_\ell^\omega(L,\rho)}_{=\overset{-}{\alpha}_\ell^\omega(L)},$$

where

$$\alpha_\ell^\omega(L,\rho) = \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} = \ell | I_0^\omega = 1) = \mu^\omega(Z_{*\Gamma_\rho}^{\omega,L} = \ell - 1 | I_0^\omega = 1) = \mu^\omega(\tau_{\Gamma_\rho}^{\omega,\ell-1} \leqslant L < \tau_{\Gamma_\rho}^{\omega,\ell} | I_0^\omega = 1). \quad (5)$$

Notice that $\{\tau_{\Gamma_\rho}^{\omega,\ell} \leqslant L\} \subset \{\tau_{\Gamma_\rho}^{\omega,\ell-1} \leqslant L\}$ implies $\{\tau_{\Gamma_\rho}^{\omega,\ell-1} \leqslant L < \tau_{\Gamma_\rho}^{\omega,\ell}\} = \{\tau_{\Gamma_\rho}^{\omega,\ell-1} \leqslant L\} \backslash \{\tau_{\Gamma_\rho}^{\omega,\ell} \leqslant L\}$ and so

$$\alpha_\ell^\omega(L,\rho) = \mu^\omega(\tau_{\Gamma_\rho}^{\omega,\ell-1} \leqslant L | I_0^\omega = 1) - \mu^\omega(\tau_{\Gamma_\rho}^{\omega,\ell} \leqslant L | I_0^\omega = 1) = \hat{\alpha}_\ell^\omega(L,\rho) - \hat{\alpha}_{\ell+1}^\omega(L,\rho). \quad (6)$$

Moreover, when $\hat{\alpha}_\ell^\omega$'s are defined, one has

$$\lim_{L\to\infty} \underline{\lim}_{\rho\to 0} \hat{\alpha}_\ell^\omega(L,\rho) - \lim_{L\to\infty} \overline{\lim}_{\rho\to 0} \hat{\alpha}_{\ell+1}^\omega(L,\rho) \leqslant \lim_{L\to\infty} \underline{\lim}_{\rho\to 0} \alpha_\ell^\omega(L,\rho)$$

$$\lim_{L\to\infty} \overline{\lim}_{\rho\to 0} \alpha_\ell^\omega(L,\rho) \leqslant \lim_{L\to\infty} \overline{\lim}_{\rho\to 0} \hat{\alpha}_\ell^\omega(L,\rho) - \lim_{L\to\infty} \underline{\lim}_{\rho\to 0} \hat{\alpha}_{\ell+1}^\omega(L,\rho)$$

$$\Rightarrow \alpha_\ell^\omega \text{ exists and } \alpha_\ell^\omega = \hat{\alpha}_\ell^\omega - \hat{\alpha}_{\ell+1}^\omega.$$

V)

$$\hat{\alpha}_\ell = \lim_{L\to\infty} \overline{\underline{\lim}}_{\rho\to 0} \hat{\alpha}_\ell(L,\rho) = \lim_{L\to\infty} \underbrace{\overline{\lim}_{\rho\to 0} \hat{\alpha}_\ell(L,\rho)}_{=\overset{+}{\hat{\alpha}}_\ell(L)} = \lim_{L\to\infty} \underbrace{\underline{\lim}_{\rho\to 0} \hat{\alpha}_\ell(L,\rho)}_{=\overset{-}{\hat{\alpha}}_\ell(L)},$$

where

$$\begin{aligned} \hat{\alpha}_\ell(L,\rho) &= \hat{\mu}(Z_{\Gamma_\rho}^L \geqslant \ell | I_0 = 1) = \frac{\int_\Omega \mu^\omega(Z_{\Gamma_\rho}^{\omega,L} \geqslant \ell, I_0^\omega = 1) d\nu(\omega)}{\int_\Omega \mu^\omega(\Gamma_\rho(\omega)) d\nu(\omega)} \\ &= \int_\Omega \hat{\alpha}_\ell^\omega(L,\rho) \frac{\mu^\omega(\Gamma_\rho(\omega))}{\int_\Omega \mu^\omega(\Gamma_\rho(\omega)) d\nu(\omega)} d\nu(\omega) = \int_\Omega \hat{\alpha}_\ell^\omega(L,\rho) d\nu_{0,\rho}(\omega), \end{aligned} \quad (7)$$

with

$$d\nu_{0,\rho}(\omega) := \frac{\mu^\omega(\Gamma_\rho(\omega))}{\int_\Omega \mu^\omega(\Gamma_\rho(\omega)) d\nu(\omega)} d\nu(\omega).$$

Notice that, by L-monotonicity, the outer limits always exist provided that the inner ones do.

VI)

$$\alpha_\ell = \lim_{L\to\infty} \overline{\underline{\lim}}_{\rho\to 0} \alpha_\ell(L,\rho) = \lim_{L\to\infty} \underbrace{\overline{\lim}_{\rho\to 0} \alpha_\ell(L,\rho)}_{=\overset{+}{\alpha}_\ell(L)} = \lim_{L\to\infty} \underbrace{\underline{\lim}_{\rho\to 0} \alpha_\ell(L,\rho)}_{=\overset{-}{\alpha}_\ell(L)},$$

where

$$\begin{aligned}\alpha_\ell(L,\rho) &= \hat{\mu}(Z^L_{\Gamma_\rho}=\ell|I_0=1) = \frac{\int_\Omega \mu^\omega(Z^{\omega,L}_{\Gamma_\rho}=\ell, I^\omega_0=1)d\nu(\omega)}{\int_\Omega \mu^\omega(\Gamma_\rho(\omega))d\nu(\omega)} \\ &= \int_\Omega \alpha^\omega_\ell(L,\rho)\frac{\mu^\omega(\Gamma_\rho(\omega))}{\int_\Omega \mu^\omega(\Gamma_\rho(\omega))d\nu(\omega)}d\nu(\omega) = \int_\Omega \alpha^\omega_\ell(L,\rho)d\nu_{0,\rho}(\omega),\end{aligned} \tag{8}$$

with

$$d\nu_{0,\rho}(\omega) := \frac{\mu^\omega(\Gamma_\rho(\omega))}{\int_\Omega \mu^\omega(\Gamma_\rho(\omega))d\nu(\omega)}d\nu(\omega).$$

Notice that, when $\hat{\alpha}_\ell$'s are defined, one has

$$\begin{gathered}\lim_{L\to\infty}\varliminf_{\rho\to 0}\hat{\alpha}_\ell(L,\rho) - \lim_{L\to\infty}\varlimsup_{\rho\to 0}\hat{\alpha}_{\ell+1}(L,\rho) \leqslant \lim_{L\to\infty}\varliminf_{\rho\to 0}\alpha_\ell(L,\rho) \\ \lim_{L\to\infty}\varlimsup_{\rho\to 0}\alpha_\ell(L,\rho) \leqslant \lim_{L\to\infty}\varlimsup_{\rho\to 0}\hat{\alpha}_\ell(L,\rho) - \lim_{L\to\infty}\varliminf_{\rho\to 0}\hat{\alpha}_{\ell+1}(L,\rho) \\ \Rightarrow \alpha_\ell \text{ exists and } \alpha_\ell = \hat{\alpha}_\ell - \hat{\alpha}_{\ell+1}.\end{gathered}$$

Now, on top of the features prescribed to the objects in our system throughout section 2.1, we'll consider the following hypotheses.

H1 (Invertibility features)**.**

1.1 (Degree)**.** $\forall\omega\in\Omega, \forall n\geqslant 1, \forall x\in M : (T^n_\omega)^{-1}(\{x\})$ *is at most countable*

1.2 (Covering)**.** $\exists R>0, \mathcal{N}\geqslant 1, \iota>0, \forall\omega\in\Omega, \forall n\geqslant 1, \exists(y^{\omega,n}_k)_{k\in K_{\omega,n}}$ *with* $\# K_{\omega,n}<\infty$*:*

- $M\backslash\bigcup_{k\in K_{\omega,n}} B(y^{\omega,n}_k, R)$ *is at most countable,*
- $\inf_{k\in K_{\omega,n}} \mu^{\theta^n\omega}(B(y^{\omega,n}_k,R)) > \iota$*,*
- $(B(y^{\omega,n}_k,R))_{k\in K_{\omega,n}}$ *has at most* $\mathcal{N}$ *overlaps.*

1.3 (Inverse branches)**.** $\forall\omega\in\Omega, \forall n\geqslant 1, \forall k\in K_{\omega,n}$,

$$\mathrm{IB}^{\omega,n}_k = \{\varphi : B(y^{\omega,n}_k,R)\to M \textit{ diffeomorphic onto its image with } T^n_\omega\circ\varphi = id\}$$

is non-empty, at most countable and so that $\varphi,\psi\in\mathrm{IB}^{\omega,n}_k, \varphi\neq\psi \Rightarrow \varphi(\mathrm{dom}(\varphi))\cap\psi(\mathrm{dom}(\psi)) = \varnothing$*. In particular, the set* $\mathrm{IB}(T^n_\omega) = \bigcup_{k\in K_{\omega,n}}\mathrm{IB}^{\omega,n}_k$ *is countable and so that* $\varphi,\psi\in\mathrm{IB}(T^n_\omega), \mathrm{dom}(\varphi)\cap\mathrm{dom}(\psi) = \varnothing \Rightarrow \varphi(\mathrm{dom}(\varphi))\cap\psi(\mathrm{dom}(\psi)) = \varnothing$*.*

The next item follows from the ones above, but we list it together for convenience.

1.4 (Cylinders)**.** $\forall\omega\in\Omega, \forall n\geqslant 1$, $C^\omega_n = \{\xi = \varphi(\mathrm{dom}(\varphi)) : \varphi\in\mathrm{IB}(T^n_\omega)\}$ *is countable and satisfies*

- $M\backslash\bigcup_{\xi\in C^\omega_n}\xi$ *is at most countable,*
- *the cover* C^ω_n *has at most* $\mathcal{N}$ *overlaps.*

H2 (Hyperbolicity features)**.**

2.1 (Good/bad sets)**.** $\forall\omega\in\Omega, \forall n\geqslant 1$, C^ω_n *splits like* $C^\omega_n = \overset{+}{C}{}^\omega_n \bigsqcup \overset{-}{C}{}^\omega_n$ *in such a way to form the measurable function* $\mathcal{G}_n(\omega,x) = \begin{cases}1, & \textit{if } x\in\bigcup_{\xi\in\overset{+}{C}{}^\omega_n}\xi \\ 0, & \textit{otherwise}\end{cases}$ *(whose complement in* M *is denoted by* $\bar{\mathcal{G}}_n$*). Not including the shrinking size of bad sets.*

2.2 (Distortion on good sets). $\exists \mathfrak{d} > 0, \exists C > 1$, $\forall \omega \in \Omega, \forall n \geqslant 1, \forall \varphi \in \mathrm{IB}(T^n_\omega)$ *(i.e., $\xi = \varphi(\mathrm{dom}(\varphi)) \in C^\omega_n$) with $\xi \subset \mathcal{G}^\omega_n$, $\forall x, y \in \xi$:*

$$\frac{J_\varphi(x)}{J_\varphi(y)} \leqslant Cn^{\mathfrak{d}}, \text{ where } J_\varphi(x) = \frac{d\varphi_* \mu^{\theta^n\omega}|_{\mathrm{dom}(\varphi)}}{d\mu^\omega|_{\varphi(\mathrm{dom}(\varphi))}}(x) = \frac{d\varphi_*\mu^{\theta^n\omega}|_{T^n_\omega \xi}}{d\mu^\omega|_\xi}(x).$$

2.3 (Backward contraction on good sets). $\exists \kappa > 1, D > 1, \forall \omega \in \Omega, \forall n \geqslant 1, \forall \varphi \in \mathrm{IB}(T^n_\omega)$ *(i.e., $\xi = \varphi(\mathrm{dom}(\varphi)) \in C^\omega_n$) with $\xi \subset \mathcal{G}^\omega_n$:*

$$D^{-1}e^{-n} \leqslant \inf_{x\in\mathrm{dom}(\varphi)} \|D\varphi(x)\|_{\mathrm{co}} \leqslant \sup_{x\in\mathrm{dom}(\varphi)} \|D\varphi(x)\| \leqslant Dn^{-\kappa} \ (\kappa' > \kappa), \ \mathrm{diam}(\xi) \leqslant Dn^{-\kappa},$$

where

$$\|D\varphi(x)\| = \sup_{\substack{v\in T_xM \\ \|v\|=1}} \|D\varphi(x)v\| \text{ and } \|D\varphi(x)\|_{\mathrm{co}} = \inf_{\substack{v\in T_xM \\ \|v\|=1}} \|D\varphi(x)v\|$$

are, respectively, the operator norm of the derivative map and its associated conorm.

In particular,

$$a_L := \sup_{\omega\in\Omega} \sup_{\varphi\in\mathrm{IB}(T^L_\omega)} \sup_{x\in\mathrm{dom}(\varphi)} \|D\varphi(x)\|_{\mathrm{co}}^{-1} = 1/\inf_{\omega\in\Omega} \inf_{\varphi\in\mathrm{IB}(T^L_\omega)} \inf_{x\in\mathrm{dom}(\varphi)} \|D\varphi(x)\|_{\mathrm{op}} \leqslant De^n.$$

H3 (Small overlap between target and bad sets).

3.1 (Averaged separation). *It holds that*

$$\lim_{L\to\infty} \overline{\lim_{\rho\to 0}} \sum_{n=L}^{\infty} \frac{\hat\mu(\bar{\mathcal{G}}_n \cap \Gamma_\rho)}{\hat\mu(\Gamma_\rho)} = 0.$$

3.2 (Quenched seperation). $\forall L \geqslant 1, \exists \rho_{\mathrm{sep}}(L) > 0, \forall \rho \leqslant \rho_{\mathrm{sep}}(L), \forall \omega \in \Omega$: $\Gamma_{3/2\rho}(\omega) \subset \mathcal{G}^\omega_L$.

Notice: I've included 2.3 to take care of the Lipschitz constant of approximations used lemma 3 and R1. The objective is to ensure that we are working with neighborhoods which are inside "differentiability regions"

Notice: The previous hypothesis, being valid even for $L = 1$, says that the starting covering presented in hypothesis (H1.2) (and therefore the associated covering with cylinder presented in hypothesis (H1.4)) is sufficiently refined relatively to the target set Γ, in such a way that one can select bad cylinders as in (H2.1) as to avoid them including target points.

Notice: Property 3.2 kills 3.1. Also: here we don't have much dynamical coherence in the C^ω_n's in respect to how cylinders compare when n increases. Do we need such coherence if we want to include Nicolai's "beta" cut (the one he used for the Pomeau-Maneville map) at this level of generality in the paper? I'll double check if $y^{\omega,n}_k$'s in 1.2 really need to change in both ω and n. Also, I've added norms and conorms near the derivatives in 2.3 to account for the Lipschitz constant of the approximations in higher dimension (as used in lemma 3 and R1

Notice: I still need to change some $\forall\omega \in \Omega$ to a.s. And change uniform bounds to L_∞ counterparts.

H4 (Measure regularity).

4.1 (Atomless). $\forall \omega \in \Omega$, *$\mu^\omega$ is atomless.*

4.2 (Ball regular). $\exists 0 < d_0 \leqslant d_1 < \infty, \exists C_0, C_1 > 0, \exists \rho_{\dim} \leqslant 1, \forall \rho \leqslant \rho_{\dim}, \forall \omega \in \Omega$:

$$C_1 \rho^{d_1} \leqslant \mu^\omega(\Gamma_\rho(\omega)) \leqslant C_0 \rho^{d_0}.$$

4.3 (Annulus regular). $\exists \eta \geqslant \beta > 0, \exists E > 0, \exists \rho_{\dim} \leqslant 1, \forall \rho \leqslant \rho_{\dim}, \forall r \in (0, \rho/2), \forall \omega \in \Omega$:

$$\frac{\mu^\omega(\Gamma_{\rho+r}(\omega) \backslash \Gamma_{\rho-r}(\omega))}{\mu^\omega(\Gamma_\rho(\omega))} \leqslant E \frac{r^\eta}{\rho^\beta}.$$

H5 (Decay of correlations). $\exists \mathfrak{p} > 1$ *so that*

5.1 (Quenched). $\forall \omega \in \Omega, \forall G \in \mathrm{Lip}_{d_M}(M, \mathbb{R}), \forall H \in L^\infty(M, \mathbb{R}), \forall n \geqslant 1$:

$$\left| \int_M G \cdot (H \circ T^n_\omega) d\mu^\omega - \mu^\omega(G) \mu^{\theta^n \omega}(H) \right| \lesssim n^{-\mathfrak{p}} \|G\|_{\mathrm{Lip}_{d_M}} \|H\|_\infty.$$

5.2 (Annealed). $\forall \omega \in \Omega, \forall G \in \mathrm{Lip}_{d_{\Omega \otimes M}}(\Omega \times M, \mathbb{R}), \forall H \in L^\infty(\Omega \times M, \mathbb{R}), \forall n \geqslant 1$:

$$\left| \int_{\Omega \times M} G \cdot (H \circ S^n) d\hat{\mu} - \hat{\mu}(G) \hat{\mu}(H) \right| \lesssim n^{-\mathfrak{p}} \|G\|_{\mathrm{Lip}(d_{\Omega \otimes M})} \|H\|_\infty,$$

where

$$d_{\Omega \otimes M}\left((\omega_1, x_1), (\omega_2, x_2)\right) = \begin{cases} \infty & \text{, if } \omega_1 \neq \omega_2 \\ d_M(x_1, x_2), & \text{if } \omega_1 = \omega_2 \end{cases} \Rightarrow \mathrm{Lip}_{d_{\Omega \otimes M}}(G) = \sup_{\omega \in \Omega} \mathrm{Lip}_{d_M}(G_\omega).$$

H6 (Hitting regular).

$$\exists (\lambda_\ell)_{\ell \geqslant 1}, \sum\nolimits_{\ell=1}^\infty \lambda_\ell = 1, \sum\nolimits_{\ell=1}^\infty \ell^3 \lambda_\ell < \infty.$$

H7 (Return regular).

$$\exists (\alpha_\ell)_{\ell \geqslant 1}, \alpha_1 > 0, \sum_{\ell=1}^\infty \alpha_\ell = 1, \sum_{\ell=1}^\infty \ell^2 \alpha_\ell < \infty.$$

We call α_1 *the extremal index.*

H7' (Pre return regular). *It holds that*

$$\exists (\hat{\alpha}_\ell)_{\ell \geqslant 1}, \hat{\alpha}_1 - \hat{\alpha}_2 > 0, \sum_{\ell=1}^\infty \ell \hat{\alpha}_\ell < \infty.$$

Using the final implication of item VI), it is immediate that (H7') $\Rightarrow$ (H7), because $\alpha_1 = \hat{\alpha}_1 - \hat{\alpha}_2 > 0$, $\sum_{\ell=1}^\infty \alpha_\ell = \hat{\alpha}_1 = 1$, and $\sum_{\ell=1}^\infty \ell^2 \alpha_\ell \leqslant 2 \sum_{\ell=1}^\infty \ell \hat{\alpha}_\ell < \infty$.

Moreover, for technical conditions, we assume that the quantities appearing in the previous hypotheses harmonize so that the following constraints hold. Mostly, they hold when (polynomial) decay is sufficiently fast.

H8 (Parametric constraints). *It holds that*

8.1. $d_0(\mathfrak{p} - 1) > \frac{\left(\frac{\beta + d_1}{\eta} \vee 1\right) + d_1}{d_0/d_1}$,

8.2. $d_0 \mathfrak{p} > \left(\frac{\beta + d_1}{\eta} \vee 1\right) + d_1$,

8.3. $d_0 > \max\{\frac{\beta+d_1(1+\eta)}{\mathfrak{p}\eta}, \frac{1+d_1}{\mathfrak{p}}\}$, *(I should discard this constraint soon)*

8.4. $0 \leqslant \mathfrak{d} < \frac{d_0\eta(\mathfrak{p}-1)-\beta-d_0}{\beta+d_0(1+\eta)}$, *(I should discard this constraint soon)*

8.5. $\mathfrak{d} < \kappa d_0 - 1$.

2.3. **Main results.** The first result to be presented, theorem 1, valid in the general setup of section 2.1, expresses hitting statistics (λ_ℓ's) in terms of return statistics (α_ℓ's). Although important on its own, it actually plays an auxiliary role within the paper, serving the following two purposes (the essential one being the second):

1) Technical: help the proof of our main result, theorem 2, via its use in the proof of lemma 2. Be aware that the statement of this lemma can be phrased as to avoid the dependence on theorem 1, both in the hypothesis and conclusion, but the dependence on return statistics is still present in the hypothesis. We believe one could bypass both aforementioned dependencies, in such a way as to write a spin-off of theorem 2 using exclusively hitting statistics (instead of return statistics), either in statement or proof.

2) Examples: even if the said 'spin-off' could occur, to handle examples, one will always need theorem 1 to compute the hitting statistics specified in theorem 2 (or its hypothetical 'spin-off'). This is because return statistics are generally much easier to compute than hitting statistics, so, whenever facing a concrete example, we calculate the former to obtain the latter.[1]

Theorem 1. *Let $(\theta, \nu, T_\omega, \mu^\omega, \Gamma)$ be a system as described in section 2.1.*

Then

$$\text{(H7')} \;\Rightarrow\; \lambda_\ell = \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1} \; (\ell \geqslant 1) \text{ and (H6)}.$$

The essential part[2] of this theorem is to conclude the equality, which will be proven in section 3. It implies that $\alpha_1 = (\sum_{\ell=1}^{\infty} \ell\lambda_\ell)^{-1}$.

Let us now formulate our main result. It says that the targeted random dynamical systems being considered have quenched limit entry distributions in the compound Poisson class.

Theorem 2. *Let $(\theta, \nu, T_\omega, \mu^\omega, \Gamma)$ be a system satisfying hypotheses (H1)-(H5),(H7') (so (H6), by theorem 1) with the parametric constraints (H8.1)-(H8.5).*

Then: $\forall t > 0, \forall n \geqslant 0, \forall (\rho_m)_{m\geqslant 1} \searrow 0$ with $\sum_{m\geqslant 1} {\rho_m}^q < \infty$ (for some $0 < q < q(d_0, d_1, \eta, \beta, \mathfrak{p})$[3]) one has

$$\mu^\omega(Z_{\Gamma_{\rho_m}}^{\omega, \lfloor t/\hat{\mu}(\Gamma_{\rho_m})\rfloor} = n) \xrightarrow[m\to\infty]{\nu\text{-a.s.}} \mathrm{CPD}_{t\alpha_1, (\lambda_\ell)_\ell}(\{n\}), \tag{9}$$

where $\mathrm{CPD}_{s,(\lambda_\ell)_\ell}$ is the compound Poisson distribution with parameter s and cluster size distribution $(\lambda_\ell)_\ell$ (see below).

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[1]The general philosophy is that hitting-related quantities are theoretical and inaccessible, so assumptions about them should be minimized; whereas return-related quantities are accessible and assumptions about them can be adopted and should be checked/computed in concrete cases.

[2](H6) follows from the previous equality and (H7) because $\sum_{\ell=1}^{\infty} \lambda_\ell = \frac{\sum_{\ell=1}^{\infty} \alpha_\ell - \alpha_{\ell+1}}{\alpha_1} = 1$ and $\sum_{\ell=1}^{\infty} \ell^3\lambda_\ell = (\alpha_1)^{-1}\sum_{\ell=1}^{\infty} \ell^3(\alpha_\ell - \alpha_{\ell+1}) = (\alpha_1)^{-1}\left(1^3\alpha_1 + \sum_{\ell=2}^{\infty} \ell^3\alpha_\ell - \sum_{\ell=2}^{\infty}(\ell-1)^3\alpha_\ell\right) = (\alpha_1)^{-1}\left(1^3\alpha_1 + \sum_{\ell=2}^{\infty}(3\ell^2 - 3\ell + 1)\alpha_\ell\right) \leqslant (\alpha_1)^{-1}\left(1 + \sum_{\ell=1}^{\infty} 7\ell^2\alpha_\ell\right) < \infty$.

[3]A quantity to be introduced in lemma 3.

Note: If the system has exponential asymptotics in (H5) and (H2.3), the previous conclusion is still true, but, actually, with fewer parametric conditions being required: instead of (H8.1)-(H8.5), only $\kappa d_0 > 1$ is needed.

Recall: The compound Poisson distribution with parameter $s \in \mathbb{R}_{>0}$ and cluster size distribution $(\lambda_\ell)_{\ell\in\mathbb{N}_{\geqslant 1}} \in \mathcal{P}(\mathbb{N}_{\geqslant 1})$, $\sum_{\ell=1}^{\infty} \ell\lambda_\ell < \infty$, denoted $\mathrm{CPD}_{s,(\lambda_\ell)_\ell} \in \mathcal{P}(\mathbb{N}_{\geqslant 0})$, is the distribution of a random variable $M : (\Omega, \mathscr{F}, \mathbb{P}) \to \mathbb{N}_{\geqslant 0}$ given by $M(\omega) = \sum_{j=1}^{N(\omega)} Q_j(\omega)$, where N is a $\mathbb{N}_{\geqslant 0}$-valued random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ having Poisson distribution with parameter γ and $(Q_j)_{j\in\mathbb{N}_{j\geqslant 1}}$ is a sequence of $\mathbb{N}_{\geqslant 1}$-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ which are iid, independent of N and whose entries have distribution $\mathbb{P}(Q_j = \ell) = \lambda_\ell$ $(j, \ell \in \mathbb{N}_{\geqslant 1})$. Denote $R_l = \sum_{j=1}^{l} Q_j$. Then the probability mass function of $\mathrm{CPD}_{s,(\lambda_\ell)_\ell}$ is given indirectly by

$$\mathrm{CPD}_{\gamma,(\lambda_\ell)_\ell}(\{n\}) = \sum_{l=1}^{n} \mathbb{P}(N = l)\mathbb{P}(R_l = n) = \sum_{l=1}^{n} \frac{s^l e^{-s}}{l!} \sum_{\substack{(n_1,\dots,n_l)\in\mathbb{N}_{\geqslant 1}^l \\ n_1+\dots+n_l=n}} \prod_{i=1}^{l} \lambda_{n_i}. \tag{10}$$

Structure of the paper. The rest of the paper is organized into two parts:

I) The first one, until section 6, is headed towards the proof of theorem 2 (section 6), accomplishing required preliminary results: theorem 1 (section 3), theorem 3 (section 4) and lemmas 2, 3 and 4 (section 5).

Theorem 1, as we said, has two uses: 1) technical: assist in the proof of lemma 2, 2) examples (see (II) below): calculate hitting statistics in terms of return statistics, which are more accessible.

Theorem 3 provides the skeleton of the proof of theorem 2, describing the asymptotics we are after with a leading term and an error.

Finally, the lemmas are used to tame the above-mentioned leading (section 6.6) and error terms (sections 6.2-6.4).

II) The second part applies the theory developed in (I) to examples. We consider certain random expanding maps of the interval, casting new light on the well-known deterministic dichotomy between periodic and aperiodic points and recovering the Polya-Aeppli case for general Bernoulli-driven systems.

3. Proof of theorem 1

Let us note that theorem 2 from [16] is generalized by theorem 1 presented in section 2.3. The proof is very similar but presented here for the convenience of the reader.

In the scope of this section, let arbitrarily chosen $\ell \in \mathbb{N}_{\geqslant 1}$ and $\gamma \in \mathbb{N}_{>1}$ be fixed. These will be used in the forthcoming proof of theorem 1 and lemma 1.

Lemma 1. *It holds that*

I) $\forall \eta > 0, \exists L_2(\eta), \forall L' > L \geqslant L_2(\eta), \exists \rho_2(\eta, L, L'), \forall \rho \leqslant \rho_2(\eta, L, L')$:

$$\hat{\mu}\left(Z_{\Gamma_\rho}^{L'-L} \circ S^L > 0, I_0 = 1\right) \leqslant \eta\hat{\mu}(\Gamma_\rho).$$

II) $\forall \eta > 0, \exists L_2(\eta), \forall L', L$ *so that* $L' - L \geqslant L_2(\eta)$, $\exists \rho_2(\eta, L, L') > 0, \forall \rho \leqslant \rho_2(\eta, L, L')$:

$$\hat{\mu}\left(Z_{\Gamma_\rho}^{L} > 0, I_{L'} = 1\right) \leqslant \eta\hat{\mu}(\Gamma_\rho).$$

Proof. Manipulating the definitions of $\hat{\alpha}_\ell$'s, the associated limits and the finiteness of the series in the hypothesis, it can be shown that: $\forall\epsilon > 0$

i) $\exists k_0(\epsilon) \geqslant 1$ so that $\sum_{k=k_0(\epsilon)}^{\infty} k\hat{\alpha}_k \leqslant \epsilon$.

ii) $\exists L_0(\epsilon) \geqslant [2k_0(\epsilon) \vee 2\ell]\,(\epsilon^{-1})^\gamma, \forall L \geqslant L_0(\epsilon)$, $\exists \rho_0(\epsilon, L) > 0$, $\forall \rho \leqslant \rho_0(\epsilon, L)$ one has

a) $\forall H \in [L, 2L^\gamma] : \sum_{k=k_0(\epsilon)}^{\infty} k\hat{\alpha}_k(H,\rho) \leqslant 2\epsilon$, $\left|\sum_{k=1}^{k_0(\epsilon)} \hat{\alpha}_k - \sum_{k=1}^{k_0(\epsilon)} \hat{\alpha}_k(H,\rho)\right| \leqslant 2\epsilon$.

b) $\forall q \in [1, 2k_0(\epsilon) \vee 2\ell], \forall H \in [L, 2L^\gamma] : |\hat{\alpha}_q - \hat{\alpha}_q(H,\rho)| \leqslant \epsilon, |\alpha_q - \alpha_q(H,\rho)| \leqslant \epsilon$.

iii) $\exists L_1(\epsilon) \geqslant L_0(\epsilon), \forall L' > L \geqslant L_1(\epsilon), \exists \rho_1(\epsilon, L, L') \in \left(0, \bigwedge_{H=L}^{L'} \rho_0(\epsilon, H)\right)$, $\forall \rho \leqslant \rho_1(\epsilon, L, L')$ one has $\sum_{k=1}^{\infty} |\hat{\alpha}_k(L',\rho) - \hat{\alpha}_k(L,\rho)| \leqslant 6\epsilon$.

To justify i) use that $\sum_{k=1}^{\infty} k\hat{\alpha}_k < \infty$.

To justify ii.a) start noticing that $\forall\epsilon > 0, \exists L_0(\epsilon), \forall L \geqslant L_0(\epsilon)$:

$$0 \leqslant \hat{\alpha}_k - \overset{\pm}{\hat{\alpha}}_k(H) \leqslant \frac{\epsilon}{k_0(\epsilon)} \ (\forall k \in [1, k_0(\epsilon)], \forall H \in [L, 2L^\gamma]),$$

$$\Rightarrow 0 \leqslant \sum_{k=1}^{k_0(\epsilon)} \hat{\alpha}_k - \sum_{k=1}^{k_0(\epsilon)} \overset{\pm}{\hat{\alpha}}_k(H) \leqslant \epsilon \ (\forall H \in [L, 2L^\gamma]),$$

because $\hat{\alpha}_k = \lim_L \overset{\pm}{\hat{\alpha}}_k(L)$ occurs monotonically increasing in L.

Then consider that $\forall\epsilon > 0, \forall L \geqslant 0, \exists \rho_0(\epsilon, L), \forall \rho \leqslant \rho_0(\epsilon, L)$:

$$\overset{-}{\hat{\alpha}}_k(H) - \frac{\epsilon}{k_0(\epsilon)^2(2L^\gamma)^2} \leqslant \hat{\alpha}_k(H,\rho) \leqslant \overset{+}{\hat{\alpha}}_k(H) + \frac{\epsilon}{k_0(\epsilon)^2(2L^\gamma)^2}$$
$$(\forall k \in [1, k_0(\epsilon)^2(2L^\gamma)^2], \forall H \in [L, 2L^\gamma])$$

implying

$$\sum_{k=1}^{k_0(\epsilon)} \overset{-}{\hat{\alpha}}_k(H) - \epsilon \leqslant \sum_{k=1}^{k_0(\epsilon)} \hat{\alpha}_k(H,\rho) \leqslant \sum_{k=1}^{k_0(\epsilon)} \overset{+}{\hat{\alpha}}_k(H) + \epsilon \ (\forall H \in [L, 2L^\gamma]),$$

and

$$\begin{aligned}\sum_{k=k_0(\epsilon)}^{\infty} k\hat{\alpha}_k(H,\rho) = \sum_{k=k_0(\epsilon)}^{H} k\hat{\alpha}_k(H,\rho) &\leqslant \sum_{k=k_0(\epsilon)}^{H} k\, \overset{+}{\hat{\alpha}}_k(H) + \sum_{k=k_0(\epsilon)}^{H} k\frac{\epsilon}{k_0(\epsilon)^2(2L^\gamma)^2}\\ &\leqslant \sum_{k=k_0(\epsilon)}^{\infty} k\hat{\alpha}_k + (2L^\gamma)^2\frac{\epsilon}{k_0(\epsilon)^2(2L^\gamma)^2}\\ &\leqslant \epsilon + \epsilon = 2\epsilon \ (\forall H \in [L, 2L^\gamma]).\end{aligned}$$

Finally, combining the conditions and conclusions of the two previous paragraphs: $\forall\epsilon > 0, \exists L_0(\epsilon), \forall L \geqslant L_0(\epsilon), \exists \rho_0(\epsilon, L) > 0, \forall \rho \leqslant \rho_0(\epsilon, L), \forall H \in [L, 2L^\gamma]$:

$$\sum_{k=k_0(\epsilon)}^{\infty} k\hat{\alpha}_k(H,\rho) \leqslant 2\epsilon \text{ and } \sum_{k=1}^{k_0(\epsilon)} \hat{\alpha}_k - 2\epsilon \leqslant \sum_{k=1}^{k_0(\epsilon)} \hat{\alpha}_k(H,\rho) \leqslant \sum_{k=1}^{k_0(\epsilon)} \hat{\alpha}_k + 2\epsilon,$$

as desired.

To justify ii.b) one can adapt the argument used above to show the second inequality in ii.a).

Finally, to justify iii) start noticing that $\forall \epsilon > 0, \exists L_1(\epsilon) \geqslant L_0(\epsilon), \forall L' > L \geqslant L_1(\epsilon)$:

$$\left| \overset{*}{\hat{\alpha}}_k(L') - \overset{*}{\hat{\alpha}}_k(L) \right| \leqslant \frac{\epsilon}{4k_0(\epsilon)} \ (\forall *, * \in \{+, -\}),$$

and therefore $\forall \epsilon > 0, \exists L_1(\epsilon) \geqslant L_0(\epsilon), \forall L' > L \geqslant L_1(\epsilon), \exists \rho_1(\epsilon, L, L') \in \left(0, \bigwedge_{H=L}^{L'} \rho_0(\epsilon, H)\right)$, $\forall \rho \leqslant \rho_1(\epsilon, L, L'), \forall k \in [1, k_0(\epsilon)]$:

$$\begin{aligned}
\overset{-}{\hat{\alpha}}_k(L') - \frac{\epsilon}{4k_0(\epsilon)} \leqslant \hat{\alpha}_k(L', \rho) \leqslant \overset{+}{\hat{\alpha}}_k(L') + \frac{\epsilon}{4k_0(\epsilon)} \ (\forall k \in [1, k_0(\epsilon)]) \\
\overset{-}{\hat{\alpha}}_k(L) - \frac{\epsilon}{4k_0(\epsilon)} \leqslant \hat{\alpha}_k(L, \rho) \leqslant \overset{+}{\hat{\alpha}}_k(L) + \frac{\epsilon}{4k_0(\epsilon)} \ (\forall k \in [1, k_0(\epsilon)]) \\
\text{so that } -\frac{\epsilon}{k_0(\epsilon)} \leqslant \hat{\alpha}_k(L', \rho) - \hat{\alpha}_k(L, \rho) \leqslant \frac{\epsilon}{k_0(\epsilon)} \ (\forall k \in [1, k_0(\epsilon)]) \\
\Rightarrow \sum_{k=1}^{k_0(\epsilon)} |\hat{\alpha}_k(L', \rho) - \hat{\alpha}_k(L, \rho)| \leqslant \epsilon,
\end{aligned}$$

and, since the quantifiers were subordinated to those of (ii), we actually get that

$$\begin{aligned}
\textstyle\sum_{k=1}^{\infty} |\hat{\alpha}_k(L', \rho) - \hat{\alpha}_k(L, \rho)| &\leqslant \textstyle\sum_{k=1}^{k_0(\epsilon)} |\hat{\alpha}_k(L', \rho) - \hat{\alpha}_k(L, \rho)| + \sum_{k=k_0(\epsilon)}^{\infty} |\hat{\alpha}_k(L', \rho) - \hat{\alpha}_k(L, \rho)| \\
&\leqslant \epsilon + \textstyle\sum_{k=k_0(\epsilon)}^{\infty} k\hat{\alpha}_k(L', \rho) + \sum_{k=k_0(\epsilon)}^{\infty} k\hat{\alpha}_k(L', \rho) \\
&\leqslant \epsilon + 2\epsilon + 2\epsilon \leqslant 6\epsilon,
\end{aligned}$$

as desired

Now we prove I).

Let $L_2(\eta) := L_1(\eta/6)$ and $\rho_2(\eta, L, L') := \rho_1(\eta/6, L, L')$. Consider $L' > L \geqslant L_2(\eta)$ and $\rho \leqslant \rho_2(\eta, L, L')$. Then

$$\begin{aligned}
&\hat{\mu}\left(Z_{\Gamma_\rho}^{L'-L} \circ S^L > 0, I_0 = 1\right) \\
&= \int_\Omega \sum_{k=1}^{\infty} \mu^\omega \left(Z_{\Gamma_\rho}^{\theta^L\omega, L'-L} \circ T_\omega^L > 0, I_0^\omega = 1, Z_{\Gamma_\rho}^{\omega, L'} = k\right) d\nu(\omega) \\
&\leqslant \int_\Omega \sum_{k=1}^{\infty} \mu^\omega \left(\Gamma_\rho(\omega) \cap \left[\{Z_{\Gamma_\rho}^{\omega, L'} \geqslant k\} \backslash \{Z_{\Gamma_\rho}^{\omega, L} \geqslant k\}\right]\right) d\nu(\omega) \\
&= \sum_{k=1}^{\infty} \left[\int_\Omega \mu^\omega(\Gamma_\rho(\omega)) \hat{\alpha}_k^\omega(L', \rho) d\nu(\omega) - \int_\Omega \mu^\omega(\Gamma_\rho(\omega)) \hat{\alpha}_k^\omega(L, \rho) d\nu(\omega)\right] \\
&= \sum_{k=1}^{\infty} \hat{\alpha}_k(L', \rho)\hat{\mu}(\Gamma_\rho) - \hat{\alpha}_k(L, \rho)\hat{\mu}(\Gamma_\rho) \leqslant 6\eta/6\,\hat{\mu}(\Gamma_\rho) = \eta\hat{\mu}(\Gamma_\rho), \text{ by (iii).}
\end{aligned}$$

This concludes the proof of I).

Now we prove II).

Again let $L_2(\eta) := L_1(\eta/6)$ and $\rho_2(\eta, L, L') := \rho_1(\eta/6, L, L')$. Consider L', L so that $L' - L \geqslant L_2(\eta)$ and $\rho \leqslant \rho_2(\eta, L, L')$. Then

$$\hat{\mu}\left(Z_{\Gamma_\rho}^L > 0, I_{L'} = 1\right) \leqslant \int_\Omega \sum_{i=0}^{L} \mu^\omega \left(I_i^\omega = 1, I_{L'}^\omega = 1\right) d\nu(\omega) = \int_\Omega \sum_{i=0}^{L} \mu^\omega \left(I_0^\omega = 1, I_{L'-i}^\omega = 1\right) d\nu(\omega)$$

$$= \int_\Omega \mathbb{E}_{\mu^\omega}\left(I_0^\omega \sum_{i=L'-L}^{L'} I_i^\omega\right) d\nu(\omega) = \int_\Omega \mathbb{E}_{\mu^\omega}\left(I_0^\omega\left[Z_{\Gamma_\rho}^{\omega,L'} - Z_{\Gamma_\rho}^{\omega,L'-L}\right]\right) d\nu(\omega)$$

$$= \int_\Omega \sum_{k=0}^\infty k\left[\mu^\omega\left(Z_{\Gamma_\rho}^{\omega,L'} = k, I_0^\omega = 1\right) - \mu^\omega\left(Z_{\Gamma_\rho}^{\omega,L'-L} = k, I_0^\omega = 1\right)\right] d\nu(\omega)$$

$$= \int_\Omega \sum_{k=0}^\infty k\left[\alpha_k^\omega(L',\rho)\mu^\omega(\Gamma_\rho(\omega)) - \alpha_k^\omega(L'-L,\rho)\mu^\omega(\Gamma_\rho(\omega))\right] d\nu(\omega),$$

but $\sum_{k=1}^\infty k\alpha_k^\omega(T,\rho) = \sum_{k=1}^\infty k[\hat\alpha_k^\omega(T,\rho) - \hat\alpha_{k+1}^\omega(T,\rho)] = \sum_{k=1}^\infty \hat\alpha_k^\omega(T,\rho)$, so

$$= \int_\Omega \Big[\sum_{k=0}^\infty \hat\alpha_k^\omega(L',\rho) - \hat\alpha_k^\omega(L'-L,\rho)\Big]\mu^\omega(\Gamma_\rho(\omega)) d\nu(\omega)$$

$$= \sum_{k=0}^\infty \left[\hat\alpha_k(L',\rho) - \hat\alpha_k(L'-L,\rho)\right]\hat\mu(\Gamma_\rho) \leqslant 6\eta/6\,\hat\mu(\Gamma_\rho) = \eta\hat\mu(\Gamma_\rho), \text{ by (iii).}$$

This concludes the proof of II). ■

Proof of theorem 1. Let $\epsilon \in (0, \alpha_1/26)$ and consider a function $\eta(\epsilon)$ to be chosen in due time.

Set $L_3(\epsilon)$ be large so that $L \geqslant L_3(\epsilon) \Rightarrow L^\gamma - L > L \geqslant 2\left[L_1(\epsilon) \vee L_2(\eta(\epsilon))\right]$.

Set $\rho_3(\epsilon, L)$ to

$$\rho_1(\epsilon, L, L^\gamma) \wedge \rho_1(\epsilon, L^\gamma - L, L^\gamma) \wedge$$
$$\bigwedge_{i \in [L, 2L^\gamma - 2L - 2]} \begin{pmatrix} \rho_2(\eta(\epsilon), L+1, 2L^\gamma - i) \wedge \rho_2(\eta(\epsilon), L/2+1, 2L^\gamma - i) \\ \wedge \rho_2(\eta(\epsilon), L/2, i) \qquad\qquad \wedge \rho_2(\eta(\epsilon), i - L/2, i) \end{pmatrix}.$$

Consider $L \geqslant L_3(\epsilon)$ and $\rho \leqslant \rho_3(\epsilon, L)$. We evaluate the numerator which appears when expanding λ_ℓ.

$$\hat\mu\left(Z_{\Gamma_\rho}^{2L^\gamma} = \ell\right) = \int_\Omega \ell^{-1} \sum_{i=0}^{2L^\gamma} \mathbb{E}_{\mu^\omega}\left(\mathbb{1}_{\left\{Z_{\Gamma_\rho}^{\omega,2L^\gamma} = \ell\right\}} I_i^\omega\right) d\nu(\omega)$$

$$\Rightarrow \left|\hat\mu\left(Z_{\Gamma_\rho}^{2L^\gamma} = \ell\right) - \ell^{-1} \sum_{i=L}^{2L^\gamma - 2L - 2} \int_\Omega \mu^\omega\left(Z_{\Gamma_\rho}^{\omega,2L^\gamma} = \ell, I_i^\omega = 1\right) d\nu(\omega)\right|$$
$$\leqslant \ell^{-1} 4L\hat\mu(\Gamma_\rho). \tag{11}$$

We establish some notation. When $i \in [L, 2L^\gamma - 2L - 2]$ and $k \in [0, \ell-1]$, write:

$$D_{\omega,i}^{L,L^\gamma} = \left\{\sum\nolimits_{u=i+L+1}^{2L^\gamma} I_u^\omega > 0, I_i^\omega = 1\right\};$$

$$F_{\omega,i}^{L/2} = \left\{\sum\nolimits_{u=0}^{L/2} I_u^\omega > 0, I_i^\omega = 1\right\};$$

$$R_{\omega,\ell,k}^{i,L} = \left\{\sum\nolimits_{u=0}^{i-1} I_u^\omega = k, \sum\nolimits_{u=i}^{i+L} I_u^\omega = \ell - k, I_i^\omega = 1\right\};$$

$$R_{\omega,\ell,k}^{i,L}(j) = R_{\omega,\ell,k}^{i,L} \cap \left\{I_j^\omega = 1, I_a^\omega = 0\ \forall a \in [0,j)\right\}, \text{ for } j \in [0,i];$$

$$S_{\omega,\ell,k}^{i,L}(j) = R_{\omega,\ell,k}^{i,L} \cap \left\{I_j^\omega = 1, I_b^\omega = 0\ \forall b \in (j,i)\right\}, \text{ for } j \in [0,i-1].$$

To update the estimate in equation (11), we'll apply many approximation steps, to be identified with uppercase roman letters and justified only at the very end.

A) For $i \in [L, 2L^\gamma - 2L - 2]$:

$$\left|\int_\Omega \mu^\omega\left(Z^{\omega,2L^\gamma}_{\Gamma_\rho} = \ell, I^\omega_i = 1\right)d\nu(\omega) - \int_\Omega \mu^\omega\left(Z^{\omega,i+L}_{\Gamma_\rho} = \ell, I^\omega_i = 1\right)d\nu(\omega)\right| \leqslant \eta(\epsilon)\hat{\mu}(\Gamma_\rho)$$

Combining equation (11) with approximation (A), while observing that second integrand above equals $\mu^\omega(\bigcup_{k=0}^{\ell-1} R^{i,L}_{\omega,\ell,k})$, gives

$$\left|\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} = \ell\right) - \ell^{-1}\sum_{i=L}^{2L^\gamma-2L-2}\sum_{k=0}^{\ell-1}\int_\Omega \mu^\omega(R^{i,L}_{\omega,\ell,k})d\nu(\omega)\right|$$
$$\leqslant \ell^{-1}[4L + (2L^\gamma - 3L - 1)\eta(\epsilon)]\hat{\mu}(\Gamma_\rho). \tag{12}$$

B) For $i \in [L, 2L^\gamma - 2L - 2]$, $k \in [0, \ell - 1]$:

$$\left|\int_\Omega \mu^\omega(R^{i,L}_{\omega,\ell,0})d\nu(\omega) - \int_\Omega \mu^\omega(R^{i,L}_{\omega,\ell,k})d\nu(\omega)\right| \leqslant 3\eta(\epsilon)\hat{\mu}(\Gamma_\rho).$$

C) For $i \in [L, 2L^\gamma - 2L - 2]$:

$$\left|\int_\Omega \mu^\omega\left(R^{i,L}_{\omega,\ell,0}\right)d\nu(\omega) - \int_\Omega \mu^\omega\left(R^{L,L}_{\omega,\ell,0}\right)d\nu(\omega)\right| \leqslant \eta(\epsilon)\hat{\mu}(\Gamma_\rho).$$

Combining equation (12) with approximations (B) and (C), gives

$$\left|\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} = \ell\right) - (2L^\gamma - 3L - 1)\int_\Omega \mu^\omega\left(R^{L,L}_{\omega,\ell,0}\right)d\nu(\omega)\right|$$
$$\leqslant [4L + 5(2L^\gamma - 3L - 1)\eta(\epsilon)]\hat{\mu}(\Gamma_\rho). \tag{13}$$

Now we look to the other side of the equality we are trying to prove.

Notice that

$$\alpha_\ell(L,\rho) - \alpha_{\ell+1}(L,\rho) = \hat{\mu}(\Gamma_\rho)^{-1}\int_\Omega \begin{bmatrix} \mu^\omega\left(\sum_{u=L}^{2L} I^\omega_u = \ell, I^\omega_L = 1\right) \\ -\mu^\omega\left(\sum_{u=L}^{2L} I^\omega_u = \ell+1, I^\omega_L = 1\right)\end{bmatrix} d\nu(\omega)$$

$$= \hat{\mu}(\Gamma_\rho)^{-1}\begin{bmatrix} \sum_{k=0}^{k_0(\epsilon)}\int_\Omega \mu^\omega(R^{L,L}_{\omega,\ell+k,k}) - \mu^\omega(R^{L,L}_{\omega,\ell+k+1,k})d\nu(\omega) \\ +\sum_{k=k_0(\epsilon)+1}^{\infty(L)}\int_\Omega \mu^\omega(R^{L,L}_{\omega,\ell+k,k}) - \mu^\omega(R^{L,L}_{\omega,\ell+k+1,k})d\nu(\omega)\end{bmatrix} \tag{14}$$

$$\overset{(\ast)}{=} \hat{\mu}(\Gamma_\rho)^{-1}\begin{bmatrix} \sum_{k=0}^{k_0(\epsilon)}\int_\Omega \mu^\omega(R^{L,L}_{\omega,\ell+k,k}) - \left(\mu^\omega(R^{L,L}_{\omega,\ell+k+1,k+1}) + \mathcal{O}\left(\eta(\epsilon)\hat{\mu}(\Gamma_\rho)\right)\right)d\nu(\omega) \\ +\sum_{k=k_0(\epsilon)+1}^{\infty(L)}\int_\Omega \mu^\omega(R^{L,L}_{\omega,\ell+k,k}) - \mu^\omega(R^{L,L}_{\omega,\ell+k+1,k})d\nu(\omega)\end{bmatrix}$$

$$\Rightarrow \left|(\alpha_\ell(L,\rho) - \alpha_{\ell+1}(L,\rho))\hat{\mu}(\Gamma_\rho) - \int_\Omega \mu^\omega\left(R^{L,L}_{\omega,\ell,0}\right)d\nu(\omega)\right|$$
$$\leqslant 2k_0(\epsilon)\mathcal{O}\left(\eta(\epsilon)\hat{\mu}(\Gamma_\rho)\right) + 2\sum_{k=k_0(\epsilon)+1}^{\infty(L)}\int_\Omega \mu^\omega(R^{L,L}_{\omega,\ell+k,k}) + \mu^\omega(R^{L,L}_{\omega,\ell+k+1,k})d\nu(\omega),$$

$$\overset{(\circledast)}{\leqslant} 2k_0(\epsilon)\mathcal{O}\left(\eta(\epsilon)\hat{\mu}(\Gamma_\rho)\right) + 7(k_0(\epsilon) + \ell)\hat{\alpha}_{k_0(\epsilon)+\ell}(2L,\rho)\hat{\mu}(\Gamma_\rho) \tag{15}$$

where steps $(\ast)$ and $(\circledast)$ are justified, respectively, with the following two approximations.

D) For $i \in [L, 2L^\gamma - 2L - 2]$, $k \in [1, \ell-1]$ (for other k's, zeroes pop up):

$$\left|\int_\Omega \mu^\omega\left(R^{i,L}_{\omega,\ell,k-1}\right) d\nu(\omega) - \int_\Omega \mu^\omega\left(R^{i,L}_{\omega,\ell,k}\right) d\nu(\omega)\right| \leqslant 3\eta(\epsilon)\hat{\mu}(\Gamma_\rho).$$

E)

$$\sum_{k=k_0(\epsilon)+1}^{\infty(L)} \int_\Omega \mu^\omega(R^{L,L}_{\omega,\ell+k,k}) d\nu(\omega) \leqslant (k_0(\epsilon)+\ell)\hat{\alpha}_{k_0(\epsilon)+\ell}(2L,\rho)\hat{\mu}(\Gamma_\rho).$$

Now choose $\eta(\epsilon) = \epsilon/(k_0(\epsilon)+1)$. Combining equations (13) & (15), using (ii.a) from the proof of lemma 1, and factoring L^γ out (notice $L^{1-\gamma} \leqslant \epsilon$), gives:

$$\left|\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} = \ell\right) - 2L^\gamma\big(\alpha_\ell(L,\rho) - \alpha_{\ell+1}(L,\rho)\big)\hat{\mu}(\Gamma_\rho)\right| \leqslant 84L^\gamma\epsilon\hat{\mu}(\Gamma_\rho). \tag{16}$$

We can finally evaluate the denominator which appears when expanding λ_ℓ.

$$\begin{aligned}
\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right) &= \sum_{h=1}^{k_0(\epsilon)+1}\left[\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} = h\right)\right] + \hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} \geqslant k_0(\epsilon)+2\right) \\
&= \sum_{h=1}^{k_0(\epsilon)+1}\left[\int_\Omega h^{-1}\mathbb{E}_{\mu^\omega}\left(\mathbb{1}_{\left\{Z^{\omega,2L^\gamma}_{\Gamma_\rho} = h\right\}} Z^{\omega,2L^\gamma}_{\Gamma_\rho}\right) d\nu(\omega)\right] + \hat{\alpha}_{k_0(\epsilon)+2}(2L^\gamma,\rho)\hat{\mu}(\Gamma_\rho) \\
&= \sum_{i=0}^{2L^\gamma}\sum_{h=1}^{k_0(\epsilon)+1}\left[h^{-1}\sum_{k=0}^{h-1}\int_\Omega \mu^\omega\left(R^{i,2L^\gamma-i}_{\omega,h,k}\right) d\nu(\omega)\right] + \mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)),
\end{aligned} \tag{17}$$

where the last line applied (ii.a) from the proof of lemma 1.

Then we consider the following approximation.

F) For $h \in [1, k_0(\epsilon)+1]$, $i \in [L, 2L^\gamma - 2L - 2]$, $k \in [0, h-1]$:

$$\left|\int_\Omega \mu^\omega\left(R^{i,2L^\gamma-i}_{\omega,h,k}\right) d\nu(\omega) - \int_\Omega \mu^\omega\left(R^{i,L}_{\omega,h,k}\right) d\nu(\omega)\right| \leqslant \eta(\epsilon)\hat{\mu}(\Gamma_\rho).$$

Starting from equation (17), splitting the i-sum into middle ($i \in [L, 2L^\gamma - 2L - 2]$) plus tail terms and applying (F,B,C,B) to the middle ones, gives

$$\begin{aligned}
\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right) &= (2L^\gamma - 3L - 1)\left[\sum_{h=1}^{k_0(\epsilon)+1}\left(\int_\Omega \mu^\omega\left(R^{L,L}_{\omega,h,h-1}\right) d\nu(\omega)\right) + \mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho))\right] \\
&\quad + \mathcal{O}\Big(4L(k_0(\epsilon)+1)\epsilon^{-1}\epsilon\hat{\mu}(\Gamma_\rho)\Big) + \mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)) \\
&\Rightarrow \left|\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right) - 2L^\gamma\sum_{h=1}^{k_0(\epsilon)+1}\int_\Omega \mu^\omega\left(R^{L,L}_{\omega,h,h-1}\right) d\nu(\omega)\right| \\
&\leqslant 3L^\gamma\mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)) + \mathcal{O}(4L\cdot L^{\gamma-1}\epsilon\hat{\mu}(\Gamma_\rho)) + \mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)) + 4L\sum_{h=1}^{k_0(\epsilon)+1}\int_\Omega \mu^\omega\left(R^{L,L}_{\omega,h,h-1}\right) d\nu(\omega) \\
&\leqslant 8L^\gamma\mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)) + 4L\sum_{h=1}^{k_0(\epsilon)+1}\int_\Omega \mu^\omega\left(R^{L,L}_{\omega,h,h-1}\right) d\nu(\omega),
\end{aligned}$$

where it was used that $L^{\gamma-1} \geqslant \epsilon^{-1}$.

To take care of the summations on both sides of the previous inequality we observe that, when $\ell = 1$ is given to the "$\alpha_\ell(L,\rho)$ side of" equation (14), one gets

$$\left|\alpha_1(L,\rho)\hat{\mu}(\Gamma_\rho) - \sum_{h=1}^{k_0(\epsilon)+1}\int_\Omega \mu^\omega(R^{L,L}_{\omega,h,h-1})d\nu(\omega)\right| = \sum_{h=k_0(\epsilon)+1}^{\infty(L)}\int_\Omega \mu^\omega(R^{L,L}_{\omega,1+h,h})d\nu(\omega)$$

$$\leqslant (k_0(\epsilon)+1)\hat{\alpha}_{k_0(\epsilon)+1}(2L,\rho)\hat{\mu}(\Gamma_\rho) \leqslant 2\epsilon\hat{\mu}(\Gamma_\rho),$$

where (E) and (ii.a) from the proof of lemma 1 are applied.

Therefore

$$\begin{aligned}&\left|\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right) - 2L^\gamma\alpha_1(L,\rho)\hat{\mu}(\Gamma_\rho)\right|\\ &\leqslant 8L^\gamma\mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)) + 4L\big(\alpha_1(L,\rho)\epsilon^{-1}\epsilon\hat{\mu}(\Gamma_\rho) + 2\epsilon\hat{\mu}(\Gamma_\rho)\big) + 4L^\gamma\epsilon\hat{\mu}(\Gamma_\rho)\\ &\leqslant 20L^\gamma\mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)) + 4\alpha_1(L,\rho)LL^{\gamma-1}\epsilon\hat{\mu}(\Gamma_\rho) \leqslant 12\cdot 2L^\gamma\mathcal{O}(\epsilon\hat{\mu}(\Gamma_\rho)). \qquad (18)\end{aligned}$$

Since $\alpha_1(L,\rho)$ is ϵ-close to α_1 and $\epsilon \in (0,\alpha_1/26)$, the previous equation sets $\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)$ far from zero, so we can really work with $\hat{\mu}(Z^{2L^\gamma}_{\Gamma_\rho} > 0)$ as a denominator.

Combining the estimates given in equations (16) and (18) gives:

$$\left|\frac{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} = \ell\right)}{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)} - \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1}\right|$$

$$\leqslant \left|\frac{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} = \ell\right) - \frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)}\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)}{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)}\right| + \left|\frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)} - \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1}\right|$$

$$\leqslant \frac{\left|\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} = \ell\right) - 2L^\gamma\hat{\mu}(\Gamma_\rho)(\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho))\right| + \left|2L^\gamma\hat{\mu}(\Gamma_\rho)(\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)) - \frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)}\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)\right|}{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)}$$
$$+ \left|\frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)} - \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1}\right|$$

$$\overset{(16)}{\underset{(18)}{\leqslant}} \frac{42\cdot 2L^\gamma\epsilon\hat{\mu}(\Gamma_\rho) + \frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)}12\cdot 2L^\gamma\epsilon\hat{\mu}(\Gamma_\rho)}{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)} + \left|\frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)} - \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1}\right|$$

$$\begin{aligned}&\leqslant 42\epsilon\left(\frac{1}{\alpha_1} + \frac{\mathcal{O}(\epsilon)}{(\alpha_1 - \mathcal{O}(\epsilon))^2}\right) + \frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)}12\epsilon\left(\frac{1}{\alpha_1} + \frac{\mathcal{O}(\epsilon)}{(\alpha_1 - \mathcal{O}(\epsilon))^2}\right)\\ &\qquad + \left|\frac{\alpha_\ell(L,\rho)-\alpha_{\ell+1}(L,\rho)}{\alpha_1(L,\rho)} - \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1}\right| \qquad (19)\end{aligned}$$

where the last inequality applied the control

$$\left|\frac{2L^\gamma\hat{\mu}(\Gamma_\rho)}{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)} - \frac{1}{\alpha_1}\right| \leqslant \left(\sup_{z \geqslant \alpha_1 - \mathcal{O}(\epsilon)} z^{-2}\right)\left|\frac{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho} > 0\right)}{2L^\gamma\hat{\mu}(\Gamma_\rho)} - \alpha_1\right| \leqslant \frac{\mathcal{O}(\epsilon)}{(\alpha_1 - \mathcal{O}(\epsilon))^2},$$

which is due to equation (18).

Passing $\lim_{\epsilon\to 0}\overline{\lim}_{L\to\infty}\overline{\lim}_{\rho\to 0}$ over equation (19), we observe the RHS going to zero and we find that

$$\overline{\lim_{L\to\infty}}\,\overline{\lim_{\rho\to 0}}\frac{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho}=\ell\right)}{\hat{\mu}\left(Z^{2L^\gamma}_{\Gamma_\rho}>0\right)}=\frac{\alpha_l-\alpha_{l+1}}{\alpha_1}.$$

Alternating between lim sup's and lim inf's lets us reach the desired conclusion.

Now we prove each of the approximations used above. Many of them rely on initial inclusions which are indicated and whose justification is left to the reader.

Proof of A) One can check that

$$\left\{Z^{\omega,i+L}_{\Gamma_\rho}=\ell,I^\omega_i=1\right\}\cap(D^{L,L^\gamma}_{\omega,i})^c\subset\left\{Z^{\omega,2L^\gamma}_{\Gamma_\rho}=\ell,I^\omega_i=1\right\}$$
$$\subset\left\{Z^{\omega,i+L}_{\Gamma_\rho}=\ell,I^\omega_i=1\right\}\cup D^{L,L^\gamma}_{\omega,i},$$

which gives an inequality that, after integration, can bound the LHS of A) by $\int_\Omega\mu^\omega(D^{L,L^\gamma}_{\omega,i})d\nu(\omega)$.

Then we calculate

$$\int_\Omega\mu^\omega(D^{L,L^\gamma}_{\omega,i})d\nu(\omega)=\int_\Omega\mu^\omega\left(Z^{\theta^{L+1}\omega,2L^\gamma-i-L-1}_{\Gamma_\rho}\circ T^{L+1}_\omega>0,I^\omega_0=1\right)d\nu(\omega),$$

but $2L^\gamma-i>L+1\Leftrightarrow 2L^\gamma-L-2\geqslant i$, $L+1\geqslant L_3(\epsilon)\geqslant L_2(\eta(\epsilon))$ and $\rho\leqslant\rho_3(\epsilon,L)\leqslant\rho_2(\eta(\epsilon),L+1,2L^\gamma-i)$, so (I, lemma 1) applies, implying that the last integral is bounded above by $\eta(\epsilon)\hat{\mu}(\Gamma_\rho)$. □

Proof of B) One can check that

$$R^{i,L}_{\omega,l,0}\cap(D^{L/2,L^\gamma}_{\omega,i}\cup F^{L/2}_{\omega,i})^c\subset\bigcup_{j=i-L/2}^{i}(T^{i-j}_\omega)^{-1}R^{i,L}_{\theta^{i-j}\omega,l,k}(j)\subset R^{i,L}_{\omega,l,0}\cup(D^{L/2,L^\gamma}_{\omega,i}\cup F^{L/2}_{\omega,i}).$$

Then we calculate

$$\int_\Omega\mu^\omega\left(F^{L/2}_{\omega,i}\right)d\nu(\omega)=\int_\Omega\mu^\omega\left(Z^{\omega,L/2}_{\Gamma_\rho}>0,I^\omega_i=1\right)d\nu(\omega)\leqslant\eta(\epsilon)\hat{\mu}(\Gamma_\rho),$$

where the inequality follows from an application of (II, lemma 1), since $i-{}^{L}\!/_{2}\geqslant L-{}^{L}\!/_{2}={}^{L}\!/_{2}\geqslant L_2(\eta(\epsilon))$ and $\rho\leqslant\rho_3(\epsilon,L)\leqslant\rho_2(\eta(\epsilon),{}^{L}\!/_{2},i)$.

And also

$$\int_\Omega\mu^\omega\left(D^{L/2,L^\gamma}_{\omega,i}\right)d\nu(\omega)=\int_\Omega\mu^\omega\left(Z^{\theta^{L/2+1}\omega,2L^\gamma-i-L/2-1}_{\Gamma_\rho}\circ T^{L/2+1}_\omega>0,I^\omega_0=1\right)d\nu(\omega)\leqslant\eta(\epsilon)\hat{\mu}(\Gamma_\rho),$$

where the inequality follows from an application of (I, lemma 1): $2L^\gamma-i>{}^{L}\!/_{2}+1\Leftrightarrow 2L^\gamma-{}^{L}\!/_{2}-2\geqslant i$, ${}^{L}\!/_{2}+1\geqslant L_2(\eta(\epsilon))$ and $\rho\leqslant\rho_3(\epsilon,L)\leqslant\rho_2(\eta(\epsilon),{}^{L}\!/_{2}+1,2L^\gamma-i)$.

The consequence is the approximation

$$\left|\int_\Omega\mu^\omega\left(R^{i,L}_{\omega,l,0}\right)d\nu(\omega)-\int_\Omega\mu^\omega\left(\bigsqcup_{j=i-L/2}^{i}(T^{i-j}_\omega)^{-1}R^{i,L}_{\theta^{i-j}\omega,l,k}(j)\right)d\nu(\omega)\right|\leqslant 2\eta(\epsilon)\hat{\mu}(\Gamma_\rho).$$

However,

$$\int_\Omega \mu^\omega\left(\bigsqcup_{j=i-L/2}^{i}(T_\omega^{i-j})^{-1}R^{i,L}_{\theta^{i-j}\omega,l,k}(j)\right)d\nu(\omega)=\int_\Omega \mu^\omega\left(\bigsqcup_{j=i-L/2}^{i}R^{i,L}_{\omega,l,k}(j)\right)d\nu(\omega)$$

$$\leqslant \int_\Omega \mu^\omega\left(R^{i,L}_{\omega,l,k}\right)d\nu(\omega)\int_\Omega \mu^\omega\left(R^{i,L}_{\omega,l,k}\right)d\nu(\omega)\leqslant\int_\Omega \mu^\omega\left(\bigsqcup_{j=i-L/2}^{i}R^{i,L}_{\omega,l,k}(j)\right)d\nu(\omega)+\eta(\epsilon)\hat{\mu}(\Gamma_\rho),$$

where the last inequality follows from the inclusion

$$R^{i,L}_{\omega,l,k}\subset\bigcup_{j=i-L/2}^{i}R^{i,L}_{\omega,l,k}(j)\cup F^{i-\frac{L}{2}}_{\omega,i}$$

and the estimate

$$\int_\Omega \mu^\omega\left(F^{i-L/2}_{\omega,i}\right)d\nu(\omega)=\int_\Omega \mu^\omega\left(Z^{\omega,i-L/2}_{\Gamma_\rho}>0,I_i^\omega=1\right)d\nu(\omega)\leqslant\eta(\epsilon)\hat{\mu}(\Gamma_\rho),$$

with the inequality following from another application of (II, lemma 1): $i-(i-L/2)=L/2\geqslant L_2(\eta(\epsilon))$ and $\rho\leqslant\rho_3(\epsilon,L)\leqslant\rho_2(\eta(\epsilon),i-{}^L\!/_2,i)$.

The conclusion follows from aforementioned approximation and the previous control. □

Proof of C) One can check that

$$R^{i,L}_{\omega,l,0}\subset(T_\omega^{i-L})^{-1}R^{L,L}_{\theta^{i-L}\omega,l,0},\ (T_\omega^{i-L})^{-1}R^{L,L}_{\theta^{i-L}\omega,l,0}\backslash R^{i,L}_{\omega,l,0}\subset F^{i-L/2}_{\omega,i},$$

which gives an inequality that, after integration, can bound the LHS of the expression we need to control by $\int_\Omega\mu^\omega(F^{i-L/2}_{\omega,i})d\nu(\omega)\leqslant\eta(\epsilon)\hat{\mu}(\Gamma_\rho)$, where the later estimate is identical to that obtained at the end of the proof of B). □

Proof of D) One can check that

$$R^{i,L}_{\omega,l,k-1}\cap(D^{L/2,L^\gamma}_{\omega,i}\cup F^{L/2}_{\omega,i})^c\subset\bigsqcup_{j=i-L/2}^{i-1}(T_\omega^{i-j})^{-1}S^{i,L}_{\theta^{i-j},l,k}(j)\subset R^{i,L}_{\omega,l,k-1}\cup(D^{L/2,L^\gamma}_{\omega,i}\cup F^{L/2}_{\omega,i}).\quad(20)$$

The corrective sets $F^{L/2}_{\omega,i}$ and $D^{L/2,L^\gamma}_{\omega,i}$ are treated again as in the proof of B), implying that

$$\left|\int_\Omega\mu^\omega\left(R^{i,L}_{\omega,l,k-1}\right)d\nu(\omega)-\int_\Omega\mu^\omega\left(\bigsqcup_{j=i-L/2}^{i-1}(T_\omega^{i-j})^{-1}S^{i,L}_{\theta^{i-j}\omega,l,k}(j)\right)d\nu(\omega)\right|\leqslant 2\eta(\epsilon)\hat{\mu}(\Gamma_\rho).\quad(21)$$

However,

$$\int_\Omega \mu^\omega\left(\bigcup_{j=i-L/2}^{i-1}(T_\omega^{i-j})^{-1}S^{i,L}_{\theta^{i-j}\omega,l,k}(j)\right)d\nu(\omega)=\int_\Omega \mu^\omega\left(\bigcup_{j=i-L/2}^{i-1}S^{i,L}_{\omega,l,k}(j)\right)d\nu(\omega)$$

$$\leqslant\int_\Omega\mu^\omega\left(R^{i,L}_{\omega,l,k}\right)d\nu(\omega)\leqslant\int_\Omega \mu^\omega\left(\bigcup_{j=i-L/2}^{i}S^{i,L}_{\omega,l,k}(j)\right)d\nu(\omega)+\eta(\epsilon)\hat{\mu}(\Gamma_\rho),$$

where the last inequality follows from the inclusion

$$R^{i,L}_{\omega,l,k} \subset \bigcup_{j=i-L/2}^{i} S^{i,L}_{\omega,l,k}(j) \cup F^{i-\frac{L}{2}}_{\omega,i},$$

and the respective estimate of the corrective set using (II, lemma 1), precisely as in the end of the proof of B). □

Proof of E) One can check that

$$\bigcup_{j=1}^{L} \bigcup_{k=k_0(\epsilon)}^{\infty(L)} (T^j_\omega)^{-1} R^{L,L}_{\theta^j\omega,l+k,k}(L-j) \subset (T^L_\omega)^{-1}\left\{Z^{\theta^L\omega,2L}_{\Gamma_\rho} \geqslant k_0(\epsilon)+\ell, I^\omega_L = 1\right\}.$$

Therefore, integrating, manipulating and using invariance over and over:

$$\begin{aligned}\int_\Omega \mu^\omega\left(\bigcup_{j=1}^{L} \bigcup_{k=k_0(\epsilon)}^{\infty(L)} (T^j_\omega)^{-1} R^{L,L}_{\theta^j\omega,l+k,k}(L-j)\right) d\nu(\omega) &= \sum_{k=k_0(\epsilon)}^{\infty(L)} \int_\Omega \mu^\omega\left(R^{L,L}_{\omega,l+k,k}\right) d\nu(\omega) \\ &\leqslant \int_\Omega \mu^\omega\left(Z^{\omega,2L}_{\Gamma_\rho} \geqslant k_0(\epsilon)+\ell, I^\omega_0 = 1\right) d\nu(\omega) \\ &= \hat\alpha_{k_0(\epsilon)+\ell}(2L,\rho)\hat\mu(\Gamma_\rho) \leqslant (k_0(\epsilon)+\ell)\hat\alpha_{k_0(\epsilon)+\ell}(2L,\rho)\hat\mu(\Gamma_\rho).\end{aligned}$$

□

Proof of F) One can check that

$$R^{i,L}_{\omega,h,k} \cap (D^{L,L^\gamma}_{\omega,i})^c \subset R^{i,2L^\gamma-i}_{\omega,h,k} \subset R^{i,L}_{\omega,h,k} \cup D^{L,L^\gamma}_{\omega,i}.$$

It remains to reapply the justification used in the proof of (A) to get that $\int_\Omega \mu^\omega(D^{L,L^\gamma}_{\omega,i})d\nu(\omega) \leqslant \eta(\epsilon)\hat\mu(\Gamma_\rho)$. □

■

4. An abstract approximation theorem

The following theorem approximates the probability distribution of an arbitrary sum of binary variables in terms of the distribution of a suitable sum of independent random variables. More precisely, to build the 'suitable' independent random variables, one splits the first sum into smaller block-sums, and each of them is distributionally mimicked by a new random variable, with the collection of new ones being taken to be independent.

Theorem 3. *Consider $n \geqslant 0$, $L \geqslant 1$, $N \in \mathbb{N}_{\geqslant 3}$ large enough so that $L \leqslant \lfloor \frac{N+1}{3} \rfloor$, and $(X_i)_{i=0}^N$ arbitrary $\{0,1\}$-valued random variables on $(\mathcal{X}, \mathscr{X}, \mathbb{Q})$. Denote $N'_L := \frac{N+1}{L} \in \mathbb{N}_{\geqslant 3}$*[4] *and $(Z^L_j)_{j=0}^{N'_L-1}$ given by $Z^L_j := \sum_{i=jL}^{(j+1)L-1} X_i$*[5]*.*

[4] Although L need not divide $N+1$, we pretend this is the case, for simplification purposes, i.e. to neglect possible remainder terms associated with the fractional part — which shouldn't play a role in the asymptotics (of either the error and leading terms).

[5] This is the first instance in the text where the letter L is used to measure block-size, where L is fixed. Before, L was iteration-time and eventually sent to infinity, as in definitions (I-VI) of section 2.2. However, through the text, these use cases merge, in the sense that the block of size L iterates the maps

Let $(\tilde{Z}_j^L)_{j=0}^{N'_L-1}$ *be an independency of* $\mathbb{N}_{\geqslant 0}$*-valued random variables on* $(X, \mathcal{X}, \mathbb{Q})$[6] *satisfying* $\tilde{Z}_j^L \sim Z_j^L$ $(j = 0, \dots, N'_L - 1)$ *and* $(\tilde{Z}_j^L)_{j=0}^{N'_L-1} \perp (Z_j^L)_{j=0}^{N'_L-1}$.

Denote $\tilde{W}_{a,b}^L := \sum_{j=a}^b \tilde{Z}_j^L$ $(0 \leqslant a \leqslant b \leqslant N'_L - 1)$ *and* $\tilde{W}^L := \tilde{W}_{0,N'_L-1}^L$. *Similarly notation with* $\sim$*'s erased is adopted, in which case* W^L *coincides with* $W := \sum_{i=0}^N X_i$.

Then:

$$\left|\mathbb{Q}(W = n) - \mathbb{Q}(\tilde{W}^L = n)\right| \lesssim \left(\mathcal{R}^1(N, L, \Delta) + \mathcal{R}^2(N, L, \Delta) + \mathcal{R}^3(N, L)\right),$$

where

$$\mathcal{R}^1(N, L, \Delta) = \sum_{j=0}^{N'_L-1} \max_{q\in[2,L]} \sum_{u=1}^{q-1} \left|\mathbb{Q}\Big(Z_j^L = u, W_{j+\Delta,N'_L-1}^L = q-u\Big) - \mathbb{Q}\Big(Z_j^L = u\Big)\mathbb{Q}\Big(W_{j+\Delta,N'_L-1}^L = q-u\Big)\right|,$$

$$\mathcal{R}^2(N, L, \Delta) = \sum_{j=0}^{N'_L-1} \mathbb{Q}\left(Z_j^L \geqslant 1, W_{j+1,j+\Delta-1}^L \geqslant 1\right) \text{ and}$$

$$\mathcal{R}^3(N, L) = \sum_{i=0}^{N} \sum_{q=0\vee(i-\Delta L)}^{i} \mathbb{Q}(X_i = 1)\mathbb{Q}(X_q = 1),$$

with the convention that, for $b > a$, $W_{b,a}^L \equiv 0$ *and* $\mathbb{Q}(W_{b,a}^L \geqslant 1) = 0$.

Proof. Notice that by using a telescopic sum and the given independence, one has

$$\left|\mathbb{Q}(W = n) - \mathbb{Q}(\tilde{W}^L = n)\right| \leqslant \sum_{j=0}^{N'_L-1} \left|\mathbb{Q}(\tilde{W}_{0,j-1}^L + W_{j,N'_L-1}^L = n) - \mathbb{Q}(\tilde{W}_{0,j}^L + W_{j+1,N'_L-1}^L = n)\right|$$

$$\leqslant \sum_{j=0}^{N'_L-1} \sum_{l=0}^{n} \mathbb{Q}(\tilde{W}_{0,j-1}^L = l) \left|\mathbb{Q}(W_{j,N'_L-1}^L = n - l) - \mathbb{Q}(\tilde{Z}_j^L + W_{j+1,N'_L-1}^L = n - l)\right|.$$

We now estimate

$$\left|\mathbb{Q}(W_{j,N'_L-1}^L = q) - \mathbb{Q}(\tilde{Z}_j^L + W_{j+1,N'_L-1}^L = q)\right|$$

$$\leqslant \sum_{u=0}^{q} \left|\mathbb{Q}(Z_j^L = u, W_{j+1,N'_L-1}^L = q - u) - \mathbb{Q}(\tilde{Z}_j^L = u, W_{j+1,N'_L-1}^L = q - u)\right|$$

$$= \sum_{u=0}^{q} \left|\mathbb{Q}(Z_j^L = u, W_{j+1,N'_L-1}^L = q - u) - \mathbb{Q}(Z_j^L = u)\mathbb{Q}(W_{j+1,N'_L-1}^L = q - u)\right| =: \sum_{u=0}^{q} |\mathcal{R}_j(u)|.$$

L times: much before the L-limit we are inclined to see L as a fixed block-size, much closer to the L-limit we are inclined to see it as growing iteration-time. It is implicit that every time this merger occurs we'll eventually want to take the L-limit. There is a special situation where this is not the case, to be seen in lemma 3 and lemma 4 item (3), where we'll be equally interested in $L = 1$ and $L \to \infty$

[6]For the statement of the theorem, it seems unimportant that the domain of the mimicking random variables is that of the original ones, but this is used in the proof. Of course, $(\mathcal{X}, \mathscr{X}, \mathbb{Q})$ then has to be a rich enough space in order to accommodate the existence of such mimicking random variables. This won't be an issue in our application.

We single out $u \in \{0, q\}$ from the previous sum,

$$\begin{aligned}
|\mathcal{R}_j(0)| &= \left|\mathbb{Q}(Z_j^L = 0, W^L_{j+1,N'_L-1} = q) - \mathbb{Q}(Z_j^L = 0)\mathbb{Q}(W^L_{j+1,N'_L-1} = q)\right| \\
&= \Big| \Big(\mathbb{Q}(W^L_{j+1,N'_L-1} = q) - \mathbb{Q}(Z_j^L \geqslant 1, W^L_{j+1,N'_L-1} = q)\Big) \\
&\qquad - \Big(\mathbb{Q}(W^L_{j+1,N'_L-1} = q) - \mathbb{Q}(Z_j^L = 0)\mathbb{Q}(W^L_{j+1,N'_L-1} = q)\Big)\Big| \\
&= \left|\mathbb{Q}(Z_j^L = 0)\mathbb{Q}(W^L_{j+1,N'_L-1} = q) - \mathbb{Q}(Z_j^L \geqslant 1, W^L_{j+1,N'_L-1} = q)\right| \\
&\leqslant \sum_{u=1}^{q} |\mathcal{R}_j(u)|,
\end{aligned}$$

and, similarly, for the other end

$$\begin{aligned}
|\mathcal{R}_j(q)| &= \left|\mathbb{Q}(Z_j^L = q)\mathbb{Q}(W^L_{j+1,N'_L-1} \geqslant 1) - \mathbb{Q}(Z_j^L = q, W^L_{j+1,N'_L-1} \geqslant 1)\right| \\
&\leqslant \sum_{u=1}^{q-1} |\mathcal{R}_j(u)|,
\end{aligned}$$

to conclude that

$$\sum_{u=0}^{q} |\mathcal{R}_j(u)| \leqslant 4 \sum_{u=q}^{q-1} |\mathcal{R}_j(u)|.$$

For $u = 1, \ldots, q-1$, we expand $|\mathcal{R}_j(u)|$ using the triangular inequality, where we include intermediate terms using the time gap Δ, to get the following three components

$$\begin{aligned}
|\mathcal{R}_j(u)| &\leqslant \left|\mathbb{Q}(Z_j^L = u, W^L_{j+1,N'_L-1} = q-u) - \mathbb{Q}(Z_j^L = u, W^L_{j+\Delta,N'_L-1} = q-u)\right| \\
&+ \left|\mathbb{Q}(Z_j^L = u, W^L_{j+\Delta,N'_L-1} = q-u) - \mathbb{Q}(Z_j^L = u)\mathbb{Q}(W^L_{j+\Delta,N'_L-1} = q-u)\right| \\
&+ \left|\mathbb{Q}(Z_j^L = u)\mathbb{Q}(W^L_{j+\Delta,N'_L-1} = q-u) - \mathbb{Q}(Z_j^L = u)\mathbb{Q}(W^L_{j+1,N'_L-1} = q-u)\right|,
\end{aligned}$$

where the entries in the RHS are denoted, respectively, by $|\mathcal{R}_j^2(u)|$, $|\mathcal{R}_j^1(u)|$ and $|\mathcal{R}_j^3(u)|$.

It follows that

$$\left|\mathbb{Q}(W = n) - \mathbb{Q}(\tilde{W}^L = n)\right| \leqslant \sum_{j=0}^{N'_L-1} \left|\mathbb{Q}(\tilde{W}^L_{0,j-1} + W^L_{j,N'_L-1} = n) - \mathbb{Q}(\tilde{W}^L_{0,j} + W^L_{j+1,N'_L-1} = n)\right|$$

is $\lesssim$-bounded by the sum of the three components evaluated in the next paragraphs.

First:

$$\begin{aligned}
\sum_{j=0}^{N'_L-1} \sum_{q=0}^{n} \sum_{u=0}^{q} |\mathcal{R}_j^1(u)| &\leqslant 4n \sum_{j=0}^{N'_L-1} \max_{q \in [2,L]} \sum_{u=1}^{q-1} |\mathcal{R}_j^1(u)| \\
&\lesssim \sum_{j=0}^{N'_L-1} \max_{q \in [2,L]} \sum_{u=1}^{q-1} |\mathcal{R}_j^1(u)| =: \mathcal{R}^1(N, L, \Delta),
\end{aligned}$$

where n is incorporated into $\lesssim$ because it is a constant in the sense that no limits in n will be taken at all.

Second:

$$\sum_{j=0}^{N'_L-1}\sum_{q=0}^{n}\sum_{u=0}^{q}|\mathcal{R}_j^2(u)| \leqslant 4n\sum_{j=0}^{N'_L-1}\max_{q\in[2,L]}\sum_{u=1}^{q-1}|\mathcal{R}_j^2(u)|$$

$$\lesssim \sum_{j=0}^{N'_L-1}\mathbb{Q}(Z_j^L\geqslant 1, W_{j+1,j+\Delta-1}^L\geqslant 1) =: \mathcal{R}^2(N,L,\Delta),$$

where the $\lesssim$ step incorporated n and used that

$$A_u := \{Z_j^L = u, W_{j+1,N'_L-1}^L = q-u\}, B_u := \{Z_j^L = u, W_{j+\Delta,N'_L-1}^L = q-u\}$$

$$\Rightarrow A_u\backslash B_u, B_u\backslash A_u \subset \{Z_j^L = u, W_{j+1,j+\Delta-1}^L\geqslant 1\}$$

$$\Rightarrow \sum_{u=1}^{q-1}|\mathcal{R}_j^2(u)| = \sum_{u=1}^{q-1}|\mathbb{Q}(A_u)-\mathbb{Q}(B_u)| \leqslant \sum_{u=1}^{q-1}\mathbb{Q}(Z_j^L = u, W_{j+1,j+\Delta-1}^L\geqslant 1)$$

$$\leqslant \mathbb{Q}(Z_j^L\geqslant 1, W_{j+1,j+\Delta-1}^L\geqslant 1).$$

Third:

$$\sum_{j=0}^{N'_L-1}\sum_{q=0}^{n}\sum_{u=0}^{q}|\mathcal{R}_j^3(u)| \leqslant 4n\sum_{j=0}^{N'_L-1}\max_{q\in[2,L]}\sum_{u=1}^{q-1}|\mathcal{R}_j^3(u)|$$

$$\lesssim \sum_{j=0}^{N'_L-1}\sum_{l=jL}^{(j+1)L-1}\sum_{i=(j+1)L}^{(j+\Delta+1)L-1}\mathbb{Q}(X_i=1)\mathbb{Q}(X_l=1) = \sum_{i=0}^{N+\Delta L+L}\sum_{l=0\vee(i-L-\Delta L)}^{i-L}\mathbb{Q}(X_l=1)\mathbb{Q}(X_i=1)$$

$$\leqslant \sum_{i=0}^{N}\sum_{l=0\vee(i-\Delta L)}^{i}\mathbb{Q}(X_l=1)\mathbb{Q}(X_i=1),$$

where the $\lesssim$ step incorporated n and used the following: (with $q' = q-u$)

$$\mathbb{Q}(W_{j+1,N'_L-1}^L = q') = \mathbb{Q}(Z_{j+1}^L\geqslant 1, W_{j+1,N'_L-1}^L = q') + \mathbb{Q}(Z_{j+1}^L = 0, W_{j+1,N'_L-1}^L = q')$$

$$\mathbb{Q}(Z_{j+1}^L = 0, W_{j+1,N'_L-1}^L = q') = \mathbb{Q}(Z_{j+1}^L = 0, W_{j+2,N'_L-1}^L = q')$$

$$= \mathbb{Q}(W_{j+2,N'_L-1}^L = q') - \mathbb{Q}(Z_{j+1}^L\geqslant 1, W_{j+2,N'_L-1}^L = q')$$

$$\Rightarrow \begin{array}{c}|\mathbb{Q}(W_{j+1,N'_L-1}^L = q')\\ -\mathbb{Q}(W_{j+2,N'_L-1}^L = q')|\end{array} = \begin{array}{c}|\mathbb{Q}(Z_{j+1}^L\geqslant 1, W_{j+1,N'_L-1}^L = q')\\ -\mathbb{Q}(Z_{j+1}^L\geqslant 1, W_{j+2,N'_L-1}^L = q')|\end{array}$$

but, with $A := \{Z_{j+1}^L\geqslant 1, W_{j+1,N'_L-1}^L = q'\}$ and $B := \{Z_{j+1}^L\geqslant 1, W_{j+2,N'_L-1}^L = q'\}$, one has $A\backslash B, B\backslash A \subset \{Z_{j+1}^L\geqslant 1\}$, implying

$$|\mathbb{Q}(W_{j+1,N'_L-1}^L = q') - \mathbb{Q}(W_{j+2,N'_L-1}^L = q')| \leqslant \mathbb{Q}(Z_{j+1}^L\geqslant 1)$$

$$\Rightarrow |\mathbb{Q}(W_{j+l,N'_L-1}^L = q') - \mathbb{Q}(W_{j+l+1,N'_L-1}^L = q')| \leqslant \mathbb{Q}(Z_{j+l}^L\geqslant 1) \leqslant \sum_{i=(j+l)L}^{(j+l+1)L-1}\mathbb{Q}(X_i=1)$$

$$\Rightarrow |\mathbb{Q}(W_{j+1,N'_L-1}^L = q') - \mathbb{Q}(W_{j+\Delta,N'_L-1}^L = q')| \leqslant \sum_{l=1}^{\Delta-1}\mathbb{Q}(Z_{j+l}^L\geqslant 1) \leqslant \sum_{i=(j+1)L}^{(j+\Delta)L-1}\mathbb{Q}(X_i=1)$$

$$\begin{aligned}\Rightarrow \sum_{u=1}^{q-1}|\mathcal{R}_j^3(u)| &\leqslant \sum_{u=1}^{q-1}\mathbb{Q}(Z_j^L=u)\sum_{i=(j+1)L}^{(j+\Delta)L-1}\mathbb{Q}(X_i=1)\\ &\leqslant \sum_{l=jL}^{(j+1)L-1}\sum_{i=(j+1)L}^{(j+\Delta)L-1}\mathbb{Q}(X_l=1)\mathbb{Q}(X_i=1).\end{aligned}$$

■

5. Borel-Cantelli type lemmata

The objective of this section is its final lemma 4, which will be used in the proof of theorem 2. Lemma 4 will follow from lemmas 2 and 3. The later lemmas are essentially independent, although lemma 2 uses return statistics in its hypothesis and relies on theorem 1 in its proof. We believe that the dependencies in the last sentence might not be intrinsic and could be untied.

Lemma 2. *Let $(\theta,\nu,T_\omega,\mu^\omega,\Gamma)$ be a system satisfying hypothesis (H7') (so (H6), by theorem 1) . Then:*

$$\lim_{L\to\infty}\overline{\lim_{\rho\to 0}}\frac{\hat{\mu}(Z_{\Gamma_\rho}^L\geqslant 1)}{(L+1)\hat{\mu}(\Gamma_\rho)}=(\textstyle\sum_{\ell=1}^\infty \ell\lambda_\ell)^{-1}=\alpha_1 \tag{22}$$

and

$$\lim_{L\to\infty}\overline{\lim_{\rho\to 0}}\frac{\hat{\mu}(Z_{\Gamma_\rho}^L=n)}{(L+1)\hat{\mu}(\Gamma_\rho)}=(\textstyle\sum_{\ell=1}^\infty \ell\lambda_\ell)^{-1}\lambda_n=\alpha_1\lambda_n\ (n\geqslant 1) \tag{23}$$

Proof. Using (H7') (for the following items (i.b-ii)) and (H6) (items (i.a,iii-iv)), it holds that: $\forall\epsilon>0$

i) $\exists\ell_0(\epsilon)\geqslant 1$ so that
   a) $\sum_{\ell=\ell_0(\epsilon)}^\infty \ell^3\lambda_\ell\leqslant\epsilon$,
   b) $\forall L\geqslant 1$: $\sum\limits_{\ell=\ell_0(\epsilon)}^\infty \ell\overset{+}{\hat{\alpha}}_\ell(L)\leqslant\sum\limits_{\ell=\ell_0(\epsilon)}^\infty \ell\hat{\alpha}_\ell\leqslant\epsilon$.

ii) $\forall L\geqslant 1,\exists\rho_1(\epsilon,L),\forall\rho\leqslant\rho_1(\epsilon,L)$:

$$\overset{-}{\hat{\alpha}}_\ell(L)-\epsilon/(L+1)^2\leqslant\hat{\alpha}_\ell(L,\rho)\leqslant\overset{+}{\hat{\alpha}}_\ell(L)+\epsilon/(L+1)^2\ (\forall\ell=1,\dots,L+1)$$

$$\Rightarrow\sum_{\ell=\ell_0(\epsilon)}^{L+1}\ell\hat{\alpha}_\ell(L,\rho)\leqslant\sum_{\ell=\ell_0(\epsilon)}^{L+1}\ell\left(\overset{+}{\hat{\alpha}}_\ell(L)+\epsilon/(L+1)^2\right)\leqslant 2\epsilon\text{ by (i).}$$

iii) $\forall L\geqslant 1,\exists\rho_3(\epsilon,L),\forall\rho\leqslant\rho_3(\epsilon,L)$:

$$\overset{-}{\lambda}_\ell(L)-\epsilon/(\ell_0(\epsilon))^2\leqslant\lambda_\ell(L,\rho)\leqslant\overset{+}{\lambda}_\ell(L)+\epsilon/(\ell_0(\epsilon))^2\ (\forall\ell=1,\dots,\ell_0(\epsilon)).$$

iv) $\exists L_0(\epsilon)>\ell_0(\epsilon),\forall L\geqslant L_0(\epsilon)$:

$$|\lambda_\ell-\overset{-}{\lambda}_\ell(L)|\leqslant\epsilon/(\ell_0(\epsilon))^2,\ |\lambda_\ell-\overset{+}{\lambda}_\ell(L)|\leqslant\epsilon/(\ell_0(\epsilon))^2\ (\forall\ell=1,\dots,\ell_0(\epsilon))$$

$$\Rightarrow|\overset{+}{\lambda}_\ell(L)-\overset{-}{\lambda}_\ell(L)|\leqslant 2\epsilon/(\ell_0(\epsilon))^2\ (\forall\ell=1,\dots,\ell_0(\epsilon)).$$

v) (due to items (iv-v)) $\exists L_0(\epsilon), \forall L \geqslant L_0(\epsilon), \exists \rho_3(\epsilon, L), \forall \rho \leqslant \rho_3(\epsilon, L)$:

$$|\lambda_\ell(L,\rho) - \overset{*}{\lambda}_\ell(L)| \leqslant 3\epsilon/(\ell_0(\epsilon))^2 \ (\forall \ell = 1, \ldots, \ell_0(\epsilon), \forall * \in \{-,+\})$$

$$\Rightarrow |\lambda_\ell(L,\rho) - \lambda_\ell| \leqslant 4\epsilon/(\ell_0(\epsilon))^2 \ (\forall \ell = 1, \ldots, \ell_0(\epsilon), \forall * \in \{-,+\})$$

$$\Rightarrow \left| \sum_{\ell=1}^{\ell_0(\epsilon)} (\ell-1)\lambda_\ell(L,\rho) - \sum_{\ell=1}^{\ell_0(\epsilon)} (\ell-1)\lambda_\ell \right| \leqslant \sum_{\ell=1}^{\ell_0(\epsilon)} \ell_0(\epsilon) 4\epsilon/(\ell_0(\epsilon))^2 \leqslant 4\epsilon.$$

Now, considering any $\epsilon < 1/5 \sum_{\ell=1}^{\infty} \ell\lambda_\ell$, $L \geqslant L_0(\epsilon)$ and $\rho \leqslant \rho_1(\epsilon, L) \wedge \rho_2(\epsilon) \wedge \rho_3(\epsilon, L)$, we evaluate the quantity of interest, $\hat{\mu}(Z^L_{\Gamma_\rho} \geqslant 1)/(L+1)\hat{\mu}(\Gamma_\rho)$, starting with its numerator:

$$\hat{\mu}(Z^L_{\Gamma_\rho} \geqslant 1) = \int_\Omega \mu^\omega(Z^{\omega,L}_{\Gamma_\rho} \geqslant 1) d\nu(\omega) = \int_\Omega \mu^\omega \left(\bigcup_{j=0}^{L} (T^j_\omega)^{-1} \Gamma_\rho(\theta^j \omega) \right) d\nu(\omega)$$

$$\overset{(\star)}{=} \int_\Omega \sum_{j=0}^{L} \mu^\omega((T^j_\omega)^{-1}\Gamma_\rho(\theta^j\omega)) d\nu(\omega) - \int_\Omega \sum_{\ell=0}^{L} \ell \mu^\omega(Z^{\omega,L}_{\Gamma_\rho} = \ell+1) d\nu(\omega)$$

$$= (L+1)\hat{\mu}(\Gamma_\rho) - \int_\Omega \left(\sum_{\ell=0}^{L} \ell \lambda^\omega_{\ell+1}(L,\rho) \right) \mu^\omega(Z^{\omega,L}_{\Gamma_\rho} > 0) d\nu(\omega)$$

$$\begin{aligned} = (L+1)\hat{\mu}(\Gamma_\rho) \quad &- \int_\Omega \left(\sum_{\ell=0}^{\ell_0(\epsilon)-1} \ell \lambda^\omega_{\ell+1}(L,\rho) \right) \mu^\omega(Z^{\omega,L}_{\Gamma_\rho} > 0) d\nu(\omega) \\ &- \int_\Omega \left(\sum_{\ell=\ell_0(\epsilon)}^{\infty} \ell \lambda^\omega_{\ell+1}(L,\rho) \right) \mu^\omega(Z^{\omega,L}_{\Gamma_\rho} > 0) d\nu(\omega) \end{aligned}$$

$$= (L+1)\hat{\mu}(\Gamma_\rho) - \sum_{\ell=0}^{\ell_0(\epsilon)-1} \ell \hat{\mu}(Z^L_{\Gamma_\rho} = \ell+1) - \int_\Omega \left(\sum_{\ell=\ell_0(\epsilon)}^{\infty} \ell \lambda^\omega_{\ell+1}(L,\rho) \right) \mu^\omega(Z^{\omega,L}_{\Gamma_\rho} > 0) d\nu(\omega)$$

$$\begin{aligned} = (L+1)\hat{\mu}(\Gamma_\rho) \quad &- \quad \left(\sum_{\ell=1}^{\ell_0(\epsilon)} (\ell-1)\lambda_\ell(L,\rho) \right) \hat{\mu}(Z^L_{\Gamma_\rho} > 0) \\ &- \quad \int_\Omega \left(\sum_{\ell=\ell_0(\epsilon)+1}^{\infty} (\ell-1)\lambda^\omega_\ell(L,\rho) \right) \mu^\omega(Z^{\omega,L}_{\Gamma_\rho} > 0) d\nu(\omega) \end{aligned}$$

where $(\star)$ applied a typical Venn diagram argument using overcounting and correction.

Then we consider the following two estimates.

First, we have that:

$$\sum_{\ell=1}^{\ell_0(\epsilon)} (\ell-1)\lambda_\ell(L,\rho) \overset{(v)}{\leqslant} \sum_{\ell=1}^{\ell_0(\epsilon)} (\ell-1)\lambda_\ell + 4\epsilon \leqslant \sum_{\ell=1}^{\infty} (\ell-1)\lambda_\ell + 5\epsilon \text{ and}$$

$$\begin{aligned}\sum_{\ell=1}^{\ell_0(\epsilon)}(\ell-1)\lambda_\ell(L,\rho) &\overset{\text{(v)}}{\geqslant} \sum_{\ell=1}^{\ell_0(\epsilon)}(\ell-1)\lambda_\ell - 4\epsilon = \sum_{\ell=1}^{\infty}(\ell-1)\lambda_\ell - \sum_{\ell=\ell_0(\epsilon)+1}^{\infty}(\ell-1)\lambda_\ell - 4\epsilon\\ &\overset{\text{(i.a)}}{\geqslant} \sum_{\ell=1}^{\infty}(\ell-1)\lambda_\ell - 5\epsilon.\end{aligned}$$

Second, with $\upsilon^\omega_{\Gamma_\rho}(x) = \inf\{j \geqslant 0 : T^j_\omega \in \Gamma_\rho(\theta^j\omega)\}$, we have that:

$$0 \leqslant \int_\Omega \left(\sum_{\ell=\ell_0(\epsilon)+1}^{\infty}(\ell-1)\lambda^\omega_\ell(L,\rho)\right)\mu^\omega(Z^{\omega,L}_{\Gamma_\rho} > 0)d\nu(\omega) \leqslant \sum_{\ell=\ell_0(\epsilon)+1}^{L+1}\ell\hat\mu(Z^L_{\Gamma_\rho} = \ell)$$

$$= \sum_{\ell=\ell_0(\epsilon)+1}^{L+1}\ell\sum_{j=0}^{L}\hat\mu(Z^L_{\Gamma_\rho} = \ell, \upsilon_{\Gamma_\rho} = j) \leqslant \sum_{\ell=\ell_0(\epsilon)+1}^{L+1}\ell\sum_{j=0}^{L}\hat\mu(Z^{L-j}_{\Gamma_\rho}\circ S^j = \ell, (S^j)^{-1}\Gamma_\rho)$$

$$= \sum_{\ell=\ell_0(\epsilon)+1}^{L+1}\ell\sum_{j=0}^{L}\alpha_\ell(L-j,\rho)\hat\mu(\Gamma_\rho) = \left(\sum_{j=0}^{L}\sum_{\ell=\ell_0(\epsilon)+1}^{L+1}\ell\alpha_\ell(L-j,\rho)\right)\hat\mu(\Gamma_\rho)$$

$$\leqslant \left[\sum_{j=0}^{L}\left(\sum_{\ell=\ell_0(\epsilon)+1}^{L+1}\hat\alpha_\ell(L-j,\rho)\right) + \ell_0(\epsilon)\hat\alpha_{\ell_0(\epsilon)}(L-j,\rho) - (L+1)\hat\alpha_{L+2}(L-j,\rho)\right]\hat\mu(\Gamma_\rho)$$

$$\leqslant \left[\sum_{j=0}^{L}\sum_{\ell=\ell_0(\epsilon)+1}^{L+1}\ell\hat\alpha_\ell(L-j,\rho)\right]\hat\mu(\Gamma_\rho) \overset{\text{(ii)}}{\leqslant} 2\epsilon(L+1)\hat\mu(\Gamma_\rho)$$

Combining what we got so far, it follows that:

$$\begin{aligned}\frac{\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat\mu(\Gamma_\rho)} &\leqslant \frac{(L+1)\hat\mu(\Gamma_\rho) - \left(\sum_{\ell=1}^{\infty}(\ell-1)\lambda_\ell - 5\epsilon\right)\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat\mu(\Gamma_\rho)}\\ &= 1 - \left(\sum_{\ell=1}^{\infty}\ell\lambda_\ell - 1 - 5\epsilon\right)\frac{\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat\mu(\Gamma_\rho)}\\ \Rightarrow \frac{\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat\mu(\Gamma_\rho)} &\leqslant \frac{1}{\sum_{\ell=1}^{\infty}\ell\lambda_\ell - 5\epsilon}\end{aligned}$$

and

$$\begin{aligned}\frac{\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat\mu(\Gamma_\rho)} &\geqslant \frac{(L+1)\hat\mu(\Gamma_\rho) - \left(\sum_{\ell=1}^{\infty}(\ell-1)\lambda_\ell + 5\epsilon\right)\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1) - 2\epsilon(L+1)\hat\mu(\Gamma_\rho)}{(L+1)\hat\mu(\Gamma_\rho)}\\ &= 1 - \left(\sum_{\ell=1}^{\infty}\ell\lambda_\ell - 1 + 5\epsilon\right)\frac{\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat\mu(\Gamma_\rho)} - 2\epsilon\\ \Rightarrow \frac{\hat\mu(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat\mu(\Gamma_\rho)} &\geqslant \frac{1-2\epsilon}{\sum_{\ell=1}^{\infty}\ell\lambda_\ell + 5\epsilon}\end{aligned}$$

Considering the final two inequalities and passing $\lim_{\epsilon\to 0}\overline{\lim}_{L\to\infty}\overline{\lim}_{\rho\to 0}$ we observe that

$$\overline{\lim_{L\to\infty}}\;\overline{\lim_{\rho\to 0}}\;\frac{\hat{\mu}(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat{\mu}(\Gamma_\rho)} = \left(\textstyle\sum_{k=1}^{\infty} k\lambda_k\right)^{-1} = \alpha_1$$

Alternating between lim sup's and lim inf's lets us reach the first desired conclusion. Finally, to take care of the second desired conclusion, it suffices to note that

$$\frac{\hat{\mu}(Z^L_{\Gamma_\rho}=n)}{(L+1)\hat{\mu}(\Gamma_\rho)} = \frac{\hat{\mu}(Z^L_{\Gamma_\rho}\geqslant 1)}{(L+1)\hat{\mu}(\Gamma_\rho)}\,\frac{\hat{\mu}(Z^L_{\Gamma_\rho}=n)}{\hat{\mu}(Z^L_{\Gamma_\rho}>0)},$$

then take the appropriate limits and apply the first conclusion we've just proved (to obtain α_1), together with the definition of λ_n. ■

Lemma 3. *Let $(\theta,\nu,T_\omega,\mu^\omega,\Gamma)$ be a system satisfying hypotheses (H2.3), (H3.2), (H4.2), (H4.3), (H5.1) and (H5.2) with the parametric constraint (H8.1).*

Then: $\forall t>0, \forall n\geqslant 1$, $\forall L\geqslant 1$[7], $\exists\rho_{\mathrm{var}}(L)>0$, $\forall\rho\leqslant\rho_{\mathrm{var}}(L)$ small enough so that $N_\rho := \lfloor\frac{t}{\hat{\mu}(\Gamma_\rho)}\rfloor\geqslant 3$ and $N'_{\rho,L} := \frac{N_\rho+1}{L}\in\mathbb{N}_{\geqslant 3}$[8], one has:

$$\mathrm{var}_\nu(\mathfrak{W}^{L,n}_\rho)\leqslant C_{t,L}\cdot\rho^q,\ \forall q\in\big(0,q(d_0,d_1,\eta,\beta,\mathfrak{p})\big),$$

where

$$\mathfrak{W}^{L,n}_\rho(\omega) := \sum_{j=0}^{N'_{\rho,L}-1}\mu^\omega(Z^{\omega,\rho,L}_j=n),\ Z^{\omega,\rho,L}_j := {}^{9}\sum_{l=jL}^{(j+1)L-1} I^{\omega,\rho}_l,\ I^{\omega,\rho}_l := \mathbb{1}_{\Gamma_\rho(\theta^l\omega)}\circ T^l_\omega$$

and $q(d_0,d_1,\eta,\beta,\mathfrak{p})$ is a positive quantity to be presented in the proof (which can be written explicitly).

Proof. Let t,n and L be as in the statement. Fix $\alpha\in(0,1)$. Set $\rho_{\mathrm{var}}(L)\leqslant\rho_{\mathrm{sep}}(L)\wedge\rho_{\dim}$ small enough so that ${N_\rho}^\alpha < N'_{\rho,L}$. Consider $\rho\leqslant\rho_{\mathrm{var}}(L)$ as in the statement and $\omega\in\Omega$.

[7] See footnote 5.

[8] See footnote 4.

[9] The notation $Z^{\omega,\rho,L}_j$ is in parallel to that of Z^L_j in theorem 3. They, on purpose, resemble that $Z^{\omega,L}_{\Gamma_\rho}$ introduced in equation (1). However notice the slight difference: the latter sums $L+1$ terms while the other sums have L terms.

For a given $j \in [0, N'_{\rho,L} - 1]$, write $\omega' = \theta^{jL}\omega$ and notice that

$$\mathbb{E}_\nu(\mathfrak{W}_\rho^{L,n}) = \sum_{j=0}^{N'_{\rho,L}-1} \mathbb{E}_\nu\left(\mu^\omega(Z_j^{\omega,\rho,L} = n)\right) = \sum_{j=0}^{N'_{\rho,L}-1} \mathbb{E}_\nu\left(\mu^\omega(\textstyle\sum_{l=jL}^{(j+1)L-1} \mathbb{1}_{\Gamma_\rho(\theta^l\omega)} \circ T_\omega^l = n)\right)$$

$$= \sum_{j=0}^{N'_{\rho,L}-1} \mathbb{E}_\nu\left(\mu^\omega(\textstyle\sum_{i=0}^{L-1} \mathbb{1}_{\Gamma_\rho(\theta^i\omega')} \circ T_{\omega'}^i \circ T_\omega^{jL} = n)\right) = \sum_{j=0}^{N'_{\rho,L}-1} \mathbb{E}_\nu\left(\mu^{\omega'}(\textstyle\sum_{i=0}^{L-1} \mathbb{1}_{\Gamma_\rho(\theta^i\omega')} \circ T_{\omega'}^i = n)\right)$$

$$= \sum_{j=0}^{N'_{\rho,L}-1} \mathbb{E}_\nu\left(\mu^\omega(\textstyle\sum_{i=0}^{L-1} \mathbb{1}_{\Gamma_\rho(\theta^i\omega)} \circ T_\omega^i = n)\right) = \sum_{j=0}^{N'_{\rho,L}-1} \mathbb{E}_\nu(\mu^\omega(Z_0^{\omega,\rho,L} = n))$$

$$= \sum_{j=0}^{N'_{\rho,L}-1} \hat{\mu}(Z_0^{\rho,L} = n) = N'_{\rho,L}\hat{\mu}(Z_0^{\rho,L} = n).$$

Now fix $\Delta := {N_\rho}^\alpha < N'_{\rho,L}$. Then:

$$\begin{aligned}
\mathbb{E}_\nu((\mathfrak{W}_\rho^{L,n})^2) &= \sum_{i,j=0}^{N'_{\rho,L}-1} \int_\Omega \mu^\omega(Z_i^{\omega,\rho,L} = n)\mu^\omega(Z_j^{\omega,\rho,L} = n)d\nu(\omega) \\
&= 2\sum_{i=0}^{N'_{\rho,L}-1} \sum_{j=i}^{(i+\Delta)\wedge(N'_{\rho,L}-1)} \int_\Omega \mu^\omega(Z_i^{\omega,\rho,L} = n)\mu^\omega(Z_j^{\omega,\rho,L} = n)d\nu(\omega) \\
&+ 2\sum_{i=0}^{N'_{\rho,L}-1} \sum_{j=(i+\Delta)\wedge(N'_{\rho,L}-1)+1}^{N'_{\rho,L}-1} \int_\Omega \mu^\omega(Z_i^{\omega,\rho,L} = n)\mu^\omega(Z_j^{\omega,\rho,L} = n)d\nu(\omega) \\
&=: (I) + (II).
\end{aligned}$$

Immediately we get that

$$\mu^\omega(Z_j^{\omega,\rho,L} = n) \leqslant \mu^\omega(Z_j^{\omega,\rho,L} \geqslant 1) \leqslant \sum_{l=jL}^{(j+1)L-1} \mu^{\theta^l\omega}(\Gamma_\rho(\theta^l\omega)) \overset{\text{(H4.2)}}{\lesssim} L\rho^{d_0}$$

$$\Rightarrow (I) \lesssim L\rho^{d_0}\Delta\mathbb{E}_\nu(\mathfrak{W}_\rho^{L,k}) = \Delta\rho^{d_0}N_\rho\hat{\mu}(Z_0^{\rho,L} = n).$$

Most of the remaining work is to control component (II).

Fix $\omega \in \Omega$ and, for a given $i \in [0, N'_{\rho,L} - 1]$, write $\omega' = \theta^{iL}\omega$. Moreover, consider $r \in (0, \rho/2)$, $v \in [0, L-1]$ and denote by

$$U_{v,\omega'} = \Gamma_\rho(\theta^v\omega'), \quad \overset{-}{U}_{v,r,\omega'} = B({U_{v,\omega'}}^c, r)^c, \quad \overset{+}{U}_{v,r,\omega'} = B(U_{v,\omega'}, r), \tag{24}$$

respectively, the ρ-sized target with seed ω' v-steps ahead; its diminishment by radius r; and its enlargment by radius r. They relate as $\overset{-}{U}_{v,r,\omega'} \subset U_{v,\omega'} \subset \overset{+}{U}_{v,r,\omega'}$.

Moreover, dynamical counterparts of those in equation (24) are denote by

$$\{Z_0^{\omega',\rho,L}=n\} = \mathcal{U}_{\omega'} = \bigsqcup_{0\leqslant v_1<\ldots<v_n\leqslant L-1}\left(\bigcap_{l=1}^{n}(T_{\omega'}^{v_l})^{-1}U_{v_l,\omega'} \cap \bigcap_{\substack{v\in[0,L-1]\\ \setminus\{v_l:l=1,\ldots,n\}}}(T_{\omega'}^{v})^{-1}{U_{v,\omega'}}^c\right),$$

$$\bar{\mathcal{U}}_{r,\omega'} = \bigsqcup_{0\leqslant v_1<\ldots<v_n\leqslant L-1}\left(\bigcap_{l=1}^{n}(T_{\omega'}^{v_l})^{-1}\,\bar{U}_{v_l,\omega'} \cap \bigcap_{\substack{v\in[0,L-1]\\ \setminus\{v_l:l=1,\ldots,n\}}}(T_{\omega'}^{v})^{-1}{\overset{+}{U}_{v,\omega'}}^{c}\right),$$

$$\overset{+}{\mathcal{U}}_{r,\omega'} = \bigsqcup_{0\leqslant v_1<\ldots<v_n\leqslant L-1}\left(\bigcap_{l=1}^{n}(T_{\omega'}^{v_l})^{-1}\,\overset{+}{U}_{v_l,\omega'} \cap \bigcap_{\substack{v\in[0,L-1]\\ \setminus\{v_l:l=1,\ldots,n\}}}(T_{\omega'}^{v})^{-1}{\bar{U}_{v,\omega'}}^{c}\right),$$

describing

- the locus of points which hit the ρ-sized target exactly k times during the time interval $[0, L-1]$ when given the random seed ω';
- the diminishment of the first by radius r, in the sense that hits are considered in a r-stringent way (at least r-inside the ρ-sized target) and non-hits are considered in a r-stringent way (at least r-away from the ρ-sized target);
- the enlargment of the first by radius r, in the sense that hits are considered in a r-permissive way (at most r-away from the ρ-sized target) and non-hits are considered in a r-permissive way (at most r-inside the ρ-sized target).

They relate as $\bar{\mathcal{U}}_{r,\omega'}\subset\mathcal{U}_{\omega'}\subset\overset{+}{\mathcal{U}}_{r,\omega'}$.

Finally, define

$$\bar{\phi}_r^{\omega'}(x)=\begin{cases}1, x\in\bar{\mathcal{U}}_{r,\omega'}\\ 0, x\in{\mathcal{U}_{\omega'}}^c\\ \text{linear interp., otherwise}\end{cases} \quad\text{and}\quad \overset{+}{\phi}_r^{\omega'}(x)=\begin{cases}1, x\in\mathcal{U}_{\omega'}\\ 0, x\in{\overset{+}{\mathcal{U}}_{r,\omega'}}^c\\ \text{linear interp., otherwise}\end{cases}.$$

They relate as $\bar{\phi}_r^{\omega'}\leqslant\mathbb{1}_{\mathcal{U}_{\omega'}}\leqslant\overset{+}{\phi}_r^{\omega'}$.

The Lipschitz constant of $\overset{+}{\phi}_r^{\omega'}$ is bounded by the inverse of $d\big(\mathcal{U}_{\omega'},{\overset{+}{\mathcal{U}}_{r,\omega'}}^c\big)$. On the other hand, for a point $x\in\mathcal{U}_{\omega'}$ to be minimally-displaced in such a way as to reach ${\overset{+}{\mathcal{U}}_{r,\omega'}}^c$, either: a) some of the hits in its finite-orbit is consequently-displaced to an extent which now makes it at least r-away from associated ρ-sized target, or b) some of the non-hits in its finite-orbit is consequently-displaced to an extent which now makes it at least r-inside the associated ρ-sized target. In either cases, the associated image point of x has to be consequently-displaced by distance at least r. When the said image point being consequently-displaced happens to be the last one in the orbit of x, i.e., its $L-1$ iterate, by the expanding feature of the system (H2.3), this is when x has to be displaced the least: no more than r/a_{L-1} (see hypothesis (H3.2) to recall the definition of a_L). Therefore

$r/a_{L-1} \leqslant d(\mathcal{U}_{\omega'}, \overset{+}{\mathcal{U}}_{r,\omega'}{}^c)$, so

$$\mathrm{Lip}_{d_M}(\overset{+}{\phi}{}_r^{\omega'}) \leqslant (a_{L-1})/r \leqslant De^{L-1}/r$$
$$\|\overset{+}{\phi}{}_r^{\omega'}\|_{Lip_{d_M}} = \|\overset{+}{\phi}{}_r^{\omega'}\|_\infty \vee \mathrm{Lip}_{d_M}(\overset{+}{\phi}{}_r^{\omega'}) = 1 \vee \mathrm{Lip}_{d_M}(\overset{+}{\phi}{}_r^{\omega'}) = \mathrm{Lip}_{d_M}(\overset{+}{\phi}{}_r^{\omega'}) \leqslant (De^{L-1})/r,$$

where the last equality follows from ρ sufficiently small.

Now we start looking at (II) directly:

$$\left|\int_\Omega \mu^\omega(Z_j^{\omega,\rho,L} = n)\mu^\omega(Z_i^{\omega,\rho,L} = n)d\nu(\omega) - \int_\Omega \mu^\omega(Z_j^{\omega,\rho,L} = n)\mu^{\omega'}(\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega)\right|$$
$$= \left|\int_\Omega \mu^\omega(Z_j^{\omega,\rho,L} = n)\mu^{\omega'}(\mathbb{1}_{\mathcal{U}_{\omega'}})d\nu(\omega) - \int_\Omega \mu^\omega(Z_j^{\omega,\rho,L} = n)\mu^{\omega'}(\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega)\right|$$
$$\lesssim \int_\Omega \mu^{\theta^j\omega}(Z_0^{\theta^j\omega,\rho,L} = n)L\frac{r^\eta}{\rho^\beta}d\nu(\omega) = L\frac{r^\eta}{\rho^\beta}\hat{\mu}(Z_0^{\rho,L} = n),$$

where the last inequality is because

$$\mu^{\omega'}(\overset{+}{\phi}{}_r^{\omega'}) \leqslant \mu^{\omega'}(\overset{+}{\mathcal{U}}_{r,\omega'} \setminus \overset{-}{\mathcal{U}}_{r,\omega'}) \leqslant \sum_{v=0}^{L-1}\mu^{\theta^v\omega}(\overset{+}{U}_{v,r,\omega'} \setminus \overset{+}{U}_{v,r,\omega'}) \overset{\text{(H4.3)}}{\lesssim} L\frac{r^\eta}{\rho^\beta}.$$

The approximating term that appeared above is transformed as follows:

$$\left|\int_\Omega \mu^\omega(Z_j^{\omega,\rho,L} = n)\mu^{\omega'}(\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega) - \int_\Omega \mu^{\omega'}(\mathbb{1}_{\{Z_{j-i}^{\omega',\rho,L}=n\}}\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega)\right|$$
$$= \left|\int_\Omega \mu^{\omega'}(Z_{j-i}^{\omega',\rho,L} = n)\mu^{\omega'}(\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega) - \int_\Omega \mu^{\omega'}(\mathbb{1}_{\{Z_0^{\theta^{(j-i)L}\omega',\rho,L}\circ T_{\omega'}^{(j-i)L}=n\}}\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega)\right|$$
$$= \int_\Omega \left|\mu^{\theta^{(j-i)L}\omega'}(Z_0^{\theta^{(j-i)L}\omega',\rho,L} = n)\mu^{\omega'}(\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega) - \mu^{\omega'}(\mathbb{1}_{\{Z_0^{\theta^{(j-i)L}\omega',\rho,L}=n\}}\circ T_{\omega'}^{(j-i)L}\overset{+}{\phi}{}_r^{\omega'})\right|d\nu(\omega)$$
$$\overset{(H5.1)}{\lesssim} \int_\Omega ((j-i)L)^{-\mathfrak{p}}\|\overset{+}{\phi}{}_r^{\omega'}\|_{\mathrm{Lip}_{d_M}}1d\nu(\omega) \leqslant ((j-i)L)^{-\mathfrak{p}}De^{L-1}/r.$$

Whereas the new approximating term which appeared above is transformed as follows:

$$\int_\Omega \mu^{\omega'}(\mathbb{1}_{\{Z_{j-i}^{\omega',\rho,L}=n\}}\overset{+}{\phi}{}_r^{\omega'})d\nu(\omega) = \int_\Omega \mu^\omega(\mathbb{1}_{\{Z_{j-i}^{\omega,\rho,L}=n\}}\overset{+}{\phi}{}_r^{\omega})d\nu(\omega)$$
$$= \int_{\Omega\times M}\mathbb{1}_{\{Z_{j-i}^{\rho,L}=n\}}\overset{+}{\phi}{}_r d\hat{\mu} = \int_{\Omega\times M}\mathbb{1}_{\{Z_0^{\rho,L}=n\}}\circ S^{(j-i)L}\overset{+}{\phi}{}_r d\hat{\mu} \tag{25}$$

and

$$\left|\int_{\Omega\times M}\mathbb{1}_{\{Z_0^{\rho,L}=n\}}\circ S^{(j-i)L}\overset{+}{\phi}{}_r d\hat{\mu} - \int_{\Omega\times M}\mathbb{1}_{\{Z_0^{\rho,L}=n\}}d\hat{\mu}\cdot\int_{\Omega\times M}\overset{+}{\phi}{}_r d\hat{\mu}\right|$$
$$\overset{(H5.2)}{\lesssim} ((j-i)L)^{-\mathfrak{p}}\|\overset{+}{\phi}{}_r\|_{\mathrm{Lip}_{d_{\Omega\otimes M}}} \leqslant ((j-i)L)^{-\mathfrak{p}}De^{L-1}/r.$$

Finally, we notice that

$$\left|\hat{\mu}(Z_0^{\rho,L} = n)\hat{\mu}(\overset{+}{\phi}{}_r) - \hat{\mu}(Z_0^{\rho,L} = n)^2\right| \leqslant \hat{\mu}(Z_0^{\rho,L} = n)\int_\Omega \mu^\omega(\overset{+}{\phi}{}_r^\omega - \mathbb{1}_{\mathcal{U}_\omega})d\nu(\omega) \overset{\text{(H4.3)}}{\lesssim} L\frac{r^\eta}{\rho^\beta}\hat{\mu}(Z_0^{\rho,L} = n).$$

Combining the previous four steps, we arrive at

$$\left|\int_\Omega \mu^\omega(Z_j^{\omega,\rho,L}=n)\mu^\omega(Z_i^{\omega,\rho,L}=n)d\nu(\omega)-\hat\mu(Z_0^{\rho,L}=n)^2\right| \lesssim L\frac{r^\eta}{\rho^\beta}\hat\mu(Z_0^{\rho,L}=n)+((j-i)L)^{-\mathfrak{p}}\frac{e^{L-1}}{r},$$

which implies

$$\begin{aligned}(II) &\lesssim \sum_{i=0}^{N'_{\rho,L}-1}\sum_{j=(i+\Delta)\wedge(N'_{\rho,L}-1)+1}^{N'_{\rho,L}-1}\left(\hat\mu(Z_0^{\rho,L}=n)^2+L\frac{r^\eta}{\rho^\beta}\hat\mu(Z_0^{\rho,L}=n)+((j-i)L)^{-\mathfrak{p}}\frac{e^{L-1}}{r}\right)\\ &\lesssim N'_{\rho,L}(N'_{\rho,L}-\Delta)\left(\hat\mu(Z_0^{\rho,L}=n)^2+L\frac{r^\eta}{\rho^\beta}\hat\mu(Z_0^{\rho,L}=n)\right)+N'_{\rho,L}\frac{e^{L-1}}{r}(\Delta L)^{-\mathfrak{p}+1}.\end{aligned}$$

Then we can conclude the following about the variance:

$$\begin{aligned}\mathrm{var}_\nu(\mathfrak{W}_\rho^{L,n}) &= \mathbb{E}_\nu((\mathfrak{W}_\rho^{L,n})^2)-(\mathbb{E}_\nu(\mathfrak{W}_\rho^{L,n}))^2\\ &\lesssim \Delta\rho^{d_0}N_\rho\hat\mu(Z_0^{\rho,L}=n)\\ &+ N'_{\rho,L}(N'_{\rho,L}-\Delta)\left(\hat\mu(Z_0^{\rho,L}=n)^2+L\frac{r^\eta}{\rho^\beta}\hat\mu(Z_0^{\rho,L}=n)\right)+N'_{\rho,L}\frac{e^{L-1}}{r}(\Delta L)^{-\mathfrak{p}+1}\\ &- {N'_{\rho,L}}^2\hat\mu(Z_0^{\rho,L}=n)^2.\\ &\lesssim \Delta\rho^{d_0}N_\rho\hat\mu(Z_0^{\rho,L}=n)+{N'_{\rho,L}}^2L\frac{r^\eta}{\rho^\beta}\hat\mu(Z_0^{\rho,L}=n)+N'_{\rho,L}\frac{e^{L-1}}{r}(\Delta L)^{-\mathfrak{p}+1}\\ &\overset{(\star)}{\lesssim} {N_\rho}^\alpha L\rho^{d_0}+N_\rho\frac{r^\eta}{\rho^\beta}+e^{L-1}L^{-\mathfrak{p}}\frac{N_\rho {N_\rho}^{\alpha(-\mathfrak{p}+1)}}{r}\\ &\overset{(\star\star)}{\lesssim} \frac{t^\alpha}{\hat\mu(\Gamma_\rho)^\alpha}L\rho^{d_0}+\frac{t}{\hat\mu(\Gamma_\rho)}\rho^{w\eta-\beta}+e^{L-1}L^{-\mathfrak{p}}\frac{t}{\hat\mu(\Gamma_\rho)}\frac{t^{\alpha(-\mathfrak{p}+1)}}{\hat\mu(\Gamma_\rho)^{\alpha(-\mathfrak{p}+1)}}\rho^{-w}\\ &\overset{\text{(H4.2)}}{\lesssim} L\rho^{d_0-\alpha d_1}+\rho^{w\eta-\beta-d_1}+e^{L-1}L^{-\mathfrak{p}}\rho^{\alpha d_0(\mathfrak{p}-1)-w-d_1}\\ &\overset{(\star\star\star)}{\lesssim} \rho^{d_0-\alpha d_1}+\rho^{w\eta-\beta-d_1}+\rho^{\alpha d_0(\mathfrak{p}-1)-w-d_1},\end{aligned}$$

where $(\star)$ uses $N'_{\rho,L}\hat\mu(Z_0^{\rho,L}=n)\leqslant(N_\rho+1)L^{-1}\hat\mu(Z_0^{\rho,L}\geqslant 1)\leqslant 2N_\rho L^{-1}L\hat\mu(\Gamma_\rho)\lesssim t$ and t is incorporated into the $\lesssim$ sign; $(\star\star)$ uses the choice $r:=\rho^w$ for a given $w>1$; and $(\star\star\star)$ incorporates L dependent quantities on $\lesssim$. Notice that t and L dependent constants being incorporated inside $\lesssim$ is associated to the use of a constant $C_{t,L}$ in the statement.

Finally, we need to choose $(\alpha,w)\in(0,1)\times(1,\infty)$ so that

$$\begin{cases}d_0>\alpha d_1\\ w\eta>\beta+d_1\\ \alpha d_0(\mathfrak{p}-1)>w+d_1\end{cases}\quad\text{i.e.}\quad\begin{cases}\alpha<\frac{d_0}{d_1}\wedge 1=\frac{d_0}{d_1}\\ w>\frac{\beta+d_1}{\eta}\vee 1\\ w<\alpha d_0(\mathfrak{p}-1)-d_1\end{cases},$$

which admits a solution if, and only if,

$$\frac{\beta+d_1}{\eta}\vee 1=\frac{d_0}{d_1}d_0(\mathfrak{p}-1)-d_1\Leftrightarrow d_0(\mathfrak{p}-1)>\frac{\frac{\beta+d_1}{\eta}\vee 1+d_1}{d_0/d_1}.$$

This is guaranteed by the parametric constraint (H8.1), so there exists some solution (α_*, w_*) to the system. Actually, the space of solutions forms a triangle and one can select (α_*, w_*) as its incenter, a function of d_0, d_1, η, β and $\mathfrak{p}$, whereas the strictly positive margin this choice opens in the inequalities of the original system is denoted by $q(d_0, d_1, \eta, \beta, \mathfrak{p})$. With such a choice, we obtain that

$$\mathrm{var}_\nu(\mathfrak{W}_\rho^{L,n}) \leqslant C_{t,L} \cdot \rho^{q(d_0,d_1,\eta,\beta,\mathfrak{p})} \leqslant C_{t,L} \cdot \rho^q \ , \forall q \in (0, q(d_0, d_1, \eta, \beta, \mathfrak{p}))\,.$$

■

Lemma 4. *Let $(\theta, \nu, T_\omega, \nu^\omega, \Gamma)$ be a system satisfying the hypotheses (H2.3), (H3.2), (H4.2), (H4.3), (H5.1), (H5.2) and (H7') with the parametric constraint (H8.1).*

Then: $\forall t > 0, \forall n \geqslant 1, \forall (\rho_m)_{m\geqslant 1} \searrow 0$ with $\sum_{m\geqslant 1} {\rho_m}^q < \infty$ (for some $0 < q < q(d_0, d_1, \eta, \beta, \mathfrak{p})$), denoting $N_\rho := \lfloor \frac{t}{\hat\mu(\Gamma_\rho)} \rfloor$ and $N'_{\rho,L} = \frac{N_\rho+1}{L}$[10], one has:

1)

$$\lim_{L\to\infty} \overline{\lim_{m\to\infty}} \sum_{j=0}^{N'_{\rho_m,L}-1} \mu^\omega(Z_j^{\omega,\rho_m,L} = n) = t\alpha_1\lambda_n,\ \nu\text{-a.s.}$$

2)

$$\lim_{L\to\infty} \overline{\lim_{m\to\infty}} \sum_{j=0}^{N'_{\rho_m,L}-1} \mu^\omega(Z_j^{\omega,\rho_m,L} \geqslant 1) = t\alpha_1,\ \nu\text{-a.s.}$$

3)[11]

$$\lim_{m\to\infty} \sum_{j=0}^{N_{\rho_m}-1} \mu^{\theta^j\omega}(\Gamma_\rho(\theta^j\omega)) = t,\ \nu\text{-a.s.},$$

Proof. Let t, n and $(\rho_m)_{m\geqslant 1}$ be as in the statement. Consider $L \geqslant 1$ and m large enough so that $\rho_m \leqslant \rho_{\mathrm{var}}(L)$, $N_{\rho_m} \geqslant 3$ and $N'_{\rho_m,L} \geqslant 3$. Denote also $\mathfrak{W}_\rho^{L,n}(\omega) = \sum_{j=0}^{N'_{\rho,L}-1} \mu^\omega(Z_j^{\omega,\rho,L} = n)$.

Using Chebycheff's inequality combined with lemma 3, we get that

$$\nu\big(\big|\mathfrak{W}_\rho^{L,n} - \mathbb{E}_\nu(\mathfrak{W}_\rho^{L,n})\big| > a\big) \leqslant \frac{\mathrm{var}_\nu(\mathfrak{W}_\rho^{L,n})}{a^2} \leqslant \frac{C_{t,L}}{a^2}\rho^q,$$

and therefore, since $\sum_{m\geqslant 1} {\rho_m}^q < \infty$, Borel-Cantelli lemma let us conclude that

$$\lim_{m\to\infty} \big|\mathfrak{W}_{\rho_m}^{L,n} - \mathbb{E}_\nu(\mathfrak{W}_{\rho_m}^{L,n})\big| = 0,\ \nu\text{-a.s.}$$

On the other hand,

$$\mathbb{E}_\nu(\mathfrak{W}_{\rho_m}^{L,n}) = \frac{1}{L}\left(\frac{t}{\hat\mu(\Gamma_{\rho_m})} + 1\right)\hat\mu(Z_0^{\rho_m,L} = n) = t\frac{\hat\mu(Z_{\Gamma_{\rho_m}}^{L-1} \geqslant 1)}{L\hat\mu(\Gamma_{\rho_m})}\frac{\hat\mu(Z_{\Gamma_{\rho_m}}^{L-1} = n)}{\hat\mu(Z_{\Gamma_{\rho_m}}^{L-1} \geqslant 1)} + \frac{\hat\mu(Z_{\Gamma_{\rho_m}}^{L-1} = n)}{L},$$

[10] See footnote 4.
[11] See footnote 5.

so, by lemma 2, the definition of λ_n and noting that (H4.2) implies $\frac{\hat{\mu}(Z^{L-1}_{\Gamma_{\rho_m}}=n)}{L} \leqslant LC_0{\rho_m}^{d_0}$, we have that

$$\lim_{L\to\infty} \overline{\lim_{m\to\infty}}\, \mathbb{E}_\nu(\mathfrak{W}^{L,n}_{\rho_m}) = t\alpha_1\lambda_n$$

and therefore, combining the previous two centered limits, conclusion (1) follows:

$$\lim_{L\to\infty} \overline{\lim_{m\to\infty}}\, \mathfrak{W}^{L,n}_{\rho_m} = t\alpha_1\lambda_n,\ \nu\text{-a.s.}$$

For (2), it suffices to repeat the argument noticing that the new expectation will be driven by $t\frac{\hat{\mu}(Z^{L-1}_{\Gamma_{\rho_m}}\geqslant 1)}{L\hat{\mu}(\Gamma_{\rho_m})}$, whose double limit is $t\alpha_1$.

For (3), it suffices to fix $L = 1$ and $n = 1$ in the above argument, and after the Borel-Cantelli step, notice

$$\mathbb{E}_\nu(\mathfrak{W}^{1,1}_{\rho_m}) = t\frac{\hat{\mu}(\Gamma_{\rho_m})}{\hat{\mu}(\Gamma_{\rho_m})} + \hat{\mu}(\Gamma_\rho) \overset{m\to\infty}{\to} t.$$

■

6. Proof of theorem 2

6.1. **Applying the abstract approximation theorem.** Let $t > 0$, $n \geqslant 1$ ($n = 0$ is the leftover) and $\omega \in \Omega$ be any. Actually, at finitely many instances of the argument, we will restrict ω to be taken in a set of full measure. To be seen in due time.

Fix, once and for all, $(\rho_m)_{m\geqslant 1} \searrow 0$ fast enough so that $\sum_{m\geqslant 1}(\rho_m)^q < \infty$, for some $0 < q < q(d_0, d_1, \eta, \beta, \mathfrak{p})$. For example, $\rho_m = m^{-2/q}$ is adapted to q (but not $q/2$) while $\rho_m = e^{-m}$ is adapted to any positive q.

Fix $L \geqslant 1$. We won't choose it as a function of other variables, i.e., it will consist of a new free variable.

Define $N_m := \left\lfloor \frac{t}{\hat{\mu}(\Gamma_{\rho_m})} \right\rfloor$. Let $v \in (0, d_0)$ and set $\Delta_m := {\rho_m}^{-v}$. We'll consider m large enough (depending on L) so that $N_m \geqslant 3$, $\Delta_m \geqslant 2$, $\rho_m \leqslant \rho_{\text{var}}(L)$, $L \leqslant \lfloor \frac{N_m+1}{3} \rfloor$ and $\Delta_m < N'_{m,L}$. Lastly, define $N'_{m,L} := \frac{N_m+1}{L} \in \mathbb{N}_{\geqslant 3}$[12].

We want to study

$$\mu^\omega(Z^{\omega,N_m}_{\Gamma_{\rho_m}} = n) = \mu^\omega(\textstyle\sum_{i=0}^{N_m} I^{\omega,m}_i = n) = \mu^\omega\left(\displaystyle\sum_{j=0}^{N'_{m,L}-1}\sum_{i=jL}^{(j+1)L-1} I^{\omega,m}_i + \sum_{i=N'_{m,L}L}^{N_m} I^{\omega,m}_i = n\right),$$

where $I^{\omega,m}_i = \mathbb{1}_{\Gamma_{\rho_m}(\theta^i\omega)} \circ T^i_\omega$.

To harmonize with the notation of theorem 3, we write

$$I^{\omega,m}_i =: X^{\omega,m}_i : (M, \mathscr{B}_M, \mu^\omega) \to \{0,1\}\ (i \in [0, N_m] \cap \mathbb{N}_{\geqslant 0}) \text{ and}$$

$\sum_{i=jL}^{(j+1)L-1} I^{\omega,m}_i = \sum_{i=jL}^{(j+1)L-1} X^{\omega,m}_i =: Z^{\omega,m,L}_j : (M, \mathscr{B}_M, \mu^\omega) \to \mathbb{N}_{\geqslant 0}\ (j \in [0, N'_{m,L}-1] \cap \mathbb{N}_{\geqslant 0})$.

Then one can plug in the variables here to those of the theorem 3, namely

$N{:=}N_m$, $(\mathcal{X}, \mathscr{X}, \mathbb{Q}){:=}(M, \mathscr{B}_M, \mu^\omega)$, $X_i{:=}X^{\omega,m}_i$, $L{:=}L$, $\Delta{:=}\Delta_m$, $N'_L{:=}N'_{m,L}$, $Z^L_j{:=}Z^{\omega,m,L}_j$,

[12]See footnote 4.

to obtain that

$$\left|\mu^{\omega}\left(Z^{\omega,N_m}_{\Gamma_{\rho_m}}=n\right)-\mu^{\omega}\left(\sum_{j=0}^{N'_{m,L}-1}\tilde{Z}_j^{\omega,m,L}=n\right)\right|$$

$$\leqslant 4\left(\mathcal{R}^1_{\omega,m}(N_m,L,\Delta_m)+\mathcal{R}^2_{\omega,m}(N_m,L,\Delta_m)+\mathcal{R}^3_{\omega,m}(N_m,L)\right),$$

where objects being invoked are presented in theorem 3 and, since N_m and Δ_m are (or will be) functions of m, we'll actually write $\mathcal{R}^1_{\omega,m}(N_m,L,\Delta_m)=\mathcal{R}^1(\omega,m,L)$, and, similarly, $\mathcal{R}^2(\omega,m,L)$ and $\mathcal{R}^3(\omega,m,L)$.

6.2. **Estimating the error $\mathcal{R}^1$.** Recall that

$$\mathcal{R}^1_{\omega,m}(N_m,L,\Delta_m)=$$

$$\sum_{j=0}^{N'_{m,L}-1}\max_{q\in[2,L]}\sum_{u=1}^{q-1}\left|\mu^{\omega}\left(Z_j^{\omega,m,L}{=}u,\sum_{k=j+\Delta_m}^{N'_{m,L}-1}Z_k^{\omega,m,L}{=}q{-}u\right)-\mu^{\omega}\left(Z_j^{\omega,m,L}{=}u\right)\mu^{\omega}\left(\sum_{k=j+\Delta_m}^{N'_{m,L}-1}Z_k^{\omega,m,L}{=}q{-}u\right)\right|.$$

Recycling the construction and notation used in the proof of lemma 3 to control the term (II): for a given $j\in[0,N'_{m,L}-1]$, writing $\omega'=\theta^{jL}\omega$ and considering $r\in(0,\rho_m/2)$, $v\in[0,L-1]$, we once again have the objects: $U_{v,\omega'},\overset{-}{U}_{v,r,\omega'},\overset{+}{U}_{v,r,\omega'},\mathcal{U}_{\omega'},\overset{-}{\mathcal{U}}_{r,\omega'},\overset{+}{\mathcal{U}}_{r,\omega'},\overset{-}{\phi}{}_r^{\omega'}$ and $\overset{+}{\phi}{}_r^{\omega'}$. Then:

$$\left|\mu^{\omega}\left(Z_j^{\omega,m,L}{=}u,\sum_{k=j+\Delta_m}^{N'_{m,L}-1}Z_k^{\omega,m,L}{=}q{-}u\right)-\mu^{\omega}\left(Z_j^{\omega,m,L}{=}u\right)\mu^{\omega}\left(\sum_{k=j+\Delta_m}^{N'_{m,L}-1}Z_k^{\omega,m,L}{=}q{-}u\right)\right|$$

$$=\left|\mu^{\omega}\left(\sum_{i=jL}^{(j+1)L-1}I_i^{\omega,m}=u,\sum_{i=(j+\Delta_m)L}^{N_m}I_i^{\omega,m}=q-u\right)\right.$$

$$\left.-\mu^{\omega}\left(\sum_{i=jL}^{(j+1)L-1}I_i^{\omega,m}=u\right)\mu^{\omega}\left(\sum_{i=(j+\Delta_m)L}^{N_m}I_i^{\omega,m}=q-u\right)\right|$$

$$=\left|\mu^{\omega'}\left(\sum_{i=0}^{L-1}I_i^{\omega',m}=u,\sum_{i=\Delta_mL}^{N_m-jL}I_i^{\omega',m}=q-u\right)\right.$$

$$\left.-\mu^{\omega'}\left(\sum_{i=0}^{L-1}I_i^{\omega',m}=u\right)\mu^{\theta^{\Delta_mL}\omega'}\left(\sum_{i=0}^{N_m-(j+\Delta_m)L}I_i^{\theta^{\Delta_mL}\omega',m}=q-u\right)\right|.$$

$$=\left|\mu^{\omega'}\left(\mathbb{1}_{\mathcal{U}_{\omega'}}\mathbb{1}_{\{V_j^{\omega,m,L,\Delta_m}=q-u\}}\circ T_{\omega'}^{\Delta_mL}\right)-\mu^{\omega'}\left(\mathbb{1}_{\mathcal{U}_{\omega'}}\right)\mu^{\theta^{\Delta_mL}\omega'}\left(\mathbb{1}_{\{V_j^{\omega,m,L,\Delta_m}=q-u\}}\right)\right|$$

where we used that $V_j^{\omega,m,L,\Delta_m}:=\sum_{i=0}^{N_m-(j+\Delta_m)L}I_i^{\theta^{\Delta_mL}\omega',m}$, and thus $\sum_{i=\Delta_mL}^{N_m-jL}I_i^{\omega',m}=V_j^{\omega,m,L,\Delta_m}\circ T_{\omega'}^{\Delta_mL}$,

$$\leqslant\left|\mu^{\omega'}\left(\overset{\pm}{\phi}{}_r^{\omega'}\mathbb{1}_{\{V_j^{\omega,m,L,\Delta_m}=q-u\}}\circ T_{\omega'}^{\Delta_mL}\right)-\mu^{\omega'}\left(\mathbb{1}_{\mathcal{U}_{\omega'}}\right)\mu^{\theta^{\Delta_mL}\omega'}\left(\mathbb{1}_{\{V_j^{\omega,m,L,\Delta_m}=q-u\}}\right)\right|,$$

where $\overset{\pm}{\phi}{}^{\omega'}_r$ means that either $\overset{+}{\phi}{}^{\omega'}_r$ or $\overset{-}{\phi}{}^{\omega'}_r$ will make the inequality true,

$$\begin{aligned}
&\leqslant \left| \mu^{\omega'}\left(\overset{\pm}{\phi}{}^{\omega'}_r \mathbb{1}_{\{V_j^{\omega,m,L,\Delta_m}=q-u\}} \circ T_{\omega'}^{\Delta_m L} \right) - \mu^{\omega'}\left(\overset{\pm}{\phi}{}^{\omega'}_r\right) \mu^{\theta^{\Delta_m L}\omega'}\left(\mathbb{1}_{\{V_j^{\omega,m,L,\Delta_m}=q-u\}} \right) \right| \\
&+ \left| \left[\mu^{\omega'}\left(\overset{\pm}{\phi}{}^{\omega'}_r\right) - \mu^{\omega'}\left(\mathbb{1}_{\mathcal{U}_{\omega'}}\right) \right] \mu^{\theta^{\Delta_m L}\omega'}\left(\mathbb{1}_{\{V_j^{\omega,m,L,\Delta_m}=q-u\}} \right) \right| \\
&=: \ (A) + (B).
\end{aligned}$$

Now notice that

$$(A) \overset{\text{(H5.1)}}{\lesssim} (\Delta_m L)^{-\mathfrak{p}} \left\| \overset{\pm}{\phi}{}^{\omega'}_r \right\|_{\mathrm{Lip}_{d_M}} 1 \underset{\text{(H3.2)}}{\overset{\text{(H2.3)}}{\lesssim}} (\Delta_m L)^{-\mathfrak{p}} e^{L-1}/r$$

and

$$(B) \leqslant \mu^{\theta^{\Delta_m L}\omega'}\left(V_j^{\omega,m,L,\Delta_m} = q-u \right) \mu^{\omega'}(\overset{+}{\mathcal{U}}_{r,\omega'} \setminus \overset{-}{\mathcal{U}}_{r,\omega'}) \overset{\text{(H4.3)}}{\lesssim} \mu^{\theta^{\Delta_m L}\omega'}\left(V_j^{\omega,m,L,\Delta_m} = q-u \right) L \frac{r^\eta}{{\rho_m}^\beta}.$$

Therefore

$$\begin{aligned}
\mathcal{R}^1_{\omega,m}(N_m, L, \Delta_m) &\lesssim \sum_{j=0}^{N'_{m,L}-1} \max_{q\in[2,L]} \sum_{u=1}^{q-1} \left[(\Delta_m L)^{-\mathfrak{p}} e^{L-1}/r + \mu^{\theta^{\Delta_m L}\omega'}\left(V_j^{\omega,m,L,\Delta_m} = q-u \right) L \frac{r^\eta}{{\rho_m}^\beta} \right] \\
&\leqslant \sum_{j=0}^{N'_{m,L}-1} \sum_{u=1}^{L-1} (\Delta_m L)^{-\mathfrak{p}} e^{L-1}/r + L \frac{r^\eta}{{\rho_m}^\beta} \sum_{j=0}^{N'_{m,L}-1} \max_{q\in[2,N_m]} \mu^{\theta^{\Delta_m L}\omega'}\left(V_j^{\omega,m,L,\Delta_m} \in [0,q] \right) \\
&\lesssim N_m (\Delta_m L)^{-\mathfrak{p}} e^{L-1}/r + N_m \frac{r^\eta}{{\rho_m}^\beta},
\end{aligned}$$

because $V_j^{\omega,m,L,\Delta_m}$ takes values between 0 and $N_m - (j+\Delta_m)L + 1 \leqslant N_m - \Delta_m L + 1 \leqslant N_m$.

6.3. **Estimating the error $\mathcal{R}^2$.** To start

$$\begin{aligned}
\mathcal{R}^2_{\omega,m}(N_m, L, \Delta_m) &= \sum_{j=0}^{N'_{m,L}-1} \mu^\omega(Z_j^{\omega,m,L} \geqslant 1, \sum_{k=j+1}^{j+\Delta_m-1} Z_k^{\omega,m,L} \geqslant 1) \\
&\leqslant \sum_{j=0}^{N'_{m,L}-1} \sum_{k=j+1}^{j+\Delta_m-1} \mu^\omega(Z_j^{\omega,m,L} \geqslant 1, Z_k^{\omega,m,L} \geqslant 1)
\end{aligned}$$

where we reverse the double sum and single out the $k = j+1$ terms

$$\begin{aligned}
&= \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \mu^\omega(Z_j^{\omega,m,L} \geqslant 1, Z_k^{\omega,m,L} \geqslant 1) + \sum_{k=1}^{N'_{m,L}} \mu^\omega(Z_{k-1}^{\omega,m,L} \geqslant 1, Z_k^{\omega,m,L} \geqslant 1) \\
&=: (I) + (II)
\end{aligned}$$

To estimate (I) we notice that:

$$(I) \leqslant \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \sum_{i=jL}^{(j+1)L-1} \sum_{l=kL}^{(k+1)L-1} \mu^\omega\left((T_\omega^i)^{-1}\Gamma_{\rho_m}(\theta^i\omega) \cap (T_\omega^l)^{-1}\Gamma_{\rho_m}(\theta^l\omega) \right) \ (l > i)$$

$$\begin{aligned}
&\leqslant \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \sum_{i=jL}^{(j+1)L-1} \sum_{l=kL}^{(k+1)L-1} \mu^{\omega'}\left(\Gamma_{\rho_m}(\omega')\cap (T^{l-i}_{\omega'})^{-1}\Gamma_{\rho_m}(\theta^{l-i}\omega')\cap \mathcal{G}^{\omega'}_{l-i}\right)\\
&+ \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \sum_{i=jL}^{(j+1)L-1} \sum_{l=kL}^{(k+1)L-1} \mu^{\omega'}\left(\Gamma_{\rho_m}(\omega')\cap (T^{l-i}_{\omega'})^{-1}\Gamma_{\rho_m}(\theta^{l-i}\omega')\cap \bar{\mathcal{G}}^{\omega'}_{l-i}\right)\\
&=: \ (I_{\text{good}}) + (I_{\text{bad}})
\end{aligned}$$

where $\omega' := \theta^i\omega$.

To estimate (I_{good}) we begin evaluating the following:

$$\begin{aligned}
&\mu^{\omega'}\left(\mathcal{G}^{\omega'}_{l-i}\cap\Gamma_{\rho_m}(\omega')\cap (T^{l-i}_{\omega'})^{-1}\Gamma_{\rho_m}(\theta^{l-i}\omega')\right)\\
&\leqslant \sum_{\substack{\xi=\varphi(\text{dom}(\varphi))\in \overset{+}{C}{}^{\omega'}_{l-i}\\ \varphi\in \text{IB}(T^{l-i}_{\omega'}):\\ \xi\cap\Gamma_{\rho_m}(\omega')\neq\varnothing}} \frac{\mu^{\omega'}|_\xi\left(\xi\cap (T^{l-i}_{\omega'})^{-1}\Gamma_{\rho_m}(\theta^{l-i}\omega')\right)}{\mu^{\omega'}|_\xi(\xi)}\mu^{\omega'}(\xi),
\end{aligned}$$

where, from (H2.2), $\varphi\in \text{IB}(T^{l-i}_{\omega'})$ implies $\mu^{\omega'}|_{\varphi(\text{dom}(\varphi))} = {J_\varphi}^{-1}\left[\varphi_*(\mu^{\theta^{l-i}\omega'}|_{\text{dom}(\varphi)})\right]$, and so

$$\leqslant \sum_{\xi \text{ as above}} \frac{\left[{J_\varphi}^{-1}\left[\varphi_*(\mu^{\theta^{l-i}\omega'}|_{\text{dom}(\varphi)})\right]\right]\left(\varphi(\text{dom}(\varphi))\cap (T^{l-i}_{\omega'})^{-1}\Gamma_{\rho_m}(\theta^{l-i}\omega')\right)}{\left[{J_\varphi}^{-1}\left[\varphi_*(\mu^{\theta^{l-i}\omega'}|_{\text{dom}(\varphi)})\right]\right](\varphi(\text{dom}(\varphi)))}\mu^{\omega'}(\xi)$$

$$\leqslant \sum_{\xi \text{ as above}} \frac{\sup_{x\in\xi} {J_\varphi}^{-1}(x)}{\inf_{x\in\xi} {J_\varphi}^{-1}(x)} \frac{\mu^{\theta^{l-i}\omega'}|_{\text{dom}(\varphi)}\left(\text{dom}(\varphi)\cap \varphi^{-1}(T^{l-i}_{\omega'})^{-1}\Gamma_{\rho_m}(\theta^{l-i}\omega')\right)}{\mu^{\theta^{l-i}\omega'}|_{\text{dom}(\varphi)}(\text{dom}(\varphi))}\mu^{\omega'}(\xi)$$

$$\begin{aligned}
&\underset{\text{(H1.2)}}{\overset{\text{(H2.2)}}{\lesssim}} (l-i)^{\mathfrak{d}}\iota^{-1}\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \sum_{\xi \text{ as above}}\mu^{\omega'}(\xi) \overset{\text{(H1.2)}}{\leqslant} (l-i)^{\mathfrak{d}}\iota^{-1}\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega))\mathcal{N}\mu^{\omega'}\left(\bigcup_{\xi \text{ as above}}\xi\right)\\
&\overset{\text{(H2.3)}}{\leqslant} (l-i)^{\mathfrak{d}}\iota^{-1}\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega))\mathcal{N}\mu^{\omega'}\left(B(\Gamma_{\rho_m}(\omega'), D(l-i)^{-\kappa})\right)\\
&\overset{\text{(H4.2)}}{\leqslant} (l-i)^{\mathfrak{d}}\iota^{-1}\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega))\mathcal{N}C_0(\rho_m+D(l-i)^{-\kappa})^{d_0} \lesssim \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega))(l-i)^{\mathfrak{d}}\left[{\rho_m}^{d_0}+(l-i)^{-\kappa d_0}\right].
\end{aligned}$$

Then

$$\begin{aligned}
(I_{\text{good}}) &\leqslant \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \sum_{i=jL}^{(j+1)L-1} \sum_{l=kL}^{(k+1)L-1} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega))(l-i)^{\mathfrak{d}}\left[{\rho_m}^{d_0}+(l-i)^{-\kappa d_0}\right]\\
&= \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \sum_{l=kL}^{(k+1)L-1} \left(\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \sum_{i=jL}^{(j+1)L-1}(l-i)^{\mathfrak{d}}\left[{\rho_m}^{d_0}+(l-i)^{-\kappa d_0}\right]\right),
\end{aligned}$$

where, for each l fixed, as i runs, we have $l-i\in[kL-jL-L+1, kL-jL+L-1]$, so

$$\leqslant \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \sum_{l=kL}^{(k+1)L-1} \left(\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \sum_{s=kL-jL-L+1}^{kL-jL+L-1} s^{\mathfrak{d}}\left[{\rho_m}^{d_0}+s^{-\kappa d_0}\right]\right)$$

$$= \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \left[\left(\sum_{l=kL}^{(k+1)L-1} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \right) \left(\sum_{s=kL-jL-L+1}^{kL-jL+L-1} s^{\mathfrak{d}} \left[{\rho_m}^{d_0} + s^{-\kappa d_0} \right] \right) \right]$$

$$\leqslant \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \left(\sum_{l=kL}^{(k+1)L-1} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \right) \left(\sum_{j=(k-\Delta_m+1)\vee 0}^{(k-2)\wedge(N'_{m,L}-1)} \sum_{s=kL-jL-L+1}^{kL-jL+L-1} s^{\mathfrak{d}} \left[{\rho_m}^{d_0} + s^{-\kappa d_0} \right] \right)$$

where $s \in [L+1, 3\Delta_m L]$[13], so

$$\lesssim \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \left(\sum_{l=kL}^{(k+1)L-1} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \right) \left(\sum_{u=L+1}^{3\Delta_m L} u^{\mathfrak{d}} \left[u^{-\kappa d_0} + {\rho_m}^{d_0} \right] \right)$$

$$\lesssim \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \left(\sum_{l=kL}^{(k+1)L-1} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \right) \left(L^{\mathfrak{d}-\kappa d_0+1} + (\Delta_m L)^{\mathfrak{d}+1} {\rho_m}^{d_0} \right),$$

where for the first term in the square bracket we've used that, for $\alpha > 1$, $\sum_{n=m}^{\infty} n^{-\alpha} \lesssim m^{-\alpha+1}$ together with $\mathfrak{d} - \kappa d_0 < -1$, which is guaranteed by (H8.5), whereas for the second we've used that $u^{\mathfrak{d}}$ is increasing and the summation interval is bounded above by $3\Delta_m L$.

We'll leave (I_{bad}) to the end.

For (II), we consider $L' < L$ and proceed as follows

$$\sum_{k=1}^{N'_{m,L}} \mu^{\omega}(Z_{k-1}^{\omega,m,L} \geqslant 1, Z_k^{\omega,m,L} \geqslant 1)$$

$$= \sum_{k=1}^{N'_{m,L}} \mu^{\omega} \left(\sum_{i=(k-1)L}^{kL-1} I_i^{\omega,m,L} \geqslant 1, \sum_{l=kL}^{kL+L'-1} I_l^{\omega,m,L} + \sum_{l=kL+L'}^{(k+1)L-1} I_l^{\omega,m,L} \geqslant 1 \right)$$

$$\leqslant \sum_{k=1}^{N'_{m,L}} \mu^{\omega} \left(\sum_{i=(k-1)L}^{kL-1} I_i^{\omega,m,L} \geqslant 1, \sum_{l=kL}^{kL+L'-1} I_l^{\omega,m,L} \geqslant 1 \right) + \mu^{\omega} \left(\sum_{i=(k-1)L}^{kL-1} I_i^{\omega,m,L} \geqslant 1, \sum_{l=kL+L'}^{(k+1)L-1} I_l^{\omega,m,L} \geqslant 1 \right)$$

and, denoting $\omega' = \theta^i\omega$,

$$\begin{aligned}
&\leqslant \sum_{k=1}^{N'_{m,L}} \sum_{l=kL}^{kL+L'-1} \mu^{\theta^l\omega} \left(\Gamma_{\rho_m}(\theta^l\omega) \right) \\
&+ \sum_{k=1}^{N'_{m,L}+\Delta_m-2} \sum_{i=(k-1)L}^{kL-1} \sum_{l=kL+L'}^{(k+1)L-1} \mu^{\omega'} \left(\Gamma_{\rho_m}(\omega') \cap (T_{\omega'}^{l-i})^{-1} \Gamma_{\rho_m}(\theta^{l-i}\omega') \cap \mathcal{G}_{l-i}^{\omega'} \right) \\
&+ \sum_{k=1}^{N'_{m,L}} \sum_{i=(k-1)L}^{kL-1} \sum_{l=kL+L'}^{(k+1)L-1} \mu^{\omega'} \left(\Gamma_{\rho_m}(\omega') \cap (T_{\omega'}^{l-i})^{-1} \Gamma_{\rho_m}(\theta^{l-i}\omega') \cap \bar{\mathcal{G}}_{l-i}^{\omega'} \right) \\
&=: (II_{\text{rest}}) + (II_{\text{good}}) + (II_{\text{bad}}).
\end{aligned}$$

[13]The interval where s ranges basically has length $2L$ and it is translated by L when j moves one unit, therefore the original and the new interval overlap by half, so eventual repetitions are more than compensated by a factor of two.

The term (II_{rest}) won't be improved, whereas the term (II_{good}) is approached just like (I_{good}), as follows:

$$(II_{\text{good}}) \lesssim \sum_{k=1}^{N'_{m,L}} \sum_{l=kL+L'}^{(k+1)L-1} \left(\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \sum_{i=(k-1)L}^{kL-1} (l-i)^{\mathfrak{d}} \left[{\rho_m}^{d_0} + (l-i)^{-\kappa d_0} \right] \right)$$

where, for each l fixed, as i runs, we have $l-i \in [L'+1, 2L-1]$, so

$$\leqslant \sum_{k=1}^{N'_{m,L}} \left(\sum_{l=kL+L'}^{(k+1)L-1} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \right) \left(\sum_{u=L'+1}^{2L-1} u^{\mathfrak{d}} \left[{\rho_m}^{d_0} + u^{-\kappa d_0} \right] \right)$$

$$\lesssim \sum_{k=1}^{N'_{m,L}} \left(\sum_{l=kL}^{(k+1)L-1} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \right) \left({L'}^{\mathfrak{d}-\kappa d_0+1} L^{\mathfrak{d}+1} {\rho_m}^{d_0} \right).$$

Now we combine (I_{bad}) and (II_{bad}) and their domain of summation[14] to see that

$$\begin{aligned}(I_{\text{bad}}) + (II_{\text{bad}}) &\lesssim \sum_{i=0}^{N_m} \sum_{l=i+L+L'}^{i+\Delta_m L} \mu^{\theta^i\omega}(\bar{\mathcal{G}}_{l-i}^{\theta^i\omega} \cap \Gamma_{\rho_m}(\theta^i\omega)) = \sum_{i=0}^{N_m} \sum_{s=L+L'}^{\Delta_m L} \mu^{\theta^i\omega}(\bar{\mathcal{G}}_{s}^{\theta^i\omega} \cap \Gamma_{\rho_m}(\theta^i\omega)) \\ &\leqslant \sum_{s=L'}^{\Delta_m L} \sum_{i=0}^{N_m} \mu^{\theta^i\omega}(\bar{\mathcal{G}}_{s}^{\theta^i\omega} \cap \Gamma_{\rho_m}(\theta^i\omega)).\end{aligned}$$

Combining the bounds of (I_{good}) and (II_{good}), we conclude that

$$\begin{aligned}\mathcal{R}^2_{\omega,m}(N_m, L, \Delta_m) &\lesssim \sum_{l=0}^{5N_m} \mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) \left({L'}^{\mathfrak{d}-\kappa d_0+1} + (\Delta_m L)^{\mathfrak{d}+1} {\rho_m}^{d_0} \right) \\ &\quad + \sum_{k=1}^{N'_{m,L}} \sum_{l=kL}^{kL+L'-1} \mu^{\theta^l\omega}\left(\Gamma_{\rho_m}(\theta^l\omega)\right) + \sum_{s=L'}^{\Delta_m L} \sum_{i=0}^{N_m} \mu^{\theta^i\omega}(\bar{\mathcal{G}}_{s}^{\theta^i\omega} \cap \Gamma_{\rho_m}(\theta^i\omega)).\end{aligned}$$

6.4. **Estimating the error $\mathcal{R}^3$.** Here we use (H4.2) to see that

$$\begin{aligned}\mathcal{R}^3_{\omega,m}(N_m, L, \Delta_m) &= \sum_{i=0}^{N_m} \sum_{\ell=0 \vee (i-\Delta_m L)}^{i} \mu^{\theta^i\omega}(\Gamma_{\rho_m}(\theta^i\omega)) \mu^{\theta^\ell\omega}(\Gamma_{\rho_m}(\theta^\ell\omega))) \\ &\lesssim \Delta_m L \; {\rho_m}^{d_0} \sum_{i=0}^{N_m} \mu^{\theta^i\omega}(\Gamma_{\rho_m}(\theta^i\omega)),\end{aligned}$$

which, noticing that $\Delta_m L \leqslant (\Delta_m L)^{\mathfrak{d}+1}$, reveals to be bounded above by $\mathcal{R}^2_{\omega,m}(N_m, L, \Delta_m)$.

[14]Notice that the initial L'-strip of the first component of the original summation has already been singled out inside (II_{rest}).

6.5. **Controlling the total error.** Put $r = {\rho_m}^w$ $(w > 1)$ and $L' = L^\alpha$ $(0 < \alpha < 1)$. Then

$$\left|\mu^\omega\left(Z^{\omega,N_m}_{\Gamma_{\rho_m}} = n\right) - \mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1}\tilde{Z}^{\omega,m,L}_j = n\right)\right|$$
$$\lesssim e^{L-1}{\rho_m}^{\mathfrak{p}v-w-d_1} + {\rho_m}^{w\eta-\beta-d_1}$$
$$+\sum_{l=0}^{5N_m}\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega))\left({L'}^{\mathfrak{d}-\kappa d_0+1} + L^{\mathfrak{d}+1}{\rho_m}^{d_0-v(\mathfrak{d}+1)}\right)$$
$$+\sum_{k=1}^{N'_{m,L}}\sum_{l=kL}^{kL+L'-1}\mu^{\theta^l\omega}\left(\Gamma_{\rho_m}(\theta^l\omega)\right) + \sum_{s=L'}^{\Delta_m L}\sum_{i=0}^{N_m}\mu^{\theta^i\omega}(\bar{\mathcal{G}}^{\theta^i\omega}_s \cap \Gamma_{\rho_m}(\theta^i\omega)).$$

Until this point, parameters v (accompanying Δ_m, see section 6), w (accompanying r), and α (accompanying L'), which are local to the proof, were not fine-tuned.

In the last equation, we need the exponents accompanying ρ to be strictly positive. In particular, we need

$$w > \frac{\beta + d_1}{\eta} \vee 1,\ pv - w - d_1 > 0 \text{ and } d_0 - v(\mathfrak{d}+1) > 0.$$

The space of solutions $(w, v) \in (1,\infty)\times(0, d_0)$ to those inequalities is non-empty if $\mathfrak{p} > \frac{\left(\frac{\beta+d_1}{\eta}\vee 1\right)+d_1}{d_0}$, which is guaranteed by (H8.2).

We'll take double limits of the type $\overline{\lim}_{L\to\infty}\overline{\lim}_{m\to\infty}$ on the RHS. Initially, taking the $\overline{\lim}_{m\to\infty}$, we use that, by lemma 4,

$$\lim_{m\to\infty}\sum_{l=0}^{5N_m}\mu^{\theta^l\omega}(\Gamma_{\rho_m}(\theta^l\omega)) = 5t,\ \nu\text{-a.s.}$$

and, by similar arguments[15],

$$\lim_{m\to\infty}\sum_{k=1}^{N'_{m,L}}\sum_{l=kL}^{kL+L'-1}\mu^{\theta^l\omega}\left(\Gamma_{\rho_m}(\theta^l\omega)\right) = tL^{\alpha-1},\ \nu\text{-a.s.}.$$

Finally, using hypothesis (H3.1) and noticing that $\mathfrak{d}-\kappa d_0+1 < 0$ (by (H8.5)) and $\alpha-1 < 0$ (by design), we conclude that the RHS under the double limit $\overline{\lim}_{L\to\infty}\overline{\lim}_{m\to\infty}$ goes to 0. The same thing occurs if we adopt the double limits $\underline{\lim}_{L\to\infty}\overline{\lim}_{m\to\infty}$, $\overline{\lim}_{L\to\infty}\underline{\lim}_{m\to\infty}$ and $\underline{\lim}_{L\to\infty}\underline{\lim}_{m\to\infty}$. Therefore

$$\lim_{L\to\infty}\overline{\lim_{m\to\infty}}\left|\mu^\omega\left(Z^{\omega,N_m}_{\Gamma_{\rho_m}} = n\right) - \mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1}\tilde{Z}^{\omega,m,L}_j = n\right)\right| = 0,\ \nu\text{-a.s.}.$$

[15]Adapting the argument of lemma 4 item (III) to the new term, we see that the new ν-expectation is $tL^{\alpha-1}$, but the variance lemma used therein, lemma 3, would need to be adapted as well, what we omitted.

6.6. Convergence of the leading term to the compound Poisson distribution. It remains to show that $\mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1}\tilde{Z}_j^{\omega,m,L}=n\right)$ to $\mathrm{CPD}_{t\alpha_1,(\lambda_\ell)_\ell}(\{n\})$.

Due to the independence and distributional properties of the $\tilde{Z}_j^{\omega,m,L}$'s (see theorem 3):

$$\mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1}\tilde{Z}_j^{\omega,m,L}=n\right)$$

$$=\sum_{l=1}^{n}\sum_{0\leqslant j_1<\ldots<j_l\leqslant N'_{m,L}-1}\left(\prod_{\substack{j\in[0,N'_{m,L}-1]\\ \setminus\{j_i:i=1,\ldots,l\}}}\mu^\omega(Z_j^{\omega,m,L}=0)\cdot\sum_{\substack{(n_1,\ldots,n_l)\in\mathbb{N}^l_{\geqslant 1}\\ n_1+\ldots+n_l=n}}\prod_{i=1}^{l}\mu^\omega(Z_j^{\omega,m,L}=n_i)\right)$$

$$\overset{(\star)}{=}(1+o(1))\prod_{j=0}^{N'_{m,L}-1}\mu^\omega(Z_j^{\omega,m,L}=0)\sum_{l=1}^{n}\frac{1}{l!}\sum_{\substack{j_i\in[0,N'_{m,L}-1]\\ i=1,\ldots,l}}\sum_{\substack{(n_1,\ldots,n_l)\in\mathbb{N}^l_{\geqslant 1}\\ n_1+\ldots+n_l=n}}\prod_{i=1}^{l}\mu^\omega(Z_{j_i}^{\omega,m,L}=n_i)$$

$$\overset{(\star\star)}{=}(1+o(1))\prod_{j=0}^{N'_{m,L}-1}\mu^\omega(Z_j^{\omega,m,L}=0)\sum_{l=1}^{n}\frac{1}{l!}\sum_{\substack{(n_1,\ldots,n_l)\in\mathbb{N}^l_{\geqslant 1}\\ n_1+\ldots+n_l=n}}\prod_{i=1}^{l}\left(\sum_{j=0}^{N'_{m,L}-1}\mu^\omega(Z_j^{\omega,m,L}=n_i)\right),$$

where i) $o(1)$ refers to a function $g(\omega,m,L)$ so that $\overline{\lim}_{L\to\infty}\overline{\lim}_{m\to\infty}|g(\omega,m,L)|=0$, ν-a.s.; ii) equality $(\star)$ included $1/l!$ to account for j_i's not being anymore increasing and used that the error terms that come from different j_i's being equal are small, as one can see in the case when two j_i agree; and iii) equality $(\star\star)$ uses that a product of sums distributes as a sum of products.

We then notice that, by lemma 4,

$$\lim_{L\to\infty}\overline{\lim_{m\to\infty}}\sum_{j=0}^{N'_{\rho m,L}-1}\mu^\omega(Z_j^{\omega,\rho_m,L}=n_i)=t\alpha_1\lambda_{n_i},\ \nu\text{-a.s.}$$

and

$$\lim_{L\to\infty}\overline{\lim_{m\to\infty}}\prod_{j=0}^{N'_{m,L}-1}\mu^\omega(Z_j^{\omega,m,L}=0)=\lim_{L\to\infty}\overline{\lim_{m\to\infty}}\exp\left(\sum_{j=0}^{N'_{m,L}-1}\ln\left(1-\mu^\omega(Z_j^{\omega,m,L}\geqslant 1)\right)\right)$$

$$=\lim_{L\to\infty}\overline{\lim_{m\to\infty}}\exp\left(\sum_{j=0}^{N'_{m,L}-1}-\mu^\omega(Z_j^{\omega,m,L}\geqslant 1)+o(1)\right)=e^{-t\alpha_1},\ \nu\text{-a.s..}$$

Therefore

$$\lim_{L\to\infty}\overline{\lim_{m\to\infty}}\left|\mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1}\tilde{Z}_j^{\omega,m,L}=n\right)-e^{-t\alpha_1}\sum_{l=1}^{n}\frac{(t\alpha_1)^l}{l!}\sum_{\substack{(n_1,\ldots,n_l)\in\mathbb{N}^l_{\geqslant 1}\\ n_1+\ldots+n_l=n}}\prod_{i=1}^{l}\lambda_{n_i}\right|=0,\ \nu\text{-a.s.}$$

$$\Leftrightarrow \lim_{L\to\infty} \overline{\lim_{m\to\infty}} \left| \mu^\omega \left(\sum_{j=0}^{N'_{m,L}-1} \tilde{Z}_j^{\omega,m,L} = n \right) - \mathrm{CPD}_{t\alpha_1,(\lambda_\ell)_\ell}(\{n\}) \right| = 0, \ \nu\text{-a.s.},$$

where the equivalence is because the former term is precisely the density of such a compound Poisson distribution (see equation (H10)).

As a consequence,

$$\begin{aligned}
\left|\mu^\omega(Z^{\omega,N_m}_{\Gamma_{\rho m}} = n) - \mathrm{CPD}_{t\alpha_1,(\lambda_\ell)_\ell}(\{n\})\right| &\overset{\forall L\geqslant 1}{\leqslant} \left|\mu^\omega(Z^{\omega,N_m}_{\Gamma_{\rho m}} = n) - \mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1} \tilde{Z}_j^{\omega,m,L} = n\right)\right| \\
&+ \left|\mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1} \tilde{Z}_j^{\omega,m,L} = n\right) - \mathrm{CPD}_{t\alpha_1,(\lambda_\ell)_\ell}(\{n\})\right| \\
\Rightarrow \overline{\lim_{m\to\infty}} \left|\mu^\omega(Z^{\omega,N_m}_{\Gamma_{\rho m}} = n) - \mathrm{CPD}_{t\alpha_1,(\lambda_\ell)_\ell}(\{n\})\right| & \\
\leqslant \lim_{L\to\infty} \overline{\lim_{m\to\infty}} &\left|\mu^\omega(Z^{\omega,N_m}_{\Gamma_{\rho m}} = n) - \mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1} \tilde{Z}_j^{\omega,m,L} = n\right)\right| \\
+ \lim_{L\to\infty} \overline{\lim_{m\to\infty}} &\left|\mu^\omega\left(\sum_{j=0}^{N'_{m,L}-1} \tilde{Z}_j^{\omega,m,L} = n\right) - \mathrm{CPD}_{t\alpha_1,(\lambda_\ell)_\ell}(\{n\})\right| \\
= 0, \ \nu\text{-a.s.} &
\end{aligned}$$

We then conclude that $\lim_{m\to\infty} \left|\mu^\omega(Z^{\omega,N_m}_{\Gamma_{\rho m}} = n) - \mathrm{CPD}_{t\alpha_1,(\lambda_\ell)_\ell}(\{n\})\right| = 0$, ν-a.s., as desired.

7. Examples

Consider a system $(\theta, \nu, T_\omega, \mu^\omega, \Gamma)$ as described in section 2.1.

Define

$$(m \in \mathbb{N}_{\geqslant 1}) \ \ \mathrm{Per}_m(\Gamma) := \{\omega \in \Omega \mid T^m_\omega \Gamma(\omega) \cap \Gamma(\theta^m \omega) \neq \varnothing, \nexists m' < m : T^{m'}_\omega \Gamma(\omega) \cap \Gamma(\theta^{m'}\omega) \neq \varnothing\}, \tag{26}$$

the set of random seeds which bring the target to itself in m iterates (and not earlier).

Define

$$\mathrm{Per}(\Gamma) := \bigsqcup_{m\geqslant 1} \mathrm{Per}_m(\Gamma), \tag{27}$$

the set of random seeds which bring the target to itself at some point in time.

Define

$$\mathcal{P}\mathrm{er}(\Gamma) := \bigsqcup_{(m_k)_{k=0}^\infty \subset \mathbb{N}_{\geqslant 1}} \bigcap_{k=0}^{\infty} \theta^{-M_{k-1}} \mathrm{Per}_{m_k}(\Gamma), \text{ where } M_{k-1} = \sum_{i=0}^{k-1} m_i, \tag{28}$$

the set of random seeds which bring the target to itself infinitely many times. The periods taken for such returns to occur are prescribed in $(m_k)_k$.

Define
$$\mathcal{AP}\mathrm{er}(\Gamma) := \Omega \backslash \mathrm{Per}(\Gamma), \tag{29}$$
the set of random seeds which never accomplish even a single return.

Define
$$(K \in \mathbb{N}_{\geqslant 1})\ \mathcal{E}\mathrm{v}_K\mathcal{AP}\mathrm{er}(\Gamma) := \bigsqcup_{(m_k)_{k=0}^{K-1} \subset \mathbb{N}_{\geqslant 1}} \left(\bigcap_{k=0}^{K-1} \theta^{-M_{k-1}} \mathrm{Per}_{m_k}(\Gamma) \bigcap \theta^{-M_k} \mathcal{AP}\mathrm{er}(\Gamma) \right), \tag{30}$$
the set of random seeds which produce K returns (with periods $(m_k)_{k=0}^{K-1}$), but no returns thereafter.

Convention: we stretch K to also take values $K \in \{0, \infty\}$, by letting
$$\mathcal{E}\mathrm{v}_\infty\mathcal{AP}\mathrm{er}(\Gamma) := \mathcal{P}\mathrm{er}(\Gamma) \text{ and } \mathcal{E}\mathrm{v}_0\mathcal{AP}\mathrm{er}(\Gamma) := \mathcal{AP}\mathrm{er}(\Gamma).$$

Notice that
$$\Omega = \bigsqcup_{K \in \{0,\infty\} \cup \mathbb{N}_{\geqslant 1}} \mathcal{E}\mathrm{v}_K\mathcal{AP}\mathrm{er}(\Gamma).$$

Associated to a random seed ω, there are some quantities of interest, entailed by the previous construction, that we'll want to keep track of. For $\omega \in \Omega$, let
$$K(\omega) := K, \text{ when } \omega \in \mathcal{E}\mathrm{v}_K\mathcal{AP}\mathrm{er}(\Gamma)\ (K \in \{0, \infty\} \cup \mathbb{N}_{\geqslant 1}) \tag{31}$$
be the amount of returns ω produces, and
$$(m_k(\omega))_{k=0}^{K(\omega)} := \begin{cases} (m_0(\omega) := \infty) & \text{, if } K(\omega) = 0, \\ (m_0(\omega), m_1(\omega), \ldots, m_{K(\omega)-1}(\omega), m_{K(\omega)}(\omega) := \infty), & \text{if } K(\omega) \in \mathbb{N}_{\geqslant 1}, \\ (m_0(\omega), m_1(\omega), \ldots) & \text{, if } K(\omega) = \infty. \end{cases} \tag{32}$$
be the associated vector of periods, with ∞ appended at the end (if it ends), where, of course, $(m_k(\omega))_{k=0}^{K(\omega)-1}$ is obtained from the sequence of m_k's appearing in the union within equations (30) or (28), respectively, in such a way as to determine where ω belongs to. Note that we put ∞ at the end of a tuple (if it ends) to mean that the orbit went to a no-return trip. Also, observe that the quantities $K(\omega)$ and $(m_k(\omega))_{k=0}^{K(\omega)}$ are also dependent of Γ, but we omit this from our notation.

Agreeing that $\sup \varnothing = 0$, let
$$M_\Gamma(\omega) := \sup\{m_k(\omega) : k \in [0, K(\omega))\} \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}, \tag{33}$$
$$M_\Gamma := \sup\{M_\Gamma(\omega) : \omega \in \Omega\} = \sup\{m \geqslant 1 : \mathrm{Per}_m(\Gamma) \neq \varnothing\} \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}, \tag{34}$$
$$K_\Gamma :=^{16} \left\{ K(\omega) \geqslant 0 : \omega \in \bigsqcup_{K \in \{0\} \cup \mathbb{N}_{\geqslant 1}} \mathcal{E}\mathrm{v}_K\mathcal{AP}\mathrm{er}(\Gamma) \right\} \in \mathbb{N}_{\geqslant 0} \cup \{\infty\}. \tag{35}$$

The previous definitions were introduced in very general terms. From now on, we adopt additional conditions which let us conclude that α_ℓ's exist and can be represented with

[16]The quantity K_Γ can turn out to be unimportant in relevant cases. For example, with a full shift random driving and maps $T_\omega = T_{\pi_0(\omega)}$, one has $K_\Gamma \in \{0, \infty\}$, because if there is some concatenation of maps producing a return, they can be concatenated over and over. But we include this definition for completeness.

explicit formulas. The intent is not yet to put conditions which imply (H1)-(H5),(H7), but only (H7).

C 1. *Consider finitely many maps of the unit interval (or circle), $T_v : M \to M$, for $v \in \{0, \dots, u-1\}$. For ease of exposition, say that $u = 2$.*

C 2. *Consider that the maps carry a family of open intervals $\mathcal{A}_v = (O_{v,i})_{i=1}^{I_v}$ ($I_v \leqslant \infty$) so that $M \backslash \bigcup_{i=1}^{I_v} O_{v,i}$ is at most countable and $T_v|_{O_{v,i}}$ is invertible onto its image and differentiable with* $\inf\{T_v{}'(x) > 1 : x \in O_{v,i}\} \geqslant 1$.

C3. *Let $\Omega = {\mathbb{N}_{\geqslant 0}}^{\mathbb{Z}}$ and consider that $\mathbb{N}_{\geqslant 0} = \mathbb{N}^0 \sqcup \mathbb{N}^1$, with $\mathbb{N}^0$ and $\mathbb{N}^1$ being the set of even and odd numbers, respectively. Set $T_\omega := T_{\tilde{\pi}_0(\omega)}$, where $\tilde{\pi}_j(\omega) = \begin{cases} 0, \text{ if } \omega_j \in \mathbb{N}^0 \\ 1, \text{ if } \omega_j \in \mathbb{N}^1 \end{cases}$ $(j \in \mathbb{Z})$. As usual, consider $\theta : \Omega \to \Omega$ to be the shift map.*

Note: The usual case where $\Omega = \{0, \dots, u-1\}^{\mathbb{Z}}$ and $T_\omega = T_{\omega_0}$ is basically a simpler version of the present one, so it can be treated with the exact same arguments and lead to the exact same results.

For the moment, we assume nothing else about the measure ν, which is only considered to be θ invariant. Important examples are Bernoulli and Markov measures, but also what we call restricted Bernoulli measures, given by $\nu = (w(0)\eta^0 + w(1) + \eta^1)^{\mathbb{Z}}$, where $w(0) + w(1) = 1$, $\eta^0 \in \mathcal{P}(\mathbb{N}^0)$, $\eta^1 \in \mathcal{P}(\mathbb{N}^1)$ and $\eta^0(2n) = \eta(n) = \eta^1(2n+1)$ ($\forall n \geqslant 0$), for some $\eta \in \mathcal{P}(\mathbb{N}_{\geqslant 0})$.

For $n \geqslant 1$, let $\mathcal{A}_n^\omega = \bigvee_{j=0}^{n-1} (T_\omega^j)^{-1} \mathcal{A}_{\tilde{\pi}_j(\omega)}$. For $n = 0$, we adopt the convention $\mathcal{A}_0^\omega = \{(0,1)\}$ ($\forall \omega \in \Omega$). Write $\mathcal{O}_n^\omega = \bigcup_{O \in \mathcal{A}_n^\omega} O$ (co-countable) and, for $x \in \mathcal{O}_n^\omega$, denote by $\mathcal{A}_n^\omega(x)$ the element of $\mathcal{A}_n^\omega$ containing x. In particular, $x \in \mathcal{O}_n^\omega(x)$ implies that x is a point of differentiability for T_ω^n. As said before, now we aren't interested in showing (H1), but to avoid confusion we notice that taking $B(y_k^{\omega,n}, R) \equiv B(0.5, 0, 5) = (0,1)$ makes $\mathcal{A}_n^\omega = C_n^\omega$ ($\forall n \geqslant 1$).

C4. *Consider that there exists $K, Q > 1$ and $\beta \in (0,1]$ so that $\mu^\omega = h_\omega \operatorname{Leb}$ satisfies: i) $(\omega, x) \mapsto h_\omega(x)$ is measurable, ii) $K^{-1} \leqslant h_\omega \leqslant K$ ($\forall \omega \in \Omega$), and iii) $h_\omega \in \operatorname{Hol}_\beta(M)$ with $H_\beta(h_\omega) := \sup_{\substack{x,y \in M \\ x \neq y}} \dfrac{|h_\omega(x) - h_\omega(y)|}{d_M(x,y)^\beta} \leqslant Q$ ($\forall \omega \in \Omega$).*

C5. *Consider $x : \Omega \to M$ a random point so that*

$$\nu\left(\left\{\omega \in \Omega : x(\theta^i \omega) \in \bigcap_{l=1}^{\infty} \bigcap_{n=0}^{\infty} \mathcal{O}_l^{\theta^n \omega}, \forall i \geqslant 0\right\}\right) = 1, \tag{36}$$

where the intersection appearing above is, for every $\omega \in \Omega$, a co-countable set.

Important examples are what we'll call projective random points x, given by $x(\omega) = x_{\mathfrak{m}(\pi_0(\omega))}$, where $\pi_j(\omega) = \omega_j$ ($j \in \mathbb{Z}$), $(x_n)_{n=0}^N$ ($N \leqslant \infty$) is a sequence of points in M and $\mathfrak{m} : \mathbb{N}_{\geqslant 0} \to \{0, \dots, N\}$ is a function. If $N = \infty$, one could choose $\mathfrak{m} = \operatorname{id}$.

C6. *With $\Gamma(\omega) = \{x(\omega)\}$ ($\omega \in \Omega$), consider that $M_\Gamma < \infty$.*

To illustrate, for deterministic targets $x(\omega) \equiv x$, two noticeable cases occur:

i) *Pure periodic points* x, i.e., when there is some $m_* = m_*(x) \geqslant 1$ so that x is (minimally) fixed by any concatenations of m_* maps in $(T_v)_{v=0}^{u-1}$. In this case, $\mathrm{Per}_{m_*}(\Gamma) = \Omega$, $M_\Gamma = m_*$ and $K_\Gamma = 0$. These examples can be constructed explicitly.

ii) *Pure aperiodic points* x, i.e., when x isn't fixed by any finite concatenation of maps in $(T_v)_{v=0}^{u-1}$. In this case, $\Omega = \mathcal{AP}\mathrm{er}(\Gamma)$, $M_\Gamma = 0$ and $K_\Gamma = 0$. These examples are not necessarily easy to be constructed explicitly, but, once the maps are fixed, the set of pure aperiodic x's is generic, because it is given by

$$M \backslash \bigcup_{p \geqslant 1} \bigcup_{(v_0, \ldots, v_{p-1}) \in \{0, \ldots, u-1\}^p} \mathrm{Fix}(T_{v_{p-1}} \circ \ldots \circ T_{v_0}),$$

which is co-countable.

A) Calculation of $(\alpha_\ell)_{\ell \in \mathbb{N}_{\geqslant 1}}$

Now we calculate α_ℓ's for systems $(\theta, \nu, T_\omega, \mu^\omega, \Gamma)$ as described in section 2.1 satisfying conditions (C1)-(C6).

Consider $\ell \geqslant 1$ and $\omega \in \Omega$. Actually, at finitely many instances of the argument, we will restrict ω to be taken in a set of full measure. To be seen in due time.

Consider

$$L \geqslant \sum_{k=0}^{(\ell-1) \wedge (K(\omega)-1)} m_k(\omega) = M_{\ell \wedge K(\omega)}(\omega). \tag{37}$$

and $\rho_0(\omega, L) = \rho_0(\tilde{\pi}_0(\omega), \ldots, \tilde{\pi}_L(\omega))$ so that $\rho \leqslant \rho_0(\omega, L)$ implies

$$\forall i \in [1, L] \backslash \{M_k(\omega) : k \in [1, K(\omega)]\} \text{ one has } T_\omega^i B(x(\omega), \rho) \cap B(x(\theta^i \omega), \rho) = \varnothing, \tag{38}$$

which can be guaranteed noticing that

a) returns occur precisely in the instants $\{M_k(\omega) : k \in [1, K(\omega)]\}$ and not in between (by minimality),

b) T_ω^i is continuous on $x(\omega)$ ($\forall i \geqslant 1$), a.s., because, by (C2) and (C5), one has $x(\omega) \in \mathcal{O}_i^\omega$.

Because of the previous constraint, one could actually have started with L's of the form $L = M_{q_L \wedge K(\omega)}(\omega)$, $q_L \geqslant \ell$ (so still satisfying equation (37)), in the sense that other choices of L are superfluous from the viewpoint of the quantity we'll study, $Z_{*\Gamma_\rho}^{\omega, L}$. Then one could consider $\rho_1(\omega, L) = \rho_1(\tilde{\pi}_0(\omega), \ldots, \tilde{\pi}_L(\omega)) < \rho_0(\omega, L)$ so that $\rho \leqslant \rho_1(\omega, L)$ implies: $\forall k' \leqslant k, k', k \in [0, q_L \wedge K(\omega)]$ one has

$$T_{\theta^{M_{k'}(\omega)}\omega}^{\overbrace{M_k(\omega) - M_{k'}(\omega)}^{M_{k-k'}\left(\theta^{M_{k'}(\omega)}\omega\right)}} B(x(\theta^{M_{k'}(\omega)}\omega), \rho) \subset \mathcal{A}_{M_{q_L \wedge K(\omega)}(\omega) - M_k(\omega)}^{\theta^{M_k(\omega)}\omega} \left(x(\theta^{M_k(\omega)}\omega)\right), \tag{39}$$

which can be guaranteed noticing that

a) $T_{\theta^{M_{k'}(\omega)}\omega}^{M_{k-k'}\left(\theta^{M_{k'}(\omega)}\omega\right)} x(\theta^{M_{k'}(\omega)}\omega) = x(\theta^{M_{k-k'}\left(\theta^{M_{k'}(\omega)}\omega\right)}\theta^{M_{k'}(\omega)}\omega) = x(\theta^{M_k(\omega)}\omega) \in \mathcal{O}_{M_{q_L \wedge K(\omega)}(\omega) - M_k(\omega)}^\omega$, a.s., where the first equality is due to the return times within $M_k(\omega)$'s, the second one is due to the equality in the overbrace above and the inclusion follows from (C2) and (C5),

b) $T_{\theta^{M_{k'}(\omega)}\omega}^{M_{k-k'}(\theta^{M_{k'}(\omega)}\omega)}$ is continuous at $x(\theta^{M_{k'}(\omega)}\omega)$, a.s., because, again by (C2) and (C5), one has $x(\theta^{M_{k'}(\omega)}\omega) \in \mathcal{O}^{\theta^{M_{k'}(\omega)}\omega}_{M_{k-k'}(\theta^{M_{k'}(\omega)}\omega)}$.

The point with condition (39) is to say that for whatever intermediate starting point in time, $M_{k'}(\omega)$, the ball that lives there is small enough so that its image into any other further intermediate point in time, $M_k(\omega)$, fits inside the injectivity domain of the map which evolves the system for the remaining time, from $M_k(\omega)$ to $M_{q_K \wedge K(\omega)}(\omega)$. In particular, under iteration, the ball at time zero grows inside the interval but never enough to wrap around or break injectivity. Therefore, if a initial condition in the ball at time zero iterates as to miss the target at some intermediate point in time, it won't revisit it in the remaining time considered (until $M_{q_L \wedge K(\omega)(\omega)}$), because, due to expanding feature of the maps, only a break in injectivity would allow for such a revisit to happen. In other words, given an initial condition, if we are to code the sequence of hits within times $M_1(\omega), \dots, M_{q_L \wedge K(\omega)}(\omega)$, they will be formed by 1's in a row (possibly none) followed by 0's in a row (possibly none).

Then, for ω, L and ρ as above, one has:

$$
\begin{aligned}
\hat{\alpha}_\ell^\omega(L,\rho)\mu^\omega(\Gamma_\rho(\omega)) &= \mu^\omega(Z^{\omega,L}_{*\Gamma_\rho} \geqslant \ell-1, I_0^\omega = 1) \\
&\overset{(38)}{=} \mu^\omega\left(\sum_{j\in\{M_k(\omega):k\in[1,q_L\wedge K(\omega)]\}} I_j^\omega \geqslant \ell-1, I_0^\omega = 1\right) \\
&\overset{(39)}{=} \begin{cases} \mu^\omega\left(I^\omega_{M_1(\omega)} = 1, \dots, I^\omega_{M_{\ell-1}(\omega)} = 1\right), & \text{if } \ell-1 \leqslant K(\omega) \\ 0 & \text{, otherwise} \end{cases} \\
&\overset{(39)}{=} \begin{cases} \mu^\omega\left(I^\omega_{M_{\ell-1}(\omega)} = 1\right), & \text{if } \ell-1 \leqslant K(\omega) \\ 0 & \text{, otherwise} \end{cases},
\end{aligned}
$$

so that

$$
\alpha_\ell^\omega(L,\rho)\frac{\mu^\omega(\Gamma_\rho(\omega))}{\hat{\mu}(\Gamma_\rho)} \overset{(6)}{=} \begin{cases} \dfrac{\mu^\omega\left((T_\omega^{M_{\ell-1}(\omega)})^{-1}\Gamma_\rho(\theta^{M_{\ell-1}(\omega)}\omega)\right)}{\hat{\mu}(\Gamma_\rho)} - \dfrac{\mu^\omega\left((T_\omega^{M_\ell(\omega)})^{-1}\Gamma_\rho(\theta^{M_\ell(\omega)}\omega)\right)}{\hat{\mu}(\Gamma_\rho)}, & \text{if } \ell \leqslant K(\omega), \\ \dfrac{\mu^\omega\left((T_\omega^{M_{\ell-1}(\omega)})^{-1}\Gamma_\rho(\theta^{M_{\ell-1}(\omega)}\omega)\right)}{\hat{\mu}(\Gamma_\rho)} & \text{, if } \ell = K(\omega)+1, \\ 0 & \text{, if } \ell \geqslant K(\omega)+2. \end{cases}
$$

Moreover (for ω and L chosen as above), for all $\epsilon > 0$, exists $\rho_2(\omega, L, \epsilon) = \rho_2(\tilde{\pi}_0(\omega), \dots, \tilde{\pi}_L(\omega), \epsilon)$ so that for all $\rho \leqslant \rho_2(\omega, L, \epsilon)$

$$
\frac{\mathrm{Leb}\left((T_\omega^{M_{\ell-1}(\omega)})^{-1}\Gamma_\rho(\theta^{M_{\ell-1}(\omega)}\omega)\right)}{\mathrm{Leb}(\Gamma_\rho(\omega))} = \left[JT_\omega^{M_{\ell-1}(\omega)}(x(\omega))\right]^{-1} + \mathcal{O}(\epsilon),
$$

and there exists $\rho_3(\omega,\epsilon) = (\epsilon/H_\beta(h_\omega))^{1/\beta}$ so that for all $\rho \leqslant \rho_3(\omega,\epsilon)$

$$
h_\omega(z) = h_\omega(x(\omega)) + \mathcal{O}(\epsilon), \ \forall z \in B(x(\omega),\rho).
$$

We can use (C1), (C4) and (C6) to pass to controls that are uniform on ω and then integrate: for any $\epsilon > 0$, $L \geqslant L_* := \ell M_\Gamma = \sup_{\omega\in\Omega} M_{\ell\wedge K(\omega)}(\omega)$ and

$$\rho \leqslant \rho_*(L,\epsilon) := \min_{(v_0,\dots,v_L)\in\ \{0,1\}^{L+1}} \rho_1(v_0,\dots,v_L) \wedge \min_{(v_0,\dots,v_L)\in\ \{0,1\}^{L+1}} \rho_2(v_0,\dots,v_L,\epsilon) \wedge \left(\frac{\epsilon}{Q}\right)^{1/\beta},$$

one has

$$\alpha_\ell(L,\rho) = \int_\Omega \begin{cases} \dfrac{h_\omega(x(\omega)) + \mathcal{O}(\epsilon)}{\int_\Omega h_\omega(x(\omega)) + \mathcal{O}(\epsilon) d\nu(\omega)} \left[\left(JT_\omega^{M_{\ell-1}(\omega)}(x(\omega))\right)^{-1} + \mathcal{O}(\epsilon) - \left(JT_\omega^{M_\ell(\omega)}(x(\omega))\right)^{-1} - \mathcal{O}(\epsilon)\right], & \text{if } \ell \leqslant K(\omega) \\ \dfrac{h_\omega(x(\omega)) + \mathcal{O}(\epsilon)}{\int_\Omega h_\omega(x(\omega)) + \mathcal{O}(\epsilon) d\nu(\omega)} \left[\left(JT_\omega^{M_{\ell-1}(\omega)}(x(\omega))\right)^{-1} + \mathcal{O}(\epsilon)\right] & , \text{if } \ell = K(\omega)+1 \\ 0 & , \text{if } \ell \geqslant K(\omega)+2 \end{cases} d\nu(\omega),$$

then taking iterated limits of the type $\lim_\epsilon \lim_L \overline{\lim}_\rho$ one finds that

$$\alpha_\ell = \int_\Omega \begin{cases} \dfrac{h_\omega(x(\omega))}{\int_\Omega h_\omega(x(\omega)) d\nu(\omega)} \left[\left(JT_\omega^{M_{\ell-1}(\omega)}(x(\omega))\right)^{-1} - \left(JT_\omega^{M_\ell(\omega)}(x(\omega))\right)^{-1}\right], & \text{if } \ell \leqslant K(\omega) \\ \dfrac{h_\omega(x(\omega))}{\int_\Omega h_\omega(x(\omega)) d\nu(\omega)} \left[\left(JT_\omega^{M_{\ell-1}(\omega)}(x(\omega))\right)^{-1}\right] & , \text{if } \ell = K(\omega)+1 \\ 0 & , \text{if } \ell \geqslant K(\omega)+2 \end{cases} d\nu(\omega). \tag{40}$$

The following diagram helps one to visualize how the integrand in equation (40), with the prefactor $\frac{h_\omega(x(\omega))}{\int_\Omega h_\omega(x(\omega))d\nu(\omega)}$ suppressed, changes

a) when ω is found in each of the portions making up $\Omega = \bigsqcup_{K\in\{0,\infty\}\cup\mathbb{N}_{\geqslant 1}} \mathcal{E}\mathrm{v}_K\mathcal{AP}\mathrm{er}(\Gamma)$ (read the different lines),

b) as ℓ grows (read the different columns).

$$\begin{array}{rcccc} & \ell=1 & \ell=2 & \ell=3 & \\ \omega\in\mathcal{P}\mathrm{er}(\Gamma)\colon & \Big(1-1/JT_\omega^{m_0(\omega)}(x), & \dfrac{1-1/JT^{m_1(\omega)}_{\theta^{m_0(\omega)}\omega}(x)}{JT_\omega^{m_0(\omega)}(x)}, & \dfrac{1-1/JT^{m_2(\omega)}_{\theta^{m_0(\omega)+m_1(\omega)}\omega}(x)}{JT_\omega^{m_0(\omega)}(x)JT^{m_1(\omega)}_{\theta^{m_0(\omega)}\omega}(x)}, & \dots\Big) \\ \omega\in\mathcal{AP}\mathrm{er}(\Gamma)\colon & (\quad 1, & 0, & 0, & ,\overset{\bar{0}}{\dots}) \\ \omega\in\mathcal{E}\mathrm{v}_1\mathcal{AP}\mathrm{er}(\Gamma)\colon & \Big(1-1/JT_\omega^{m_0(\omega)}(x), & \dfrac{1}{JT_\omega^{m_0(\omega)}(x)}, & 0, & ,\overset{\bar{0}}{\dots}\Big) \\ \omega\in\mathcal{E}\mathrm{v}_2\mathcal{AP}\mathrm{er}(\Gamma)\colon & \Big(1-1/JT_\omega^{m_0(\omega)}(x), & \dfrac{1-1/JT^{m_1(\omega)}_{\theta^{m_0(\omega)}\omega}(x)}{JT_\omega^{m_0(\omega)}(x)}, & \dfrac{1}{JT_\omega^{m_0(\omega)}(x)JT^{m_1(\omega)}_{\theta^{m_0(\omega)}\omega}(x)}, & ,\overset{\bar{0}}{\dots}\Big). \end{array} \tag{41}$$

Now we want to consider what else can be said about equation (40) when more structure is assumed. Let us adopt the following additional hypothesis.

C7. *Consider that* $\nu(\mathcal{AP}\mathrm{er}(\Gamma)) = 1$.

The takeaway is that adding condition (C7) to (C1)-(C6) makes one conclude that $(\alpha_\ell)_\ell$ is pure Poisson.

That is because, in this case, considering the second line of equation (41) (with $\frac{h_\omega(x(\omega))}{\int_\Omega h_\omega(x(\omega))d\nu(\omega)}$ re-factored in), one has simply

$$\alpha_\ell = \begin{cases} 1, \text{ if } \ell = 1 \\ 0, \text{ if } \ell \geqslant 2 \end{cases} . \tag{42}$$

In the opposite case, we substitute condition (C7) with the following one.

C8. *Consider that ν is Bernoulli, $\nu(\mathcal{P}\mathrm{er}(\Gamma)) = 1$ and that $x : \Omega \to M$ is a projective random point. Moreover, assume that $h_\omega(x(\omega)) \perp \left(JT^{m_j(\omega)}_{\theta^{M_j(\omega)}\omega}(x(\theta^{M_j(\omega)}\omega))\right)_j$, which happens, for example, when $h_\omega \equiv 1$ ($\forall \omega \in \Omega$). This can always be the case in the Bernoulli context because it's highly expected that the μ^ω's are dependent on the past only (Markov measures), and that past and future are independent (Markov system).*

The takeaway is that adding condition (C8) to (C1)-(C6) makes one conclude that $(\alpha_\ell)_\ell$ is Polya-Aeppli.

Notice that $\nu(\mathcal{P}\mathrm{er}(\Gamma)) = 1$ and the independence of $h_\omega(x(\omega))$ from the rest implies

$$\alpha_\ell = \int_\Omega \prod_{j=0}^{\ell-2} \left[JT^{m_j(\omega)}_{\theta^{M_j(\omega)}\omega}(x(\theta^{M_j(\omega)}\omega))\right]^{-1} d\nu(\omega) - \int_\Omega \prod_{j=0}^{\ell-1} \left[JT^{m_j(\omega)}_{\theta^{M_j(\omega)}\omega}(x(\theta^{M_j(\omega)}\omega))\right]^{-1} d\nu(\omega),$$

then, after we make the point in I) that $\left(\omega \mapsto JT^{m_j(\omega)}_{\theta^{M_j(\omega)}\omega}(x(\theta^{M_j(\omega)}\omega))\right)_j$ is independent under ν, we will find that

$$\alpha_\ell = \prod_{j=0}^{\ell-2} \int_\Omega \left[JT^{m_j(\omega)}_{\theta^{M_j(\omega)}\omega}(x(\theta^{M_j(\omega)}\omega))\right]^{-1} d\nu(\omega) - \prod_{j=0}^{\ell-1} \int_\Omega \left[JT^{m_j(\omega)}_{\theta^{M_j(\omega)}\omega}(x(\theta^{M_j(\omega)}\omega))\right]^{-1} d\nu(\omega),$$

which, we'll argue in II), equals

$$\alpha_\ell = \prod_{j=0}^{\ell-2} \int_\Omega \left[JT^{m_0(\omega)}_\omega x(\omega)\right]^{-1} d\nu(\omega) - \prod_{j=0}^{\ell-1} \int_\Omega \left[JT^{m_0(\omega)}_\omega x(\omega)\right]^{-1} d\nu(\omega) = (D-1)D^{-\ell},$$

where $D^{-1} := \int_\Omega [JT^{m_0(\omega)}_\omega x(\omega)]^{-1} d\nu(\omega)$, as desired.

Let us make the points that are missing.

I) Notice first that

$$\begin{aligned} \nu(m_0(\omega) = i_0, m_1(\omega) = i_1) &= \nu(m_0(\omega) = i_0, m_0(\theta^{i_0}\omega) = i_1) = \nu(\mathbb{1}_{\mathrm{Per}_{i_0}(\Gamma)} \mathbb{1}_{\theta^{-i_0}\,\mathrm{Per}_{i_1}(\Gamma)}) \\ &= \nu(\mathbb{1}_{\mathrm{Per}_{i_0}(\Gamma)})\nu(\mathbb{1}_{\theta^{-i_0}\,\mathrm{Per}_{i_1}(\Gamma)}) = \nu(m_0(\omega) = i_0)\nu(m_0(\omega) = i_1), \end{aligned}$$

where the first equality in the second line is because $(\tilde{\pi}_j)$'s are independent under ν and the indicator functions can be expressed in terms of disjoint blocks of $(\tilde{\pi}_j)$'s, namely $\tilde{\pi}_0, \dots, \tilde{\pi}_{i_0-1}$ and $\tilde{\pi}_{i_0}, \dots, \tilde{\pi}_{i_0+i_1-1}$. On the other hand

$$\nu(m_1(\omega) = i_1) = \sum_{i_0} \nu(m_0(\omega) = i_0, m_1(\omega) = i_1)$$

$$= \sum_{i_0} \nu(m_0(\omega) = i_0)\nu(m_0(\omega) = i_1) = \nu(m_0(\omega) = i_1).$$

So combining the two previous chains of equality, we find that m_0 and m_1 are independent, i.e., $m_0 \perp m_1$.

Once again, since $(\pi_j)_j$ is an independency under ν, whenever two random variables X and Y can be expressed as $X = \phi \circ (\pi_0, \ldots, \pi_{i_0-1})$ and $Y = \psi \circ (\pi_{i_0}, \ldots, \pi_{i_0+i_1-1})$, then $X \perp Y$. Similarly for $\tilde{\pi}$ instead of π. This is the case for $(JT_{\cdot}^{i_0}(x(\cdot)), \mathbb{1}_{m_0(\cdot)=i_0}) \perp (JT_{\theta^{i_0}\cdot}^{i_1}(x \circ \theta^{i_0}(\cdot)), \mathbb{1}_{m_1(\cdot)=i_1})$.

Therefore

$$\nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = a, \left[JT_{\theta^{m_0(\omega)}\omega}^{m_1(\omega)}(x(\theta^{m_0(\omega)}\omega))\right]^{-1} = b\right\}\right)$$

$$= \sum_{i_0}\sum_{i_1} \nu\left(\left\{\omega : \left[JT_\omega^{i_0}(x(\omega))\right]^{-1} = a, \left[JT_{\theta^{i_0}\omega}^{i_1}(x(\theta^{i_0}\omega))\right]^{-1} = b, m_0(\omega) = i_0, m_0(\theta^{i_0}\omega) = i_1\right\}\right)$$

$$= \sum_{i_0}\sum_{i_1} \left[\nu\left(\left\{\omega : \left[JT_\omega^{i_0}(x(\omega))\right]^{-1} = a, m_0(\omega) = i_0\right\}\right) \nu\left(\left\{\omega : \left[JT_{\theta^{i_0}\omega}^{i_1}(x(\theta^{i_0}\omega))\right]^{-1} = b, m_0(\theta^{i_0}\omega) = i_1\right\}\right)\right]$$

$$= \left[\sum_{i_0} \nu\left(\left\{\omega : \left[JT_\omega^{i_0}(x(\omega))\right]^{-1} = a, m_0(\omega) = i_0\right\}\right)\right]\left[\sum_{i_1} \nu\left(\left\{\omega : \left[JT_\omega^{i_1}(x(\omega))\right]^{-1} = b, m_0(\omega) = i_1\right\}\right)\right]$$

$$= \nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = a\right\}\right) \nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = b\right\}\right).$$

On the other hand

$$\nu\left(\left\{\omega : \left[JT_{\theta^{m_0(\omega)}\omega}^{m_1(\omega)}(x(\theta^{m_0(\omega)}\omega))\right]^{-1} = b\right\}\right)$$

$$= \sum_a \nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = a, \left[JT_{\theta^{m_0(\omega)}\omega}^{m_1(\omega)}(x(\theta^{m_0(\omega)}\omega))\right]^{-1} = b\right\}\right)$$

$$= \sum_a \nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = a\right\}\right) \nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = b\right\}\right)$$

$$= \nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = b\right\}\right).$$

So combining the two previous chains of equality, we find that

$$JT_{\cdot}^{m_0(\cdot)}(x(\cdot)) \perp JT_{\theta^{m_0(\cdot)}\cdot}^{m_1(\cdot)}(x(\theta^{m_0(\cdot)}\cdot)),$$

as desired.

II) Notice that

$$\int_\Omega \left[JT_{\theta^{m_0(\omega)}\omega}^{m_1(\omega)} x(\theta^{m_0(\omega)}\omega)\right]^{-1} d\nu(\omega) = \sum_b b\nu\left(\left\{\omega : JT_{\theta^{m_0(\omega)}\omega}^{m_1(\omega)} x(\theta^{m_0(\omega)}\omega)\right]^{-1} = b\right\}\right)$$

$$= \sum_b b\nu\left(\left\{\omega : \left[JT_\omega^{m_0(\omega)}(x(\omega))\right]^{-1} = b\right\}\right) = \int_\Omega \left[JT_\omega^{m_0(\omega)} x(\omega)\right]^{-1} d\nu(\omega),$$

where we have used the last equality in I).

B) Check if $\alpha_1 > 0$ and $\sum_{\ell=1}^\infty \ell^2 \alpha_\ell < \infty$

For the moment, this section considers the uniformly expanding case with $\inf\{{T_v}'(x) > 1 : x \in O_{v,i}\} \geqslant d_{\min} > 1$.

It holds that $\alpha_1 > 0$ because the quantity found in the first column of diagram (41) is bounded below by $1 - 1/d_{\min} > 0$.

Moreover, considering the integrand of equation (40), we see that α_ℓ is at most $(1/d_{\min})^{\ell-1}$, therefore

$$\sum_{\ell=1}^{\infty} \ell^2 \hat{\alpha}_\ell \leqslant \sum_{\ell=1}^{\infty} \ell^2 \left({}^1\!/_{d_{\min}{}^1}\right)^{\ell-1} < \infty,$$

since $d_{\min} > 1$.

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