Convergence of the natural approximations of piecewise monotone interval maps

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Abstract

We consider piecewise monotone interval mappings which are topologically mixing and satisfy the Markov property. It has previously been shown that the invariant densities of the natural approximations converge exponentially fast in uniform pointwise topology to the invariant density of the given map provided it's derivative is piecewise Lipshitz continuous. We provide an example of a map which is Lipshitz continuous and for which the densities converge in the bounded variation norm at a logarithmic rate. This shows that in general one cannot expect exponential convergence in the bounded variation norm. Here we prove that if the derivative of the interval map is Hölder continuous and its variation is well approximable (γ-uniform variation for $\gamma > 0$), then the densities converge exponentially fast in the norm.

In this paper we study the approximability of the densities of absolutely continuous invariant measures of expanding interval maps. This classical problem is of particular interest for numerical simulations where it is important to know at what rate and in which sense one obtains convergence of the approximating densities. In particular we provide an example that illustrates the difficulties that can arise in this context.

1 Introduction

The dynamics of (mixing) expanding piecewise monotone maps has been extensively studied, in particular the spectral properties of the associated transfer operators have been established by notably Hofbauer and Keller [3] [5] and Rychlik [9]. On the Banach space of functions of bounded variations the transfer operator has a simple largest real eigenvalue whose eigenfunction is the density of

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the invariant measure. The remainder of the spectrum has then strictly smaller radius which in particular implies that the correlation functions decay exponentially fast. For a piecewise affine map which satisfies the Markov property the transfer operator can be written as a matrix, which makes is more practical to find its invariant density [10]. In [7] it was shown the densities of affine approximations of an expanding piecwise monotone map indeed converge to the actual density of the invariant measure if the map is piecewise twice differentiable. This result was by Gora and Boyarsky [2] strengthened to piecewise differentiable maps whose derivatives have bounded variations. In [4] a bound for the rate of approximation is given for the absolutely continuous invariant measure for C^2 interval maps that are expanding. It is shown that the approximating probability vectors converge exponentially fast (with the notation of the present paper) in the L^1 sense. Recently Liverani [8] has studied these approximations emphasising the numerical aspect.

If the approximations are "natural" (piecewise linear Markov approximations in the terminology of [2]), then [1] the convergence is in fact exponentially fast pointwise uniformly provided the map has (piecewise) Lipshitz continuous derivatives.

In a recent paper Keller and Liverani [6] proved a more general result on the convergence of transfer operators that generalises the approach used in previous results on the approximability of densities. They prove that in the convergence in the supremum norm is determined by the approximation of the map, which in this context is exponential. In this paper we are however interested in the convergence in the bounded variation norm about which the paper by Keller and Liverani does not make any conclusion and in fact cannot: Indeed, in section 5 we provide an example which shows that under these conditions the convergence cannot in general expected to be exponential in the bounded variation norm. However, our main result, Theorem 8, shows that the approximating densities convergence in the Bounded Variation norm exponentially fast if one assumes that the variation of the map's derivative can sufficiently well be approximated by variations over partitions (the derivative has uniform variation—see Definition 4).

In section 2 we introduce uniform variation and define the associated norm. In section 3 we formulate the maintheorem (Theorem 8) and prove some properties for the variation of the derivative of an interval map provided it satisfies the uniform variation property. In section 4 we present a direct proof of Theorem 8. In section 5 we give the example of a map whose derivative has bounded variation but not uniformly bounded variation and whose associated invariant densities converge in the BV-norm at no more than logarithmic speed. In the Appendix (section 6) we give a proof of the Lasota Yorke inequality for the uniform variation norm also show that the unitball is precompact in the BV-norm. The reader will be able to use these two two results to get a shorter and possibly more general proof of Theorem 8. In this paper however I prefer the direct approach of section 4 to the general proof because the involved estimates can then be used in section 5 to construct an example of a nice map with very slow convergence properties.

Let T be a piecewise $C^{1+\gamma}$ continuous map (that is its first derivative is γ-Hölder continuous) of the unit interval $[0,1]$ to itself such that its restriction to the atoms of a finite partition $\mathcal{A} = \{(a_{j-1}, a_j) : j = 1, \ldots, J\}$ are monotone with Hölder continuous derivative T', where $0 = a_0 < a_1 < a_2 < \cdots < a_J = 1$. We shall furthermore make the following assumptions:
(i) There exists a constant $\rho < 1$ so that $|T'| \geq \rho^{-1}$ on every atom $A \in \mathcal{A}$.
(ii) At the endpoints a_j of the atoms of $\mathcal{A}$ the map T shall assume one or the other value of the continuous extensions of T to the closures of the adjacent intervals.
(iii) T has the Markov property, that is if $T(A) \cap A' \neq \emptyset$ for some $A, A' \in \mathcal{A}$ then $A' \subset T(A)$.
(iv) We require that there is a positive integer N so that the branches of monotonicity of T^N are onto $[0,1]$. Without loss of generality we can assume that $N = 1$.

We can now define the transfer operator $\mathcal{L}$ by:

$$\mathcal{L}\phi(x) = \sum_{y \in T^{-1}x} \frac{\phi(y)}{|T'(y)|},$$

for suitable functions $\phi : [0,1] \to \mathbf{C}$. On the space BV of functions with bounded variation the spectral radius of $\mathcal{L} : \mathrm{BV} \to \mathrm{BV}$ is equal to 1, and 1 is a simple eigenvalue with a strictly positive eigenfunction h which we assume to be normalised. Moreover the essential spectral radius is $\vartheta < 1$, where

$$\vartheta = \limsup_{n \to \infty} \frac{1}{|(T^n)'|^{1/n}}.$$

If we put

$$\tau = \max\{|\lambda| : \lambda \neq 1 \text{ in the spectrum of } \mathcal{L} : \mathrm{BV} \to \mathrm{BV}\},$$

(spectral gap), then the convergence of $\mathcal{L}$ to the eigenspace spanned by h is exponential at rate $\tau < 1$, that is

$$\|\mathcal{L}^n \phi - h\mu(\phi)\|_{\mathrm{BV}} \leq \mathrm{const.}\tau'^n \|\phi\|_{\mathrm{BV}},$$

for every $\tau' > \tau$, where μ is the Lebesgue measure on $[0,1]$ and $\|\phi\|_{\mathrm{BV}} = |\phi|_\infty + \mathrm{var}_{[0,1]}\phi$ is the bounded variation norm.

Definition 1 *The* nth piecewise affine approximation *of T is the transform $T_n : [0,1] \to [0,1]$ such that $T|_A$ is affine for all $A \in \mathcal{A}^n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{A}$ (the n-th join) and matches with T at the endpoints of the atoms A.*

Denote by $\mathcal{L}_n$ the transfer operator for the n-th affine approximation:

$$\mathcal{L}_n\phi(x) = \sum_{y\in T_n^{-1}x} \frac{\phi(y)}{|T_n'(y)|}, \quad \phi \in \mathrm{BV}.$$

Clearly 1 is a simple eigenvalue of $\mathcal{L}_n$ with strictly positive eigenfunction h_n. Assume h_n is normalised.

For the transfer operator for the map T we obtain for all k that

$$\mathcal{L}^k\mathbf{1}(x) = \sum_{y\in T^{-k}x} \frac{1}{|(T^k)'(y)|} \leq \frac{1}{\inf h} \sum_{y\in T^{-k}x} \frac{h(y)}{|(T^k)'(y)|} = \frac{h(x)}{\inf h} \leq \frac{|h|_\infty}{\inf h}.$$

Since the eigenfunction h depends continuously on the map T, we conclude that

$$|\mathcal{L}_n^k\mathbf{1}|_\infty \leq c_1,$$

for all k and n, where c_1 is some positive constant.

Theorem 2 *[1] Let T be a piecewise monotone interval map which satisfies the conditions (i)–(iv) (for some $\gamma > 0$). There exists a constant C_1 such that*

$$\mathrm{var}\, h_n \leq C_1, \ \forall n.$$

For every $\tau' > \tau^{1/6}$ there exists a constant C_2 such that

$$|h_n - h|_\infty \leq C_2 \tau'^n, \quad n = 1, 2, 3, \ldots.$$

We shall prove that the convergence is indeed exponential in the BV-norm if T satisfies the stronger property of having uniform variation. In section 3 we shall then produce an example of a piecewise C^{1+L} map (i.e. the derivative of the map is piecewise Lipshitz continuous) for which $\mathrm{var}\,(h - h_{n_j}) \geq \frac{\mathrm{const.}}{\log n_j}$ for a suitable sequence of integers n_j. That is, for that example the convergence of the densities is very slow in the bounded variation norm.

2 Uniform variation

If a finite partition $\mathcal{P}$ of the unit interval consists of the intervals between the points $0 = p_0 < p_1 < p_2 < \cdots < p_m = 1$, then we put

$$\mathrm{var}\,(\phi, \mathcal{P}) = \sum_{j=1}^{m} |\phi(p_{j-1}) - \phi(p_j)|$$

for the variation of the function ϕ with respect to the partition $\mathcal{P}$. Clearly, $\mathrm{var}\,\phi = \sup_{\mathcal{P}} \mathrm{var}\,(\phi, \mathcal{P})$, where the supremum is over all partitions $\mathcal{P}$ of the unit interval. (Observe that $\operatorname{var}\phi = \operatorname{var}(\phi, \mathcal{P})$, for all partitions $\mathcal{P}$, if ϕ is monotonically increasing or decreasing.) Put

$$\operatorname{Var}(\phi, \mathcal{P}) = |\operatorname{var}\phi - \operatorname{var}(\phi, \mathcal{P})|\,,$$

and denote by $\operatorname{diam}\mathcal{P} = \max_j (p_j - p_{j-1})$ the *diameter* of the partition $\mathcal{P}$.

Let $\mathcal{P}$ be a partition of I given by the points $0 = p_0 < p_1 < \cdots < p_m = 1$ and denote by $\phi_{\mathcal{P}}$ the piecewise constant approximation of ϕ on the partition $\mathcal{P}$ with discontinuities at the points p_j and at the points where ϕ is discontinuous. For instance, if ϕ has no discontinuity on $(p_j, p_{j+1}]$ then we can put $\phi_{\mathcal{P}}(x) = \phi(p_{j+1})$ for $p_j < x \leq p_{j+1}$, and similarly if ϕ has discontinuities.

Lemma 3 *For every partition $\mathcal{P}$ one has*

$$\operatorname{Var}(\phi, \mathcal{P}) \leq \operatorname{var}(\phi - \phi_{\mathcal{P}}).$$

Proof. If the partition $\mathcal{P}$ is given by the points $p_0, p_1, \ldots, p_n$, then

$$\operatorname{Var}(\phi, \mathcal{P}) = \sum_{j=1}^{n} \left(\operatorname{var}\ \phi - |\phi(p_j) - \phi(p_{j-1})|\right).$$

Fix some j and let $\mathcal{Q} = \{q_0, q_1, \ldots, q_m\}$ be a partition of the interval $[p_{j-1}, p_j]$ (naturally $q_0 = p_{j-1}, q_m = p_j$). Then

$$\operatorname{var}_{[p_{j-1},p_j]}(\phi - \phi_{\mathcal{P}}) = \sum_{i=1}^{m} |a_i - b_i|$$

where

$$\begin{aligned} a_i &= \phi(q_i) - \phi(q_{i-1}), \\ b_i &= \phi_{\mathcal{P}}(q_i) - \phi_{\mathcal{P}}(q_{i-1}), \end{aligned}$$

$i = 1, 2, \ldots, m$. Naturally $\sum_i |b_i| = |\phi(p_j) - \phi(p_{j-1})|$. (Note that all the b_i have the same sign.) Then, since $|a_i - b_i| \geq ||a_i| - |b_i|| \geq |a_i| - |b_i|$, we obtain

$$\begin{aligned} \operatorname{var}_{[p_{j-1},p_j]}(\phi - \phi_{\mathcal{P}}) &\geq \sum_{i=1}^{m} (|a_i| - |b_i|) \\ &= \operatorname{var}_{[p_{j-1},p_j]}(\phi, \mathcal{Q}) - |\phi(p_j) - \phi(p_{j-1})|. \end{aligned}$$

Since the partition $\mathcal{Q}$ was arbitrary, we conclude

$$\operatorname{var}_{[p_{j-1},p_j]}(\phi - \phi_{\mathcal{P}}) \geq \operatorname{var}_{[p_{j-1},p_j]}\phi - |\phi(p_j) - \phi(p_{j-1})|.$$

Summation over j yields

$$\operatorname{var}(\phi - \phi_{\mathcal{P}}) = \sum_{j=1}^{n} \operatorname{var}_{[p_{j-1},p_j]}(\phi - \phi_{\mathcal{P}}) \geq \operatorname{Var}\ (\phi, \mathcal{P}).$$

$\square$

The inequality in Lemma 3 is a strict one if ϕ is not monotone on the atoms of the partition $\mathcal{P}$.

Definition 4 *We say a function ϕ on the unit interval has* γ-uniform variation *for some $\gamma > 0$ if there exists a constant U so that*

$$\mathrm{Var}\,(\phi, \mathcal{P}) \leq U(\mathrm{diam}\,\mathcal{P})^\gamma,$$

for all partitions $\mathcal{P}$ of $[0,1]$. We shall put $U_\gamma(\phi)$ for the smallest possible choice of the constant U.

Continuous functions whose variation is γ-uniform are not necessarily γ-Hölder continuous as the example $\phi = x^{\gamma/2}$, $x \in [0,1]$ shows. Since ϕ is increasing, $\mathrm{var}\phi = \mathrm{var}(\phi, \mathcal{P})$ and therefore $\mathrm{Var}(\phi, \mathcal{P}) = 0$ for every partition $\mathcal{P}$ of the unit interval. Thus ϕ has γ-uniform variation although it is not γ-Hölder continuous.

If ψ is piecewise Hölder continuous with exponent $\gamma > 0$ then we denote by $|\psi|_\gamma$ the Hölder constant (applied to the intervals of continuity).

Note that for a monotone function ϕ one has $\mathrm{var}\,\phi = \mathrm{var}\,(\phi, \mathcal{P})$ and therefore $\mathrm{Var}\,(\phi, \mathcal{P}) = 0$ for all partitions $\mathcal{P}$.

Obviously $\mathrm{Var}\,(\phi + \psi) \leq \mathrm{Var}\,\psi + \mathrm{Var}\,\phi$. In the following lemma we shall use the well-known inequality $\mathrm{var}\,\phi\psi \leq |\phi|_\infty\,\mathrm{var}\,\psi + |\psi|_\infty\,\mathrm{var}\,\phi$.

Lemma 5 *Let ϕ and ψ be two piecewise γ-Hölder continuous functions of bounded variation on the interval I. Then*

$$\mathrm{Var}\,(\phi\psi, \mathcal{P}) \leq |\psi|_\infty\,\mathrm{Var}\,(\phi, \mathcal{P}) + |\phi|_\infty\,\mathrm{Var}\,(\psi, \mathcal{P}) + (\mathrm{diam}\,\mathcal{P})^\gamma(|\phi|_\gamma \mathrm{var}\,\psi + |\psi|_\gamma \mathrm{var}\,\phi),$$

for all partitions $\mathcal{P}$ of I.

Proof. Let $\mathcal{P}$ be a partition of I and denote by $\phi_{\mathcal{P}}$ is piecewise constant approximation introduced above preceeding Lemma 3. Then

$$\mathrm{Var}\,(\phi, \mathcal{P}) \leq \mathrm{var}\,(\phi - \phi_{\mathcal{P}}),$$

and

$$|\phi - \phi_{\mathcal{P}}|_\infty \leq |\phi|_\gamma(\mathrm{diam}\,\mathcal{P})^\gamma$$

Similarly $\psi_{\mathcal{P}}$ is the piecewise constant approximation of ψ on $\mathcal{P}$ and $\psi_{\mathcal{P}}\phi_{\mathcal{P}} = (\psi\phi)_{\mathcal{P}}$ is the piecewise constant approximation of $\psi\phi$.

Then

$$\begin{aligned}
\mathrm{Var}\,(\psi\phi, \mathcal{P}) &\leq \mathrm{var}\,((\psi\phi - \psi_{\mathcal{P}}\phi_{\mathcal{P}})) \\
&\leq \mathrm{var}\,(\psi_{\mathcal{P}}(\phi - \phi_{\mathcal{P}})) + \mathrm{var}\,(\phi(\psi - \psi_{\mathcal{P}})) \\
&\leq |\psi_{\mathcal{P}}|_\infty\,\mathrm{var}\,(\phi - \phi_{\mathcal{P}}) + |\phi - \phi_{\mathcal{P}}|_\infty \mathrm{var}\,\psi_{\mathcal{P}} \\
&\quad + |\phi|_\infty\,\mathrm{var}\,(\psi - \psi_{\mathcal{P}}) + |\psi - \psi_{\mathcal{P}}|_\infty \mathrm{var}\,\phi \\
&\leq |\psi|_\infty\,\mathrm{Var}\,(\phi, \mathcal{P}) + |\phi|_\gamma(\mathrm{diam}\,\mathcal{P})^\gamma \mathrm{var}\,\psi \\
&\quad + |\phi|_\infty\,\mathrm{Var}\,(\psi, \mathcal{P}) + |\psi|_\gamma(\mathrm{diam}\,\mathcal{P})^\gamma \mathrm{var}\,\phi
\end{aligned}$$

since $|\phi_{\mathcal{P}}|_\infty \leq |\phi|_\infty$ and $|\psi_{\mathcal{P}}|_\infty \leq |\psi|_\infty$. □

If f and g are functions on the unit interval with finite U_γ values, then it is easily seen that $U_\gamma(f+g) \leq U_\gamma(f)+U_\gamma(g)$ and $U_\gamma(cf) = |c|U_\gamma(f)$ for constants c. For constant functions c we have $U_\gamma(c) = 0$, and similarly $U_\gamma(\phi) = 0$ for a monotone function ϕ.

Let us define a norm $\|\cdot\|_\gamma$ ($\gamma > 0$) by

$$\|\phi\|_\gamma = |\phi|_\infty + U_\gamma(\phi) + |\phi|_\gamma$$

($|\phi|_\gamma$ is the Hölder constant on the intervals of continuity), and introduce for some partition $\mathcal{P}$ of the unit interval the function space

$$\mathrm{UV}_\gamma = \{\phi \in C^\gamma(\mathcal{P}) : \ \|\phi\|_\gamma < \infty\}$$

of functions with γ-uniform variation, where $C^\gamma(\mathcal{P})$ consists of all functions on $[0,1]$ which are γ-Hölder continuous on the atoms $\mathcal{P}$. The space $(\mathrm{UV}_\gamma, \|\cdot\|_\gamma)$ is a Banach space. Clearly $\mathrm{UV}_\gamma \subset \mathrm{UV}_{\gamma'}$ for $\gamma' \leq \gamma$, and moreover $\mathrm{UV}_\gamma \subset \mathrm{BV}$ for all positive γ. (Naturally $U_{\gamma'}(\phi) \leq U_\gamma(\phi)$ if $\gamma' \leq \gamma$.) Observe that since $|\mathrm{var}\,\phi - \mathrm{var}\,(\phi, \{I\})| \leq U_\gamma(\phi)$ (as $\mathrm{diam}\, I = 1$), one obtains that

$$\mathrm{var}\,\phi \leq \mathrm{var}\,(\phi, \{I\}) + U_\gamma(\phi) \leq 2|\phi|_\infty + U_\gamma(\phi).$$

Also note that for the locally constant approximants ϕ_n of a piecewise continuous function ϕ by Lemma 7 $\|\phi_n\|_\gamma \leq 3\|\phi\|_\gamma$.

Let us denote by S_k the inverse branches of T^k and put A_φ for the range of the inverse branches $\varphi \in S_k$. Note that $A_\varphi \in \mathcal{A}^k$.

Lemma 6 *Let T be a transformation of the unit interval to itself. Assume $|T'| \in \mathrm{UV}_\gamma$ ($\gamma > 0$), then $\mathcal{L}$ maps UV_γ into itself.*

Proof. Clearly $|\mathcal{L}\phi|_\infty \leq |\phi|_\infty |\mathcal{L}\mathbf{1}|_\infty \leq c_1|\phi|_\infty$, for some constant c_1. To estimate the variation we proceed as follows

$$\begin{aligned} \mathrm{Var}\,(\mathcal{L}\phi, \mathcal{P}) &= |\mathrm{var}\,\mathcal{L}\phi - \mathrm{var}\,(\mathcal{L}\phi, \mathcal{P})| \\ &\leq \sum_{\varphi \in S_1} \left| \mathrm{var}\,\frac{\phi\varphi}{|T'\varphi|} - \mathrm{var}\,\left(\frac{\phi\varphi}{|T'\varphi|}, \mathcal{P}\right)\right| \\ &\leq |S_1|\,\mathrm{Var}\,\left(\frac{\phi}{|T'|}, T^{-1}\mathcal{P}\right). \end{aligned}$$

This implies that $\|\mathcal{L}\phi\|_\gamma \leq c_2\|\phi\|_\gamma$ for some constant c_2. □

For the transfer operator and its affine approximations one can now deduce the Lasota-Yorke inequality

$$\mathrm{U}_\gamma(\mathcal{L}^k\phi) + |\mathcal{L}^k\phi|_\gamma \leq \mathrm{const.}\,|\phi|_\infty + \varsigma^k(\mathrm{U}_\gamma(\phi) + |\phi|_\gamma), \tag{1}$$

for some $\varsigma < 1$, for all k and functions $\phi \in \mathrm{UV}_\gamma$ ($\gamma > 0$). For a proof see the Appendix. As a consequence of it one obtains yet another spectral theorem for the transfer operator on the slightly different function space UV_γ (Exercise).

3 Piecewise approximation of the map T

Let ϕ be a piecewise Hölder continous function on $[0,1]$ so that $U_\gamma(\phi) < \infty$ and denote by ϕ_n the locally constant function which is constant on intervals of continuity in the atoms of $\mathcal{A}^n$. For simplicity's sake let us assume that ϕ is continuous. Then ϕ_n is constant on the atoms of $\mathcal{A}^n$ and if $\mathcal{P} = \{p_0, p_1, \ldots, \}$ is a partition so that each subinterval in $\mathcal{A}^n$ contains at least one point p_j then $\mathrm{Var}\,(\phi_n, \mathcal{P}) = 0$.

Lemma 7 *Assume $U_\gamma(\phi) < \infty$ and put $\phi_n = \phi_{\mathcal{A}^n}$. Then $U_\gamma(\phi_n) \leq 3U_\gamma(\phi)$.*

Proof. Let $\mathcal{B}$ be the partition of $[0,1]$ by the discontinuity points of ϕ and put $\mathcal{Q} = \mathcal{B} \vee \mathcal{A}^n$. Then ϕ_n is constant on the subintervals in $\mathcal{Q}$. Let $\mathcal{P} = \{p_0, p_1, \ldots\}$ be a partition and put

$$J = \{j : |(p_{j-1}, p_j] \cap \{q_0, q_1, \ldots\}| \geq 2\},$$

where $\{q_0, q_1, \ldots\}$ are the cutting points of the partition $\mathcal{Q}$. In other words, J indexes the subintervals of $\mathcal{P}$ that 'jump' entire subintervals of $\mathcal{Q}$. These are the only places that will contribute towards $\mathrm{Var}\,(\phi_n, \mathcal{P})$. We thus obtain

$$\begin{aligned} \mathrm{Var}\,(\phi_n, \mathcal{P}) &\leq \sum_{j \in J} |\phi_n(p_{j-1}) - \phi_n(p_j)| \\ &= \sum_{j \in J} |\phi(\bar{p}_{j-1}) - \phi(\bar{p}_j)| \\ &\leq U_\gamma(\phi)(3\mathrm{diam}\,\mathcal{P})^\gamma \end{aligned}$$

as $|\bar{p}_k - p_k|, |\bar{p}_{k-1} - p_{k-1}| \leq \mathrm{diam}\,\mathcal{P} \ \ \forall k \in J$, where $\bar{p}_k$ is the right endpoint of the subinterval $A \in \mathcal{Q}$ which contains p_k. □

It follows that the approximating densities h_n in fact lie in UV_γ for all $\gamma > 0$. The following theorem, which is proven in the following section, asserts that convergence of the approximating densities is in fact exponential if we assume that T' has γ-uniform variation.

Theorem 8 *Let T be a transformation of the interval which satisfies the conditions (i)–(iv) and let $\mathcal{L}$ be the associated transfer operator acting on the Banach space of functions of bounded variation. If we assume that T' has γ-uniform variation for some positive γ, then there exists a constant C_4 and a $\sigma \in (0,1)$ so that $\|h - h_n\|_{\mathrm{BV}} \leq C_4 \sigma^n \ \ \forall n$. In particular $\mathrm{var}\,(h - h_n) \leq C_4 \sigma^n$, for all n.*

The proof of this theorem is the subject of the next section. We will need the following tow lemmata in the proof of Theorem 8 as well as in the Appendix where we outline how to use the standard approach to get the convergence using the theorem of Ionescu Tulcea-Marinescu with the two Banach space norms $\|\cdot\|_{BV} \le \|\cdot\|_\gamma$ and the inequality of Lemma 11.

Lemma 9 *There exists a constant C_5 (depending on ρ, $\operatorname{var}\log|T'|$ and $U_\gamma(\log|T'|)$) so that for every interval $A \subset [0,1]$ on which T^k is one-to-one (similar statements hold true for T_n):*
(I) For every partition $\mathcal{P}$ of A one has

$$\operatorname{Var}_A\left(\frac{1}{|(T^k)'|}, \mathcal{P}\right) \le C_5 \sup_A \frac{1}{|(T^k)'|}(\operatorname{diam} T^k\mathcal{P})^\gamma.$$

(II)

$$\operatorname{var}_A \frac{1}{|(T^k)'|} \le kC_5 \sup_A \frac{1}{|(T^k)'|}$$

where the constant C_5 depends continuously on ρ and $U_\gamma(\log|T'|)$.

Proof. Let $\mathcal{Q}$ be a partition of A. Assume $\mathcal{Q}$ is given by the points $y_0 < y_1 < \cdots < y_Q$. Since $|(T^k)'|$ is γ-Hölder continuous, i.e. $|(T^k)'(y) - (T^k)'(y')| \le c_1|T^k y - T^k y'|^\gamma$ for $y, y' \in A$ (where c_1 is some constant that depends on the Hölder constant of $|T'|$ on intervals of continuity and $|T'|_\infty$), we get that the variation over the partition $\mathcal{Q}$ is then (for $\operatorname{diam}\mathcal{Q}$ small enough)

$$\begin{aligned}
\operatorname{var}\left(\frac{1}{|(T^k)'|}, \mathcal{Q}\right) &= \sum_{j=1}^{Q}\left|\frac{1}{|(T^k)'(y_{j-1})|} - \frac{1}{|(T^k)'(y_j)|}\right| \\
&= \sum_{j=1}^{Q}\frac{1}{|(T^k)'(y_j)|}\left|1 - \frac{|(T^k)'(y_j)|}{|(T^k)'(y_{j-1})|}\right| \\
&= \sum_{j=1}^{Q}\frac{1+q_j}{|(T^k)'(y_j)|}\left|\log\frac{|(T^k)'(y_j)|}{|(T^k)'(y_{j-1})|}\right|, \qquad (2)
\end{aligned}$$

with some numbers q_j for which $|q_j| \le c_1(\operatorname{diam} T^k\mathcal{Q})^\gamma$. Hence, letting $\operatorname{diam}\mathcal{Q}$ go to zero, one obtains

$$\begin{aligned}
\operatorname{var}\frac{1}{|(T^k)'|} &\le \sup_A\left|\frac{1}{(T^k)'}\right| \operatorname{var}_A \log|(T^k)'| \\
&\le \sup_A\left|\frac{1}{(T^k)'}\right| \sum_{\ell=0}^{k-1} \operatorname{var}_{T^\ell A} \log|T'| \\
&\le \sup_A\left|\frac{k}{(T^k)'}\right| \operatorname{var}\log|T'|,
\end{aligned}$$

This proves the statement (II) of the lemma.

To prove the inequality (I), let $\mathcal{P}$ be some partition of A. Equation (2) yields an upper bound for the variation (as diam $\mathcal{Q} \to 0$)

$$\operatorname{var} \frac{1}{|(T^k)'|} \leq \sum_{P \in \mathcal{P}} \sup_{P \cap A} \left| \frac{1}{(T^k)'} \right| \operatorname{var}_{P \cap A} \log |(T^k)'|$$

and a lower bound for the variation with respect to $\mathcal{P}$

$$\operatorname{var} \left(\frac{1}{|(T^k)'|}, \mathcal{P} \right) \geq \left(1 - c_1 (\operatorname{diam} T^k \mathcal{P})^\gamma\right) \sum_{P \in \mathcal{P}} \inf_{P \cap A} \left| \frac{1}{(T^k)'} \right| \operatorname{var}_{P \cap A} \left(\log |(T^k)'|, \mathcal{P} \right).$$

Using Hölder continuity, the inf term in the lower estimate works out to be

$$\inf_{P \cap A} \left| \frac{1}{(T^k)'} \right| \geq \left(1 - c_1 (\operatorname{diam} T^k P)^\gamma\right) \sup_{P \cap A} \left| \frac{1}{(T^k)'} \right|,$$

for $P \in \mathcal{P}$. We can finally estimate as follows (as diam $P \leq$ diam $\mathcal{P}$):

$$\begin{aligned} \operatorname{Var} \left(\frac{1}{|(T^k)'|}, \mathcal{P} \right) &\leq \sum_{P \in \mathcal{P}} \sup_{P \cap A} \left| \frac{1}{(T^k)'} \right| \Big(\operatorname{var}_{P \cap A} \log |(T^k)'| \\ &\qquad\qquad - \operatorname{var}_{P \cap A} \left(\log |(T^k)'|, \mathcal{P} \right) \Big) \\ &\quad + c_2 (\operatorname{diam} T^k \mathcal{P})^\gamma \sup_A \left| \frac{1}{(T^k)'} \right| \operatorname{var}_A \left(\log |(T^k)'|, \mathcal{P} \right) \\ &\leq \sup_A \left| \frac{1}{(T^k)'} \right| \left(\operatorname{Var}_A \left(\log |(T^k)'|, \mathcal{P} \right) + c_3 (\operatorname{diam} T^k \mathcal{P})^\gamma \right) \\ &\leq C_5 \sup_A \left| \frac{1}{(T^k)'} \right| (\operatorname{diam} T^k \mathcal{P})^\gamma, \end{aligned}$$

for some constants c_2, c_3. In the last two inequalities we used that with some constant $c_4 = U_\gamma(\log |T'|)$

$$\operatorname{Var}_A \left(\log |(T^k)'|, \mathcal{P} \right) \leq \sum_{\ell=1}^{k} \operatorname{Var}_{T^k A} \left(\log |T'|, T^\ell \mathcal{P} \right) \leq c_4 (\operatorname{diam} T^k \mathcal{P})^\gamma \frac{1}{1 - \rho},$$

since T^k is one-to-one on A and $\operatorname{diam} T^\ell \mathcal{P} \leq \rho^{k-\ell} \operatorname{diam} T^k \mathcal{P}$. Similarly $\operatorname{var}_A \left(\log |(T^k)'|, \mathcal{P} \right) \leq \frac{c_4}{1-\rho} (\operatorname{diam} T^k \mathcal{P})^\gamma$.

Put $C_5 = \max(c_3 + c_4/(1-\rho), \operatorname{var} \log |T'|)$. □

Let S_k denote the inverse branches of T^k. If $\varphi \in S_k$, then we denote its range by A_φ which in fact is an atom in $\mathcal{A}^k$. If $k < n$ then every inverse branch $\varphi \in S_k$ has a corresponding inverse branch $\tilde{\varphi}$ of T_n^k whose range $A_{\tilde{\varphi}} \in \mathcal{A}^k$ coincides with A_φ.

Lemma 10 *For every $\vartheta' > \vartheta$ there exists a constant C_6 (depending continuously on T) so that for all $k < n$:*
(I)

$$\left|1 - \frac{|(T_n^k)'\tilde{\varphi}|}{|(T^k)'\varphi|}\right|_\infty \leq nC_6\vartheta'^{(n-k)\gamma}.$$

(II)

$$\left|\frac{1}{|(T^k)'\varphi|}\right|_\gamma \leq C_6\left|\frac{1}{|(T^k)'\varphi|}\right|_\infty$$

and similarly for $\tilde{\varphi}$ instead of φ and T_n instead of T.

Proof. Given $\vartheta' > \vartheta$ then for any $\vartheta'' \in (\vartheta, \vartheta')$ we can find c_1 so that $\frac{1}{|(T^k)'|} \leq c_1\vartheta''^k$ for all k. Since T^k is one-to-one on the atoms of $\mathcal{A}^k$ we conclude that $\operatorname{diam}\mathcal{A}^k \leq c_1\vartheta''^k$ for all k. Now let $k < n$ and observe that $\varphi(x) = \tilde{\varphi}(x)$ on the endpoints x of atoms in $\mathcal{A}^{n-k}$. Therefore, by the mean value theorem,

$$\begin{aligned}
\left|1 - \frac{|(T_n^k)'\tilde{\varphi}|}{|(T^k)'\varphi|}\right|_\infty &\leq c_2\left|\log\frac{|(T_n^k)'\tilde{\varphi}|}{|(T^k)'\varphi|}\right|_\infty \\
&\leq c_2\max_{B\in\mathcal{A}^{n-k}}\left(\operatorname{var}_{T^kB}\log|(T^k)'| + \operatorname{var}_{T_n^kB}\log|(T_n^k)'|\right) \\
&\leq c_2\max\log|(T^k)'|\,2\,(\operatorname{diam}\mathcal{A})^{(n-k)\gamma} \\
&\leq (n-k)c_3\vartheta''^{(n-k)\gamma}
\end{aligned}$$

for some constants c_2, c_3.

To proof part (II) we proceed similarly

$$\begin{aligned}
\left|\frac{1}{|(T^k)'\varphi x|} - \frac{1}{|(T^k)'\varphi x'|}\right| &\leq c_2\frac{1}{|(T^k)'\varphi x|}\sum_{j=0}^{k-1}\left|\log|T'T^j\varphi x| - \log|T'T^j\varphi x'|\right| \\
&\leq \frac{c_2}{1-\rho^\gamma}\frac{1}{|(T^k)'\varphi x|}\left|\log|T'|\right|_\gamma |x-x'|^\gamma.
\end{aligned}$$

The statements now follow if we choose $C_6 = \max(c_3, c_2\left|\log|T'|\right|_\gamma/(1-\rho^\gamma))$. $\square$

In the following we shall use the fact that h and h_n are γ-Hölder continuous. Let C_7 be a constant so that the Hölder constants $|h|_\gamma, |h_n|_\gamma$ are bounded by C_7.

4 Proof of Theorem 8

We shall present a direct proof of Theorem 8 although it can also be proven using the usual method involving the Ioncesu Tulcea-Marinescu inequality. In the next section where we present an example with logarithmic convergence speed, we will need the explicit estimates for the terms I, II and III.

We have to show that $\mathrm{var}(h-h_n)\le c_1\sigma^n$ for some $\sigma<1$ and some constant c_1. Assume $k<n$ and given an inverse branch $\varphi\in S_k$ let, as above, $\tilde\varphi$ be the associated inverse branch of T_n^k which has the same range $A_\varphi=A_{\tilde\varphi}\in\mathcal{A}^k$. Then, for any $k<n$,

$$\begin{aligned}
\mathrm{var}(h-h_n) &= \mathrm{var}\,(\mathcal{L}^k h-\mathcal{L}_n^k h_n)\\
&= \mathrm{var}\sum_{\varphi\in S_k}\left(\frac{h\varphi}{|(T^k)'\varphi|}-\frac{h_n\tilde\varphi}{|(T_n^k)'\tilde\varphi|}\right)\\
&\le \mathrm{var}\sum_{\varphi\in S_k}\frac{1}{|(T_n^k)'\varphi|}(h\varphi-h_n\tilde\varphi)\\
&\quad +\mathrm{var}\sum_{\varphi\in S_k}h_n\tilde\varphi\left(\frac{1}{|(T_n^k)'\tilde\varphi|}-\frac{1}{|(T^k)'\varphi|}\right)\\
&\le I+II+III+IV,
\end{aligned}$$

where S_k are the inverse branches of T^k and their ranges are the atoms of $\mathcal{A}^k$. We use the identity $\mathrm{var}\,fg\le|f|_\infty\mathrm{var}\,g+|g|_\infty\mathrm{var}\,f$ to split the estimate into the four terms which we estimate separately. Notice that $\mathrm{var}\,\mathcal{L}_n^k(h-h_n)\le I+II$ and $\mathrm{var}\,(\mathcal{L}^k-\mathcal{L}_n^k)h_n\le III+IV$.

For the first term we get

$$\begin{aligned}
I &= \sum_{\varphi\in S_k}\left|\frac{1}{(T_n^k)'\varphi}\right|_\infty \mathrm{var}_{\,T^kA_\varphi}\,(h\varphi-h_n\tilde\varphi)\\
&\le \vartheta'^k\sum_{\varphi\in S_k}\left(\mathrm{var}_{A_\varphi}h+\mathrm{var}_{A_\varphi}h_n\right)\\
&\le c_2\vartheta'^k.
\end{aligned}$$

Using Lemma 9(II) we estimate the second term as follows

$$\begin{aligned}
II &= \sum_{\varphi\in S_k}|h\varphi-h_n\tilde\varphi|_\infty\mathrm{var}_{T^kA_\varphi}\left(\frac{1}{|(T_n^k)'\tilde\varphi|}\right)\\
&\le c_3\kappa^n\sum_{\varphi\in S_k}\mathrm{var}_{A_\varphi}\left(\frac{1}{|(T_n^k)'|}\right)\\
&\le c_3C_5\kappa^n\sum_{\varphi\in S_k}\max_{A_\varphi}\frac{1}{|(T_n^k)'\tilde\varphi x|}\\
&\le c_4\kappa^n.
\end{aligned}$$

Using Lemma 10, the third term becomes

$$III = \sum_{\varphi\in S_k}\left|1-\frac{|(T_n^k)'\tilde\varphi|}{|(T^k)'\varphi|}\right|_\infty\mathrm{var}_{T^kA_\varphi}\frac{h_n\tilde\varphi}{|(T_n^k)'\tilde\varphi|}$$

$$
\begin{aligned}
&\le \quad nC_6\vartheta'^{n-k}\sum_{\varphi\in S_k}\mathrm{var}_{A_\varphi}\frac{h_n}{|(T_n^k)'|}\\
&\le \quad nC_6\vartheta'^{n-k}\sum_{\varphi\in S_k}\left(\left|\frac{1}{(T_n^k)'}\right|_\infty \mathrm{var}_{A_\varphi} h_n + |h_n|_\infty \mathrm{var}_{A_\varphi}\frac{1}{|(T_n^k)'|}\right)\\
&\le \quad nC_6\vartheta'^{n-k}\left(\vartheta'^k \mathrm{var}\, h_n + |h_n|_\infty C_5 c_5 |h_n|_\infty\right)\\
&\le \quad nc_6\vartheta'^{n-k},
\end{aligned}
$$

for some constants c_5, c_6, where we used Lemma 9(II) once more.

For the fourth term we have to use that $|T'|$ has uniform variation. As before

$$
IV = \sum_{\varphi\in S_k}\left|\frac{h_n}{(T_n^k)'}\circ\tilde\varphi\right|_\infty \mathrm{var}_{T^k A_\varphi}\left(1-\frac{|(T_n^k)'\varphi|}{|(T^k)'\tilde\varphi|}\right),
$$

where

$$
\begin{aligned}
\mathrm{var}_{T^k A_\varphi}\left(1-\frac{|(T_n^k)'\tilde\varphi|}{|(T^k)'\varphi|}\right) &= \mathrm{var}_{A_\varphi}\left(1-\frac{|(T_n^k)'\hat\varphi_0|}{|(T^k)'|}\right)\\
&\le c_7\,\mathrm{var}_{A_\varphi}\log\frac{|(T^k)'\hat\varphi_0|}{|(T_n^k)'|}\\
&= c_7\sum_{\ell=0}^{k-1}\mathrm{var}_{T^\ell A_\varphi}\log\frac{|T'|}{|T_n'\hat\varphi_\ell|}
\end{aligned}
$$

with some c_7, where $\hat\varphi_\ell = T_n^\ell\tilde\varphi T^{k-\ell}$. In order to estimate the variation term inside the sum for a given value of ℓ, consider on $T^\ell A_\varphi$ the partition $\mathcal{Q}_\ell$ induced by $T^\ell\mathcal{A}^n$. Notice that $|T_n'|$ and $|T_n'\hat\varphi_\ell|$ are both constant on the atoms of $\mathcal{Q}_\ell$. In every atom $Q\in\mathcal{Q}_\ell$ there is (by the mean value theorem) a point y_Q so that so that $|T'(y_Q)| = |T_n'|_Q|$. The points $\{y_Q : Q\in\mathcal{Q}_\ell\}$ form a partition of $T^\ell A_\varphi$ which we shall call $\mathcal{R}_\ell$. As $\log|T'|$ has γ-uniform variation, one obtains:

$$
\mathrm{var}_{T^\ell A_\varphi}\log\frac{|T'|}{|T_n'\hat\varphi_\ell|} = \mathrm{Var}_{T^\ell A_\varphi}\left(\log|T'|,\mathcal{R}_\ell\right) \le \mathrm{U}_\gamma(\log|T'|)\left(\mathrm{diam}\,\mathcal{R}_\ell\right)^\gamma .
$$

Moreover, since

$$
\mathrm{diam}\,\mathcal{R}_\ell \le 2\,\mathrm{diam}\,\mathcal{Q}_\ell \le 2\,\mathrm{diam}\,T^\ell\mathcal{A}^n \le 2\,\mathrm{diam}\,\mathcal{A}^{n-\ell} \le 2\rho^{n-\ell},
$$

we deduce that

$$
\begin{aligned}
\sum_{\varphi\in S_k}\left|\frac{h_n}{(T_n^k)'}\circ\tilde\varphi\right|_\infty \mathrm{var}_{T^\ell A_\varphi}\log\frac{|T'|}{|T_n'\hat\varphi_\ell|} &\le \vartheta'^{k-\ell}\sum_{\psi\in S_{k-\ell}}\mathrm{Var}_{A_\psi}\left(\log|T'|,\mathcal{R}_\ell\right)\\
&\le c_8\vartheta'^{k-\ell}\rho^{\gamma(n-\ell)},
\end{aligned}
$$

and, with some c_9:

$$IV \le c_8 \sum_{\ell=0}^{k-1} \vartheta'^{k-\ell} \rho^{\gamma(n-\ell)} \le k c_9 \theta^{n-k},$$

where $\theta \ge \max(\vartheta', \rho^\gamma)$, $\theta < 1$.

Finally with a suitable choice for k, say $k = n/2$, we get that

$$\operatorname{var}(h - h_n, \mathcal{P}) \le I + II + III + IV \le c_{11} \left(\vartheta'^k + \kappa^k + n\vartheta'^{n-k} + k\theta^k \right) \le c_1 \sigma^n,$$

for all n and for some $\sigma < 1$. □

5 Example

In this section we present a map T of the unit interval to itself which is piecewise C^{1+L} and for which $\operatorname{var}(h - h_{n_j})$ converges subexponentially fast to zero as the sequence n_j goes to infinity. As before h_n are the densities for the n-th natural affine approximation T_n of T.

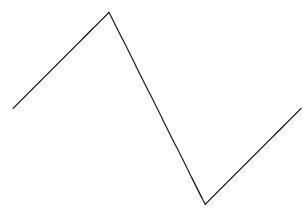
As in the proof of Theorem 8 let us consider (for a suitable sequence of n-values)

$$\operatorname{var}(h - h_n) = \operatorname{var}(\mathcal{L}^k h - \mathcal{L}_n^k h_n) \le I + II + III + IV,$$

where as above $I + II + III \le c_1(\vartheta^k + \kappa^n + n\vartheta^{n-k})$ for $k < n$, where the numbers $\vartheta, \kappa < 1$ are as above. We shall now construct T so that the fourth term IV can be estimated from below (polynomially in $\log n_j$). We shall proceed in four steps. We shall construct the map (i); then estimate the conditional variation $\log T'$ from below for a certain partition (ii) and from above for some finer partitions (iii); and finally we shall obtain lower bounds on the variation $\operatorname{var}(h - h_n)$ (iv).

(i) We approximate T in the following way in C^1 by a sequence of piecewise C^{1+L}-maps V_j, $j = 1, 2, \ldots$. Let P be some large integer which is to be chosen later and put V_1 for the piecewise linear map $V_1(x) = Px \bmod 1$, for $x \in [0, 1]$. Then V_1' is constant equal to P on P subintervals of length $1/P$ that form a partition $\mathcal{A}$ of $[0, 1]$. The quantities $\vartheta' = (P - \frac{1}{3P})^{-1}$ and $\vartheta = (P + \frac{1}{3P})^{-1}$ will be a lower respectively upper bound for the contraction rates for the inverse branches of the maps V_j and T. Choose $\beta \ge 2\frac{\log \vartheta}{\log \vartheta'}$ ($\beta > 1$), and define a sequence of integers $n(m) = [\beta^m]$ ($[\cdot]$ denotes integer part). That is $m \sim \log_\beta n(m)$. The sequence of positive numbers $\alpha_m = \vartheta^{n(m)}$, $m = 1, 2, \ldots$, is at least exponentially fast decreasing to zero ($\alpha_m \sim \vartheta^{\beta^m}$). Note that $\vartheta'^{n(m)/2} \le \alpha_{m-1}$.

Suppose we already constructed the maps $V_1, V_2, \ldots, V_{m-1}$, and let us now proceed to find V_m. First, denote by H_j the union of (open) intervals on which the derivative V_j' is constant (in fact equal to P). Then H_{m-1} is the disjoint union of (finitely many) intervals, say $I_{m-1,1}, I_{m-1,2}, \ldots$. Each of these intervals $I_{m-1,\ell}$ is divided into $[|I_{m-1,\ell}|\frac{1}{m\alpha_m}]+1$ ($|\cdot|$ denotes the length of the interval) many pieces and on the middle of each of these pieces we replace the constant value P of the derivative V'_{m-1} by a *squiggle* which is a piecewise affine curve whose left and right thirds both have slope 1 while the middle third has slope -2 (that is, the second derivative V''_m is on 1 on the left and right thirds of the interval which we modify and $V''_m = -2$ on the middle third)



Squiggle

of length α_m (the length of a squiggle is the euclidean distance between the left and right endpoints) so that the left and right endpoints of the squiggle connect with the constant value P of the derivative V'_{m-1}. In this way we find the derivative V'_m of the map V_m. This completes the construction of V_m.

Since the squiggle over its length rises by the same amount above the constant value P as it dips below P, its average (over its length) is equal to P. This implies that the integral V_m of V'_m assumes the same value on the endpoints of the squiggle as V_{m-1} did before we replaced the straight linesegment of its derivative V'_{m-1} by the squiggle. This implies that $V_m(x) = V_{m-1}(x)$ for x outside the squiggles which were introduced at the step m (i.e. for $x \in [0,1] \setminus (H_m \setminus H_{m-1})$).

Also observe that $V'_m \geq P - 2$, since we modified the derivative by introducing the squiggles, thus lowering the given constant derivative P by at most the value 2. In particular, when we choose the value of P large enough (i.e. larger than 3) than all the approximates V_m will be expanding interval maps.

For the Lebesgue measure (sum of the lengths of subintervals) of the 'linear set' H_m of V_m (i.e. $H_m = \{x \in [0,1] : V'_m(x) = P\}$) we obtain

$$\begin{aligned}
|H_m| &= |H_{m-1}| - \sum_{\ell} \left(\left[|I_{m-1,\ell}| \frac{1}{m\alpha_m} \right] + 1 \right) \alpha_m \\
&\geq |H_{m-1}|(1 - \frac{1}{m}) \\
&= |H_{m-1}| \frac{m-1}{m} \\
&\geq |H_1| \frac{1}{m}.
\end{aligned}$$

Thus $|H_m| \sim 1/m$. Let us furthermore observe that the number M_m of squiggles of length α_m is

$$M_m \sim |H_m| \frac{1}{m\alpha_m} \sim \frac{\vartheta^{-\beta^m}}{m^2} \sim \frac{\vartheta^{-n}}{m^2}, \tag{3}$$

and the total length of the α_m-squiggles is $|H_{m-1} \setminus H_m| \sim \frac{1}{m^2}$.

It follows from the construction that all the maps V_m are C^{1+L} on the atoms of $\mathcal{A}$ (in fact 2 is a Lipshitz constant for all the derivatives V_m'), and their derivatives satisfy $1/\vartheta \leq V_m' \leq 1/\vartheta'$. By Arzela-Ascoli the maps V_m converge in C^1 to a limit T which is piecewise of class C^{1+L}.

Clearly, T has the Markov property and is topologically mixing (in fact $N = 1$). It thus satisfies the conditions (i)–(iv) (with $\rho = \vartheta'$).

If $\mathcal{A}^n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{A}$ denotes the n-th join of $\mathcal{A}$, then $\vartheta'^{-n} \leq |\mathcal{A}^n| \leq \vartheta^{-n}$ (cardinality) and $\vartheta^n \leq |A| \leq \vartheta'^n$ (length) for every atom $A \in \mathcal{A}^n$.

(ii) Let us next estimate $\mathrm{Var}_{T^{k-1}A}(\log T', \mathcal{R}_{k-1})$, for $A \in \mathcal{P}^k$, $k < n$, where $\mathcal{R}_{k-1}$ is a partition of A whose diameter $\vartheta \leq 2\vartheta'^{n-k}$ and whose cardinality $|\mathcal{R}_{k-1}|$ is bounded from above by $\vartheta^{-(n-k)}$. Since in our case $T^{k-1}A = [0,1]$ for all $A \in \mathcal{A}^k$, we therefore get that (recall that by construction $T' > 1$)

$$\mathrm{Var}_{T^{k-1}A}(\log T', \mathcal{R}_{k-1}) = \mathrm{Var}(\log T', \mathcal{R}_{k-1}).$$

If we put $k = [n/2]$ and $n = n(m)$, then $|\mathcal{A}^{n-k}| \leq M_m$ by (3) for large enough m. We can therefore conclude that the total length of those squiggles of lengths $\leq \alpha_m$ which are subsets of single atoms of the partition $\mathcal{R}_{k-1}$ is at least $|H_{m+1}| \sim \frac{1}{m+1}$. Every such squiggle contributes $\frac{4}{3}$-times its length to the variation $\mathrm{Var}(\log T', \mathcal{R}_{k-1})$. Hence

$$\mathrm{Var}(\log T', \mathcal{R}_{k-1}) \geq \frac{4}{3}|H_{m+1}| \geq \frac{1}{m}. \tag{4}$$

(iii) Now let $\ell < k-1$, and let us find upper estimates for $\mathrm{Var}_{T^\ell A}(\log T', \mathcal{R}_\ell)$ for $A \in \mathcal{A}^k$, where $\mathcal{R}_\ell$ are partitions whose diameters are bounded above by $2\vartheta'^{n-\ell}$. Clearly, whenever $A \in \mathcal{A}^k$, then

$$\vartheta^{l-1-\ell} \leq |T^\ell A| \leq \vartheta'^{k-1-\ell}.$$

Suppose we have a squiggle of some length s. If we approximate by a locally constant function which is constant on intervals of lengths $< s/3$ say, then the conditional variation can only be produced by those intervals which contain corners (kinks) of the squiggle (of which there are four in each squiggle). Hence if we only consider squiggles of size $\geq \alpha_{m-1}$ we get

$$\mathrm{Var}_{T^\ell A \cap ([0,1]\setminus H_m)}(\log T', \mathcal{R}_\ell) \leq \mathrm{diam}(\mathcal{R}_\ell) 4 N_m 2 |T^\ell A| (1 - |H_m|), \tag{5}$$

(since all squiggles of lengths $\le \alpha_m$ dwell on H_m) where N_m is the total number of squiggles of lengths $\ge \alpha_{m-1}$. We estimate ($\beta > 1$ large enough):

$$N_m = \sum_{k=2}^{m-1} M_k \sim \sum_{k=2}^{m-1} \frac{1}{k^2 \alpha_k} \le \frac{1}{m\alpha_{m-1}}.$$

On the other hand for squiggles $\le \alpha_m$ we get as before

$$\mathrm{Var}_{T^\ell A \cap H_m} (\log T', \mathcal{R}_\ell) \le \frac{4}{3} |T^\ell A| \cdot |H_m| \sim \frac{4}{3m} |T^\ell A| \tag{6}$$

(the variation of a squiggle is $4/3$ times its length). As $|T^\ell A| \le \vartheta'^{k-1-\ell}$ if $A \in \mathcal{A}^k$, we now obtain from equations (5) and (6)

$$\begin{aligned} \mathrm{Var}_{T^\ell A} (\log T', \mathcal{R}_\ell) &\le \frac{4}{3m} |T^\ell A| + 8 |T^\ell A| N_m \vartheta'^{n-\ell} \\ &\le \vartheta'^{k-1-\ell} \frac{1}{m} \left(\frac{4}{3} + \frac{4\vartheta'^{n-\ell}}{\alpha_{m-1}} \right), \end{aligned}$$

as $N_m \le 1/m\alpha_{m-1}$. Hence, since by assumption $\vartheta'^{n-\ell} \alpha_{m-1}^{-1} \le 1$, (as $\ell < k = [n/2]$) we get

$$\sum_{\ell=0}^{k-2} \mathrm{Var}_{T^\ell A} (\log T', \mathcal{R}_\ell) \le \frac{6}{m} \sum_{\ell=0}^{k-2} \vartheta'^{k-1-\ell} \le \frac{6}{m} \frac{\vartheta'}{1-\vartheta'}. \tag{7}$$

(iv) Let us now complete the estimate of the variation in our example. As above, let $A_\varphi \in \mathcal{A}^k$ be the ranges of the inverse branches $\varphi \in S_k$ of T^k, and consider on $T^\ell A_\varphi$ the partition $\mathcal{Q}_\ell$ induced by $T^\ell \mathcal{A}^n$. (As above, $|T_n'|$ and $|T_n' \hat{\varphi}_\ell|$ are both constant on the atoms of $\mathcal{Q}_\ell$.) In every atom $Q \in \mathcal{Q}_\ell$ there is (by the mean value theorem) a point y_Q so that $|T'(y_Q)| = |T_n'|_Q|$. The points $\{y_Q : Q \in \mathcal{Q}_\ell\}$ form a partition of $T^\ell A_\varphi$ which we shall call $\mathcal{R}_\ell$. One obtains for $\ell = 0, \ldots, k-1$:

$$\mathrm{var}_{T^\ell A_\varphi} \log \frac{|T'|}{|T_n' \hat{\varphi}_\ell|} = \mathrm{Var}_{T^\ell A_\varphi} (\log |T'|, \mathcal{R}_\ell),$$

and

$$\operatorname{diam} \mathcal{R}_\ell \le 2 \operatorname{diam} \mathcal{Q}_\ell \le 2 \operatorname{diam} T^\ell \mathcal{A}^n \le 2 \operatorname{diam} \mathcal{A}^{n-\ell} \le 2\vartheta'^{n-\ell}.$$

Therefore, if we use the estimates (4) and (7) which we deduced above, we obtain:

$$\begin{aligned} \mathrm{var}_{A_\varphi} \log \left| \frac{(T^k)'}{(T_n^k)'} \right| &= \mathrm{var}_{A_\varphi} \sum_{\ell=0}^{k-1} \log \left| \frac{T' T^\ell}{T_n' T_n^\ell} \right| \\ &\ge \mathrm{var}_{A_\varphi} \log \left| \frac{T' T^{k-1}}{T_n' T_n^{k-1}} \right| - \mathrm{var}_{A_\varphi} \sum_{\ell=0}^{k-2} \log \left| \frac{T' T^\ell}{T_n' T_n^\ell} \right| \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{Var}_{T^{k-1}A_\varphi}(\log T', \mathcal{R}_{k-1}) - \sum_{\ell=0}^{k-2} \operatorname{Var}_{T^\ell A_\varphi}(\log T', \mathcal{R}_\ell) \\
&\geq \frac{1}{m} - \frac{6}{m}\frac{\vartheta'}{1-\vartheta'}.
\end{aligned}
$$

If we choose P large enough so that $\frac{6\vartheta'}{1-\vartheta'} \leq \frac{1}{2}$, then

$$
\operatorname{var}_{A_\varphi} \log \left| \frac{(T^k)'}{(T_n^k)'} \right| \geq \frac{1}{2m}.
$$

This finally allows us to estimate the term IV from below by

$$
\mathrm{IV} \geq c_1 \frac{1}{2m} \sum_{\varphi \in S_k} \frac{h_n \varphi}{|(T_n^k)'\tilde{\varphi}|} \geq c_1 \frac{1}{2m} \inf h_n \geq \frac{c_1}{4m} \inf h,
$$

for large enough n and some positive constant c_1. Consequently, since we chose $k = [n/2]$, we obtain

$$
\operatorname{var}(h - h_n) \geq IV - (I + II + III) \geq \frac{c_2}{m} - c(\vartheta^k + \kappa^n + n\vartheta^{n-k}) \geq \frac{c_3}{m(n)} \geq \frac{c_4}{\log n}
$$

for some positive constants c_2, c_3, c_4.

6 Appendix

In this section I would like to present the main ingredients that are necessary to prove Theorem 8 in a more general functional analytic way. That approach as recently become the preferred way to obtain such convergence results and here I would like to take the opportunity to show how the space of functions with uniform variation fits into that framework.

The main part of the argument invokes the two-norm theorem of Ionescu Tulcea-Marinescu which is used to prove the spectral gap for the transfer operator on the space UV_γ. Then Theorem 8 follows using [6] (provided the unitball in UV_γ is precompact which is shown in Lemma 12). In the following two lemmas we thus prove the 'Lasota-Yorke inequality' (Lemma 11) and precompactness of the unitball in UV_γ in BV.

Lemma 11 *Let T be a map on the interval satisfying the conditions (i) to (iv) and so that $T' \in \mathrm{UV}_\gamma$ for some $\gamma > 0$. Then for all $\phi \in \mathrm{UV}_\gamma$ and all large enough k:*

$$
\mathrm{U}_\gamma(\mathcal{L}^k \phi) \leq C_7 |\phi|_\infty + \varsigma^k (\mathrm{U}_\gamma(\phi) + |\phi|_\gamma),
$$

where $\varsigma < 1$ and $C_7 > 0$.

Proof. For an arbitary partition $\mathcal{P}$ (refining the partition $\mathcal{A}$) of the unit interval we obtain for $\vartheta' \in (\vartheta, 1)$ by Lemma 5

$$\begin{aligned}
\mathrm{Var}\,(\mathcal{L}^k\phi, \mathcal{P}) &= \mathrm{Var}\left(\sum_{\varphi\in S_k} \frac{\phi\varphi}{|(T^k)'\varphi|}, \mathcal{P}\right) \\
&\leq \sum_{\varphi\in S_k} \mathrm{Var}\left(\frac{\phi\varphi}{|(T^k)'\varphi|}, \mathcal{P}\right) \\
&\leq I + II,
\end{aligned}$$

where by Lemma 9(I)

$$\begin{aligned}
I &= \sum_{\varphi\in S_k} \sup \frac{1}{|(T^k)'\varphi|} \mathrm{Var}_{A_\varphi}(\phi, T^{-k}\mathcal{P}) + |\phi|_\infty \sum_{\varphi\in S_k} \mathrm{Var}\left(\frac{1}{|(T^k)'\varphi|}, \mathcal{P}\right) \\
&\leq c_1\vartheta'^k \mathrm{Var}\,(\phi, T^{-k}\mathcal{P}) + C_5|\phi|_\infty \sum_{\varphi\in S_k} \sup_{A_\varphi} \frac{1}{|(T^k)'|} (\mathrm{diam}\ \mathcal{P})^\gamma \\
&\leq c_1\vartheta'^k \mathrm{U}_\gamma(\phi)(\mathrm{diam}\, T^{-k}\mathcal{P})^\gamma + c_2|\phi|_\infty (\mathrm{diam}\ \mathcal{P})^\gamma,
\end{aligned}$$

where in the last step we used in the second term that

$$\sup_{A_\varphi} \frac{1}{|(T^k)'|} \leq c_3 \inf_{A_\varphi} \frac{1}{|(T^k)'|}$$

(as T' is γ-Hölder continuous) for some c_3 and that the eigenfunction h is bounded away from 0. For the other term we get

$$II = (\mathrm{diam}\,\mathcal{P})^\gamma \sum_{\varphi\in S_k} \left(\left|\frac{1}{|(T^k)'\varphi|}\right|_\gamma \mathrm{var}\ \phi\varphi + |\phi\varphi|_\gamma \mathrm{var}_{A_\varphi} \frac{1}{|(T^k)'\varphi|} \right).$$

To estimate the Hölder norm of $\frac{1}{|(T^k)'\varphi|}$ we get for any x, x':

$$\begin{aligned}
\left|\frac{1}{|(T^k)'|}(\varphi x) - \frac{1}{|(T^k)'|}(\varphi x')\right| &\leq \frac{1}{|(T^k)'|}(\varphi x) \left|1 - \frac{|(T^k)'|(\varphi x)}{|(T^k)'|(\varphi x')}\right| \\
&\leq \frac{c_3}{|(T^k)'|}(\varphi x) \sum_{j=0}^{k-1} \big|\log|T'T^j\varphi x| - \log|T'T^j\varphi x'|\big| \\
&\leq c_3\vartheta'^k |\log|T'||_\gamma \frac{|x-x'|^\gamma}{1-\rho^{\gamma k}}
\end{aligned}$$

and therefore ($c_4 > 0$)

$$\left|\frac{1}{|(T^k)'\varphi|}\right|_\gamma \leq c_4\vartheta'^k |T'|_\gamma. \tag{8}$$

(This estimate improves on Lemma 10(II) as the right hand side makes the dependency on T explicit.) Since $|\phi\varphi|_\gamma \le \rho^{\gamma k}|\phi|_\gamma$ we get

$$|\phi\varphi|_\gamma \mathrm{var}\, \frac{1}{|(T^k)'\varphi|} \le kC_5\rho^k|\phi|_\gamma \sup_{A_\varphi} \frac{1}{|(T^k)'|},$$

and using Lemma 9(II) we obtain (as $\mathrm{diam}\, T^{-k}\mathcal{P} \le \rho^k \mathrm{diam}\,\mathcal{P}$):

$$\begin{aligned} II &= (\mathrm{diam}\,\mathcal{P})^\gamma \left(c_4\vartheta'^k|T'|_\gamma \sum_{\varphi\in S_k} \mathrm{var}_{A_\varphi}\, \phi + \rho^{\gamma k}kC_5|\phi|_\gamma \sum_{\varphi\in S_k} \sup_{A_\varphi} \frac{1}{|(T^k)'|}\right) \\ &\le c_5(\mathrm{diam}\,\mathcal{P})^\gamma \left(\vartheta'^k \mathrm{var}\ \phi + \rho^{\gamma k}k|\phi|_\gamma\right). \end{aligned}$$

Hence

$$\mathrm{Var}\ (\mathcal{L}^k\phi, \mathcal{P}) \le c_6 \left((\vartheta'\rho^\gamma)^k \mathrm{U}_\gamma(\phi) + \vartheta'^k \mathrm{var}\ \phi + \rho^{\gamma k}k|\phi|_\gamma + |\phi|_\infty\right)(\mathrm{diam}\ \mathcal{P})^\gamma,$$

and since

$$\mathrm{var}\ \phi \le \mathrm{var}\ (\phi, \{I\}) + \mathrm{Var}\ (\phi, \{I\}) \le 2|\phi|_\infty + 2\mathrm{U}_\gamma(\phi)$$

we finally obtain

$$\mathrm{U}_\gamma(\mathcal{L}^k\phi) \le C_7|\phi|_\infty + 2kc_6\varsigma^k(\mathrm{U}_\gamma(\phi) + |\phi|_\gamma),$$

where $C_7 \le 2 + c_6$ and $\varsigma = \min(\vartheta', \rho^\gamma)$. A slightly larger value of ς will make the term kc_8 absorb for large k. □

For Hölder continuous functions on the unit interval it is easily found that in our setting:

$$|\mathcal{L}^k\phi|_\gamma \le c_1|\phi|_\infty + \varsigma^k|\phi|_\gamma,$$

for some constant $c_1 > 0$ and some $\varsigma < 1$ which we can assume to have the same value as in the previous lemma. We thus obtain the estimate that is so central to an application of the theorem of Tulcea and Marinescu, namely

$$\mathrm{U}_\gamma(\mathcal{L}^k\phi) + |\mathcal{L}^k\phi|_\gamma \le C_7\|\phi\|_\gamma + \varsigma^k(\mathrm{U}_\gamma(\phi) + |\phi|_\gamma).$$

Note that since by Lemma 7 $\|T_n\|_\gamma \le 3\|T\|_\gamma$ for all n, the inequality

$$\mathrm{U}_\gamma(\mathcal{L}_n^k\phi) \le C_7|\phi|_\infty + \varsigma^k(\mathrm{U}_\gamma(\phi) + |\phi|_\gamma),$$

holds for all n if we slightly increase the constant C_7.

Lemma 12 *The unitball of* UV_γ *is precompact in* BV.

Proof. Given a sequence $\{h_j \in \mathrm{UV}_\gamma : j = 1, 2, \ldots\}$ so that $\|h_j\|_\gamma \le 1\ \forall j$ we have to show that there is a subsequence that converges in the Bounded Variation norm. For simplicity's sake let us assume that all the h_j are continuous. Since a subsequence converges in the supremum norm we can without loss of generality assume that h_j converges in the supremum norm. Hence, for any $\varepsilon > 0$ there exists $N(\varepsilon)$ so that $|h_j - h_k|_\infty < \varepsilon$ for all $j, k \ge N(\varepsilon)$. Let us denote by $|\mathcal{P}|$ the number of intervals in a partition $\mathcal{P}$ and let $\mathcal{P}$ be a partition so that $\operatorname{diam}\mathcal{P} < \varepsilon^{\frac{1}{1+\gamma}}$ and $|\mathcal{P}| < 2/\operatorname{diam}\mathcal{P} < 3\varepsilon^{-\frac{1}{1+\gamma}}$. Since by assumption $\operatorname{var} h_j - \operatorname{var}(h_j, \mathcal{P}) \le (\operatorname{diam}\mathcal{P})^\gamma\ \ \forall\, j$, we get

$$\begin{aligned}
\operatorname{var}(h_j - h_k) &\le \operatorname{var}(h_j - h_k, \mathcal{P}) + 2(\operatorname{diam}\mathcal{P})^\gamma \\
&\le |h_j - h_k|_\infty |\mathcal{P}| + 2(\operatorname{diam}\mathcal{P})^\gamma \\
&\le \varepsilon|\mathcal{P}| + 6\varepsilon^{\frac{\gamma}{1+\gamma}} \\
&\le c_1 \varepsilon^{\frac{\gamma}{1+\gamma}}
\end{aligned}$$

for all $j, k \ge N(\varepsilon)$. Hence h_j converges in BV. □

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