

Entrance time and Rényi entropy

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For ergodic systems with generating partitions, the well-known result of Ornstein and Weiss shows that the exponential growth rate of the recurrence time is almost surely equal to the metric entropy. Here we look at the exponential growth rate of entrance times, and show that it equals the entropy, where the convergence is in probability in the product measure. This is however under the assumptions that the limiting entrance times distribution exists almost surely. This condition looks natural in the light of an example by Shields in which the limsup in the exponential growth rate is infinite almost everywhere but where the limiting entrance times do not exist. We then also consider ϕ -mixing systems and prove a result connecting the Rényi entropy to sums over the entrance times orbit segments.

Keywords: Growth rate of entrance times; ϕ -mixing; Rényi entropy.

1. Introduction

Let T be a map on a space Ω , then $\{T^i(x)\}_{i=0}^{\infty}$ defines the orbit of $x \in \Omega$. For a set $A \subset \Omega$, the *entrance time* τ_A of a point x into set A refers to the time that takes for the orbit of x to first enter set A . In particular, if $x \in A$, τ_A refers to the *return time* of the point x : the time that it takes the orbit of x to return for the first time to set A . For invariant probability measures μ the Poincaré Recurrence Theorem

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states that a point in a positive measure set returns to that set almost surely finite. In other words, $\tau_A(x) < \infty$ for almost every x in A , provided $\mu(A) > 0$. In 1946, this result was quantified by Kac who showed that for ergodic measures the expected return time is the reciprocal of the measure of the return set. If the space Ω has a generating partition \mathcal{A} then in 1993, Ornstein and Weiss [10] proved for ergodic measures μ that $\frac{1}{n} \log \tau_n$ converges to the entropy h_μ almost surely, where the n th recurrence time $\tau_n(x) = \tau_{A_n(x)}(x)$ measures the time for x to return to its initial n -cylinder $A_n(x)$.

Intuitively, the entrance time should behave similar to the return time in ergodic systems, as in such systems when a point x travels long enough it tends to forget where it started. If we assume Ω has a partition \mathcal{A} , then it is natural to consider the exponential growth rate of entrance times to the n -cylinders $A_n(z)$ centred at an arbitrary point z . However, Shields [14] in 1992 constructed an example in which $\frac{1}{n} \log \tau_{A_n(z)}(x)$ does not converge for almost every x . In fact the lim sup goes to infinity almost surely. Here we impose an additional assumption in order to get convergence in probability to the metric entropy. We require that the limiting entrance times distributions exist almost everywhere.

We then also give a condition under which the convergence is almost surely. We then also look at ϕ -mixing measures and show that they satisfy this condition and thus have almost sure convergence of exponential growth rate of entrance times. In the last theorem we consider the Rényi entropy which was first introduced by Alfréd Rényi [12] in 1961 in order to generalise the Shannon entropy. Here we generalise a result of Ko [8] which had been proven for return times to entrance times. For ϕ -mixing systems we obtain in Theorem 5 a relationship between entrance time and the Rényi entropy.

In Sec. 2, we state definitions, basic facts and the four main theorems that we will prove in this paper. Theorem 1 proves the convergence of the entrance time in probability while Theorem 2 proves the almost convergence of the entrance time under an additional assumption. Theorem 4 verifies that additional assumption for ϕ -mixing measures. Theorem 5 considers the sum of measures of n -cylinders visited by a point along its orbit until it enters a set, and proves that it converges to a constant in terms of the Rényi entropy and metric entropy for ϕ -mixing systems. The proofs of Theorems 1 and 2 are given in Sec. 3, the proof of Theorem 4 is in Sec. 4 while the proof of Theorem 5 is given in Sec. 5.

2. Main Results

Let Ω be a space with a probability measure μ and $T : \Omega \rightarrow \Omega$ be a measurable map. We assume μ is T -invariant and ergodic. Let $\mathcal{A} = \{\mathcal{P}_i\}$ be a generating partition (finite or countably infinite) and denote by $\mathcal{A}^n = \bigvee_{i=0}^{n-1} T^{-i} \mathcal{A} = \{\bigcap_{0 \leq i \leq n-1} T^{-i}(\mathcal{P}_{j_i}) : \mathcal{P}_{j_i} \in \mathcal{A}\}$ its n th join. The elements of \mathcal{A}^n are referred to as n -cylinders. We denote by $A_n(x) \in \mathcal{A}^n$ the n -cylinder which contains the point $x \in \Omega$.

The theorem of Shannon–McMillan–Breiman (see e.g. [9]) states that for any T -invariant ergodic probability measure μ and generating partition \mathcal{A} of Ω ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\log \mu(A_n(x))| = h_\mu \quad (1)$$

for almost every $x \in \Omega$, where h_μ is the measure theoretic entropy of μ . This asymptotic formula was first proven by Shannon [13] in 1948 for stationary Markov chains and then subsequently strengthened by McMillan and Breiman to its present form for finite alphabets and then extended to countably infinite alphabets (with finite entropy) by Chung [3] in 1961 and Carleson [2] in 1958. In other words, the measure of the n -cylinder which contains x decays exponentially with rate roughly the metric entropy.

For any $x \in \Omega$ and set $A \subset \Omega$, let us define now the *entrance time* of x into the set A by

$$\tau_A(x) = \min\{i \geq 1 : T^i(x) \in A\}.$$

We call $\tau_n(x) = \tau_{A_n(x)}(x)$ the n th *recurrence time* of x ; it is the first time that x returns to the n -cylinder which contains x . Ornstein and Weiss proved in [10] for finite partition, and in [11] for countably infinite partition (provided that h_μ is finite) that for almost every x ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \tau_n(x) = h_\mu \quad (2)$$

assuming μ is ergodic. Intuitively, the entrance time $\tau_{A_n(z)}(x)$ should behave similarly to the recurrence time $\tau_n(z) = \tau_{A_n(z)}(z)$ as in (2), since when points travel a long enough time in ergodic systems they tend to forget where they start and hence whether starting at the point x or z should not matter. However, Shields constructed in 1992 an example of a dynamical system in which the entrance time fails to converge [14]. Here we prove that $\frac{1}{n} \log \tau_{A_n(z)}(x)$ converges in probability to h_μ provided the system has an almost sure entrance times distribution.

In the following we adopt probability notations that for events $A, B \subset \Omega$ we denote $\mu(A)$ by $\mathbb{P}(A)$ and $\mu_B(A) = \mu(B \cap A)/\mu(B)$ by $\mathbb{P}_B(A)$ (assuming $\mu(B) > 0$). For $z \in \Omega$, $n \in \mathbb{N}$ and $t > 0$, put

$$F_z^n(t) = \mathbb{P} \left(\tau_{A_n(z)} \geq \frac{t}{\mu(A_n(z))} \right) = \mu \left(\left\{ x \in \Omega : \tau_{A_n(z)}(x) \geq \frac{t}{\mu(A_n(z))} \right\} \right)$$

and if $B = A_n(z)$ we put

$$F_B(t) = F_z^n(t).$$

We shall require that the limit $\lim_{n \rightarrow \infty} F_z^n$ exists almost everywhere. For a number of classes of positive entropy systems this limit is e^{-t} a.s. There are however examples of ergodic zero entropy systems that have other limiting distributions.

The following two theorems prove convergences of the entrance time: Theorem 1 proves the existence of the limit and convergence in probability under the assumption that the limiting distribution of entrance (or return) times exists almost everywhere. Theorem 2 gives a sufficient condition under which the convergence is almost

sure. Let us note that there are many examples when the limiting entrance/return times do not exist. The example of Shields is one of them. Also, Downarowicz [5] has given examples when the limiting distribution exists along subsequences of full density and where the limit can be made to decay arbitrarily slowly, in particular so slow as to violate the condition in Theorem 2.

Theorem 1. *Suppose for almost every $z \in \Omega$ and for $t \geq 0$, $\lim_{n \rightarrow \infty} F_z^n(t) = F_z(t)$ exists and $F_z(t) \rightarrow 0$ as $t \rightarrow \infty$. Then $\frac{1}{n} \log \tau_{A_n(z)}(x)$ converges to h_μ in probability as n goes to ∞ .*

Theorem 2. *Suppose μ is a T -invariant ergodic probability measure on Ω , and for all small enough $\epsilon > 0$ we have*

$$\sum_{n=1}^{\infty} \int_{\Omega} F_z^n(e^{n\epsilon}) d\mu(z) < \infty.$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \tau_{A_n(z)}(x) = h_\mu$$

for $\mu \times \mu$ -almost every $(x, z) \in \Omega \times \Omega$.

Remark. (i) Let us note that the summability condition of Theorem 2 is only required to get the upper bound on the limit. By Lemma 6 we get the lower bound on the limit almost surely for all ergodic measures.

(ii) Although the recurrence time $\tau_n(x) = \tau_{A_n(x)}(x)$ is a special case of the return time, Theorem 2 does not imply the asymptotic formula in (2) since the above convergence is true for $\mu \times \mu$ -almost every (x, z) which does not imply that it applies to points on the diagonal $x = z$ as the diagonal has measure 0 in the product measure.

The remainder of the paper looks at a situation in which the hypothesis of Theorem 2 is satisfied. We consider systems with some mixing property.

Definition 3. We say an invariant measure μ is ϕ -mixing if there exists a decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{R}$ so that

$$\frac{|\mu(A \cap T^{-(n+i)}(B)) - \mu(A)\mu(B)|}{\mu(A)} \leq \phi(i) \tag{3}$$

for all $A \in \mathcal{A}^n$, all $B \in \sigma(\mathcal{A}^*)$, where $\mathcal{A}^* = \cup_{n=1}^{\infty} \mathcal{A}^n$ and for all $n \in \mathbb{N}$.

The following two theorems will moreover assume that ϕ is summable, that is $\sum_{i=1}^{\infty} \phi(i) < \infty$. Let us note that the limiting entrance times distribution $F_z(t)$ for ϕ -mixing measures (with summable ϕ) is exponential almost everywhere [1], i.e. $F_z(t) = e^{-t}$ for μ -almost every $z \in \Omega$. This includes in particular measures of maximal entropy and equilibrium states for Hölder continuous potential on Axiom A systems which are ψ -mixing at an exponential rate.

Theorem 4. *Suppose μ is a T -invariant ϕ -mixing measure of Ω with summable ϕ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \tau_{A_n(z)}(x) = h_\mu$$

for $\mu \times \mu$ -almost every $(x, z) \in \Omega \times \Omega$.

For the final result we will also require that the (countably infinite) partition $\mathcal{A} = \{\mathcal{P}_i\}_{i=1}^\infty$ has an *exponentially decaying tail* if

$$\mu \left(\bigcup_{i \geq j} \mathcal{P}_i \right) = \mathcal{O}(\delta^j) \tag{4}$$

for all j and for some $\delta < 1$. If $|\mathcal{A}|$ is finite then (4) is trivially satisfied.

For $s > 0$, put

$$Z_n(s) = \sum_{A_n \in \mathcal{A}^n} \mu(A_n)^{1+s}$$

and define the *Rényi Entropy Function* [12] on $(0, \infty)$ by

$$R(s) = \lim_{n \rightarrow \infty} \frac{1}{sn} |\log Z_n(s)|$$

if the limit exists. For larger values of s , the Rényi entropy is weighted towards highest probability events. Moreover, for the value $s = 0$, the Rényi entropy typically coincides with the Shannon entropy. The Rényi entropy exists as a uniform limit in weakly ψ -mixing systems [6] and a pointwise limit under weaker assumption [8].

Theorem 5. *Suppose $T : \Omega \rightarrow \Omega$ is measurable, μ is T invariant and ϕ -mixing with summable ϕ , and \mathcal{A} has exponential tails. Suppose the Rényi entropy $R(s)$ exists for $s > 0$. Then for $\mu \times \mu$ every $(x, z) \in \Omega \times \Omega$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^{\tau_{A_n(z)}(x)} \mu(A_n(T^i(x)))^s = h_\mu - sR(s).$$

This generalises a previous result of Ko [8] in which z was assumed to be equal to x . Obviously, (3) ensures the ergodicity of μ . Furthermore, (3) implies the exponential decay of cylinders and this ensures that the metric entropy h_μ is positive. The summability of ϕ is needed to estimate the variance of the hitting time function (see Sec. 5.2). The condition (4) in particular implies that h_μ is finite (see Lemma 4 of [8]). It also allows us to control the “tail” of the partition \mathcal{A}^n in the proof of Lemma 14. From now on we will abbreviate $\tau_{A_n(z)}(x)$ by $\tau_n^z(x)$ for convenience.

3. Convergence of Entrance Time

We first prove the lower bound of Theorems 1 and 2.

Lemma 6. *Suppose μ is a T -invariant ergodic probability measure of Ω . Then*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \tau_n^z(x) \geq h_\mu$$

for $\mu \times \mu$ -almost every $(x, z) \in \Omega \times \Omega$.

Proof. Let $0 < b < c < h_\mu$, and put

$$\mathbb{E}_n = \{x : \tau_n^z(x) \leq e^{bn}\}.$$

Note that $\mathbb{E}_n = \bigcup_{j=1}^{\lfloor e^{bn} \rfloor} T^{-j}(A_n(z))$. Then we have,

$$\mu(\mathbb{E}_n) \leq \sum_{j=1}^{\lfloor e^{bn} \rfloor} \mu(T^{-j}(A_n(z))) = \mu(A_n(z))e^{bn}.$$

By (1), $\mu(A_n(z)) \leq e^{-nc}$ for almost every z . Therefore, $\mu(\mathbb{E}_n) \leq e^{-(c-b)n}$, summable on n . By the Borel Cantelli lemma, for almost every z , $\mu(\limsup \mathbb{E}_n) = 0$. In other words, this implies that for almost every z , the set of initial conditions x for which the return times to $A_n(z)$ are smaller than e^{bn} infinitely often have μ measure 0. This implies

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \tau_n^z(x) \geq h_\mu$$

for almost every x . □

Remark. Note that in the proof above we showed that for any $\varepsilon > 0$ and almost every z ,

$$\lim_{n \rightarrow \infty} \mu \left(\left\{ x : \frac{1}{n} \log \tau_n^z(x) \leq h_\mu - \varepsilon \right\} \right) = 0,$$

which is equivalent to

$$\lim_{n \rightarrow \infty} \mu \times \mu \left(\left\{ (x, z) : \frac{1}{n} \log \tau_n^z(x) \leq h_\mu - \varepsilon \right\} \right) = 0.$$

To complete the proof of Theorems 1 and 2, we obtain the other side of the inequality in Lemma 6 under certain assumptions. One might have attempted to show this by only assuming that the measure μ is T -invariant and ergodic. However, Shields [14] constructed an example of a dynamical system (on a four-element subshift) in which

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \tau_n^z(x) = \infty$$

for $\mu \times \mu$ -almost every $(x, z) \in \Omega \times \Omega$.

Lemma 7. *Suppose for all small enough $\epsilon > 0$ and $\delta > 0$,*

$$\lim_{n \rightarrow \infty} \mu(\{z \in \Omega : F_z^n(e^{n\epsilon}) > \delta\}) = 0.$$

Then $\frac{1}{n} \log \tau_n^z(x)$ converges to h_μ in probability as $n \rightarrow \infty$.

Proof. Let $\delta > 0$, $b > h_\mu$ and $\mathbb{D}_n = \{(x, z) \in \Omega \times \Omega : \tau_n^z(x) > e^{nb}\}$. We want to show that $\mu \times \mu(\mathbb{D}_n)$ is bounded from above by δ for large enough n . Further let $\epsilon \in (0, b - h_\mu)$ and $\delta' = b - (h_\mu + \epsilon)$. Put

$$\begin{aligned} \Omega_n &= \{z \in \Omega : F_z^n(e^{n\epsilon}) \leq \delta/3\}, \\ \mathcal{S}_n &= \{B \in \mathcal{A}^n : \mu(B) \geq e^{-n(h_\mu + \delta')}\} \end{aligned}$$

and

$$S_n = \bigcup_{B \in \mathcal{S}_n} B.$$

By hypothesis and (1), we can choose n large enough so that $\mu(\Omega_n^c) < \delta/3$ and $\mu(S_n^c) < \delta/3$. Put

$$\bar{\Omega}_n = \{B \in \mathcal{A}^n : B \cap \Omega_n \neq \emptyset\}.$$

As $F_z^n(t)$ is locally constant on n -cylinders B , $F_B(e^{n\epsilon}) \leq \delta/3$ for $B \in \bar{\Omega}_n$. Note also that $F_B(t)$ decreases as t increases; therefore, $F_B(\mu(B)e^{nb}) \leq F_B(e^{n\epsilon})$. It follows that for large enough n

$$\begin{aligned} \mu \times \mu(\mathbb{D}_n) &= \sum_{B \in \mathcal{A}^n} \mu(B) \mathbb{P}(\tau_B \geq e^{nb}) = \sum_{B \in \mathcal{A}^n} \mu(B) F_B(\mu(B)e^{nb}) \\ &= \sum_{B \in \Omega_n^c} \mu(B) F_B(\mu(B)e^{nb}) + \sum_{B \in \bar{\Omega}_n} \mu(B) F_B(\mu(B)e^{nb}) \\ &< \mu(\Omega_n^c) + \sum_{B \in \mathcal{S}_n^c} \mu(B) F_B(\mu(B)e^{nb}) + \sum_{B \in \bar{\Omega}_n \cap \mathcal{S}_n} \mu(B) F_B(\mu(B)e^{nb}) \\ &< \frac{\delta}{3} + \mu(S_n^c) + \sum_{B \in \bar{\Omega}_n \cap \mathcal{S}_n} \mu(B) F_B(e^{n\epsilon}) \\ &< \frac{2\delta}{3} + \frac{\delta}{3} \sum_{B \in \bar{\Omega}_n \cap \mathcal{S}_n} \mu(B) \leq \delta. \end{aligned}$$

As the above is true for any $b > h_\mu$, we showed for any $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \mu \times \mu \left(\left\{ (x, z) : \frac{1}{n} \log \tau_n^z(x) \geq h_\mu + \epsilon \right\} \right) = 0.$$

Together with the remark under Lemma 6, the proof is complete. □

Similar to the entrance time distribution, for $z \in \Omega$, $n \in \mathbb{N}$ and $t > 0$ we define the *return time distribution* as

$$\begin{aligned} \tilde{F}_z^n(t) &= \mathbb{P}_{A_n(z)} \left(\tau_{A_n(z)} \geq \frac{t}{\mu(A_n(z))} \right) \\ &= \mu \left(\left\{ x \in A_n(z) : \tau_n^z(x) \geq \frac{t}{\mu(A_n(z))} \right\} \right) / \mu(A_n(z)) \end{aligned}$$

assuming $\mu(A_n(z)) > 0$ and if $B = A_n(z)$ we put

$$\tilde{F}_B(t) = \tilde{F}_z^n(t).$$

By [7] the entrance times distribution F_B and the return times distribution \tilde{F}_B are related by the identity $F_B(t) = \int_t^\infty \tilde{F}_B(s) ds$.

Proof of Theorem 1. Let β, ϵ and δ be positive. By Lemma 7, we want to show that

$$\lim_{n \rightarrow \infty} \mu(\{z \in \Omega : F_z^n(e^{n\epsilon}) > \delta\}) = 0.$$

Put $V_N = \{z \in \Omega : F_z(N) \leq \delta/2\}$. Since $F_z(t)$ decreases to 0 by assumption, there exists $K = N_{\delta, \beta}$ such that $\mu(V_K^c) < \beta/2$. Put $U_n = \{z \in \Omega : |F_z^n(K) - F_z(K)| \leq \delta/2\}$. Since F_z^n converges to F_z for almost every z , when n is large enough, we have $\mu(U_n^c) < \beta/2$, and $e^{n\epsilon} > K$. For $z \in V_K \cap U_n$, we get

$$F_z^n(e^{n\epsilon}) \leq F_z^n(K) \leq F_z(K) + \delta/2 < \delta.$$

This shows for large n ,

$$\mu(z \in \Omega : F_z^n(e^{n\epsilon}) > \delta) \leq \mu(V_K^c) + \mu(U_n^c) < \beta/2 + \beta/2 = \beta,$$

and the proof is complete. □

Now we turn to prove the almost sure convergence of the entrance time.

Proof of Theorem 2. Let $b > h_\mu, \epsilon \in (0, b - h_\mu)$ and $\delta = b - (h_\mu + \epsilon)$. We claim that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \tau_n^z(x) \leq h_\mu$$

for $\mu \times \mu$ -almost every $(x, z) \in \Omega \times \Omega$. Put

$$S_n = \{B \in \mathcal{A}^n : \mu(B) \geq e^{-n(h_\mu + \delta)}\}$$

and

$$S_n = \bigcup_{B \in S_n} B.$$

Then as $F_B(t)$ is decreasing, if we put $\mathbb{D}_n = \{(x, z) \in \Omega \times \Omega : \tau_n^z(x) > e^{nb}\}$, we have

$$\begin{aligned} \mu \times \mu(\mathbb{D}_n \cap (\Omega \times S_n)) &= \sum_{B \in S_n} \mu(B) F_B(\mu(B) e^{nb}) \\ &\leq \sum_{B \in S_n} \mu(B) F_B(e^{-n(h_\mu + \delta)} e^{nb}) \\ &= \sum_{B \in S_n} \mu(B) F_B(e^{n\epsilon}) \\ &\leq \int_\Omega F_z^n(e^{n\epsilon}) d\mu(z) \end{aligned}$$

which is summable by our hypothesis. Applying the Borel Cantelli lemma

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} (\mathbb{D}_n \cap (\Omega \times S_n)) \right) = 0$$

and (1) gives

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} (\Omega \times S_n^c) \right) = 0,$$

because the set $\limsup_{n \rightarrow \infty} S_n^c$ only contains points on which (1) does not converge. Therefore, we conclude the following

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \mathbb{D}_n \right) \leq \mathbb{P} \left(\limsup_{n \rightarrow \infty} (\mathbb{D}_n \cap (\Omega \times S_n)) \right) + \mathbb{P} \left(\limsup_{n \rightarrow \infty} (\Omega \times S_n^c) \right) = 0.$$

As $b > h_\mu$ is arbitrary, our claim is proved. Together with Lemma 6, we proved Theorem 2. \square

Corollary 8. *Suppose for almost every $z \in \Omega$, there exists $F_z(t)$, a decreasing function on $t > 0$, and a summable sequence $a_n > 0$ such that for all small enough $\epsilon > 0$,*

- (i) $\sum_{n=1}^{\infty} \mu(\{z : |F_z^n(e^{n\epsilon}) - F_z(e^{n\epsilon})| > a_n\}) < \infty$ and
- (ii) $\sum_{n=1}^{\infty} \int_{\Omega} F_z(e^{n\epsilon}) d\mu(z) < \infty$.

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \tau_n^z(x) = h_\mu$$

for $\mu \times \mu$ -almost every $(x, z) \in \Omega \times \Omega$.

Proof. In light of Theorem 2, it is sufficient to show that our hypothesis implies

$$\sum_{n=1}^{\infty} \int_{\Omega} F_z^n(e^{n\epsilon}) d\mu(z) < \infty$$

for small enough ϵ . But

$$\begin{aligned} \int_{\Omega} F_z^n(e^{n\epsilon}) d\mu(z) &\leq \int_{\Omega} |F_z^n(e^{n\epsilon}) - F_z(e^{n\epsilon})| d\mu(z) + \int_{\Omega} F_z(e^{n\epsilon}) d\mu(z) \\ &\leq 2\mu(\{z : |F_z^n(e^{n\epsilon}) - F_z(e^{n\epsilon})| > a_n\}) + a_n + \int_{\Omega} F_z(e^{n\epsilon}) d\mu(z). \end{aligned}$$

The three terms on the right-hand side above are all summable by our hypothesis, and we are done. \square

4. Proof of Theorem 4

We shall need the following result of Abadi [1, Theorem 1]). The following is a simplified version.

Lemma 9. Let μ be a ϕ -mixing T -invariant probability measure such that ϕ is summable. Then there exist constants $M > 0, K_9 < \infty$ such that

$$\mathbb{P}\left(\tau_A > \frac{t}{\mu(A)}\right) \leq e^{-Mt} + K_9(n\mu(A) + \phi(n))$$

for all $A \in \mathcal{A}^n$ and all $n \in \mathbb{N}$.

Proof of Theorem 4. We have to prove that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \tau_n^z(x) = h_\mu$$

exists for $\mu \times \mu$ -almost every $(x, z) \in \Omega \times \Omega$ under the assumption that μ is ϕ -mixing and $\sum_i \phi(i) < \infty$. By Theorem 2, we need to show that

$$\sum_{n=1}^{\infty} \int_{\Omega} F_z^n(e^{n\epsilon}) d\mu(z) < \infty$$

for any small enough ϵ . It is well known that for a ϕ -mixing system, there exists $r > 0$ such that $\mu(A) \leq e^{-rn}$ for all n and n -cylinder $A \in \mathcal{A}^n$. Moreover by Lemma 9 we have

$$\mathbb{P}\left(\tau_{A_n(z)} > \frac{t}{\mu(A_n(z))}\right) \leq e^{-Mt} + K_9(n\mu(A_n(z)) + \phi(n))$$

for every $z \in \Omega, n \in \mathbb{N}$ and $t > 0$. Then for any $\epsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{\Omega} F_z^n(e^{n\epsilon}) d\mu(z) &= \int_{\Omega} \sum_{n=1}^{\infty} \mathbb{P}\left(\tau_{A_n(z)} > \frac{e^{n\epsilon}}{\mu(A_n(z))}\right) d\mu(z) \\ &\leq \int_{\Omega} \sum_{n=1}^{\infty} \{e^{-M \exp(n\epsilon)} + K_9 n\mu(A_n(z)) + K_9 \phi(n)\} d\mu(z) \\ &\leq \int_{\Omega} \sum_{n=1}^{\infty} \{e^{-M \exp(n\epsilon)} + K_9 n e^{-rn} + K_9 \phi(n)\} d\mu(z) < \infty \end{aligned}$$

as required. □

5. Proof of Theorem 5

From now on we will assume that the measure μ satisfies the ϕ -mixing property with summable ϕ and the partition \mathcal{A} has an exponentially decaying tail (see (3) and (4)). We separately prove the upper and lower bound on the limit. The upper is quite easy but the lower bound requires a more careful analysis of hitting numbers. For $z \in \Omega$, define

$$D^z := \left\{x : \lim_{n \rightarrow \infty} \frac{1}{n} \log \tau_n^z(x) = h_\mu\right\}$$

and put

$$D := \{z : \mu(D^z) = 1\}.$$

Theorem 4 implies that $\mu(D) = 1$. For $z \in D, \epsilon > 0$ and all $x \in D^z$, we have

$$e^{n(h_\mu - \epsilon)} < \tau_n^z(x) < e^{n(h_\mu + \epsilon)} \tag{5}$$

for large enough n . In the rest of the paper we assume $z \in D$, and for convenience we put $W_n^s(x, z) = \sum_{i=1}^{\tau_n^z(x)} \mu(A_n(T^i(x)))^s$. Note that $W_n^0(x, x) = \tau_n(x)$, and in this case Theorem 5 coincides with (2). Also the case $x = z$ and $s > 0$ of Theorem 5 was proven in [8].

5.1. Proof of the upper bound of the limit in Theorem 5

By the proof of Proposition 2.3 in [4] (see also [8] Proposition 6) for every $\epsilon > 0$ there exists $D_\epsilon \subset \Omega$ with measure 1 such that for $x \in D_\epsilon$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^{\exp(n(h_\mu + \frac{\epsilon}{3}))} \mu(A_n(T^i(x)))^s \leq h_\mu - sR(s) + \epsilon.$$

Also by (5), we know that for $z \in D$ and therefore for all $x \in D^z \cap D_\epsilon$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log W_n^s(x, z) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^{\exp(n(h_\mu + \frac{\epsilon}{3}))} \mu(A_n(T^i(x)))^s \\ &\leq h_\mu - sR(s) + \epsilon. \end{aligned}$$

Finally, as $\mu(\bigcap_{m=1}^\infty D_{1/m} \cap D^z) = 1$, this establishes the upper bound in Theorem 5. □

5.2. Hitting numbers

To prove the lower bound on the limit in Theorem 5 we need estimates on the hitting number

$$N_{U,M}(x) = \sum_{i=0}^M \chi_U \circ T^i(x)$$

of $U \in \sigma(\mathcal{A}^n)$ (unions of n -cylinders), where χ_U is the characteristic function of the set U . $N_{U,M}(x)$ counts the number of times $i \in [0, M]$ that $T^i(x) \in U$. Similarly $\nu_x^z(U) = N_{U, \tau_n^z(x)}(x)$ is the number of times that x hits the set U when it travels along its orbit segment until it returns to $A_n(z)$. Following [4] it was shown in [8] that the variance of the hitting time can be estimated by $\text{Var}(N_{U,M}) \leq c_1 M n \mu(U)$ for a constant c_1 .

The following two lemmas provide us with lower and upper bounds for the hitting time. For $z = x$ these results have been proven in [8] and here we give the modification required for the present more general setting.

Lemma 10. *Let μ be a ϕ -mixing T -invariant measure where $\phi(i)$ is summable and $U_n \in \sigma(\mathcal{A}^n)$, $n = 1, 2, \dots$, be a sequence of sets in $\sigma(\mathcal{A}^n)$. Let $\epsilon > 0$ and assume γ_n is a sequence of positive numbers so that for all n large enough ($C, a, b > 0$)*

constants): Assume one of the following two conditions are satisfied:

- (I) $\mu(U_n) \geq Ce^{-\gamma_n(h_\mu+\epsilon)}$ and $(n - \gamma_n)h_\mu - \epsilon(n + \gamma_n) \geq an^b$,
- (II) $\mu(U_n) \geq Ce^{-\gamma_n(h_\mu-\epsilon)}$ and $(n - \gamma_n)(h_\mu - \epsilon) \geq an^b$.

Then for almost every (x, z) ,

$$N_{U_n, \tau_n^z(x)}(x) \geq \frac{\mu(U_n)}{2} e^{n(h_\mu-\epsilon)}$$

for all n large enough.

Proof. (I) Put $M = [e^{n(h_\mu-\epsilon)}]$. Using the estimate on the variance of $N_{U_n, M}$ and Chebycheff's inequality it was shown in [8] that there exists a set D' , with measure 1, such that, for all $x \in D'$, and for all n large enough, it holds, $\frac{N_{U_n, M}(x)}{M} \geq \mu(U_n)/2$. Since by (5) for $z \in D, x \in D^z \cap D'$, and for n large enough, we have $\tau_n^z(x) > e^{n(h_\mu-\epsilon)}$, and therefore

$$N_{U_n, \tau_n^z(x)}(x) \geq N_{U_n, M}(x) \geq M\mu(U_n)/2 = \mu(U_n)e^{n(h_\mu-\epsilon)}/2.$$

Since $\mu(D^z \cap D') = 1$, the estimate follows.

Part (II) is proven similarly. □

Lemma 11. Let μ be as in Lemma 10 and $U_n \in \sigma(\mathcal{A}^n), n = 1, 2, \dots$, be a sequence of sets. Suppose there exists a constant $C > 0$ so that $\mu(U_n) \geq C$ for all large enough n . Then for $\epsilon > 0$ and for almost every x ,

$$N_{U_n, \tau_n^z(x)}(x) \leq \frac{3\mu(U_n)}{2} e^{n(h_\mu+\epsilon)}$$

for all n large enough.

Proof. For $M = [e^{n(h_\mu+\epsilon)}]$ it was shown in [8] that $|\frac{N_{U_n, M}(x)}{M} - \mu(U_n)| \leq \mu(U_n)/2$ for all n large enough. By (5) for $z \in D, x \in D^z \cap D'$, and for large enough n , we have $\tau_n^z(x) < e^{n(h_\mu+\epsilon)}$, and hence $N_{U_n, \tau_n^z(x)}(x) \leq N_{U_n, M}(x) \leq 3M\mu(U_n)/2 = 3\mu(U_n)e^{n(h_\mu+\epsilon)}/2$ as desired. □

Using (1) and Egoroff's Theorem, there exists a set \mathcal{E} with measure greater than $1/2$ on which $|\log \mu(A_n(x))|/n$ converges to h_μ uniformly as $n \rightarrow \infty$. Define

$$E_n := \{x : A_n(x) \cap \mathcal{E} \neq \emptyset\},$$

the union of those n -cylinders which intersect \mathcal{E} . As $E_n \in \sigma(\mathcal{A}^n)$, let us apply Lemmas 10 and 11 to obtain estimations on the hitting number of E_n .

Corollary 12. For any positive $\epsilon < h_\mu$ and almost every (x, z) ,

- (I) $\nu_x^z(E_n) \geq \frac{\mu(E_n)}{2} e^{n(h_\mu-\epsilon)}$;
- (II) $\nu_x^z(E_n^c) \leq \frac{3\mu(E_n^c)}{2} e^{n(h_\mu+\epsilon)}$

for all n large enough (where $\nu_x^z(U) = N_{U, \tau_n^z(x)}(x)$).

Proof. (I) We use Lemma 10(II) with $U_n = E_n$ and $\gamma_n = n/2$ for any $a > 0$ and $0 < b < 1$. In order to verify the first part of the condition of Lemma 10(II) note that $\mu(E_n) \geq \mu(\mathcal{E}) \geq 1/2$ for all n . Thus $\mu(E_n) \geq e^{-\gamma_n(h_\mu - \epsilon)}$ for all large enough n and Lemma 10(II) gives the desired result. (II) First suppose that $\mu(E_m) < 1$ for some integer m . Since $E_{n+1} \subseteq E_n$ (as $A_{n+1}(x) \subseteq A_n(x)$), we conclude that $\mu(E_n^c) \geq \mu(E_m^c)$ for all $n > m$. Hence by Lemma 11 (with $C = \mu(E_m^c)$) we are done. If $\mu(E_n) = 1$ for all n , then put

$$B = \bigcap_{n \geq 1} \bigcap_{i \geq 0} T^{-i}(E_n).$$

Then $\mu(B) = 1$ and for $x \in B, T^i(x) \notin E_n^c$ for any n . Hence for almost every $x, \nu_x^z(E_n^c) = 0 = \frac{3\mu(E_n^c)}{2}e^{n(h_\mu + \epsilon)}$ for all n . \square

5.3. Proof of the lower bound of the limit in Theorem 5

Let $c, \alpha \in (0, 1)$, and $\epsilon > 0$ be a small number which depends on h_μ and c (which is close to 1), and will be determined later. Define $\tilde{\gamma}_n = n - [n^\alpha]$ and $\Delta = [n^\alpha]$. Denote by $\tilde{A}_n(x) \in \mathcal{A}^{\tilde{\gamma}_n}$ and $\bar{A}_n(x) \in \mathcal{A}^{[cn]}$ the $\tilde{\gamma}_n$ -cylinder and the $[cn]$ -cylinder which contain x respectively. As $n > \tilde{\gamma}_n > cn$ (for n large enough), we have $A_n(x) \subset \tilde{A}_n(x) \subset \bar{A}_n(x)$. In the following we denote an n -cylinder by A_n or A , a $[cn]$ -cylinder by \bar{A}_n or \bar{A} , and a $\tilde{\gamma}_n$ -cylinder by \tilde{A}_n or \tilde{A} . For $\epsilon > 0$, there exists K_ϵ such that for any $n > K_\epsilon$, we have

$$e^{-n(h_\mu + \epsilon)} \leq \mu(A_n(x)) \leq e^{-n(h_\mu - \epsilon)} \tag{6}$$

for $x \in \mathcal{E}$. Note that for $n > K_\epsilon$ and if the n -cylinder $A \subset E_n$, we have $A = A_n(x)$ for some $x \in \mathcal{E}$ and hence $\mu(A)$ satisfies (6). From now on, we assume n is large enough so that $n > \tilde{\gamma}_n > cn > K_\epsilon$ and hence (6) holds with n replaced by $\tilde{\gamma}_n$ and cn . The inequality (6) shows the uniformity property of the measures of cylinders in the sense that when $x \in \mathcal{E}$, we have

$$\mu(\bar{A}_n(x)) \leq e^{-cn(h_\mu - \epsilon)} \leq \mu(A_n(x))e^{(1-c)nh_\mu + 2n\epsilon}. \tag{7}$$

If we put

$$\bar{E}_n = \{x : \bar{A}_n(x) \cap \mathcal{E} \neq \emptyset\},$$

(the union of $[cn]$ -cylinders which intersect \mathcal{E}) then $\mathcal{E} \subseteq E_n \subseteq \bar{E}_n$.

Let $\epsilon = \frac{1-\epsilon}{1+c}h_\mu$ (recall that h_μ is positive, by the comment after (4)), and from now on we choose $\epsilon < \epsilon$. When we let $\epsilon \rightarrow 0$, we have both $\epsilon \rightarrow 0$ and $c \rightarrow 1$. For convenience we also put

$$\begin{aligned} \tilde{E}_n^+ &= \{\tilde{A} \in \mathcal{A}^{\tilde{\gamma}_n} : \mu(\tilde{A}) \geq e^{-\tilde{\gamma}_n(h_\mu - \epsilon)}\}; \\ \tilde{E}_n^- &= \{\tilde{A} \in \mathcal{A}^{\tilde{\gamma}_n} : \mu(\tilde{A}) \leq e^{-\tilde{\gamma}_n(h_\mu - \epsilon)}\}. \end{aligned}$$

According to [8] the lower bound on the limit in Theorem 5 follows immediately from the following two lemmas.

Lemma 13. *There exists some constant K_{13} , which depends only on s , so that for almost every x ,*

$$W_n^s(x, z) \geq e^{K_{13}n\epsilon} e^{nh_\mu} \sum_{\tilde{A} \in \tilde{E}_n^-} \mu(\tilde{A})^{1+s}$$

for all n large enough.

Proof. We proceed in three steps:

(I) We have

$$\begin{aligned} W_n^s(x, z) &\geq \sum_{i=1}^{\tau_n^z(x)} \mu(A_n(T^i(x)))^s \chi_{E_n}(T^i(x)) \\ &\geq e^{s((c-1)nh_\mu - 2n\epsilon)} S_1 \geq e^{-4sn\epsilon} S_1 \end{aligned}$$

as $\epsilon < \epsilon$ and $c < 1$, where $S_1 = \sum_{i=1}^{\tau_n^z(x)} \mu(\bar{A}_n(T^i(x)))^s \chi_{E_n}(T^i(x))$.

(II) Put $S_2 = \sum_{i=1}^{\tau_n^z(x)} \mu(\bar{A}_n(T^i(x)))^s \chi_{\bar{E}_n}(T^i(x))$. Then $S_1 \leq S_2$ as $E_n \subseteq \bar{E}_n$. If $T^i(x) \in E_n$, then by (6) we have $\mu(\bar{A}_n(T^i(x))) \geq e^{-cn(h_\mu + \epsilon)}$ and consequently.

$$\begin{aligned} S_1 &\geq \sum_{i=1}^{\tau_n^z(x)} e^{-scn(h_\mu + \epsilon)} \chi_{E_n}(T^i(x)) \\ &= \nu_x^z(E_n) e^{-scn(h_\mu + \epsilon)} \\ &\geq \frac{\mu(E_n)}{2} e^{n(h_\mu - \epsilon)} e^{-scn(h_\mu + \epsilon)}. \end{aligned} \tag{8}$$

Meanwhile, by Corollary 12(II) we also have

$$\begin{aligned} S_2 - S_1 &= \sum_{i=1}^{\tau_n^z(x)} \mu(\bar{A}_n(T^i(x)))^s \chi_{\bar{E}_n \setminus E_n}(T^i(x)) \\ &\leq \sum_{i=1}^{\tau_n^z(x)} e^{-scn(h_\mu - \epsilon)} \chi_{E_n^c}(T^i(x)) \\ &= \nu_x^z(E_n^c) e^{-scn(h_\mu - \epsilon)} \\ &\leq \frac{3\mu(E_n^c)}{2} e^{n(h_\mu + \epsilon)} e^{-scn(h_\mu - \epsilon)}. \end{aligned} \tag{9}$$

Since $\mu(E_n^c) < 1/2 < \mu(E_n)$, we get

$$S_2 - S_1 \leq 3 \left[\frac{1}{2} \mu(E_n) e^{n(h_\mu - \epsilon)} e^{-scn(h_\mu + \epsilon)} \right] e^{2n\epsilon} e^{2scn\epsilon}.$$

By (8) the quantity in the bracket of the above inequality is less than S_1 . Consequently $S_2 - S_1 \leq 3e^{2n\epsilon(1+s\epsilon)}S_1$ and

$$S_2 \leq e^{4n\epsilon(1+s\epsilon)}S_1 \leq e^{4(1+s)\epsilon n}S_1$$

as $c < 1$ and $\epsilon < \varepsilon$.

(III) We also have

$$S_2 = \sum_{\bar{A} \subset \bar{E}_n} \nu_x^z(\bar{A})\mu(\bar{A})^s$$

using the counting function $\nu_x^z(\bar{A})$ for which we have bounds by Lemma 10(I): since for $\bar{A} \subset \bar{E}_n$, one has $\mu(\bar{A}) \geq e^{-cn(h_\mu + \epsilon)}$, and hence the first condition of Lemma 10(I) is fulfilled (with $\gamma_n = cn$). The second condition, $(n - cn)h_\mu - \epsilon(n + cn) = ((1 - c)h_\mu - \epsilon(1 + c))n > 0$ follows from $\epsilon < \varepsilon = \frac{(1 - c)}{(1 + c)}h_\mu$. Therefore, by Lemma 10(I),

$$S_2 \geq \sum_{\bar{A} \subset \bar{E}_n} \frac{\mu(\bar{A})}{2} e^{n(h_\mu - \epsilon)} \mu(\bar{A})^s = \frac{1}{2} e^{n(h_\mu - \epsilon)} \sum_{\bar{A} \subset \bar{E}_n} \mu(\bar{A})^{1+s}$$

and we conclude as in [8] that $S_2 \geq e^{n(h_\mu - \varepsilon)}S_3$, where $S_3 = \sum_{\bar{A} \subset \bar{E}_n} \mu(\bar{A})^{1+s}$. Finally we use the fact from [8] that $S_3 \geq e^{-2sn\varepsilon}S_4$, where $S_4 = \sum_{\tilde{A} \in \tilde{E}_n^-} \mu(\tilde{A})^{1+s}$.

We thus obtain for some constant c_1 independent of ε :

$$W_n^s(x, z) \geq e^{-c_1 n\varepsilon} e^{nh_\mu} S_4. \quad \square$$

Lemma 14. *There exists some constant K_{14} , which depends only on s , so that for almost every x ,*

$$W_n^s(x, z) \geq e^{-K_{14}n\varepsilon} e^{nh_\mu} \sum_{\tilde{A} \in \tilde{E}_n^+} \mu(\tilde{A})^{1+s}$$

for all n large enough.

Proof. Let $\beta > 1$ and define $(\Delta = [n^\alpha])$

$$\mathcal{G}_n = \{x : \mu(A_n(x)) \geq \exp(-\Delta^\beta)\mu(\tilde{A}_n(x))\}.$$

Then \mathcal{G}_n is a union of n -cylinders, since by definition if $x \in \mathcal{G}_n$, we have $A_n(x) \subseteq \mathcal{G}_n$. Moreover, put

$$F_{j,\Delta} = \bigcap_{i=1}^{\Delta} T^{-i} \left(\bigcup_{m=1}^{j-1} \mathcal{P}_m \right)$$

for $j \in \mathbb{N}$. Note that if $x \in F_{j,\Delta}$, then for $1 \leq i \leq \Delta$, $T^i(x) \notin \mathcal{P}_k$ for all $k \geq j$. The set $F_{j,\Delta}$ is a finite union of Δ -cylinders and consists of point x that do not hit the “tail” $\bigcup_{m=j}^{\infty} \mathcal{P}_m$ for the first Δ iterates. Obviously, $F_{j,\Delta} \subseteq F_{j+1,\Delta}$ and $F_{j,\Delta'} \subseteq F_{j,\Delta}$ for $\Delta' > \Delta$. We will consider the sets $F_{k_n,\Delta}$ for $k_n = [n^t], t > 1$.

We make use of \mathcal{G}_n to compare the summands $\mu(A_n(T^i(x)))$ and $\mu(\tilde{A}_n(T^i(x)))$ as follows:

$$\begin{aligned}
 W_n^s(x, z) &\geq \sum_{i=1}^{\tau_n^z(x)} \mu(A_n(T^i(x)))^s \chi_{\mathcal{G}_n}(T^i(x)) \\
 &\geq e^{-s\Delta^\beta} \sum_{i=1}^{\tau_n^z(x)} \mu(\tilde{A}_n(T^i(x)))^s \chi_{\mathcal{G}_n}(T^i(x)) \\
 &= e^{-s\Delta^\beta} \sum_{\tilde{A} \in \mathcal{A}^{\tilde{\gamma}_n}} \mu(\tilde{A})^s \nu_x^z(\tilde{A} \cap \mathcal{G}_n) \\
 &\geq e^{-s\Delta^\beta} \sum_{\tilde{A} \in \tilde{E}_n^+} \mu(\tilde{A})^s \nu_x^z(\tilde{A} \cap \mathcal{G}_n). \tag{10}
 \end{aligned}$$

The first inequality is true since $0 \leq \chi_{\mathcal{G}_n} \leq 1$. The second inequality follows from the definition of \mathcal{G}_n . The last inequality is valid since we restrict the sum to a subcollection of \tilde{A} . In order to apply Lemma 10(II) to obtain a lower bound of $\nu_x^z(\tilde{A} \cap \mathcal{G}_n)$, we write

$$\begin{aligned}
 \mu(\tilde{A} \cap \mathcal{G}_n) &\geq \mu(\tilde{A} \cap T^{-\tilde{\gamma}_n}(F_{k_n, \Delta}) \cap \mathcal{G}_n) \\
 &= \left[\frac{\mu(\tilde{A} \cap T^{-\tilde{\gamma}_n}(F_{k_n, \Delta}))}{\mu(\tilde{A})} - \frac{\mu(\tilde{A} \cap T^{-\tilde{\gamma}_n}(F_{k_n, \Delta}) \cap \mathcal{G}_n^c)}{\mu(\tilde{A})} \right] \mu(\tilde{A}). \tag{11}
 \end{aligned}$$

In [8] it was shown that the quantity inside the bracket goes to 1 as n tends to ∞ . (The first term converges to 1 and the second term converges to 0.) Thus $\frac{\mu(\tilde{A} \cap \mathcal{G}_n)}{\mu(\tilde{A})} \rightarrow 1$ as $n \rightarrow \infty$ and in particular for large enough n , $\mu(\tilde{A} \cap \mathcal{G}_n) \geq \mu(\tilde{A})/2 \geq e^{-\tilde{\gamma}_n(h_\mu - \epsilon)}/2$ if $\tilde{A} \in \tilde{E}_n^+$. Let us now apply Lemma 10(II) where we put $\gamma_n = \tilde{\gamma}_n$. Hence

$$\nu_x^z(\tilde{A} \cap \mathcal{G}_n) \geq \frac{\mu(\tilde{A} \cap \mathcal{G}_n)}{2} e^{n(h_\mu - \epsilon)} \geq \frac{\mu(\tilde{A})}{4} e^{n(h_\mu - \epsilon)}.$$

From (10) one thus obtains

$$W_n^s(x, z) \geq \frac{\exp(-s\Delta^\beta)}{4} \exp(-n\epsilon) \exp(nh_\mu) \sum_{\tilde{A} \in \tilde{E}_n^+} \mu(\tilde{A})^{1+s}.$$

Now let $\beta \in (1, 1/\alpha)$ so that $\alpha\beta < 1$, then n dominates $\Delta^\beta = [n^\alpha]^\beta$ and hence there exists K_{14} so that for large enough n , $e^{-n\epsilon} e^{-s\Delta^\beta} / 4 \geq e^{-K_{14}n^\epsilon}$. \square

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