The Topological Entropy of Generalized Polygon Exchanges

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1 Summary

We introduce the notion of a generalized polyhedron exchange (g.p.e.), and define the topological entropy of a g.p.e.. By our definition, the g.p.e. is a generalization to an arbitrary dimension of the interval exchange transformation [1]. Many natural dynamical systems are g.p.e.'s. For instance, the Poincare return map of a polygonal billiard, [4], is a g.p.e. in two dimensions (a generalized polygon exchange).

The (topological) entropy of a g.p.e., T, is the growth rate, under iterations, of the number of atoms in the defining paritition, $\mathcal{P}_n$, for T^n. Our main result, Theorem 3 below, says that if the length of the boundary of $\mathcal{P}_n$ grows exponentially fast at a given rate ϑ (and if two other, technical, conditions are satisfied), the entropy of T is less or equal to ϑ. In particular it follows that if the length of the boundary grows subexponentially, then entropy of T is zero.

We give two applications of this theorem. One of them, Theorem 5, says that the topological entropy for (the Poincare map of) a polygonal billiard is zero. The two previous proofs of this fact, [4, 2], required a detailed analysis

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of the dynamics of polygonal billiards. Our proof is based on the observation that the boundary length grows quadratically, hence Theorem 3 applies.

The other application, Theorem 6, is to the *directional billiards in a rational polytop* (see the definition below). It says that the topological entropy of (the Poincare map of) any directional billiard is zero.

2 Generalized Polyhedron Exchanges

By a *polyhedron* P, $\dim P = n$, we mean a compact Euclidean polyhedron in $\mathbf{R}^n$, $\mathrm{int}P \neq \emptyset$. The m-dimensional faces of P are polyhedra of dimension $m < n$. *Polygons (polytops)* are the polyhedra of dimension 2 (3). A *partition* $\mathcal{P}$ *of a polyhedron* P is a representation $P = \bigcup_{i=1}^{n} P_i$, where P_i are *subpolyhedra* of P, and $\mathrm{int}P_i \cap \mathrm{int}P_j = \emptyset$ if $i \neq j$.

We fix $r \geq 1$, and shall say that something is smooth, whenever it is of class C^r.

Definition 1 *(I) A* generalized polyhedron X *of dimension n is a closed subset of a smooth manifold M^n, and a mapping $f : X \mapsto \mathbf{R}^n$ (into) such that: 1) f extends to a diffeomorphism of an open set O, $X \subset O \subset M$, into $\mathbf{R}^n$ 2) $f(X)$ is a polyhedron.*
(II) A d-dimensional space X with a generalized polyhedral partition $\mathcal{P}$ *is a subset of a manifold M^d and a representation $X = \bigcup_{i=1}^{n} X_i$ satisfying the following conditions.*
i) The X_i are generalized polyhedra of dimension d.
ii) int$(X_i) \cap$ int$(X_j) = \emptyset$ if $i \neq j$.
iii) If $I \subset \{1, \ldots, n\}$ is such that $\bigcap_{i \in I} X_i \neq \emptyset$, the polyhedral structures on $X_i, i \in I$, agree, so that $\bigcup_{i \in I} X_i$ is a generalized polyhedron.

If $(X, \mathcal{P})$ is as above, we say that X_i are the *atoms* of $\mathcal{P}$, and that $\partial\mathcal{P} = \bigcup_{i=1}^{n} \partial X_i$ is the *boundary* of $\mathcal{P}$.

Definition 2 *Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of X, and let $T : (X, \mathcal{P}) \to (X, \mathcal{Q})$ be such that $T_i = T|_P : P_i \to Q_i$ and $T_i^{-1} : Q_i \to P_i$ are homeomorphisms and smooth on the interiors. Then we say T is a* generalized polyhedron exchange *(g.p.e.).*

We say a partition $\mathcal{Q}$ is a *refinement* of a partition $\mathcal{P}$ (write $\mathcal{P} < \mathcal{Q}$) if every atom of $\mathcal{Q}$ is a subpolyhedron of an atom of $\mathcal{P}$. If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of X, their join $\mathcal{P} \vee \mathcal{Q}$ is the partition formed by the intersections $P \cap Q$ where $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$. It turns out that the proper way to study the dynamics of a g.p.e. $T : (X, \mathcal{P}) \to (X, \mathcal{Q})$ is to consider an *inverse limit space* $\hat{X}$, which is constructed from T, $\mathcal{P}$ and $\mathcal{Q}$, and on which the map $\hat{T}$ induced by T is a homeomorphism. In that framework one can then define as usual (see, e.g., [5]) the nth join $\mathcal{P}_n = \bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}$ of the partition $\mathcal{P}$. For simplicity of exposition, we shall in §3 consider only the case $\mathcal{P} = \mathcal{Q} = \mathcal{R}$, and say that $T : (X, \mathcal{R}) \to (X, \mathcal{R})$ is a g.p.e. on the partition $\mathcal{R}$.

We say a point $x \in X$ is *n-regular* if x belongs to the interior of an atom of $\mathcal{R}_n, n \geq 0$, and *regular* if it is n-regular for all n. Since we don't assume that T_i and T_j agree on $P_i \cap P_j$, in general, the powers T^k of a g.p.e. T are well defined only on the set of regular points, which is dense in X.

When dim$X = 2$ (3) we typically speak of a *generalized polygon (polytop) exchange.*

Examples. 1. A partition of $X = [0, 1]$ is given by n intervals $P_i = [a_{i-1}, a_i]$. where $0 = a_0 < a_1 < \ldots < a_n = 1$. An interval exchange on the intervals $P_i, 1 \leq i \leq n$, (see, e.g., [1]) is then a special case of a g.p.e., and the mappings T_i are parallel translations.

2. Let X be a rectangle in $\mathbf{R}^2$, e.g. $X = [0, 1] \times [0, 1]$. Let the atoms of the partition $\mathcal{P} : X = \bigcup_{i=1}^{n} X_i$ be rectangles with the sides parallel to the coordinate axes. Let $t_i = (a_i, b_i)$ be n vectors such that the rectangles $Q_i = P_i + t_i, 1 \leq i \leq n$, form a partition $\mathcal{Q}$ of X. This defines a *rectangle exchange* $T : (X, \mathcal{P}) \to (X, \mathcal{Q})$ where the restrictions T_i, $T_i x = x + t_i$, are translations in the plane.

2'. The space $X \subset \mathbf{R}^2$ is an arbitrary polygon, the atoms of a partition $\mathcal{P} : X = \cup P_i, 1 \leq i \leq n$, are subpolygons. We define an affine polygon exchange on $\mathcal{P}$ by n affine transformations $T_i : P_i \to X$ such that $\mathcal{Q} = \{Q_i = T_i(P_i), i = 1, \ldots, n\}$, is a partition of X.

3. The obvious analog of Example 2 in 3 dimensions features the unit cube $X = [0, 1] \times [0, 1] \times [0, 1] \subset \mathbf{R}^3$ partitioned by n *rectangular parallelepipeds* P_i. The mappings T_i are parallel translations $x \mapsto x + t_i$, $t_i \in \mathbf{R}^3$, such that the parallelepipeds $Q_i = P_i + t_i$ form a partition of X.

3'. In the 3 dimensional version of Example 2', X is a polytop, $\mathcal{P}$ is a partition of X by subpolytops $P_i, 1 \leq i \leq n$, and $T_i : P_i \to X$ are affine transformations of $\mathbf{R}^3$ such that the polytops $Q_i = T_i(P_i), 1 \le i \le n$, form a partition, $\mathcal{Q}$, of X. This defines a polytop exchange $T : (X, \mathcal{P}) \to (X, \mathcal{Q})$.

3 Subexponential growth

Let $T : X \to X$ be a g.p.e. on a partition $\mathcal{R} = \{R_i : i \in I\}$. The number of atoms in a finite partition will be denoted by $|\mathcal{R}|$. The number $h(T, \mathcal{R}|) = \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{R}_n|$ is called the entropy of the g.p.e. T relative to the partition $\mathcal{R}$, and the topological entropy of the g.p.e. T is then given by $h(T) = \sup_{\mathcal{U}} h(T, \mathcal{U})$ where the supremum is over partitions $\mathcal{U}$ such that $\mathcal{R} < \mathcal{U}$ (compare with the topological entropy of a homeomorphism, see, e.g., [5]). If $\mathcal{R}$ is a generating partition (that is, the atoms of the infinite join $\bigvee_{j=-\infty}^{\infty} T^j \mathcal{R}$ consist of single points), then $h(T) = h(T, \mathcal{R})$.

We say a positive sequence $a_n, n \ge 1$, has *growth rate* $\vartheta \ge 0$ if

$$\limsup_{n\to\infty} \frac{\log a_n}{n} = \vartheta.$$

We shall assume that there is a Finsler metric (i.e., a metric induced by norms on the tangent spaces) $d(\cdot, \cdot)$ on the space X. If γ is a piecewise C^1 curve on X, we denote its length by $|\gamma|$. If $\mathcal{P}$ is a polyhedral partition, we denote by $E(\mathcal{P})$ the union of edges of the atoms of $\mathcal{P}$, and set $\ell(\mathcal{P}) = |E(\mathcal{P})|$. For $x \in X$ let $\mathcal{P}_x \subset \mathcal{P}$ be the set of atoms containing x, and set $b(\mathcal{P}) = \max_x |\mathcal{P}_x|$. Our main result is the following theorem:

Theorem 3 *Let $T : X \to X$ be a g.p.e. on a partition $\mathcal{R}$. Suppose that the partitions $\mathcal{R}_n, n > 0$, satisfy the following conditions:*
i) The atoms of $\mathcal{R}_n$ are connected.
ii) The sequence $c_n = \ell(\mathcal{R}_n)$ has growth rate $\vartheta \ge 0$.
iii) The sequence $b_n = b(\mathcal{R}_n)$ has growth rate $\le \vartheta$.
Then the growth rate of the sequence $|\mathcal{R}_n|$ is bounded by ϑ.

Sketch of Proof. We will give a proof for the special case of a rectangle exchange, that is, the target partition of any partition that consists of euclidean rectangles is made up of euclidean rectangles (e.g. if T is a Baker's transformation). Thus let $\mathcal{P}$ be a partition of a rectangle X into subrectangles P_i, on each of which the map T is a translation.

By assumption (ii) we have that $\ell(\mathcal{P}_n) \leq \text{const.}\vartheta'^n$ for any $\vartheta' > e^\vartheta$ and some const. which we can assume to be equal to 1. Asummptions (i), (iii) are always satisfied for rectangle exchanges. In fact we have that $b(\mathcal{P}_n) \leq 4$ for all n.

Suppose the sequence $a_n = |\mathcal{P}_n|$ has growth rate $\alpha > \vartheta' > \vartheta$. By considering a suitable iterate of T, we can achieve that $\vartheta' > 4$ and $\alpha > \vartheta' + 2$. For simplicity let us assume that $\alpha = \lim_n \frac{1}{n} \log a_n$.

Let δ be the smallest height or width of any of the rectangles in $\mathcal{P}$, and denote by B_n the number of 'bad' rectangles in $\mathcal{P}_n$, which are those atoms whose diameters exceed δ. In particular a 'bad' rectangle has boundary length greater than 2δ. Thus

$$B_n \leq \frac{\ell(\mathcal{P}_n)}{2\delta} \leq \frac{\vartheta'^n}{2\delta},$$

since $\ell(\mathcal{P}_n) \leq \vartheta'^n$ for large enough n. Let us now estimate $a_{n+1} = |\mathcal{P}_{n+1}| = |T^{-1}\mathcal{P}_n \vee \mathcal{P}|$. Since in taking the join $T^{-1}\mathcal{P}_n \vee \mathcal{P}$ no 'good' rectangle (i.e. with diameter $\leq \delta$) in $\mathcal{P}_n$ is cut into more than four pieces, we obtain

$$a_{n+1} \leq a_1 B_n + 4a_n \leq \frac{a_1 \vartheta'^n}{2\delta} + 4a_n \leq \vartheta'^n \cdot \text{const.} + 4\alpha^n < (\vartheta' + 1)\alpha^n$$

if n is large enough. But this contradicts our assumption that $a_{n+1} \approx \alpha^{n+1} \geq (\vartheta' + 2) \cdot \alpha^n$. $\square$

Corollary 4 *If a g.p.e.* $T : (X, \mathcal{R}) \to (X, \mathcal{R})$ *satisfies the conditions of Theorem 3 then* $h(T, \mathcal{R}) = 0$*. If* $\mathcal{R}$ *is moreover a generating partition, then* T *has topological entropy* $\leq \vartheta$*.*

4 Applications

Polygonal billiard. Theorem 3 can be applied to the Poincare return map on the billiard flow in a polygon. If Δ is a polygon in $\mathbf{R}^2$, then the Poincare return map T maps the space $X = \partial\Delta \times [0, \pi]$ onto itself, where $\partial\Delta = [0, L]$ and L is the total length of $\partial\Delta$. If (s, ϑ) is a point in X, then it corresponds to a trajectory with footpoint $s \in \partial\Delta$ and whose angle of incidence is measured by ϑ. The map T is piecewise continuous on the elements of a finite partition $\mathcal{P} = \{P_i : i\}$, where the boundary of the individual polygons P_i consists of precisely those points in X whose trajectory, when it hits $\partial\Delta$ next time, will be in a vertex of the polygon Δ. One can easily check that the Poincare return map T is indeed a g.p.e. that satisfies the first and the last condition of Theorem 3. To show that the length of the boundary set of $\mathcal{P}_n$ grows subexponentially, or actually, quadratically, one defines a Finsler metric m on the cross-section X by

$$dm = |ds| \sin\vartheta + |d\vartheta|.$$

Using the explicit form of the differential, dT, one can show that $m(\partial\mathcal{P}_n) \leq cn^2$, where c is a constant. Thus the second condition of Theorem 3 is satisfied by T, and we conclude as follows.

Theorem 5 *The Poincare return map of a polygonal billiard has zero topological entropy.*

Billiards in rational polytops. Let S be the surface of a closed connected polytop $V \subset \mathbf{R}^3$. The billiard flow in V has a natural cross-section X which consists of unit vectors v with footpoints in S, directed inward. The first return map $T : X \to X$ is the *billiard ball map* in 3 dimensions. The group $\mathrm{Aut}(\mathbf{R}^3)$ of Euclidean motions of $\mathbf{R}^3$ is a semi-direct product of the orthogonal group, $O(3)$, and the group of translations of $\mathbf{R}^3$. For any hyperplane $L \subset \mathbf{R}^3$ let $g_L \in \mathrm{Aut}(\mathbf{R}^3)$ be the symmetry about L, and let $r_L \in O(3)$ be the corresponding orthogonal reflection. If the reflection group $G = G(V)$, generated by reflections r_L about the hyperplanes L spanned by the faces of the polytop V, is finite, then we say that V is a *rational polytop*.

We shall assume that V is a rational polytop. The natural action of G on the unit sphere $S^2 \subset \mathbf{R}^3$ has a *fundamental domain*, $\Omega \subset \mathbf{R}^3$ which is a spherical polygon. We identify the points $\omega \in \Omega$ with the orbits of G in S^2, i.e. $\Omega \cong S^2/G$.

The phase space Z of the billiard flow in V consists of pairs $(v, \theta) \in V \times S^2$ and is foliated by invariant subsets $Z_\omega = \{(v, \theta) : \theta \in \omega\} \in Z$, where $\omega \in \Omega$. We shall call the restriction of the billiard flow to Z_ω the *billiard flow in direction* ω and denote it by $B^t_\omega : Z_\omega \mapsto Z_\omega$.

For $\omega \in \mathrm{int}(\Omega)$, the invariant sets Z_ω are naturally homeomorphic to a canonical space W, that depends only on the polytop V and not on the 'parameter' ω. The space W is a closed 3-dimensional topological manifold, which geometrically is obtained by glueing $|G|$ copies of V along their boundary faces. This construction of W from a rational polytop V has a close analog in two dimensions, where a rational polygon P in the plane gives, in a natural way, rise to a closed surface $S = S(P)$ which is triangulated by a finite number of copies of P, and which carries a one-parameter family of the directional billiard flows.

Denote by $F = F(W)$ the union of faces of W. We have the following result.

Theorem 6 *Let V be a rational polytop.*
(I) For any $\omega \in \Omega$, the set F is a cross-section of the directional billiard flow B^t_ω. Moreover, the first return map, T_ω, is an affine polygon exchange.
(II) For any direction ω, the topological entropy of T_ω is zero.

Directional billiard flows in a rational polytop are the analog in the three dimensions of such flows for a rational polygon (see, e.g., [3]). Thus, Theorem 6 is the three dimensional version of the well known fact that the directional billiards in a rational polygon have entropy zero (see, e.g., [1]).

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