Parabolic rational maps

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Abstract: We study the dynamics of rational maps with indifferent parabolic points by comparing their dynamical properties to those of it's 'jump transformation' which is uniformly expanding on a non-compact set with infinite Markov partition. We establish the spectral properties of a two-variables operator-valued function associated to the jump transformation and exploit their dynamical relevance to study the analytic properties of the pressure, the escape rate from a neighborhood of the Julia set and the asymptotic distribution of pre-images.

1 Introduction

Let $T : \mathbf{C} \to \mathbf{C}$ be a rational map of the Riemann sphere $\bar{\mathbf{C}}$ equipped with the spherical metric d and denote by J its Julia set. We shall assume that the point ∞ does not lie in the Julia set. This allows us to use the euclidean metric in $\mathbf{C}$ rather than the spherical metric on $\bar{\mathbf{C}}$. One says that $T|_J$ is *expansive* if there exists an expansive constant $\alpha > 0$ such that

$$\sup_{n \geq 0} d(T^n(x), T^n(y)) \geq \alpha$$

for all $x, y \in J$, $x \neq y$. This property does not depend on the metric. Moreover, one says that T is *expanding* if there exists $\beta > 1$ (an expanding

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constant) such that, for some $n \geq 1$,

$$|(T^n)'(x)| \geq \beta$$

for all $x \in J$. A rational map T is said to be *parabolic* if the restriction $T|_J$ is expansive but not expanding. Furthermore, T is parabolic if and only if the Julia set contains no critical point of T, but contains rationally indifferent periodic points [5].

Rational parabolic Julia sets have been studied by a variety of people starting with Fatou at the beginning of this century. In recent years, a sequence of papers by Denker, Urbanski and others (e.g. [1, 5, 6, 12]) has provided us with a detailed study of conformal and invariant measures on J as well as their relationships with the pressure function and the Hausdorff dimension.

In what follows we shall partially rely on these previous results by explicitly stating those which are repeatedly used throughout the paper. In particular, those dealing with the asymptotic behaviour of a rational map near parabolic points (see Propositions 1 and 2 below), the existence of a Markov partition (Proposition 3) and the continuity and monotonicity of the pressure function (first part of Theorem 15).

However in this paper we adopt a different point of view, mainly focusing on the spectrum of a two-variables operator-valued function which is related to a jump transformation $\hat{T}$ obtained by inducing on a subset of J where the map T is uniformly expanding (see Sections 2 and 3). The main results that we extract from these spectral properties concern the behaviour of the derivatives of the pressure function P_t in a neighbourhood of $t = h$ where h is the Hausdorff dimension of J (Section 4), a polynomial escape rate from a neighbourhood of J (Section 5) and the asymptotic distribution of the pre-images of points in J (Section 6).

2 Parabolic points and the jump transformation

The set Λ of all rationally indifferent periodic points of T is finite (see, e.g., [2]). One may then consider the map T^q where $q \geq 1$ is taken so that $T^q(\lambda) = \lambda$ and $(T^q)'(\lambda) = 1$ for every $\lambda \in \Lambda$. Thus, without loss of generality, we can assume that the parabolic points Λ are fixed points. We can moreover assume that $T'(\lambda) = 1$ for all $\lambda \in \Lambda$. Set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ for some $r \geq 1$.

For every $\lambda \in \Lambda$ there exists an open neighbourhood U_λ of λ of diameter not exceeding an expansive constant for T and a unique holomophic inverse branch $T_\lambda^{-1} : U_\lambda \to \mathbf{C}$ of T such that $T_\lambda^{-1}(\lambda) = \lambda$ and

$$T_\lambda^{-1}(U_\lambda \cap J) \subset U_\lambda \cap J.$$

Moreover, according to Fatou's flower theorem [2] T_λ^{-1} has the expansion

$$T_\lambda^{-1}(z) = z - a_\lambda (z-\lambda)^{p(\lambda)+1} + \text{higher order terms in } (z-\lambda)$$

with $a_\lambda \neq 0$. The integer $p(\lambda) + 1 \geq 2$ is called the *multiplicity* of the fixed point λ (see [13]) and we shall define the *characteristic value* of λ to be the number

$$\gamma(\lambda) = \frac{p(\lambda)+1}{p(\lambda)} \in \{2, \frac{3}{2}, \frac{4}{3}, \ldots\}$$

Of the following two results ([1], Section 8), both of which are consequences of Fatou's theorem, Proposition 2 involves the Köbe distortion theorem.

Proposition 1 *Let T be a parabolic rational map with Julia set J and assume that the parabolic points Λ are all fixed points of T. Then every $\lambda \in \Lambda$ has an open neighbourhood $V_\lambda \subset U_\lambda$ such that for any $z \in J \cap V_\lambda$ one has*

$$\begin{aligned} \lim_{n\to\infty} \frac{|z_n - \lambda|}{n^{-1/p(\lambda)}} &= (|a_\lambda| p(\lambda))^{-1/p(\lambda)}, \\ \lim_{n\to\infty} \frac{|z_n - z_{n+1}|}{n^{-\gamma(\lambda)}} &= |a_\lambda| (|a_\lambda| p)^{-\gamma(\lambda)}, \end{aligned}$$

where $z_n = T_\lambda^{-n}(z)$ and $\gamma(\lambda)$ is the characteristic value of λ.

Proposition 2 *Let T, J, Λ and V_λ be as in Proposition 1. Then for any $z \in J \cap V_\lambda$ there is a constant $C_1 > 1$ such that*

$$C_1^{-1} \leq \frac{|(T^n)'(z_n)|}{n^{\gamma(\lambda)}} \leq C_1.$$

A measure m on J is said to be t-conformal, $t \geq 0$, for T if

$$m(T(A)) = \int_A |T'|^t \, dm$$

for every Borel set $A \subset J$ on which T is injective. Any rational map T possesses a t-conformal measure for some $t \geq 0$ [19], and this conformal measure is unique if T is expanding. The exponent t associated to m equals the Hausdorff dimension h of J and m is equivalent to the h-dimensional Hausdorff measure on J [16]. For parabolic rational maps, the set of exponents for which conformal measures exist is infinite, even though the h-conformal measure is non-atomic and unique ([1], Theorem 8.7) and will from now on be denoted by m.

The Hausdorff dimension h of the Julia set of a parabolic rational map satisfies $1/\gamma_0 < h < 2$ ([1] Section 8), where $\gamma_0 = \min\{\gamma(\lambda) : \lambda \in \Lambda\}$ (recall that $\gamma(\lambda)$ is the characteristic value of the parabolic fixed point λ).

Proposition 3 *[5] Let T be a parabolic rational map with Julia set J and expansive constant $\alpha > 0$, so small that for every pair of rationally indifferent points $\lambda \neq \lambda' \in J$,*

$$\begin{aligned} \operatorname{dist}(T(B_{2\alpha}(\lambda)), \Lambda \setminus T(\lambda)) &> 2\alpha \qquad (1) \\ |\lambda - \lambda'| &> 4\alpha. \qquad (2) \end{aligned}$$

Then there exist $\delta \in (0, \alpha)$ and a Markov partition $\mathcal{A} = \{A_1, \ldots, A_s\}$ such that
(1) $\operatorname{diam} T(A_k) < \delta$ and $T|_{A_k}$ is 1-1 for all $k = 1, \ldots, s$.
(2) if $T(A_k) \cap (J \setminus B_\alpha(\Lambda)) \neq \emptyset$, then the inverse branches $\mathcal{S}_n$ of T^n are univalent on $B_{2\delta}(T(A_k))$ for $n \geq 1$.
(3) if $\bigcap_{0 \leq j < n} T^{-j}(A_{k_j}) \neq \emptyset$ and $T(A_{k_{n-1}}) \cap (J \setminus B_\alpha(\Lambda)) \neq \emptyset$, then there exists a unique analytic inverse branch $\varphi \in \mathcal{S}_n$ with domain $B_{2\delta}(T(A_{k_{n-1}}))$ such that

$$\varphi(T(A_{k_{n-1}})) = \bigcap_{j=0}^{n-1} T^{-j}(A_{k_j}).$$

(4) $m(\partial\mathcal{A}) = 0$.

We obtain a symbolic description of the residual set $X = J \setminus \bigcup_{n \geq 0} T^{-n}(\partial\mathcal{A})$, where $\partial\mathcal{A} = \bigcup_{i=1}^{r} A_i \setminus \operatorname{int}(A_i)$. Define an $s \times s$ transition matrix t by putting

$t_{ij} = 1$ if $\mathrm{int}(A_i) \cap \mathrm{int}(T^{-1}(A_j)) \neq \emptyset$ and $t_{ij} = 0$ otherwise. Then we put

$$\Omega = \{\omega \in \{1, \ldots, s\}^{\mathbf{N}} : \; t_{\omega_j \omega_{j+1}} = 1, \; j \geq 0\},$$

and denote by $\tau : \Omega \to \Omega$ the shift transformation. The map $\xi : \Omega \to J$ given by

$$\xi(\omega) = \bigcap_{j=0}^{\infty} T^j(A_{\omega_j})$$

is a bijection between Ω and the residual set X and semiconjugates τ with T, that is $\xi T = \tau \xi$. If $\eta = \eta_0, \ldots, \eta_{n-1}$ is a word in Ω (of length n), then we write $U(\eta)$ for the 'cylinder set' $\{\omega \in \Omega : \; \omega_j = \eta_j, \; j = 0, \ldots, n-1\}$. Put Ω_n for the set of words in Ω of lengths n, $n \in \mathbf{N}$.

Let $\mathcal{A}^n = \bigvee_{j=0}^{n-1} T^{-j}(\mathcal{A})$ be the n-th join of $\mathcal{A}$ and denote by $A_\eta = \xi(U(\eta)) \in \mathcal{A}^n$ the atom that corresponds to the word $\eta \in \Omega_n$ (of length n) in Ω. Define $\mathcal{G}_n \subset \mathcal{A}^n$ ('good' cylinders of length n) as

$$\mathcal{G}_n = \{A_\eta : \; A_{\eta_{n-1}} \cap (J \setminus B_\alpha(\Lambda)) \neq \emptyset, \; \eta \in \Omega_n\}.$$

Let us now introduce the return function $R : J \to \mathbf{N} \cup \{\infty\}$ by

$$R(z) = \inf\{n \in \mathbf{N} : \; \bigcap_{j=0}^{n-1} T^{-j}(A_{\omega_j(z)}) \in \mathcal{G}_n\}$$

(where $\inf \emptyset = \infty$) and its levelsets $\mathcal{R}_n = \{x \in X : R(x) = n\}$. If $n > 2$ we clearly have $T(\mathcal{R}_n) = \mathcal{R}_{n-1}$ and $T(\mathcal{R}_2) \subseteq \mathcal{R}_1$. Since the map is parabolic, one also has $T(\mathcal{R}_1 \setminus T(\mathcal{R}_2)) = X$.

Definition 4 *([18],[1]) The map $\hat{T} : J \to J$ given by $\hat{T}(z) = T^{R(z)}(z)$, $z \in X$ is called the* jump transformation *associated with T.*

The partition

$$\mathcal{B} = \{A \cap \mathcal{R}_n : \; A \in \mathcal{A}^n, \; n = 1, \ldots\},$$

is a Markov partition for the jump transformation $\hat{T}$. Its atoms have all diameter $< \delta$. Moreover, $\hat{T}|_B$ is a homeomorphism for every $B \in \mathcal{B}$. The return time function R is constant on the atoms B of $\mathcal{B}$ we and shall write $R(B) = n$ if $B \subset \mathcal{R}_n$.

We denote by T_A and $\hat{T}_A$ the restrictions of T and respectively $\hat{T}$ to a set $A \subset J$. As $T_{\mathcal{R}_1} = \hat{T}_{\mathcal{R}_1}$ we have that $|T'_{\mathcal{R}_1}| = |\hat{T}'_{\mathcal{R}_1}| \geq \beta$ for some $\beta > 1$. Since $\mathcal{R}_n \subset B_{2\alpha}(\Lambda)$ for $n = 2, \ldots$, we obtain using Propositions 1 and 2 the following result:

Proposition 5 *(1) There exists a* $\beta > 1$ *such that* $|\hat{T}'_{\mathcal{R}_1}| \geq \beta$.
(2) There exists a constant $C_2 \geq 1$ *such that for all* $B \in \mathcal{B}$*:*

$$C_2^{-1} \leq \frac{|\hat{T}'_B|}{R(B)^{\gamma(B)}} \leq C_2,$$

where $\gamma(B)$ *is the characteristic value of the parabolic fixed point* $\lambda(B)$.

There exists a $\hat{T}$-invariant probability measure $\rho \sim m$ such that $\phi = d\rho/dm$ is bounded away from 0 and ∞ and is Lipschitz continuous ([1], Section 9).

Proposition 6 *There exits positive constant* C_3 *and numbers* $M_\lambda,\ \lambda \in \Lambda$, *such that for all* $B \in \mathcal{B}$

$$\begin{aligned} C_3^{-1} &\leq m(B)R(B)^{h\gamma(B)} \leq C_3, \\ \rho(B) &= M_{\lambda(B)}\, m(B) \left(1 + \mathcal{O}(R(B)^{-1/p(B)})\right). \end{aligned}$$

Proof. If $B \in \mathcal{B}$ so that $B \subset \mathcal{R}_{n+2}$, then

$$m(T^n(B)) = \int_B |(T^n)'|^h \, dm = |(T^n)'\, \bar{z}_n|^h \, m(B)$$

for some $\bar{z}_n \in B$. Put $z = T^n \bar{z}_n$ (note $z \in \mathcal{R}_2$), and it follows from Proposition 2 that

$$C_1^{-1}\, m(T^n(B)) \leq m(B)\, R(B)^{h\gamma(B)} \leq C_1\, m(T^n(B)).$$

The first set of inequalities follows now if we put $C_3 = C_1(m(\mathcal{R}_2) + 1/m(\mathcal{R}_2))$.

Consider the local averages

$$\phi_B = \frac{\rho(B)}{m(B)} = \frac{1}{m(B)} \int_B \phi \, dm,$$

$B \in \mathcal{B}$, and let $z_B \in B$ be such that $\phi_B = \phi(z_B)$. Since the inverse branches of T near parabolic points are (non-uniform) contractions, we obtain

$$|\phi_B - \phi_{B'}| \leq L\, |z_B - z'_B| \leq L\, |z_B - \lambda(B)|$$

for all $B' \in \mathcal{B}$ for which $\lambda(B') = \lambda(B)$, where L is the Lipshitz constant of the density function ϕ. Thus, for every parabolic point $\lambda \in \Lambda$ we obtain a limit

$$M_\lambda = \lim_{\lambda(B)=\lambda,\, R(B)\to\infty} \phi_B,$$

where by proposition 1 as: $|M_{\lambda(B)} - \phi_B| \leq c_1 R(B)^{-1/p(B)}$, for some constant c_1. This proves the second set of inequalities. □

The previous results now easily provide a criterion for the finiteness of the natural invariant measure as follows (see also [1]).

Proposition 7 *The σ-finite measure $\nu = R\rho$ is T-invariant. Moreover, if $h\gamma_0 > 2$ then $\nu(J) < \infty$ whereas if $h\gamma_0 \leq 2$ then $\nu(J) = \infty$ (recall $\gamma_0 = \min_{\lambda \in \Lambda} \gamma(\lambda)$).*

Proof. The T-invariance of ν follows from the identity:

$$\nu(B) = \sum_{k=0}^{\infty} \rho(T^{-k} B \cap \bigcup_{n>k} \mathcal{R}_n),$$

where B are Borel subsets of J. In particular we have $\nu(\mathcal{R}_1) = \sum_{k \geq 1} \rho(\mathcal{R}_k) = 1$. The other statement of the proposition follows applying proposition 6 to the identity:

$$\nu(J) = \rho(R) = \sum_{k=1}^{\infty} k\rho(\mathcal{R}_k) = \sum_{k=1}^{\infty} km(\phi \chi_{\mathcal{R}_k}),$$

where $\chi_{\mathcal{R}_k}$ is the characteristic function of the levelset $\mathcal{R}_k$. □

We label the atoms of the (infinite) partition $\mathcal{B} = \{B_j : j = 1, 2, \ldots\}$ so that the function R is non-decreasing, that is $R(i) \leq R(j)$ if $i < j$ (here $R(j) = R(B_j)$). Since the number of parabolic points is finite we get that $c - 1 + k/|\Lambda| \leq R(k) \leq c + k/|\Lambda|$ where $c = |\{j : R(j) = 1\}|$ is the number of elements of the partition which lie in the first levelset $\mathcal{R}_1$.

Define transition matrices M and $\hat{M}$ by putting $M_{ij} = 1$ whenever $\text{int}(B_i) \cap \text{int}(T^{-1}(B_j)) \neq \emptyset$ and $M_{ij} = 0$ otherwise, and $\hat{M}_{ij} = 1$ if $\text{int}(B_i) \cap \text{int}(\hat{T}^{-1}(B_j)) \neq \emptyset$ and $\hat{M}_{ij} = 0$ otherwise. Notice that $M_{ij} = \hat{M}_{ij} = t_{ij}$ if $B_i, B_j \subset \mathcal{R}_1$. Moreover, if $B_i \subset \mathcal{R}_k$ for some $k > 2$, then there exists a unique $B_j \subset \mathcal{R}_{k-1}$ such that $M_{ij} = 1$. Define the shiftspaces

$$\begin{aligned} \Sigma &= \{x \in \mathbf{N}^{\mathbf{N}} : M_{x_j x_{j+1}} = 1 \forall j \geq 0\}, \\ \hat{\Sigma} &= \{x \in \mathbf{N}^{\mathbf{N}} : \hat{M}_{x_j x_{j+1}} = 1 \forall j \geq 0\}, \end{aligned}$$

and denote by σ and $\hat{\sigma}$ the shift transformation on the respective shift spaces. The map $\pi : \Sigma \to J$ (resp. $\hat{\pi} : \hat{\Sigma} \to J$) given by

$$\pi(x) = \bigcap_{j=0}^{\infty} T^{-j} B_{x_j}$$

(respectively by $\hat{\pi}(x) = \bigcap_{j=0}^{\infty} \hat{T}^{-j} B_{x_j}$) is a bijection between Σ (resp. $\hat{\Sigma}$) and the residual set $X_0 = J \setminus \Lambda$. In addition, it conjugates T (resp. $\hat{T}$) with the shift τ on Σ (resp. on $\hat{\Sigma}$).

3 Transfer operators.

For $\theta \in (0,1)$ we define in the usual way a metric on Σ by setting $d_\theta(y, y') = \theta^n$ where n is largest such that $y_j = y'_j$ for $0 \leq j \leq n$. Moreover, if Φ is a (continuous) complex valued function on Σ then

$$\mathrm{var}_n \Phi = \sup\{\, |\Phi(y) - \Phi(y')| : \, y_j = y'_j, \, 0 \leq j \leq n\}$$

is the n-the variation of Φ ($n = 1, 2, \ldots$) and

$$|\Phi|_\theta = \sup_{n \geq 0} \frac{\mathrm{var}_n \Phi}{\theta^n}$$

is the associated Lipshitz constant. We denote by $\mathcal{F}_\theta$ the Banach space of complex valued functions on Σ which are finite with respect to the norm $\|\Phi\|_\theta = |\Phi|_\theta + |\Phi|_\infty$. In the same fashion we define the space $\hat{\mathcal{F}}_\theta$ of Lipshitz continuous functions on the shiftspace $\hat{\Sigma}$.

Let us define the following functions on Σ and $\hat{\Sigma}$ respectively:

$$\begin{aligned} V(x) &= \log(T_{x_0})'(\pi(\sigma(x))), \\ \hat{V}(y) &= \log(\hat{T}_{y_0})'(\hat{\pi}(\hat{\sigma}(y))), \end{aligned}$$

where $T_k = T_{B_k}^{-1}$ and $\hat{T}_k = \hat{T}_{B_k}^{-1}$ for $k \in \mathbf{N}$ and $x \in \Sigma, \; y \in \hat{\Sigma}$.

Lemma 8 *For any $\theta \geq 1/\beta$, $\hat{V} \in \hat{\mathcal{F}}_\theta$.*

Proof. If $y, y' \in \hat{\Sigma}$ are chosen so that $d_\theta(y, y') = \theta^n$ for some $n \geq 1$ then the two points $z = \hat{\pi}(y)$ and $z' = \hat{\pi}(y')$ belong to the same atom B_k, $k = y_0 = y_0{}'$, of the Markov partition $\mathcal{B}$.
Since $y_j = y_j'$, $j = 1, \ldots, n$, Proposition 5(1) implies that $|z - z'| \leq c_1 \beta^{-n} \mathrm{diam}\, B_k$, for some constant $c_1 > 0$ which is independent of k. Hence, for $\bar{z} \in B_k$, we have

$$\begin{aligned} |\hat{V}(y) - \hat{V}(y')| &= |\log \hat{T}'(z) - \log \hat{T}'(z')| \\ &\leq c_2 |z - z'| \max_{\bar{z} \in B_k} \left| \frac{\hat{T}''(\bar{z})}{\hat{T}'(\bar{z})} \right| \\ &\leq c_3 \max_{\bar{z} \in B_k} \left| \frac{\hat{T}''(\bar{z})}{\hat{T}'(\bar{z})} \right| \cdot \mathrm{diam}\, B_k \cdot \beta^{-n}, \end{aligned}$$

for some constants c_2, c_3. It remains to show that the term $|\hat{T}''(\bar{z})/\hat{T}'(\bar{z})| \cdot \mathrm{diam}\, B_k$ is bounded uniformly in k. This follows from the boundedness of $|\hat{T}''(\bar{z})/(\hat{T}')^2(\bar{z})|$. An application of the chain rule yields

$$\frac{\hat{T}''(z)}{(\hat{T}')^2(z)} = \frac{(T^k)''(z)}{((T^k)')^2(z)} = \sum_{j=0}^{k-1} \frac{T''(T^j(z))}{(T')^2(T^j(z))} \frac{1}{\prod_{\ell=j+1}^{k-1} T'(T^\ell(z))},$$

$z \in B_k$. Using Proposition 2 one estimates $\prod_{\ell=j+1}^{k-1} |T'(T^\ell(z))| \geq c_4 (k - \ell)^{\gamma(B_k)}$ (for some $c_4 > 0$). Hence the finite sum is uniformly bounded. □

For $(t, z) \in \mathbf{C}^2$ we define a transfer operator $\hat{\mathcal{L}}_{t,z} : \hat{\mathcal{F}}_\theta \to \hat{\mathcal{F}}_\theta$ by

$$\hat{\mathcal{L}}_{t,z} = \sum_{k=1}^{\infty} z^{R(k)} \mathcal{K}_{t,k} = \sum_{\ell=1}^{\infty} z^\ell \sum_{R(k)=\ell} \mathcal{K}_{t,k},$$

where $\mathcal{K}_{t,k}\phi(x) = e^{t\hat{V}(kx)}\phi(kx)$, if $M_{kx_0} = 1$ and $\mathcal{K}_{t,k} = 0$ otherwise, where $\phi \in \hat{\mathcal{F}}_\theta$. For $z = 1$ and t real and positive we recover the usual transfer operator on $\hat{\mathcal{F}}_\theta$.

Lemma 9 *The power series of $\hat{\mathcal{L}}_{t,z}$ has radius of convergence bounded from below by* 1 *for every t s.t.* $\mathrm{Re}\, t > 0$ *and, moreover, converges absolutely on the circle $|z| = 1$ for* $\mathrm{Re}\, t > 1/\gamma_0$*, (where $\gamma_0 = \min_{\lambda \in \Lambda} \gamma(\lambda)$).*

Proof. Since the cardinality of the levelsets of the function R is uniformly bounded, the radius of convergence of the power series $\hat{\mathcal{L}}_{t,z}$ is, according to

Hadamard's formula, given by $\left(\lim_{k\to\infty} \|\mathcal{K}_{t,k}\|_\theta^{1/R(k)}\right)^{-1}$. If $\phi \in \hat{\mathcal{F}}_\theta$ is a 'test' function, then, by proposition 5,

$$\begin{aligned} |\mathcal{K}_{t,k}\phi|_\infty &\leq |\phi|_\infty \exp(\mathrm{Re}\, t \sup_x \hat{V}(kx)) \\ &\leq |\phi|_\infty \sup_{B_k} |\hat{T}'|^{-\mathrm{Re}\, t} \\ &\leq |\phi|_\infty C R(k)^{-\gamma_0 \mathrm{Re}\, t}. \end{aligned}$$

To estimate the Hölder constant of $\mathcal{K}_{t,k}\phi$, let $x, x' \in \hat{\Sigma}$ be so that $x_i = x'_i$ for $0 \leq i < m$. Therefore (provided $\hat{M}_{kx_0} = 1$)

$$\begin{aligned} \mathcal{K}_{t,k}\phi(x) - \mathcal{K}_{t,k}\phi(x') &= e^{t\hat{V}(kx)}\phi(kx) - e^{t\hat{V}(kx')}\phi(kx') \\ &= e^{t\hat{V}(kx)}\left((\phi(kx) - \phi(kx')) + \phi(kx')(1 - e^{t\hat{V}(kx') - t\hat{V}(kx)})\right), \end{aligned}$$

and

$$\begin{aligned} |\mathcal{K}_{t,k}\phi(x) - \mathcal{K}_{t,k}\phi(x')| &\leq e^{\mathrm{Re}\, t\, \hat{V}(kx)}\left(|\phi|_\theta + |\phi|_\infty c_1 |\hat{V}|_\theta\right)\theta^{m+1} \\ &\leq c_2 \|\phi\|_\theta \theta^{m+1} e^{\mathrm{Re}\, t\, \hat{V}(kx)}, \end{aligned}$$

for some constants c_1, c_2, which implies that

$$\|\mathcal{K}_{t,k}\|_\theta \leq c_2 \sup_x e^{\mathrm{Re}\, t\, \hat{V}(kx)} \leq c_2 \sup_{B_k} |\hat{T}'|^{-\mathrm{Re}\, t} \leq c_2 C_2 R(k)^{-\gamma_0\, \mathrm{Re}\, t}.$$

In the last inequality we made use of proposition 5. This proves the lower bound on the radius of convergence. For $|z| = 1$ the series converges absolutely if $\gamma_0\, \mathrm{Re}\, t > 1$. □

Remark. The above result implies that the operator-valued function $(t, z) \to \hat{\mathcal{L}}_{t,z}$ is holomorphic in $\{\, t \,:\, \mathrm{Re}\, t > 1/\gamma_0\} \times \{\, z \,:\, |z| < 1\}$ and is continuous in $\{\, t \,:\, \mathrm{Re}\, t > 1/\gamma_0\} \times \{\, z \,:\, |z| \leq 1\}$. In particular, for (t, z) in this domain, we have

$$[(d/dt)\hat{\mathcal{L}}_{t,z}]\psi(x) = \sum_{k=1}^{\infty} z^{R(k)} e^{t\hat{V}(kx)} \hat{V}(kx)\, \psi(kx) = \hat{\mathcal{L}}_{t,z}(\psi\, \hat{V})(x).$$

Now, for a real valued function u on Σ we define the pressure

$$P(u) = \limsup_{n\to\infty} \frac{1}{n} \log Z_n(u),$$

where $Z_n(u) = \sum_{\sigma^n x = x} e^{u^n(x)}$ is the n-th partition function and $u^n = u + u\sigma + u\sigma^2 + \cdots + u\sigma^{n-1}$ denotes the n-th ergodic sum of u. Similarly, if u is a function on $\hat{\Sigma}$ then $\hat{P}(u) = \limsup_{n\to\infty} \frac{1}{n} \log \hat{Z}_n(u)$ denotes its pressure, where $\hat{Z}_n(u) = \sum_{\hat{\sigma}^n x = x} e^{u^n(x)}$. For $0 < z \le 1$ and t real and positive set

$$\hat{P}_{t,z} = \hat{P}(t\hat{V} + R\log z).$$

Lemma 10 *For any $0 < z \le 1$ the pressure function $\hat{P}_{t,z}$ is finite for $t > 1/\gamma_0$.*

Proof. Take first $z = 1$. Denote by $\mathcal{T}_n$ the words in the subshift $\hat{\Sigma}$ of lengths n ($\mathcal{T}_1 = \mathcal{B}$). Let z be real and positive, and recall that $|\{k : R(k) = \ell\}| = |\Lambda|$ for $\ell \ge 2$. Then

$$\begin{aligned}
\hat{Z}_{n+1} &\le \sum_{\eta \in T_{n+1}} \sup_{x \in U(\eta)} e^{t\hat{V}^{n+1}(x)} \\
&\le c_1 \hat{Z}_n \sum_{\eta_n \in T_1} \sup_{x \in U(\eta_n)} e^{t\hat{V}(x)} \\
&\le c_1 \hat{Z}_n \left(c\beta + \sum_{\ell=2}^{\infty} C_2 |\Lambda| \ell^{-\gamma t} \right),
\end{aligned}$$

where $c_1 = \exp(2|t\hat{V}|_\theta/(1-\theta))$ and $c = |\{k : R(k) = 1\}|$. We used proposition 5 to estimate $e^{\hat{V}(x)} = |\hat{T}'(x)|^{-1} \le \beta$ for $R(x) = 1$, and $e^{\hat{V}(x)} = |\hat{T}'(x)|^{-1} \le C_2 \ell^{-\gamma}$ for $R(x) = \ell \ge 2$. Thus we conclude that

$$c_1 \left(c\beta^{-t} + |\Lambda| C_2^t \sum_{\ell=2}^{\infty} \ell^{-\gamma t} \right)$$

is finite if $\gamma t > 1$. Therefore

$$\hat{Z}_{n+1} \le \hat{Z}_n c_1 \left(c\beta^{-t} + \frac{|\Lambda| C_2^t}{\gamma t - 1} \right),$$

which implies that the pressure $\hat{P}_{t,1}$ is bounded by

$$\log \left(c_1 c \beta^{-t} + \frac{c_1 |\Lambda| C_2^t}{\gamma t - 1} \right).$$

The assertion now follows by noting that since $\log \hat{Z}_n$ is monotonically increasing as a function of z for $0 < z \le 1$, so is $\hat{P}_{t,z}$. □

Theorem 11 *Let* $\mathrm{Re}\, t > 1/\gamma_0$ *and* $|z| \le 1$*. Then*
(a) The spectral radius of $\hat{\mathcal{L}}_{t,z} : \hat{\mathcal{F}}_\theta \to \hat{\mathcal{F}}_\theta$ *is bounded above by* $\exp \hat{P}_{\mathrm{Re}\, t, |z|}$*.*
(b) The essential spectral radius of $\hat{\mathcal{L}}_{t,z}$ *is bounded above by* $\theta \exp \hat{P}_{\mathrm{Re}\, t, |z|}$*.*
(c) If $(t,z) \in \mathbf{R}^2_+$ *then* $\hat{\mathcal{L}}_{t,z}$ *has at most one simple eigenvalue of modulus* $\exp \hat{P}_{t,z}$ *and exactly one at* $\exp \hat{P}_{t,z}$*.*

Proof. As above denote by $\mathcal{T}_n$ the words in the subshift $\hat{\Sigma}$ of lengths n. Since for $z = 0$ one has $\hat{\mathcal{L}}_{t,0} = 0$ let us now assume that $z \ne 0$. To prove (a) let ϕ be a 'test' function in $\hat{\mathcal{F}}_\theta$ and choose some $\varepsilon > 0$. Put $s = \log |z|$, $t = u + iv$ and $R(x) = R(x_0)$ for points x in the shiftspace $\hat{\Sigma}$. Then

$$|\hat{\mathcal{L}}^n_{t,z}\phi|_\infty \le \sum_{\eta \in \mathcal{T}_n} e^{(u\hat{V}+sR)^n(x_\eta)} |\phi|_\infty \le C \sum_{\eta \in \mathcal{T}_n} e^{(u\hat{V}+sR)^n(\eta^\infty)} |\phi|_\infty \le e^{(\hat{P}_{u,|z|}+\varepsilon)n},$$

for all large enough n, where $C = \exp \frac{t|\hat{V}|_\theta}{1-\theta}$ and $x_\eta \in U(\eta)$ is chosen so that $\hat{V}^n(x_\eta) = \max_{x \in B_\eta} \hat{V}^n(x)$. The point η^∞ denotes the periodic point which is obtained by infinitely concatenating the finite word η with itself.

To estimate the variation of $\hat{\mathcal{L}}^n_{t,z}\phi$ let x, x' be two points in $\hat{\Sigma}$ so that $x_j = x'_j$ for $j = 1, \dots, k$. Hence for all large enough n,

$$\begin{aligned}
|\hat{\mathcal{L}}^n_{t,z}\phi(x) - \hat{\mathcal{L}}^n_{t,z}\phi(x')| &= |\sum_{\eta \in \mathcal{T}_n} e^{(t\hat{V}+sR)^n(\eta x)}((\phi(\eta x) - \phi(\eta x')) \\
&\qquad + \phi(\eta x')(1 - e^{t\hat{V}^n(\eta x') - t\hat{V}^n(\eta x)}))| \\
&\le \sum_{\eta \in \mathcal{T}_n} e^{(u\hat{V}+sR)^n(\eta x)} \theta^{n+k}(|\phi|_\theta + |\phi|_\infty c_1 e^{c_1}) \\
&\le \theta^k e^{(\hat{P}_{u,|z|}+\varepsilon)n} \|\phi\|_\theta,
\end{aligned}$$

where $c_1 = |u\hat{V}|_\theta$. We have thus shown that $\|\hat{L}^n_{t,z}\phi\|_\theta \le e^{(\hat{P}_{u,|z|}+\varepsilon)n}\|\phi\|_\theta$ for all large enough n. Part (a) of the lemma follows now, since this estimate applies to every positive ε.

To prove (b) we shall in the usual way (see [10]) approximate the operator $\hat{\mathcal{L}}_{t,z}$ by compact operators and use Nussbaum's essential spectral radius formula.

Let N_0 be some integer, and decompose $\mathcal{T}_n$ into the two disjoint sets $\mathcal{T}'_n$ and $\mathcal{T}''_n$, where $\mathcal{T}'_n$ consists of those words $g_0 g_1 \dots g_{n-1}$ for which $R(g_j) \le$

$N_0, j=0,\dots,n-1$, and $\mathcal{T}_n'' = \mathcal{T}_n \setminus \mathcal{T}_n'$. Let $\mathcal{P}_n : \hat{\mathcal{F}}_\theta \to \hat{\mathcal{F}}_\theta$ be the projection operator defined by

$$\mathcal{P}_n\phi(x) = \sum_{\eta\in\mathcal{T}_n'} \phi(x_\eta)\chi_\eta,$$

where χ_η is the characteristic function of the cylinder set $U(\eta) = \{y \in \hat{\Sigma} : y_j = \eta_j, j = 0,\dots,n-1\}$, and $x_\eta \in U(\eta)$ is some (arbitrarily) chosen point.

The operators $\hat{\mathcal{L}}_t\mathcal{P}_n$ are compact since their ranges are finite dimensional vectorspaces. We obtain that

$$\begin{aligned}\hat{\mathcal{L}}_{t,z}^n(\mathrm{id}-\mathcal{P}_n)\phi(x) &= \sum_{\eta\in\mathcal{T}_n'} e^{(t\hat{V}+R\log z)^n(\eta x)}(\phi(\eta x)-\phi(x_\eta)) \\ &\qquad + \sum_{\eta\in\mathcal{T}_n''} e^{(t\hat{V}+R\log z)^n(\eta x)}\phi(\eta x),\end{aligned}$$

where the summation is over those η which satisfy $\hat{M}_{\eta_{n-1}x_0} = 1$. For every $\varepsilon > 0$ we obtain

$$\begin{aligned}|\hat{\mathcal{L}}_{t,z}^n(\mathrm{id}-\mathcal{P}_n)\phi|_\infty &\le \sum_{\eta\in\mathcal{T}_n'} e^{(u\hat{V}+sR)^n(\eta x)}\theta^n|\phi|_\theta + |\phi|_\infty \sum_{\eta\in\mathcal{T}_n''} e^{(u\hat{V}+sR)^n(\eta x)} \\ &\le \theta^n|\phi|_\theta\, e^{(\hat{P}_{u,|z|}+\varepsilon)n} + \varepsilon_n|\phi|_\infty,\end{aligned}$$

for all large enough n, where

$$\varepsilon_n = \sum_{\eta\in\mathcal{T}_n''} e^{(u\hat{V}+sR)^n(\eta x)} \le \theta^n e^{\hat{P}_{u,|z|}n},$$

if we only choose $N_0 = N_0(n)$ big enough. Hence, for all large enough n

$$|\hat{\mathcal{L}}_{t,z}^n(\mathrm{id}-\mathcal{P}_n)\phi|_\infty \le \theta^n e^{(\hat{P}_{u,|z|}+\varepsilon)n}|\phi|_\theta.$$

In order to estimate the variation we proceed as in part (a):

$$\begin{aligned}\mathrm{var}_k\, \hat{\mathcal{L}}_{t,z}^n(\mathrm{id}-\mathcal{P}_n)\phi &\le \sup_{x\in\hat{\Sigma}} \sum_{\eta\in\mathcal{T}_n} e^{(u\hat{V}+sR)^n(\eta x)}(\theta^{n+k}|\phi|_\theta \\ &\qquad + \theta^k|(\mathrm{id}-\mathcal{P}_n)\phi|_\infty c_1 e^{c_1}) \\ &\le \theta^{k+n}e^{(\hat{P}_{u,|z|}+\varepsilon)n},\end{aligned}$$

for all large enough n. This implies by Nussbaum's essential spectrum formula that the essential spectrum of $\hat{\mathcal{L}}_{t,z}$ is contained in the disk $\{w \in \mathbf{C} : |w| \leq \theta e^{\hat{P}_{u,|z|}}\}$. Finally, the proof of (c) follows standard arguments. □

Let us define a one-block map $\iota : \hat{\Sigma} \to \Sigma$ as follows. If y is a point in $\hat{\Sigma}$, then the j-th symbol y_j will be mapped to the string $aa \cdots aa'$, where the element a is $R(y_j) - 1$ times repeated and a, a' are so that $\hat{\pi}(\hat{\sigma}^j(y)) \in A_a$ and $\hat{\pi}(\hat{\sigma}^{j+1}(y)) \in A_{a'}$. Hence the image of the element y_j is a block in Σ of length $R(y_j)$. The image point $\iota(y)$ is now the concatenation of the image blocks for the symbols $y_0, y_1, y_2, \ldots$. Clearly, ι is a one-to-one map. Moreover, although ι is not continuous, its inverse $\iota^{-1} : \Sigma \to \hat{\Sigma}$ is a continuous map. We have the relation

$$\iota^* \hat{V}(y) = \sum_{k=0}^{y_0 - 1} V(\sigma^k(x)).$$

For $i = 1, 2$ let us define the linear operators $\mathcal{L}_{t,i} : C(\Sigma) \to C(\Sigma)$ by $\mathcal{L}_{t,1}\phi(x) = \sum_{a:R(ax)=1} e^{tV(ax)}\phi(ax)$ and $\mathcal{L}_{t,2}\phi(x) = \sum_{a:R(ax)>1} e^{tV(ax)}\phi(ax)$. If we put

$$\mathcal{M}_{t,z} = \iota_* \hat{\mathcal{L}}_{t,z},$$

then we have

$$\mathcal{M}_{t,z} = \sum_{j=1}^{\infty} z^j \mathcal{L}_{t,1} \mathcal{L}_{t,2}^{j-1} = z\mathcal{L}_{t,1}(\mathrm{Id} - z\mathcal{L}_{t,2})^{-1}.$$

so that, denoting $\mathcal{L}_t = \mathcal{L}_{t,1} + \mathcal{L}_{t,2}$ the transfer operator for the map T, we obtain the following operator relation

Lemma 12 *Let $|z| \leq 1$ and and $\mathrm{Re}\, t > 1/\gamma_0$. Then for any $\psi \in C(\Sigma)$ such that $(Id - z\mathcal{L}_{t,2})\psi$ belongs to the space $\mathcal{E}_\theta := \iota^* \mathcal{F}_\theta$, we have*

$$(Id - \mathcal{M}_{t,z})(Id - z\mathcal{L}_{t,2})\, \psi = (Id - z\mathcal{L}_t)\, \psi.$$

This immediately yields the following result.

Proposition 13 *Let $|z| \leq 1$ and $\mathrm{Re}\, t > 1/\gamma_0$. Then $\mathcal{M}_{t,z}$ has an eigenvalue 1 if and only if z^{-1} is an eigenvalue of the same multiplicity for the operator $\mathcal{L}_t$. Moreover, if $\mathcal{M}_{t,z}\psi = \psi$ then $\mathcal{L}_t\phi = z^{-1}\phi$, where $\psi = (Id - z\mathcal{L}_{t,2})\phi$.*

4 Pressure functions and Hausdorff dimension.

A consequence of theorem 11 is that for any real $t > 1/\gamma_0$ and real $z \in (0,1]$ there is an equilibrium state $\rho_{t,z} = \phi_{t,z} m_{t,z}$ on $\hat{\Sigma}$ for the function $t\hat{V} + R \log z$, which is uniquely determined by the normalizing conditions: $\phi_{t,z} > 0$, $m_{t,z}(\phi_{t,z}) = 1$ and

$$\hat{\mathcal{L}}_{t,z}\phi_{t,z} = \hat{\lambda}_{t,z}\phi_{t,z}, \quad \hat{\mathcal{L}}^*_{t,z} m_{t,z} = \hat{\lambda}_{t,z} m_{t,z}$$

where $\hat{\lambda}_{t,z} = \exp \hat{P}_{t,z}$ is the principal eigenvalue of $\hat{\mathcal{L}}_{t,z}$. In particular $\hat{P}_{h,1} = 0$ so that $\hat{\lambda}_{h,1} = 1$ and $\rho_{h,1}$ is the lift of the unique $\hat{T}$-invariant probability measure $\rho = \phi m$ of [1], Section 9, that is $\rho = \hat{\pi}^* \rho_{h,1}$.

Lemma 14 *The function $\hat{\lambda}_{t,z} = \exp \hat{P}_{t,z}$ extends to a holomorphic function in a complex open neighborhood of $\{t : t > 1/\gamma_0\} \times [0,1]$. Moreover, let $h\gamma_0 \leq 2$ be fixed. Then $\lim_{z \to 1_-} d\hat{\lambda}_{h,z}/dz = +\infty$; whereas, if $h\gamma_0 > 2$ then the above limit is finite and equals $\rho(R) \equiv \sum_{\ell=1}^{\infty} \ell \cdot \rho_{h,1}(\mathcal{R}_\ell)$, the mean return time in the set $\mathcal{R}_1$.*

Proof. It will suffice to prove the analytic properties of $\hat{\lambda}_{t,z}$ as a function of z for each fixed $t > 1/\gamma_0$, because then the rest of the statement readily follows from theorem 11 and standard results in regular perturbation theory (see [8], Sections 7.1, 4.3). Now, from lemma 9 it follows in particular that for any fixed $t > 1/\gamma_0$, the map $z \to \hat{\mathcal{L}}_{t,z}$ is an analytic family in the sense of Kato for z in the open unit disk so that the first assertion follows from theorem 11 and Kato-Rellich Theorem (see, e.g., Thm. XII.8 in [15]).
To proceed, we now construct a sequence of compact spaces $\hat{\Sigma}_N$ whose elements are sequences $\hat{\sigma} = (\hat{\sigma}_0 \hat{\sigma}_1 \ldots)$ with $\hat{\sigma}_j \in \{1, \ldots, N\}$. Clearly, $\hat{\Sigma}_N \subset \hat{\Sigma}_{N+1} \subset \ldots \subset \hat{\Sigma}$. For any $0 < z \leq 1$ define a family of operators $\hat{\mathcal{L}}_{t,z,N} : \hat{\mathcal{F}}_\theta(\hat{\Sigma}_N) \to \hat{\mathcal{F}}_\theta(\hat{\Sigma}_N)$ by

$$\hat{\mathcal{L}}_{t,z,N} = \sum_{k=1}^{N} z^{R(k)} \mathcal{K}_{t,k}.$$

Let $\hat{P}_{t,z,N}$ be the pressure of $t\hat{V}|_{\hat{\Sigma}_N} + R|_{\hat{\Sigma}_N} \log z$, so that $\hat{\lambda}_{t,z,N} = \exp \hat{P}_{t,z,N}$ is the simple eigenvalue with largest modulus and $\phi_{t,z,N}$ and $m_{t,z,N}$ are the corresponding eigenvectors for $\hat{\mathcal{L}}_{t,z,N}$ and $\hat{\mathcal{L}}^*_{t,z,N}$, respectively. Now set $z = e^s$.

It is a standard result in the theory of equilibrium states that (see [17], Chapter 5) under the conditions assumed here the function $\hat{P}_{h,e^s,N}$ is real analytic in some neighborhood of $s = 0$ and its derivatives at $s = 0$ are related to suitable moments of ρ_N, the Gibbs state on $\hat{\Sigma}_N$ for the function $\hat{V}|_{\hat{\Sigma}_N}$. In particular we have $d\hat{\lambda}_{h,e^s,N}/ds|_{s=0} = \hat{\lambda}_{h,1,N}\ \rho_N(R)$. To conclude, we now use the fact that (see [4], Theorem 2.1) the triple $\hat{\lambda}_{t,z,N}, \phi_{t,z,N}, m_{t,z,N}$ converge uniformly to $\hat{\lambda}_{t,z}, \phi_{t,z}, m_{t,z}$ for $0 \le z \le 1$ and $t > 1/\gamma_0$ (so that, in particular, $\hat{\lambda}_{h,1,N} \to 1$ and $\rho_N(R) \to \rho(R)$), as $N \to \infty$.

□

Remark. It is easy to see that the pressure $\hat{P}_{t,z}$ is ≤ 0 if t is larger than or equal to the Hausdorff dimension h of J, and $0 < z \le 1$. Indeed, if we choose for every $\eta \in \mathcal{T}_n$ a point $x_\eta \in U(\eta)$ so that $\hat{V}^n(x_\eta) = \min_{x\in B_\eta} \hat{V}^n(x)$, then

$$\sum_{\eta\in\mathcal{T}_n} e^{t\hat{V}^n(\eta^\infty)} \le c_1 \sum_{\eta\in\mathcal{T}_n} e^{t\hat{V}^n(x_\eta)},$$

for some constant c_1. Now, as $\hat{T}^n$ maps B_η into J and $e^{t\hat{V}^n(x_\eta)} = \min_{x\in B_\eta}\left((\hat{T}^n)'\right)^{-t}$, we obtain that

$$\sum_{\eta\in\mathcal{T}_n} e^{t\hat{V}^n(\eta^\infty)} \le c_1 \sum_{\eta\in\mathcal{T}_n} (\mathrm{diam} B_\eta)^t < \infty,$$

which implies that $\hat{P}_{t,z} \le 0$ if $t \ge h$. On the other hand, since

$$\sum_{\ell=1}^{\infty} \rho_{h,1}(\mathcal{R}_\ell) = 1$$

and, by Proposition 6, the measures of the levelsets obey the asymptotic behaviour (with some c_2)

$$\rho_{h,1}(\mathcal{R}_\ell) \sim c_2\, \ell^{-h\gamma_0},$$

as $\ell \to \infty$, one concludes [1] that $h > 1/\gamma_0$.

The function $P_t = P(tV)$ satisfies the variational principle [20]

$$P_t = \sup_\beta \{h_\beta - t\chi_\beta(T)\},$$

where the supremum is over all T-invariant probability measures β on Σ, h_β is the metric entropy of β and $\chi_\beta(T) = \int \log |T'| d\beta$ is the Lyapunov exponent of T with respect to β.
One readily sees that for $t \in [0, h)$ the operator $\mathcal{L}_t$ has the positive leading simple eigenvalue $\lambda(t) = e^{P_t} > 0$. Indeed, we have $r(\mathcal{M}_{t,1}) > r(\mathcal{M}_{h,1}) = 1$ for $t < h$. Then by monotonicity of $\mathcal{M}_{t,z}$ for $z \in \mathbf{R}_+$ one finds a $0 < z(t) < 1$ such that $r(\mathcal{M}_{t,z(t)}) = 1$. By proposition 13 $\lambda(t) = 1/z(t)$ and therefore $\lambda(t) = e^{P_t}$ [17].

Theorem 15 *The function P_t is continuous and non-increasing for $t \in [0, \infty)$ and satisfies $P_t > 0$ for $t \in [0, h)$ and $P_t = 0$ for $t \in [h, \infty)$ (i.e. h is the smallest zero of P_t). Moreover, P_t is real analytic in $t \in [0, h)$ and $t \in (h, \infty)$ ($t = h$ is the only singularity of P_t on $\mathbf{R}_+$). In addition, if $h\gamma_0 \leq 2$ we have*

$$\lim_{t \to h_-} \frac{dP_t}{dt} = 0,$$

and if $h\gamma_0 > 2$, then

$$\lim_{t \to h_-} \frac{dP_t}{dt} = \frac{1}{\rho(R)} \cdot \frac{d\hat{P}_{t,1}}{dt}|_{t=h}.$$

Proof. The first statement follows from [6]. Moreover, from theorem 11 and proposition 13 we have that, for $t \in (1/\gamma_0, h)$,

$$\exp \hat{P}_{t,e^{-P_t}} = 1. \tag{3}$$

Since $\exp \hat{P}_{t,z}$ is jointly analytic in a complex open neighborhood of $0 \leq z \leq 1$ and $t > 1/\gamma_0$ (lemma 14), the second assertion of the lemma follows from the implicit function theorem. To get the announced identities we first set $z = z(t)$ and differentiate $\exp \hat{P}_{t,z(t)}$ w.r.t. t. Then take $z(t) = e^{-P_t} \to 1$ as $t \to h_-$ and use lemma 14 and the identity (3). □

Remark. Notice that

$$\frac{d\hat{P}_{t,1}}{dt}|_{t=h} = -\chi_\rho(\hat{T})$$

where $\chi_\rho(\hat{T}) = \rho(\log |\hat{T}'|) < \infty$ is the Lyapunov exponent of $\hat{T}$ with respect to ρ. Therefore, according to whether $\rho(R)$ is finite or infinite, $\lim_{t \to h_-} \frac{dP_t}{dt}$

is strictly negative or zero. In particular, if $\rho(R) < \infty$, then the variational principle implies that

$$\lim_{t\to h_-} \frac{dP_t}{dt} = -\chi_\mu(T),$$

where $\mu = \frac{1}{\rho(R)}\nu$. In this case, theorem 15 is an Abramov-like formula for the Lyapunov exponent (see [5] for related results).

5 Escape rate.

Let $\delta > 0$ be small and consider the inverse images of the neighbourhood $B_\delta(J)$ under iterates of T. Clearly $|T^{-n}B_\delta(J)| \to 0$ when $n \to \infty$, where the absolute value denotes the (2-dimensional) Lebesgue measure. We are interested in the rate of convergence which is known to be exponential in the case of hyperbolic rational maps, that is

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{|B_\delta(J)|}{|T^{-n}B_\delta(J)|} > 0,$$

if T is hyperbolic, and moreover the limit is then found to be [3] equal to $-P_2$. For parabolic maps, however we have subexponential convergence (i.e. the above limit is zero).
Denote by $\mathcal{D}_n$ the set of critical values of T^n. It is known [2] that the parabolic points are accumulation points of $\mathcal{D}_n$ as $n \to \infty$. However, given $\varepsilon > 0$, if we set $J_\varepsilon := J \setminus B_\varepsilon(\Lambda)$, we can find a $\delta > 0$ so that $B_\delta(J_\varepsilon) \cap \mathcal{D}_n$ is empty for all n.

Theorem 16 *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ so that for all small enough δ:*

$$|T^{-n}B_\delta(J_\varepsilon)| \le |B_\delta(J)| \frac{C}{n^{\gamma_0(2-h)}}.$$

The constant $C(\varepsilon)$ goes to infinity as $\varepsilon > 0$ tends to zero.

Proof. Let $\delta_0 > 0$ be chosen so that $B_{\delta_0}(J_\varepsilon) \cap \mathcal{D}_n = \emptyset$. Let $\delta < \delta_0/2$ and denote by $\mathcal{C}_j$, $j = 1, 2, \ldots, D$, the (finitely many) components of $B_\delta(J_\varepsilon)$. Let $\mathcal{S}_n$ denote the collection of inverse branches of T^n on $B_{\delta_0}(J_\varepsilon)$. Then, by the Köbe distortion theorem [7] there exists a constant $K' > 1$ so that

$K'^{-1} \le |\varphi'(z)/\varphi'(z')| \le K'$ for $z, z' \in \mathcal{C}_j, \forall j$ and for all $\varphi \in \mathcal{S}_n$. Moreover, for $j = 1, \dots, D$, we can find pairs $z_j, z'_j \in \mathcal{C}_j$, so that $z_{j+1} = T^{u_j} z'_j$, $j = 1, \dots$ (mod D), for some integers $u_1, \dots, u_D$. Together with the distortion property this implies that $K^{-1} \le |\varphi'(z)/\varphi'(z')| \le K$ for all $z, z' \in B_\delta(J_\varepsilon)$ and $\varphi \in \mathcal{S}_n$, where $K \ge K'$. (Notice that unlike K', the constant K depends on ε.)
Let us now estimate the area of $T^{-n} B_\delta(J_\varepsilon)$. Clearly

$$|T^{-n} B_\delta(J_\varepsilon)| \le \sum_{\varphi \in \mathcal{S}_n} |\varphi(B_\delta(J_\varepsilon))| \le K^2 \sum_{\varphi \in \mathcal{S}_n} |\varphi'(x)|^2,$$

where x is an arbitrary point in $B_\delta(J_\varepsilon)$. In particular, we can assume that $x \in J_\varepsilon$. This implies that $|T^{-n} B_\delta(J_\varepsilon)| \le K^2 \mathcal{L}_2^n \mathbf{1}(x)$, where

$$\mathcal{L}_2^n \mathbf{1}(x) = \sum_{y \in T^{-n}x} \frac{1}{|(T^n)'(y)|^2} = \sum_{y \in T^{-n}x} \frac{1}{|(T^n)'(y)|^{2-h}} \frac{1}{|(T^n)'(y)|^h}.$$

It thus remains to estimate the terms $|(T^n)'(y)|^{-(2-h)}$, for $y \in T^{-n}x$. If $x \in \mathcal{R}_q$, then $y \in \mathcal{R}_{k_1}$, where $k_1 = k_1(y) \le n + q$ for every $y \in T^{-n}x$. The value q is uniformly in $x \in J_\varepsilon$ bounded by some number $q_0(\varepsilon)$. More generally, there exist numbers $k_1, k_2, \dots, k_\ell \ge 2$ and $n_1, \dots, n_\ell \ge 0$, so that, if we put $y_1 = y$ and $y_{j+1} = T^{k_j + n_j} y_j$ for $j = 1, \dots, \ell$ ($y_{\ell+1} = x$), then

$$\begin{aligned} y_j &\in \mathcal{R}_{k_j}, \\ T^{k_j + i} y_j &\in \mathcal{R}_1, \quad i = 0, 1, \dots, n_j - 1 \end{aligned}$$

for all indices $j = 0, 1, \dots, \ell$ ($k_0 = n_0 = 0$). In addition we have $k_1 + \dots + k_\ell + n_1 + \dots + n_\ell = n + q$ and $n_\ell = 0$ if $q \ge 2$. Thus

$$(T^n)'(y) = (T^{k_\ell - q})'(y_\ell) \prod_{j=1}^{\ell-1} (T^{k_j})'(y_j) \prod_{j=1}^{\ell} (T^{n_j})'(T^{k_j} y_j),$$

where $|(T^{n_j})'(y)| \ge \beta^{n_j}$, $j = 1, \dots, \ell$, and, by Proposition 2, we have $(T^{k_j})'(y)| \ge C_1^{-1} k_j^{\gamma_0}$ for $j < \ell$, and

$$|(T^{k_\ell - q})'(y_\ell)| = \left| \frac{(T^{k_\ell})'(y_\ell)}{(T^q)'(y)} \right| \ge C_1^{-2} \left(\frac{k_\ell}{q} \right)^{\gamma_0}.$$

Therefore

$$|(T^n)'(y)| \ge c_1 \, \beta^{n_1 + \dots + n_\ell} k_1^{\gamma_0} \cdots k_\ell^{\gamma_0} q^{-\gamma_0} \ge c_2 \left(\frac{n+q}{q} \right)^{\gamma_0},$$

for some positive constants c_1, c_2. Since $q \le q_0$, this implies that

$$|\mathcal{L}_2^n \mathbf{1}(x)| \le c_2^{-(2-h)} \left(\frac{n+q}{q}\right)^{-\gamma_0(2-h)} \sum_{y \in T^{-n}x} \frac{1}{|(T^n)'(y)|^h} \le C n^{-\gamma_0(2-h)},$$

as the sum converges to the principal eigenfunction e of $\mathcal{L}_h$ which is uniformly bounded on J outside any open neighbourhood of the parabolic set Λ. The fact that the constant $C = C(\varepsilon)$ becomes arbitrarily large as $\varepsilon \to 0$ is evident from the singular behaviour of the principal eigenfunction of $\mathcal{L}_h$ on the parabolic set and the fact that $q_0(\varepsilon) \to \infty$ as ε goes to zero. □

6 The asymptotic distribution of pre-images.

In this Section we shall combine the construction of Sections 1 and 2 with ideas from [11] to obtain the asymptotic behaviour of some counting functions related to the distribution of the set of pre-images $T^{-k}x = \{y \in J : T^n y = x\}$. More precisely, let $U : J \to \mathbf{R}$ be non-negative and Lipschitz continuous. Having fixed $\epsilon > 0$, define $N(r, x, U)$, for $r \in \mathbf{R}_+$ and $x \in J_\varepsilon \equiv J \setminus B_\varepsilon(\Lambda)$, by

$$N(r, x, U) := \sum_{k=0}^{\infty} \sum_{y \in T^{-k}x} U(y)\, \chi^{(r)}(y),$$

where $\chi^{(r)}(y) = 1$ if $\log|(T^k)'(y)| \le r$ and $\chi^{(r)}(y) = 0$ otherwise. Notice that since $x \in J_\varepsilon$ we have $\log|(T^k)'(y)| > 0$ if $T^k y = x$. Therefore $N(r, x)$ is finite for all $r \ge 0$. Moreover $N(r, x, U)$ is nonnegative, nondecreasing in r and satisfies the *renewal equation*:

$$N(r, x, U) \;=\; U(x) \;+ \sum_{y \in T^{-1}x} N(r - \log|T'(y)|, y, U). \tag{4}$$

The next result will show that N is of exponential class in the variable r.

Lemma 17 *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ so that*

$$N(r, x, U) \le C e^{r(h + 1/\gamma_0)}.$$

Proof. Let $\varepsilon > 0$, then there exists an number K, so that $\mathcal{R}_k \subset B_\varepsilon(\Lambda)$ for all $k \geq K$. Since $x \in J_\varepsilon$ we have $x \in \mathcal{R}_j$ for some $j < K$, and for $y \in T^{-k}x$ we can find a constant $c_1 > 0$ so that

$$|(T^k)'(y)| = \frac{|(T^{k+j})'(y)|}{|(T^j)'(x)|} \geq c_1 \left(\frac{k+j}{j}\right)^{\gamma_0},$$

because y lies in some levelset $\mathcal{R}_{k'}$ where k' is at most $k+j$. The upper bound $|(T^k)'(y)| \leq e^r$ now implies that in the first sum of equation (4) the summation is over

$$k \leq j\left(\frac{e^{r/\gamma_0}}{c_2} - 1\right) \leq \frac{K}{c_2} e^{r/\gamma_0},$$

with $c_2 = c_1^{1/\gamma_0}$. We thus obtain

$$\begin{aligned} N(r,x,U) &= \sum_{k=0}^{[\frac{K}{c_2}e^{r/\gamma_0}]} \sum_{y \in T^{-k}x} U(y)\, \chi^{(r)}(y) \\ &\leq |U|_\infty \sum_{k=0}^{[\frac{K}{c_2}e^{r/\gamma_0}]} \sum_{y \in T^{-k}x} \frac{e^{hr}}{|(T^k)'(y)|^h} \\ &\leq c_3 |U|_\infty \frac{K}{c_2} e^{r(h+1/\gamma_0)} \end{aligned}$$

where $c_3 = \sup_{x \in J_\varepsilon} |e(x)|$ is finite since the principal eigenfunction e of $\mathcal{L}_h$ is regular outside the set Λ of parabolic points. Now put $C = c_3 |U|_\infty \frac{K}{c_2}$. □

We shall now investigate the asymptotic behaviour of $N(r,x,U)$ as $r \to \infty$. To this end, we consider the Fourier-Laplace transform:

$$\hat{N}(t,x,U) = \int_0^\infty e^{-tr}\, N(r,x,U)\, dr \tag{5}$$

Equation (4) transforms as (notice that $N(r,x,U) = 0$ for $r < 0$)

$$\hat{N}(t,x,U) \;=\; U(x) \;+\; \mathcal{L}_t \hat{N}(t,x,U), \tag{6}$$

where $\mathcal{L}_t$ is the transfer operator for the map T. Therefore, using proposition 12, we get[1]

[1]In what follows we shall not avoid the painless confusion between objects living on symbol space and their lifts on J.

$$\hat{N}(t,x,U) \;=\; (\mathrm{Id}-\mathcal{L}_t)^{-1}\,U(x) = (\mathrm{Id}-\mathcal{L}_{t,2})^{-1}\,(\mathrm{Id}-\mathcal{M}_t)^{-1}U(x) \qquad (7)$$

where we have used the shorthand $\mathcal{M}_t \equiv \mathcal{M}_{t,1}$. We now study the operator-valued function $t \to (\mathrm{Id}-\mathcal{M}_t)^{-1}$. We first recall that $t \to \mathcal{M}_t$ is holomorphic in the half-plane $\{\, t \,:\, \mathrm{Re}\, t \,>\, 1/\gamma_0 \,\}$. Therefore, if $(\mathrm{Id}-\mathcal{M}_t)^{-1}$ exists (as a bounded operator acting on $\mathcal{E}_\theta$) for all t in some open subset $D \subseteq \{\, t \,:\, \mathrm{Re}\, t > 1/\gamma_0 \,\}$ then $t \to (\mathrm{Id}-\mathcal{M}_t)^{-1}$ is holomorphic in D. By virtue of theorem (11) the spectral radius of $\mathcal{M}_t : \mathcal{E}_\theta \to \mathcal{E}_\theta$ is bounded above by $\exp \hat{P}_{\mathrm{Re}\, t,1}$ which is < 1 for $1/\gamma_0 < \mathrm{Re}\, t < h$. This proves the following

Lemma 18 *The function* $t \to (Id{-}\mathcal{M}_t)^{-1}$ *is holomorphic in the strip* $1/\gamma_0 < \mathrm{Re}\, t < h$.

Let $\hat{\lambda}_t \equiv \hat{\lambda}_{t,1}$ and ψ_t, m_t be such that $\mathcal{M}_t\,\psi_t = \hat{\lambda}_t\,\psi_t$ and $\mathcal{M}_t^*\,m_t = \hat{\lambda}_t\,m_t$, so that $m_h \equiv m$ is (the lift of) the unique h-conformal measure on J and $m_t(\psi_t) = m(\psi_t) = 1$. Reasoning as in the proof of lemma 14 we see that the functions $t \to \hat{\lambda}_t$, $t \to \psi_t$ and $t \to m_t$ extend to holomorphic functions in a neighbourhood N_0 of the half-line $t > 1/\gamma_0$. For $U \in \mathcal{E}_\theta$ and $t \in N_0$ we then decompose

$$\mathcal{M}_t\,U = \hat{\lambda}_t\,\psi_t, m_t(U) + \mathcal{N}_t\,U \qquad (8)$$

where $\mathcal{N}_t$ maps $\mathcal{E}_\theta$ onto the subspace $\{U \in \mathcal{E}_\theta \;:\; m_t(U) = 0\}$. Moreover, the subspace generated by ψ_t is one-dimensional and the spectral radius of $\mathcal{N}_t$ is strictly smaller than $\hat{\lambda}_{\mathrm{Re}\, t}$. Therefore, since the spectral radius is a lower semicontinuous function of t, there is a neighbourhood H of $t = h$ and $\epsilon > 0$ such that the spectral radius of $\mathcal{N}_t$ is smaller than $1-2\epsilon$ for all $t \in H \cap N_0$. Spectral radius formula then implies that $\|\mathcal{N}_t^n\|_\theta \le (1-\epsilon)^n$ for n large enough and thus $t \to (\mathrm{Id}-\mathcal{N}_t)^{-1}$ is a holomorphic operator-valued function of $t \in H \cap N_0$. Now, since $\psi_t(x)$ and $m_t(U)$ are continuous at $t = h$, we have the following

Lemma 19 *The function* $t \to (Id - \mathcal{M}_t)^{-1}$ *has a simple pole at* $t = h$. *In particular, for each* $U \in \mathcal{E}_\theta$ *and* t *in some punctured neighbourhood of* $t = h$, *we have*

$$(Id - \mathcal{M}_t)^{-1}U(x) = \frac{\psi_t(x)\, m_t(U)}{1-\hat{\lambda}_t} \; + (Id - \mathcal{N}_t)^{-1}U(x)$$

Putting together the above and formula (7) we have that $\hat{N}(t,x,U)$ has a simple pole at $t=h$ with residue

$$\frac{(\mathrm{Id}-\mathcal{L}_{h,2})^{-1}\psi_h(x)\,m(U)}{(-(d/dt)\hat{\lambda}_t)_{t=h}} = \frac{e(x)\,m(U)}{\chi_\rho(\hat{T})} = \frac{e(x)\,m(U)}{\chi_\nu(T)} =: f(x,U),$$

where, according to proposition 13, $e(x) := (\mathrm{Id}-\mathcal{L}_{h,2})^{-1}\psi_h(x)$ is the eigenfunction of $\mathcal{L}_h$ to the eigenvalue 1, and $\chi_\nu(T)$ is the Lyapunov exponent of the sigma-finite measure ν w.r.t. the map T (the identity $\chi_\rho(\hat{T}) = \chi_\nu(T)$ follows from theorem 15 if ν is finite and can be otherwise easily obtained by direct computation, using proposition 7). Let us put $s = t/h$ and rewrite equation (5) as a Laplace-Stieltjes transform,

$$\hat{N}(s,x,U) = \int_0^\infty e^{-sr}\,dF(r), \quad F(r) = \int_0^r N(\frac{u}{h},x,U)\,\frac{du}{h} \tag{9}$$

Then the above implies that $\hat{N}(s,x,U)$ converges for $\mathrm{Re}\, s > 1$ and $\hat{N}(s,x,U) - f(x,U)/h(s-1)$ is analytic at $s=1$. Ikehara's Tauberian theorem (see, e.g., [14]) along with positivity and monotonicity of $N(r,x,U)$ in r then yield the following result

Theorem 20 *Let $U : J \to \mathbf{R}$ be non-negative and Lipschitz continuous. Then we have, as $r \to \infty$,*

$$N(r,x,U) \sim f(x,U)\,e^{h\,r}$$

uniformly for $x \in J_\varepsilon$.

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