Topological entropy of generalised polygon exchanges

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February 19, 1995

Abstract

We study the topological entropy of a class of transformations with mild singularities: the generalized polygon exchanges. This class contains, in particular, polygonal billiards. Our main result is a geometric estimate, from above, on the topological entropy of generalized polygon exchanges. One of the applications of our estimate is that the topological entropy of polygonal billiards is zero. This implies the subexponential growth of various quantities associated with a polygon. Other applications are to the piecewise isometries in two dimensions, and to billiards in rational polyhedra.

[*] Mathematics Department, USC, Los Angeles, 90089-1113. The work is partially supported by NSF Grants DMS-9013220, DMS-9400295

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1 Introduction

Let $T : X \mapsto X$ be a continuous selfmapping of a compact metric space. The topological entropy is an important invariant that characterizes by a number, $h(T) \geq 0$, the exponential complexity of the orbits of T (see, e. g., [37]). If $T : M \mapsto M$ is a diffeomorphism of a compact manifold, there are estimates on $h(T)$ in terms of the geometry of T and M (see, e. g., [21]). Suppose that $T = T^t$ is the geodesic flow on a compact Riemannian manifold, N, of negative curvature. Then there are explicit bounds on $h(T)$ from above and from below in terms of the average curvature of N, its volume, diameter, etc. (see [34, 18, 27, 29]). Theorems of another type provide estimates on $h(T)$ in terms of the growth rates of various geometric quantities on M, under the iterations of T ([32, 39, 30, 10, 20, 28]). The following result of S. Newhouse served as a partial motivation for the present work.

Let $T : M \mapsto M$ be as above. For a compact C^1 curve, γ, in M, denote by $||\gamma||$ its length with respect to a Riemannian metric on M. Then

$$\lambda(T) = \sup_{\gamma} \limsup_{n\to\infty} \log ||T^n\gamma||/n \tag{1}$$

does not depend on the choice of a metric, and we say that $\lambda(T)$ is the *growth rate for the length of curves* in M.

Theorem 1.1 [30] *Let M be a compact C^2 surface, and let $T : M \mapsto M$ be a $C^{1+\epsilon}$ diffeomorphism. Then $h(T) \leq \lambda(T)$.*

A metric entropy version of this estimate (for smooth invariant measures) goes back to [24]. Let $T : M \mapsto M$ be a continuous selfmapping of a compact closed C^1 manifold. A. Manning has obtained a lower bound on $h(T)$.

Theorem 1.2 [28] *Let the setting be as above, and let $\tau_1 : H_1(M) \mapsto H_1(M)$ be the induced transformation on the first homology. Then the topological entropy of T is bounded from below by the logarithm of the spectral radius of τ_1.*

Relatively little is known about the topological entropy of the transformations with singularities, although they arise naturally, e. g., in the billiard dynamics. Let us briefly survey the related work. Pesin and Pitskel [31] have worked on the variational principle for the topological entropy for a particular class of singular transformations. Their approach is topological (continuous selfmaps of noncompact metric spaces), and goes back to Bowen [4]. There is some literature on the topological entropy (suitably interpreted) of singular transformations, arising in billiard dynamics. A. Katok [19] has shown that the topological entropy of polygonal billiards is zero (see also [9]). N. Chernov has obtained estimates on the topological entropy of Sinai (and related) billiards [6]. Chernov's results extend to the hyperbolic billiards the classical connection between the entropy of a diffeomorphism and the growth rate of the number of its periodic points [20]. Note that the metric (as opposed to topological) entropy of transformations with singularities has received more attention (see, e. g., [22, 25]).

In the present work we introduce a class of transformations with mild singularities: the *generalized polytope exchanges* (Definition 2.3). In this setting, the space X (we assume X is a manifold with a boundary) is partitioned into a finite number of *generalised polytopes*, $P_i, 1 \leq i \leq n$. The transformation $T : X \mapsto X$ is a diffeomorphism on each P_i, and, in general, T is not well defined on the singular set, $\partial\mathcal{P} = \cup_{1\leq i\leq n}\partial P_i$. A simple example of a generalized polytope exchange is a *generalized interval exchange*. In this case, X is an interval (e. g., $X = [01]$), and $P_i, 1 \leq i \leq n$, are closed subintervals.

We study the topological entropy, $h(T) \geq 0$, of the generalized polytope exchanges, which measures the exponential complexity of the regular orbits of T. The emphasis of our work is on the geometric bounds for the entropy. Our techniques are especially suitable for the two-dimensional case, i. e., for the *generalized polygon exchanges* (GPEs). Theorem 3.7 and Theorem 3.15 estimate from above the topological entropy of a GPE, $T : X \mapsto X$, by the growth rate of line elements on X under the iterations of T. They are analogs, in the present context, of the estimate of Theorem 1.1.

The framework of GPEs is wide enough to include transformations arising in very different situations. For instance, the 'baker transformation' [1], on one hand, and the Poincare maps for polygonal billiards and outer billiards [11, 16], on the other hand, are GPEs. There are many open questions about the (generalized) polytope exchanges. The one-dimensional case has been well researched. The interval exchange transformations have attracted much attention, partly because of the connection with the billiards in rational polygons (see, e. g., [36, 23]). A direct analog of interval exchange transformations in two dimensions are the *rectangle exchange transformations*. The only work on the subject known to us [17], discusses the minimality of rectangle exchange transformations, and formulates a few questions. Our Theorem 4.2 answers, in particular, a question of [17].

A few words about our techniques. With a generalized polytope exchange, $T : X \mapsto X$, we associate an increasing sequence, $\mathcal{P}_n, n \geq 1$, of partitions of X. A crucial point is to relate the growth rate, h, of the cardinality, $|\mathcal{P}_n|$, and the growth rate, λ, of the length of the singular set, $\partial\mathcal{P}_n$ (Theorem 3.6). Roughly speaking, the length limits the number of atoms in a partition. In order to implement this idea, we need to get a handle on the growth rate, β, of the maximal number of edges at a vertex of $\mathcal{P}_n$ (the *vertex complexity*). A combinatorial argument, devised for two dimensions, allows us to obtain an efficient estimate on β (Proposition 3.9). This estimate enables us to show that $h \leq \lambda$ (Theorem 3.7).

Getting an efficient bound on vertex complexity has been central in a variety of problems in dynamics. Some of them are closely related to our context [19], others are quite far away [33, 5, 35].

To illustrate our results, we apply them in §4 to a variety of transformations: piecewise isometries on surfaces, polygonal billiards, and directional billiards in rational polyhedra. We prove that their topological entropy is zero. This implies, in particular, the subexponential growth of various geometric quantities associated with a polygon or a polyhedral surface. Other proofs of the subexponential growth in this context involved a detailed analysis of the topology of billiard orbits [19, 9]. Our proof exploits the elementary observation that the length of singular sets for polygonal billiards grows quadratically.

Some of the results of the present work were announced in [14, 13]. We thank N. Chernov, A. Katok, and M. Rychlik for discussions and remarks.

2 Generalized Polytope Exchanges

2.1 The transformations

By a *polytope*, P, $\dim P = n$, we mean a compact Euclidean polytope [8] in $\mathbf{R}^n$, with a nonempty interior, $\text{int}P$. The m-dimensional faces of P are polytopes of dimension $m < n$. *Polygons (polyhedra)* are the polytopes of dimension 2 (3).

Definition 2.1 *(I) A* partition $\mathcal{P}$ of a polytope P *is a representation* $P =$

$\bigcup_{1\le i\le n} P_i$, *where* P_i *are* subpolytopes *of* P, *and* $intP_i \cap intP_j = \emptyset$ *for* $i \neq j$. *(II) A* generalized polytope X *of dimension* n *is a closed subset of a* ($C^r, r \geq 1$) *manifold,* M^n*, and a mapping* $f : X \mapsto \mathbf{R}^n$ *such that: 1)* f *extends to a diffeomorphism on an open set* O*,* $X \subset O \subset M$*; 2)* $f(X)$ *is a polytope.*

The mappings $f : X \mapsto \mathbf{R}^n$ will be implicit in what follows.

Definition 2.2 *A* space X *(of dimension* d*)* with a (generalized) polytope partition $\mathcal{P}$ *is a closed subset of a manifold* M^d*, and a representation,* $X = \bigcup_{1\le i\le n} X_i$*, satisfying the following conditions.*
i) The sets X_i *are generalized polytopes for* $1 \leq i \leq n$.
ii) For $i \neq j$*,* $intX_i \cap int\, X_j = \emptyset$.
iii) If $I \subset \{1, \ldots, n\}$ *is such that* $\bigcap_{i\in I} X_i \neq \emptyset$*, then* $\bigcup_{i\in I} X_i$ *is a generalized polytope.*

If $(X, \mathcal{P})$ is as above, we say that X_i are the *atoms* of $\mathcal{P}$, and that $\partial\mathcal{P} = \bigcup_{1\le i\le n} \partial X_i$ is the *boundary* of $\mathcal{P}$. We use notation $\mathrm{int}\mathcal{P} = X \setminus \partial\mathcal{P}$. A partition $\mathcal{Q}$ is a *refinement* of a partition $\mathcal{P}$ if every atom of $\mathcal{Q}$ is a subpolytope of an atom of $\mathcal{P}$. We denote this by $\mathcal{P} < \mathcal{Q}$. If $\mathcal{P} : X = \bigcup_{i\in I} Y_i$ and $\mathcal{Q} : X = \bigcup_{j\in J} Z_j$ are two partitions of X, their *join* $\mathcal{P} \vee \mathcal{Q}$ is the partition formed by the intersections $Y_i \cap Z_j$ such that $\mathrm{int}Y_i \cap \mathrm{int}Z_j \neq \emptyset$. Then $\mathcal{P}, \mathcal{Q} < \mathcal{P} \vee \mathcal{Q}$.

The terminology above is similar to the standard one, used in measure theory [37]. Since we are usually dealing with generalized polytope partitions, we often delete "generalized, polytope", and speak simply of partitions.

Definition 2.3 *A* generalized polytope exchange*,* $T : X \mapsto X$*, on a partition* $\mathcal{R}$*, is a collection of diffeomorphisms,* $T|_P : P \to X$ *(*$P \in \mathcal{R}$*), such that* $X = \bigcup_{P\in\mathcal{R}} T(P)$ *is a partition,* $\mathcal{R}_1 = T\mathcal{R}$*. Then* T^{-1} *is also a generalized polytope exchange.*

The *defining partition*, $\mathcal{R}$, plays an essential role in the definition above. In what follows we make a simplifying assumption (without loss of generality) that both T, T^{-1} are defined on $\mathcal{R}$. Note it does not mean $\mathcal{R}_1 = \mathcal{R}$. When $\dim X = 2$ (3) we will speak of a *generalized polygon (polyhedron) exchange*.
Examples. 1. A partition of $X = [0, 1]$ is given by n intervals $P_i = [a_{i-1}, a_i]$. where $0 = a_0 < a_1 < \ldots < a_n = 1$. An *interval exchange* on the intervals $P_i, 1 \leq i \leq n$, (see, e.g., [7]) is a special case of Definition 2.3. We obtain a

generalized interval exchange by taking n nondegenerate affine transformations T_i, such that the intervals $Q_i = T_i(P_i), 1 \leq i \leq n$, form a partition of $[0,1]$.

2. Let X be a rectangle in $\mathbf{R}^2$, e.g., $X = [0,1] \times [0,1]$. Let the atoms of the partition $\mathcal{P} : X = \bigcup_{1 \leq i \leq n} P_i$ be rectangles (with vertical and horizontal sides). Let t_i be n vectors such that the rectangles $Q_i = P_i + t_i, 1 \leq i \leq n$, form a partition of X. This defines a *rectangle exchange*.

2'. Here $X \subset \mathbf{R}^2$ is an arbitrary polygon, the atoms of a partition $\mathcal{P} : X = \bigcup_{1 \leq i \leq n} P_i$ are subpolygons. We define an affine polygon exchange on X by n nondegenerate affine transformations T_i such that $\mathcal{Q} = \{Q_i = T_i(P_i), 1 \leq i \leq n\}$ is a partition of X.

3. An obvious analog of Example 2 in 3D features the unit cube $X = [0,1] \times [0,1] \times [0,1] \subset \mathbf{R}^3$ partitioned by n *rectangular parallelepipeds* P_i. The mappings T_i are parallel translations $x \to x + t_i$ such that the parallelepipeds $Q_i = P_i + t_i$ form a partition of X.

3'. The 3D version of Example 2': X is a polyhedron, $\mathcal{P} = \bigcup_{1 \leq i \leq n} P_i$ is a polyhedral partition of X, and T_i are affine transformations of $\mathbf{R}^3$ such that the polyhedra $Q_i = T_i(P_i), 1 \leq i \leq n$, form a partition of X. This defines an affine polyhedron exchange.

2.2 A regularisation

We will associate with any generalized polytope exchange a homeomorphism of a compact. To save space, we supress the adjective "generalized", and will often use "transformation" for a polytope exchange.

Let $T : X \to X$ be a polytope exchange on a partition $\mathcal{R}$. For any pair of indices, $m, n \geq 0, m+n > 0$, we define, by induction, the partition $\mathcal{R}_{(-m,n)} = T^{-m+1}\mathcal{R} \vee \cdots \vee \mathcal{R} \vee \cdots \vee T^{n-1}\mathcal{R}$. For $n > 0$ set $\mathcal{P}_n = \mathcal{R}_{(-n,0)}, \mathcal{Q}_n = \mathcal{R}_{(0,n)}$, and $\mathcal{R}_{(0,0)}$ is the empty partition. Let $A \subset \mathbf{Z}^2$ be the set of pairs: $-m \leq 0 \leq n$. For $\alpha = (-m, n), \alpha' = (-m', n')$, we set $\alpha < \alpha'$ if $-m' \leq -m$ and $n \leq n'$. This defines a partial order on A, and A is *directed*, i.e. for any $\alpha, \beta \in A$ there exists $\gamma \in A$, such that $\alpha, \beta < \gamma$. If $\alpha = (-m, n)$ and $\beta = (-m', n')$, we set $\max(\alpha, \beta) = (-\max(m, m'), \max(n, n'))$. Let $\rho = (1, 1) \in \mathbf{Z}^2$. The following is straightforward.

Lemma 2.4 *In the notation above, we have: i) If $\alpha \leq \beta$ then $\mathcal{R}_\alpha < \mathcal{R}_\beta$; ii) If $\alpha, \beta \in A$ and $\gamma = \max(\alpha, \beta)$, then $\mathcal{R}_\alpha \vee \mathcal{R}_\beta = \mathcal{R}_\gamma$; iii) The transformation*

$T^k : X \to X$ *is a polytope exchange on* $\mathcal{R}_\alpha$*, if* $\alpha, \alpha + k\rho \in A$.

For any X with a partition $\mathcal{P} = \{P_i, i \in I\}$, the space $X_{\mathcal{P}} = \bigsqcup_{i\in I} P_i$, with the topology of a disjoint union, has a natural (disjoint) partition, $\hat{\mathcal{P}} = \{P_i, i \in I\}$. Let $\pi^{\mathcal{P}} : X_{\mathcal{P}} \mapsto X$ be the projection. A polytope exchange $T : X \to X$, on a partition $\mathcal{P}$, with $\mathcal{Q} = T(\mathcal{P})$, induces a homeomorphism, $T_{\mathcal{P}} : X_{\mathcal{P}} \to X_{\mathcal{Q}}$. If $\mathcal{P} < \mathcal{P}'$, then $\mathcal{Q} < \mathcal{Q}'$, and we denote by $\pi^{\mathcal{P}'}_{\mathcal{P}} : X_{\mathcal{P}'} \mapsto X_{\mathcal{P}}$, $\pi^{\mathcal{Q}'}_{\mathcal{Q}} : X_{\mathcal{Q}'} \mapsto X_{\mathcal{Q}}$ the natural projections. The mappings $\pi^{\mathcal{P}'}_{\mathcal{P}}, \pi^{\mathcal{Q}'}_{\mathcal{Q}}, \pi^{\mathcal{P}}, \pi^{\mathcal{Q}}, T_{\mathcal{P}}, T_{\mathcal{P}'}$ satisfy the obvious relations, e. g., $T_{\mathcal{P}}\pi^{\mathcal{P}'}_{\mathcal{P}} = \pi^{\mathcal{Q}'}_{\mathcal{Q}} T_{\mathcal{P}'}$.

Let $T : X \to X$ be a polytope exchange on $\mathcal{R}$, and let partitions $\{\mathcal{R}_\alpha, \alpha \in A\}$ be as above. For $\alpha \in A$ we denote by X_α the space $X_{\mathcal{R}_\alpha}$, and by $\pi^\alpha : X_\alpha \to X$, $\pi^\beta_\alpha : X_\beta \mapsto X_\alpha$ $(\alpha < \beta)$ the associated projections. If $\alpha, \alpha + k\rho \in A$, we denote by $T^k_\alpha : X_\alpha \to X_{\alpha+k\rho}$ the homeomorphism corresponding to T^k.

The system of compacta $\{X_\alpha : \alpha \in A\}$ and continuous maps $\{\pi^\beta_\alpha : X_\beta \mapsto X_\alpha, \alpha < \beta\}$ is an *inverse limit spectrum*, and let $\hat{X}$ be the *inverse limit space* [38]. By construction, $\hat{X}$ is endowed with the family of projections $\{\pi_\alpha : \hat{X} \mapsto X_\alpha, \alpha \in A\}$, satisfying $\pi_\alpha = \pi^\beta_\alpha \pi_\beta$ for $\alpha < \beta$. If Y is any space with a system $\{t_\alpha : Y \mapsto X_\alpha, \alpha \in A\}$ of maps satisfying the requirements above, then there is a unique surjection, $f : Y \mapsto \hat{X}$, such that $t_\alpha = \pi_\alpha f$ for all α.

Proposition 2.5 *Let* $T : X \to X$ *be a polyhedron exchange on* $\mathcal{R}$ *(and let the notation be as above). Then there is a unique homeomorphism* $\hat{T} : \hat{X} \mapsto \hat{X}$ *such that if* $\alpha, \alpha \pm \rho \in A$*, then* $T^{\pm 1}_\alpha \pi_\alpha = \pi_{\alpha \pm \rho} \hat{T}^{\pm 1}$.

Proof. A subset $B \subset A$ is *cofinal* if for any $\alpha \in A$ there is $\beta \in B$ such that $\alpha < \beta$. Any subset, $B \subset A$, inherits a partial order. If we denote by $\hat{X}_B$ the corresponding inverse limit space, then there is a canonical map (onto) $\phi_B : \hat{X} \mapsto \hat{X}_B$ which is compatible with the structures involved. If B is cofinal, then ϕ_B is a homeomorphism.

For $k \geq 0$, we denote by $A_k \subset A$ the subset $A_k = \{(-m, n) : m, n \geq k\}$. Then $A = A_0 \supset A_1 \supset \cdots \supset A_k \supset \cdots$, and every A_k is cofinal in A. Denote by $\hat{X}_k$ the inverse limit space corresponding to A_k, let $\pi_k : \hat{X}_k \to X$ be the projections, and $\phi_k : \hat{X} \to \hat{X}_k$ be the homeomorphisms. If $\alpha \in A_k$, and $|j| < k$, then $\alpha + j\rho \in A_{k-|j|}$. By preceding discussion and Lemma 2.4, this uniquely defines continuous mappings, $T^j_k : \hat{X}_k \to \hat{X}_{k-|j|}$, compatible with π_r, ϕ_s, hence the homeomorphisms, $\hat{T}^j : \hat{X} \to \hat{X}$. Finally, $\hat{T}^j = (\hat{T})^j$. $\square$

The homeomorphism $\hat{T} : \hat{X} \to \hat{X}$ depends on $\mathcal{R}$, and we use notation $\hat{T}_{\mathcal{R}} : \hat{X}_{\mathcal{R}} \to \hat{X}_{\mathcal{R}}$ for this *regularization.* If $\mathcal{R} < \mathcal{S}$, there is a unique projection, $\hat{X}_{\mathcal{S}} \to \hat{X}_{\mathcal{R}}$, compatible with $\hat{T}_{\mathcal{R}}, \hat{T}_{\mathcal{S}}$. Let $\mathcal{R} = \{R_i, i \in I\}$. A sequence of indices, $(s,n) = (i_{-n+1}, \dots, i_{n-1})$, is a *code* if the set $R_{(s,n)} = \{x : T^k x \in R_{i_k}, |k| < n\}$ has a nonempty interior. Let $S^{(n)}$ be the set of n-codes, and let $S = S_{\mathcal{R}}$ be the set of doubly infinite sequences $s = (i_k \in I : -\infty < k < \infty)$, such that for any $n > 0$ the truncated sequence $(s,n) = (i_{-n+1}, \dots, i_0, \dots, i_{n-1}) \in S^{(n)}$. Let Σ be the *full shift* on $|I|$ symbols, and let $t : \Sigma \to \Sigma$ be the left shift transformation. Then $S \subset \Sigma$ is a *subshift.* For $s \in S$ set $R_s = \bigcap_{n \geq 1} R_{(s,n)} \subset X$. Then T induces a homeomorphism of R_s onto $R_{t(s)}$.

Proposition 2.6 *Let $T : X \mapsto X$ be a polytope exchange on a partition $\mathcal{R}$, and let the notation be as above. Then: 1) there is a unique continuous semiconjugacy (onto) $\sigma : \hat{X} \mapsto S$; 2) the mapping $\pi \times \sigma : \hat{X} \mapsto X \times S$ is a homeomorphism, and its image is $\{(x,s) : x \in R_s\} \subset X \times S$.*

Proof. By construction, $R_s = \bigcap_{n \geq 0} R_{(s,n)} \neq \emptyset$. Set $X_n = \bigsqcup_{S_n} R_{(s,n)}$, and let $\sigma_n^{n+1} : S_{n+1} \mapsto S_n,\ \pi_n^{n+1} : X_{n+1} \mapsto X_n$ be the natural mappings. The set $B = \{(-n,n), n \geq 0\} \subset A$ is cofinal, hence, the inverse limit space $\lim_{\leftarrow} X_n$, as $n \to \infty$, is canonically homeomorphic to $\hat{X}$, which defines $\hat{T}$ on $\lim_{\leftarrow} X_n$. The action of $\hat{T}^{\pm 1}$ on $\lim_{\leftarrow} X_n$ is obtained from the commutative diagram:

$$\begin{tikzcd}
\hat{X} \arrow[r] & \cdots \arrow[r] & X_{n+1} \arrow[r, "\pi_n^{n+1}"] & X_n \arrow[r, "\pi_{n-1}^{n}"] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X \\
\hat{X} \arrow[u, "\hat{T}"] \arrow[r] & \cdots \arrow[r] & X_{n+1} \arrow[ur, "T_{n+1}^{\pm 1}"] \arrow[r, "\pi_n^{n+1}"] & X_n \arrow[ur, "T_n^{\pm 1}"] \arrow[r, "\pi_{n-1}^{n}"] & X_{n-1} \arrow[r] & \cdots \arrow[ur] \arrow[r] & X_1 \arrow[r] & X \arrow[u, "T"']
\end{tikzcd}$$

Diagram 1

The natural projections $\rho_n : X_n \mapsto S_n,\ n \geq 0$ define a mapping of the inverse limit spectra: $\{X_n,\ \pi_n^{n+1} : X_{n+1} \mapsto X_n, n \geq 1\} \to \{S_n,\ \rho_n^{n+1} : S_{n+1} \mapsto S_n, n \geq 1\}$. This yields a mapping of the inverse limit spaces, $\sigma : \hat{X} \mapsto S$, and the commutative diagram:

$$
\begin{tikzcd}
\hat{X} \arrow[r] \arrow[d, "\sigma"] & \cdots \arrow[r] & X_{n+1} \arrow[r, "\pi_n^{n+1}"] \arrow[d, "\sigma_{n+1}"] & X_n \arrow[r, "\pi_{n-1}^n"] \arrow[d, "\sigma_n"] & X_{n-1} \arrow[r] \arrow[d, "\sigma_{n-1}"] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, "\sigma_1"] & X \\
S \arrow[r] & \cdots \arrow[r] & S_{n+1} \arrow[r, "\rho_n^{n+1}"] & S_n \arrow[r, "\rho_{n-1}^n"] & S_{n-1} \arrow[r] & \cdots \arrow[r] & S_1 = I &
\end{tikzcd}
$$

Diagram 2

The claimed properties of $\hat{T}$ and σ are immediate from the diagrams above. $\square$

2.3 The entropy of a polytope exchange

If $\mathcal{P}$ is a finite partition (polyhedral or set-theoretic), $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$. Recall that if $F : Y \mapsto Y$ is a homeomorphism of a compact metric space, and α is an open cover of Y, then $h(F)$ $(h(F, \alpha))$ is the *topological entropy* of F (*relative to* α) [37].

Definition 2.7 *Let $T : X \to X$ be a polytope exchange on a partition $\mathcal{R}$, and let the notation be as above. We call $h(\hat{T}, \hat{\mathcal{R}})$ the* (topological) entropy *of T* relative to $\mathcal{R}$, *and $h(\hat{T})$ is the* (topological) entropy *of T. We will use notation $h(T, \mathcal{R})$ and $h(T)$ for the entropies.*

A point, $x \in X$, is *regular* if $T^k x$ is defined for all k. Let X_{reg} be the set of regular points. An *invariant measure*, μ, on X, satisfies $\mu(X_{reg}) = 1, T_*\mu = \mu$, and let $h_\mu(T)$ be the *metric entropy.* For future reference we list the basic properties of the entropy in the present context. They follow from the standard material [37].

Proposition 2.8 *Let $T : X \to X$ be a polytope exchange on $\mathcal{R}$, and let the notation be as above. Then:*
1) We have

$$h(T, \mathcal{R}) = \lim_{n\to\infty} \log |\mathcal{P}_n|/n = h(T^{-1}, \mathcal{R}) = h(t). \tag{2}$$

2) Set $\mathcal{R}_n = \mathcal{P}_n$ if $n > 0$, and $\mathcal{R}_n = \mathcal{Q}_{-n}$ if $n < 0$. Then

$$h(T^n, \mathcal{R}_n) = |n| h(T, \mathcal{R}). \tag{3}$$

3) For any partition $\mathcal{P}$ set $diam(\mathcal{P}) = \sup diam(P), P \in \mathcal{P}$ (with respect to a metric on X). If a sequence $\mathcal{R} < \mathcal{R}^{(1)} < \cdots < \mathcal{R}^{(n)} < \cdots$ of partitions is such that $diam(\mathcal{R}_n) \to 0$, then $h(T, \mathcal{R}^{(n)}) \to h(T)$ monotonically. If $\mathcal{R}$ is generating, *i.e., $diam(\vee_{n=1}^{\infty} \mathcal{P}_n) = 0$, then $h(T, \mathcal{R}) = h(T)$.*
4) For any invariant measure μ on X, we have

$$h_\mu(T) \leq h(T). \tag{4}$$

Remark. We don't claim $\sup_\mu h_\mu(T) = h(T)$ (the variational principle), because of the condition $\text{supp}\mu \subset X_{reg}$ (see [31]).

3 Estimates for Entropy

3.1 Growth rates for generalised polytope exchanges

The results of this subsection are valid in any dimension. The main result is Theorem 3.6.

Lemma 3.1 *Let p_n be a positive sequence, and let $\limsup_{n\to\infty} p_n/n = h > 0$. For any number $g, 0 \leq g < h$, and any integer $m \geq 1$, there are infinitely many n such that*

$$p_{n+m} \; - \; p_n \; \geq \; mg. \tag{5}$$

Proof. Assume the opposite, i.e., $p_{n+m} - p_n < mg$ for all $n > n_0$. Then for these n and arbitrary $k = 1, 2, \ldots$, we have $p_{n+km} - p_n < kmg$. Therefore

$$\frac{p_{n+km}}{n+km} < \frac{p_n}{n+km} + \frac{km}{n+km} g. \tag{6}$$

With n fixed and $k \to \infty$, the indices $n_k = n + km$ go to infinity along a subsequence $n_k \equiv \nu \bmod m,\ 0 \leq \nu \leq m-1$. By eq. 6, $\limsup_{k\to\infty} p_{n_k}/n_k \leq g < h$. Since there are m such subsequences ($\nu = 0, \ldots, m-1$), we have $\limsup_{r\to\infty} p_r/r \leq g < h$, which contradicts the assumption. □

Definition 3.2 *For a positive sequence, a_n, let $h = \limsup_{n\to\infty} n^{-1} \log a_n$. If $h > 0$, we say that the sequence a_n grows exponentially, at the* rate *h. If $h = 0$, then a_n* grows subexponentially.

Lemma 3.3 *Let $a_n,\ b_n,\ \ell_n,\ d_n\ (n \geq 1)$ be positive sequences such that: i) $a_{m+n} \leq a_m a_n$; ii) for all m, n*

$$a_{m+n} \ \leq \ b_m a_n \ + \ d_m \ell_n. \tag{7}$$

Let $\alpha, \beta, \lambda, \delta (\geq 0)$ be the respective growth rates. Then

$$\alpha \leq \min\{\max(\beta, \lambda),\ \max(\beta, \delta)\} = \max\{\beta,\ \min(\delta, \lambda)\}. \tag{8}$$

Proof. Suppose $\alpha > \beta$. Set $\phi_n = \log(b_n + 1)$. Then $\beta = \limsup(\phi_n/n)$. Hence there is $k \geq 1$ such that $\phi_k/k = g < \alpha$. We fix k for the rest of the proof. By Lemma 3.1, there are infinitely many n, such that $\log a_{n+k} - \log a_n \geq \phi_k$, i.e., $a_{n+k}/a_n \geq b_k + 1$. By eq. 7, $a_n(a_{n+k}/a_n - b_k) \leq d_k \ell_n$ for infinitely many n. Since, by i), $a_n \geq e^{n\alpha}$ for all n, we have $\alpha \leq \lambda$. This proves the inequality $\alpha \leq \max(\beta, \lambda)$. Fix n and divide eq. 7 by a_n, obtaining

$$\frac{a_{n+m}}{a_n} \leq b_m + \frac{\ell_n}{a_n} d_m.$$

As $m \to \infty$, the growth rate of the sequence on the left is α, while that of the sequence on the right is $\max(\beta, \delta)$. Thus $\alpha \leq \max(\beta, \delta)$. $\square$

We assume from now on that X is a metric space, and denote the distance by $d(\cdot, \cdot)$. Let $\ell(P)$ be a function on polytopes, $P \subset X$, such that $\ell(P) \geq C\mathrm{diam}(P)$ for connected P, where $C > 0$ is a constant. We say that $\ell(P)$ is a *length-type function.* In §3.3 and §4, $d(\cdot, \cdot)$ is induced by a *Finsler metric*, $|\ |$, and $\ell(P) = |\partial P|$, is the *edge length* of P. For any partition, $\mathcal{P} = \bigcup_{i \in I} P_i$, we set $\ell(\mathcal{P}) = \Sigma_{i \in I} \ell(P_i)$; $D(\mathcal{P}) = \min d(P, Q)$, the minimum over pairs of nonadjacent atoms of $\mathcal{P}$; $b(\mathcal{P}) = \max_{x \in X} |\mathcal{P}_x|$, where $\mathcal{P}_x \subset \mathcal{P}$ consists of atoms containing x. The quantity $b(\mathcal{P}) = \max_{x \in \mathcal{P}} b(x)$ measures the *vertex complexity of the graph $\mathcal{P}$.*

Proposition 3.4 *Let $T : X \mapsto X$ be a polytope exchange on a partition $\mathcal{R}$, and let $\ell(P)$ be a length-type function. For $n > 0$ set $\ell_n = \ell(\mathcal{P}_n), \bar{b}_n = b(\mathcal{Q}_n)$. Let λ and $\bar{\beta}$ be the exponential growth rates of the sequences ℓ_n and $\bar{b}_n$ respectively.*

If the atoms of $\mathcal{P}_n, n > 0$, are connected, then the growth rate of $|\mathcal{P}_n|$ is, at most, $\max(\bar{\beta}, \lambda)$.

Proof. By the material of §2, for $-m \leq k \leq n$, T^{k-1} yields a one-to-one correspondence between the atoms of $\mathcal{R}_{(-n,m)}$ and $\mathcal{R}_{(-n+k,m+k)}$. In particular, $|\mathcal{Q}_m| = a_m$, hence, for any $m, n \geq 0$:

$$a_{n+m} \;=\; |\mathcal{R}_{(-n,m)}| \;=\; |\mathcal{P}_n \vee \mathcal{Q}_m| \;\leq\; a_n a_m. \tag{9}$$

For any $m, n \geq 0$ we divide the atoms of $\mathcal{P}_n$ into two groups: the "good" and the "bad" ones. A good atom, $A \in \mathcal{P}_n$, intersects only the adjacent atoms of $\mathcal{Q}_m$. By eq. 9 and preceding remarks, A produces at most $\bar{b}_m$ atoms of $\mathcal{P}_{n+m}$. Hence, the contribution to a_{n+m} by the good atoms of $\mathcal{P}_n$ is, at most, $a_n\bar{b}_m$. The diameter of a bad atom, B, is, at least, $D(\mathcal{Q}_m) = \bar{D}_m$. Since B is connected, $\ell(B) \geq C\mathrm{diam}(B) \geq C\bar{D}_m$. Hence, the number of bad atoms in $\mathcal{P}_n$ is bounded above by $\ell_n/C\bar{D}_m$. A bad atom may intersect all atoms in $\mathcal{Q}_m$. Hence the contribution of bad atoms to $|\mathcal{P}_{n+m}|$ is, at most, $a_m\ell_n/C\bar{D}_m$. Thus

$$a_{n+m} \;\leq\; \bar{b}_m a_n \;+\; \frac{a_m}{C\bar{D}_m}\ell_n. \tag{10}$$

The claim follows, by Lemma 3.3 (with $d_m = a_m/(C\bar{D}_m)$). $\square$

The assertions below are immediate from Proposition 3.4 and §2.

Corollary 3.5 *Let $T : X \mapsto X$ be a polyhedron exchange on a partition $\mathcal{R}$, and let $t : S \to S$ be the corresponding subshift (the notation of §2). If the atoms of $\mathcal{P}_n$ are connected, and the sequences $\ell(\mathcal{P}_n), b(\mathcal{Q}_n)$ grow subexponentially, then $h(T, \mathcal{R}) = h(t) = 0$. If, in addition, $\mathcal{R}$ is generating, then $h(T) = 0$.*

Theorem 3.6 *Let $T : X \mapsto X$ be a polytope exchange on a partition $\mathcal{R}$. Let λ and $\bar{\beta}$ be the growth rates of the sequences $\ell_n = \ell(\mathcal{P}_n)$ and $\bar{b}_n = b(\mathcal{Q}_n)$ respectively.*

If the atoms of the partitions $\mathcal{P}_n$ are connected, then $h(T, \mathcal{R}) \leq \max(\bar{\beta}, \lambda)$. If, in addition, partition $\mathcal{R}$ is generating, then $h(T) \leq \max(\bar{\beta}, \lambda)$.

In general, the topological entropy of a polyhedron exchange is smaller than the upper bound of Theorem 3.6. For instance, in Yomdin's example [39] the topological entropy is strictly smaller than the growth rate of curves' lengths.

3.2 Entropy of generalised polygon exchanges

In this subsection we assume that $\dim(X) = 2$, and study the *generalised polygon exchanges* (GPEs). We will strengthen the bound on the entropy, established in Theorem 3.6. The following is the main result.

Theorem 3.7 *Let $T : X \mapsto X$ be a GPE on a partition $\mathcal{R}$, and let the notation be as in Proposition 3.4. Assume that the atoms of the partitions $\mathcal{P}_n$ are homeomorphic to the disc. Let $\ell(\cdot)$ be any length-type function, and let λ be the growth rate of the sequence $\ell(\mathcal{P}_n)$. Then $h(T, \mathcal{R}) \leq \lambda$.*

The centrepiece in the proof of Theorem 3.7 is Proposition 3.9, which gives efficient bounds on the growth rate of the vertex complexity of the sequence of graphs $\mathcal{P}_n$, generated by a GPE. In the usual fashion we will from the initial partition $\mathcal{R}$ define for every $n \geq 1$ the forward join $\mathcal{P}_n = \mathcal{R}_{(-n,0)}$ and regard its boundary set $\partial\mathcal{P}_n$ as s (two dimensional) graph with faces (atoms of the partition), edges and vertices. We shall assume that every vertex is the terminus of at least three edges (no 'false vertices' please). The following notation will apply to the remainder of this subsection: We shall denote by a_n, e_n and v_n the number of faces, edges and vertices of the partition $\mathcal{P}_n$. Their respective growthrates as n goes to infinity will accordingly be called by α, ε and ν. Moreover,let β be the growth rate of the sequence $b_n = b(\mathcal{P}_n)$ (vertex complexity of $\mathcal{P}_n$). In a similar way we shall interpret the backward join $\mathcal{Q}_n = \mathcal{R}_{(0,n)}, n \geq 1$, as a graph, and denote by $\bar{a}_n$, $\bar{e}_n$ and $\bar{v}_n$ its number of faces, edges and vertices and by $\bar{\alpha}, \bar{\varepsilon}$ and $\bar{\nu}$ their respective growthrates. We put $\bar{\beta}$ for the growth rate of sequence $\bar{b}_n = b(\mathcal{Q}_n)$.

We will need the following elementary lemma.

Lemma 3.8 *Let $b_k, k \geq 1$, be a positive sequence, such that for m, n sufficiently large*

$$b_{m+n} \leq e^{\varepsilon m} + e^{\mu n}, \tag{11}$$

with $\varepsilon, \mu > 0$. Then the exponential growth rate, β, of the sequence c_k satisfies: $\beta \leq \varepsilon\mu/(\varepsilon + \mu)$.

Proof. The function $f(x, y) = e^{\varepsilon x} + e^{\mu y}$ assumes its unique minimum on any line, $x + y = r$, at

$$(x_r, y_r) = (\frac{\mu r}{\varepsilon + \mu} + \frac{\log\mu - \log\varepsilon}{\varepsilon + \mu}, \frac{\varepsilon r}{\varepsilon + \mu} + \frac{\log\varepsilon - \log\mu}{\varepsilon + \mu}),$$

and

$$\min_{x+y=r} f(x,y) = f(x_r, y_r) = [(\mu/\varepsilon)^{\frac{\varepsilon}{\varepsilon+\mu}} + (\varepsilon/\mu)^{\frac{\mu}{\varepsilon+\mu}}]e^{\frac{\varepsilon\mu}{\varepsilon+\mu}r}.$$

Let r be a positive integer, let m_r be an integer such that $|m_r - x_r| \le 1/2$, and set $n_r = r - m_r$. Then $|y_r - n_r| \le 1/2$, and $m_r + n_r = r$. Substituting $m = m_r, n = n_r$ into eq. 11, we obtain

$$b_r \le \max(e^{\varepsilon/2}, e^{\mu/2})[(\mu/\varepsilon)^{\frac{\varepsilon}{\varepsilon+\mu}} + (\varepsilon/\mu)^{\frac{\mu}{\varepsilon+\mu}}]e^{\frac{\varepsilon\mu}{\varepsilon+\mu}r}$$

which implies the claim. □

Proposition 3.9 *Let $T : X \to X$ be a GPE on a partition $\mathcal{R}$ and consider the graphs of the partitions $\mathcal{P}_n = \mathcal{R}_{(-n,0)}$, where, as above, β and ε denote the growthrates of the quantities b_n and e_n. Set $\mu = \log M$, where $M = \min\{a_1, b_2\}$. Then*

$$\beta \le \frac{\epsilon\mu}{\epsilon + \mu}.$$

Proof. If $\mathcal{P}$ is any graph on X, and $x \in X$, we denote by $b(x) \ge 0$ the number of edges emanating from x. Thus $b(x) = 0$ (2), if x is an interior point of a face (edge) of $\mathcal{P}$, and $b(x) \ge 3$ if x is a vertex of $\mathcal{P}$ (we assume, without loss of generality, that $\mathcal{P}$ has no 'false vertices'). Note that $b(x) \ne 1$. Set $b(\mathcal{P}) = \max_{x\in\mathcal{P}} b(x)$. We denote by $b_n(x)$ the number $b(x)$ with respect to $\mathcal{P}_n$. Using the recursion $\mathcal{P}_{n+1} = \mathcal{R} \vee T^{-1}\mathcal{P}_n$, $\mathcal{P}_1 = \mathcal{R}$, we divide the vertices, x, of $\mathcal{P}_{n+1}$ into two categories:

(i) 'new vertices': x is not a vertex of $T^{-1}\mathcal{P}_n$. Thus $x = T^{-1}x'$, where $b_n(x') \le 2$, and $b_{n+1}(x) = b_1(x) + b_n(x') \le b_1 + 2$;

(ii) 'old vertices': x is a vertex of $T^{-1}\mathcal{P}_n$. Then $x = T^{-1}x'$, where x' is a vertex of $\mathcal{P}_n$. These vertices x' in $\mathcal{P}_n$ are the 'parents' of x. Since T^{-1} is a homeomorphism on the atoms of $\mathcal{R} = \mathcal{P}_1$, the maximal number of parents, an old vertex can have is, at most, a_1. On the other hand, $\mathcal{P}_2$ determines how T^{-1} puts the atoms together. Thus the number of parents is at most $\min\{a_1, b_2\} = M$.

Let $m, n \ge 1$. Let x be a vertex of $\mathcal{P}_{n+m}$. If x is an old vertex, we trace its ancestors in the 'family tree' all the way back to the generation m.

(I) For $0 < k < n$, we restrict our attention to the ancestors of x that were new vertices at the generation $m + k$. The number of those ancestors is, at most, M^{n-k}. Since an ancestor, say x', is a new vertex, it has by (i) at most b_1+2 edges. Between the generations $m+k$ and $m+n$, x' can acquire, through mergings with the vertices of $\mathcal{R}$ (or by intersection with edges of $\mathcal{R}$), at most $(n-k)b_1$ more edges (not considering mergings with other vertices). Thus the total number of edges of x these ancestors supply is at most

$$I = \sum_{k=1}^{n-1}(b_1 + 2 + (n-k)b_1)e^{(n-k)\mu} < \frac{(n+1)b_1 + 2}{M-1}M^n.$$

(II) The ancestors of x that already existed at the generation m had at most e_m edges altogether. Each ancestor acquires, between the generations m and $m+n$, at most nb_1 edges. Since there wer, at most, M^n of these ancestors, their total contribution to the edges of x is, at most,

$$II = e_m + nb_1M^n.$$

We finally obtaine the estimate

$$b_{m+n} \le I + II \le e_m + \frac{(2n+1)b_1 + 2}{M-1}e^{n\mu}. \tag{12}$$

By eq. 12, for any $\epsilon_1 > \epsilon \ge 0$ and $\mu_1 > \mu$ there exists $t = t(\epsilon_1, \mu_1)$ such that for $m, n > t$, we have $b_{m+n} < e^{\epsilon_1 m} + e^{\mu_1 n}$. By Lemma 3.8, $\beta \le \epsilon_1\mu_1/(\epsilon_1 + \mu_1)$. □

Lemma 3.10 *Let X be an arbitrary compact surface, and $\mathcal{P}$ be a (polygonal) partition of X, whose faces are homeomorphic to the disc. Let a, e, v be the numbers of faces, edges and vertices of $\mathcal{P}$ respectively, and assume that $\mathcal{P}$ has at least three edges at every vertex. Then $v \le 2a - 2\chi$, $e \le 3a - 3\chi$, where $\chi = \chi(X)$ is the Euler characteristic.*

Proof. Associating to every vertex, x, the oriented edges, 'emanating from' x, we obtain $v \le 2e/3$. This and $a + v = e + \chi$ imply the claim. □

Corollary 3.11 $\nu \le \alpha$, $\epsilon = \alpha$.

Proof. By Lemma 3.10, $\epsilon, \nu \le \alpha$. Since $a_n < e_n + \chi$, we have $\alpha \le \epsilon$. □

Proof of Theorem 3.7. Let us put $h = h(T, \mathcal{R}) = \alpha$ and $\bar{h} = \bar{\alpha}$. By assumption, the atoms of $\mathcal{P}_n$ are topological discs. Since T^{n-1} provides a homeomorphism between the atoms of $\mathcal{P}_n$ and $\mathcal{Q}_n$, $n \ge 1$, the atoms of $\mathcal{Q}_n$

are topological discs, as well. Therefore $\bar{\beta} = \beta$ and $\bar{h} = h$. If $h = 0$ then by Proposition 3.9 $\beta = 0$ which implies the theorem. If $h > 0$ then we have by Proposition 3.9 and Corollary 3.11 that $\bar{\beta} < \bar{h} = h$ which, together with Theorem 3.6 implies the theorem. $\square$

The preceding results allow to obtain efficient bounds on the asymptotic complexity of graphs generated by a GPE.

Corollary 3.12 *Let $\mu = \mu(\mathcal{R})$ be as in Proposition 3.9 and put $h(T, \mathcal{R}) = \alpha$.*
If $h = 0$, then $\epsilon = \nu = \beta = 0$.
If $h > 0$, then $\epsilon = h$, $\beta \leq h/(1 + h/\mu)$, and $h^2/(1 + h/\mu) \leq \nu \leq h$.

Proof. It suffices to prove the lower bound on ν, when $h > 0$. For any $\beta' > \beta$, and large enough n, every vertex in the graph $\mathcal{P}_n$ has at most $e^{n\beta'}$ edges. By Corollary 3.11, for any $0 < h' < h$, $e_n \geq e^{nh'}$ for large enough n. Therefore the number of vertices in $\mathcal{P}_n$ satisfies $v_n \geq 2e_n/b_n \geq 2e^{n(h'-\beta')}$. We therefore obtain $\nu \geq h - \beta$. Substituting the upper bound on β from Proposition 3.9 completes the proof. $\square$

Corollary 3.13 *Let $\ell(P)$ be a length-type function on polygons in X, and λ ($\bar{\lambda}$) be the growth rate of $\ell(\mathcal{P}_n)$ ($\ell(\mathcal{Q}_n)$). Let $\mu = \mu(\mathcal{R})$ be as in Proposition 3.9 and put $h(T, \mathcal{R}) = \alpha$, $\bar{h} = h(T^{-1}, \mathcal{R}) = \bar{\alpha}$*

Assume that the faces of $\mathcal{P}_n$ (or $\mathcal{Q}_n$), $n \geq 1$, are topological discs. Then $\epsilon = \bar{\epsilon} = h$, $h^2/(h + \mu) \leq \nu, \bar{\nu} \leq h$, and

$$\max(\beta, \bar{\beta}) < h = \bar{h} \leq \min(\lambda, \bar{\lambda}). \tag{13}$$

Proof. By Corollary 3.11, $\epsilon = \bar{\epsilon} = h$. This, and Proposition 3.9 imply $\beta, \bar{\beta} < h$. By Theorem 3.7, and $h = \bar{h}$, we have $h \leq \lambda, \bar{\lambda}$. $\square$

3.3 Lyapunov exponents and entropy

Let X is a compact Finsler surface, and let $T : X \to X$ be a GPE on a partition $\mathcal{R}$. Then the iterates, T^k, are smooth on every atom, P, in $\mathcal{R}_{(-m,n)}$, $-n < k < m$. Thus, for $x \in X_{reg}$ the differential, DT^k_x is defined for all k, and we set $L_k(x) = \log ||DT^k(x)||$, where $|| \cdot ||$ denotes the matrix norm. For $x \in X_{reg}$ we define the *Lyapunov exponent*, $L(x)$, by

$$L(x) = \limsup_{n\to\infty} L_n(x)/n = \limsup_{n\to\infty} \log ||DT^n(x)||^{\frac{1}{n}}.$$

Let $T : X \to X$ be a GPE on a partiton $\mathcal{R}$, satisfying the assumptions of Theorem 3.7. If $\mathcal{R}$ does not generate, then, in general, $h(T, \mathcal{R}) < h(T)$, and Theorem 3.7 does not imply an estimate on the entropy of T. The estimate given by the theorem below combines the growth rate of the singular set, $|\partial\mathcal{P}_n|$, and the Lyapunov exponent of T. There is a close analogy with Theorem 1.1, but because of the singularities of T, we need to make some assumptions on the regularity of the Lyapunov exponent.

Theorem 3.14 *Let $T : X \to X$ be a GPE on $\mathcal{R}$, and let $\lambda \geq 0$ be the growth rate of the sequence $\ell_n = \ell(\mathcal{P}_n)$. Assume that:*
i) there exist arbitrarily small refinements, $\mathcal{R} < \mathcal{R}^m$, such that the atoms of $\mathcal{R}^m_n = \mathcal{P}_n(\mathcal{R}^m), m, n \geq 1$, are homeomorphic to the disc;
ii) Let $L \geq sup_x L(x)$, where the supremum is over regular points, and suppose that for any $L' > L$ there is an integer $N = N(L')$ such that $L_n(x)/n \leq L'$ for $|n| \geq N$ and all $x \in X_{reg}$. Let μ be a T-invariant measure on X. Then $h_\mu(T) \leq h(T) \leq \max(\lambda, L)$.

Proof. By assumption ii), for any $L' > L$ there is a constant $C = C(L')$, such that $||DT_x^{-k}|| \leq Ce^{kL'}$ for all k and all regular points x. Let γ be a regular curve in X, and set $\gamma_n = \gamma \cup T^{-1}\gamma \cup \cdots \cup T^{-(n-1)}\gamma$. Then $|\gamma_n| \leq C_1 e^{nL'}|\gamma|$. For any partition $\mathcal{R}^m$, as in i), we have $\partial\mathcal{R}^m_n = \gamma^m_n \cup \partial\mathcal{P}_n$, for some regular curve γ^m. Therefore $|\partial\mathcal{R}^m_n|$ grows (with n) at most at the rate $\max(\lambda, L')$. Thus, by Theorem 3.7, $h(T, \mathcal{R}^m) \leq \max(\lambda, L')$. Since $L' > L$ is arbitrary, $h(T, \mathcal{R}^m) \leq \max(\lambda, L)$, for any m. Proposition 2.8 implies the claim. □

4 Applications and Examples

We apply the preceding material to a variety of geometrically defined transformations with singularities.

4.1 Piecewise isometries and related transformations

Although the notion of a piecewise isometry is quite general, we restrict our exposition to a special case.

Definition 4.1 *Let X be a compact Finsler surface (possibly, with a boundary). A GPE, $T : X \to X$, on a partition $\mathcal{R}$, is a* piecewise isometry *if $T|_P : P \to X$ is an isometry, for any atom, $P \in \mathcal{R}$.*

The boundary of a *Finsler polygon*, $P \subset X$, consists of *Finsler geodesic* segments.

Theorem 4.2 *Let X be a Finsler surface, and let a GPE $T : X \to X$ be a piecewise isometry on a partition $\mathcal{R}$, whose atoms are Finsler polygons. Then $h(T) = 0$.*

Proof. Refining $\mathcal{R}$, if necessary, we can assume that the atoms of $\mathcal{R}$ are *geodesically convex*. Recall that $P \subset X$ is geodesically convex if for any $x, y \in P$ there is a unique shortest geodesic, $\gamma(x, y)$, $|\gamma(x, y)| = d(x, y)$, joining them, and $\gamma(x, y) \subset P$. The geodesic convexity is preserved by isometries, and by taking finite intersections. Therefore, for $n \geq 1$, the atoms of $\mathcal{P}_n$ are geodesically convex, hence homeomorphic to the disc. The length, $\ell(\mathcal{P}_n)$, grows at most linearly. By Theorem 3.8, $h(T, \mathcal{R}) = 0$.

The partition $\mathcal{R}$ may not generate. By taking arbitrary refinements of $\mathcal{R}$, with geodesically convex atoms, or by invoking Theorem 3.16, we conclude that $h(T) = 0$. $\square$

A piecewise isometry of a Riemann surface, $T : X \to X$, whose singular set is a union of geodesic segments, is a special case of the setting of Theorem 4.2. If X is a *flat surface*, and the vertices of $\mathcal{R}$ are the singular points of X, then T is a Euclidean polygon exchange. When X is the unit square, and the atoms of $\mathcal{R}$ are rectangles (necessarily with vertical and horizontal sides), then T is a rectangle exchange. By Theorem 4.2, these transformations have entropy zero.

Let X, Y be Finsler surfaces. A diffeomorphism, $\phi : X \to X$, is *(Finsler) affine*, if it sends geodesics into geodesics.

Definition 4.3 *Let X be a compact Finsler surface, and let $T : X \to X$ be a GPE, on a partition $\mathcal{R} = \cup_{i \in I} P_i$. If the atoms of $\mathcal{R}$ are Finsler polygons, and $T|_{P_i} : P_i \to X, i \in I$, are Finsler affine, we say that T is a* piecewise (Finsler) affine GPE.

Theorem 4.4 *Let X be a compact Finsler surface, such that for any $x, y \in X$ there is a unique geodesic, $\gamma(x, y)$, joining them. Let $T : X \to X$ be a piecewise Finsler affine GPE, on $\mathcal{R} = \cup_{i \in I} P_i$. Set $\lambda_i = \max_{x \in P_i} ||dT(x)||$, and let $\lambda = \max_{i \in I} \lambda_i$. Then $h(T) \leq \lambda$.*

Proof. Since T preserves the geodesics, and by uniqueness of the geodesic $\gamma(x, y)$, T preserves geodesic convexity. Refining $\mathcal{R}$, if necessary, to make its atoms geodesically convex, we note that T satisfies the assumptions of Theorem 3.16. $\square$

If T is the unit squre, and the atoms of $\mathcal{R}$ are rectangles, we speak of an affine rectangle exchange. These are easier to 'construct' than the isometric rectangle exchanges. For instance, the classical 'baker transformation', T, is an affine rectangle exchange on two rectangles. Theorem 4.4 gives the estimate: $h(T) \leq 2$.

4.2 Polygonal billiards

The general billiard dynamics, in arbitrary dimensions, is an extension of the concept of the geodesic flow. We refer the reder to [7] for general information, and restrict ourselves to the case when the configuration space is a polygon in $\mathbf{R}^2$, or, more generally, a *polyhedral surface* [11]. A polyhedral surface, S, is tiled by a finite number of polygons, P_i, and the billiard on S is put together from the billiards in P_i. Thus, the billiard in a polygon is a crucial special case. To shorten the exposition, we will state our results for the billiards on general polyhedral surfaces, and prove them for polygons, leaving the general case to the reader, as an exercise. We will use the name 'polygonal billiards' for our dynamical system.

The main object of study is the polygonal *billiard flow*, T^t, on the unit tangent bundle of S. The edges of S define a natural cross-section, X, and the first return map, $T : X \to X$, retains most of the information. We call T the *polygonal billiard map*, and X its *phase space*. In the standard (arclength, angle) coordinates on X we have $T(s, \theta) = (s_1, \theta_1)$, where $0 \leq s \leq 1$ (normalized 'perimeter' of S), and $0 \leq \theta \leq pi$. The flow-invariant Liouville measure yields the *standard invariant measure*, $\omega = \sin\theta ds d\theta$, on X.

The edges and vertices of S produce two kinds of singularities of T. The former is the boundary, $\partial X = \{(s, 0)\} \cup \{(s, \pi)\}$ of X. To describe the latter, it is useful to think of elements of X as light rays in S. The light ray, corresponding to $x = (s, \theta)$ emanates from s in direction θ. A billiard trajectory is a sequence of light rays in S, transforming by the laws of geometric optics. Light rays ending (starting) at vertices of S are singular for T (T^{-1}). Let $e,\ A$ be an edge and a vertex of S. We denote by $L(e, A)$ ($L(A, e)$) the set of light rays emanating from e (A), and reaching A (e). The union of real analytic curves $L(e, A),\ L(A, e) \subset X$ (some may be empty) is the singular set of the polygonal billiard.

Theorem 4.5 *Let S be a compact polyhedral surface, and let $T : X \to X$ be the billiard map. Then: 1. there is a (generalized) polygonal partition, $\mathcal{R}$, whose boundary is the union of curves $L(e, A),\ L(A, e)$, and T is a GPE on $\mathcal{R}$; 2. the entropy of T is zero.*

Proof. The time-reversal involution, $\sigma : X \to X$, reverses the direction of light rays, and satisfies $\sigma T \sigma = T^{-1}$. By definition, $\sigma(L(e, A)) = L(A, e)$. Recall that S is a polygon, and assume, for simplicity of exposition, that S is convex. For any pair, $e \neq f$, of edges of S, let $P(e, f) \subset X$ consist of light rays that emanate from e and hit f. If e, f are disjoint, then $P(e, f)$ is a quadrilateral. Its boundary curves are $L(e, C), L(e, D), L(A, f), L(B, f) \subset \mathrm{int}(X)$, where A, B (C, D) are the endpoints of e (f). If e and f have a common vertex, A, then $P(e, f)$ degenerates into a trilateral, with one 'horizontal side' ($\theta = 0$ or $\theta = \pi$), and one 'vertical side' ($s = s(A)$). Note that $\sigma P(e, f) = P(f, e)$.

Let $e_i, 1 \leq i \leq p$, be the edges of S. Then the regions $P_{ij} = P(e_i, e_j), i \neq j$, form a (generalized) polygonal partition, $\mathcal{R}$, of X, and T is a GPE on $\mathcal{R}$. Note that $\sigma(\mathcal{R}) = \mathcal{R}$, hence σ conjugates the GPEs $T : X \to X$ and $T^{-1} : X \to X$.

By 'inserting false vertices' on the sides of S, we can treat S as a q-gon, with q arbitrarily large. Let $\mathcal{P}(q)$ be the above partition of X, defined by this representation. Then $|\mathcal{P}(q)| = q(q-1)$, and, as $q \to \infty$, the atoms of $\mathcal{P}(q)$ can be made arbitrarily small, in any metric, compatible with the natural topology on X. We will use the Finsler metric, $||\ || = \sin\theta|ds| + |d\theta|$. Let $l(s, s_1)$ be the Euclidean distance between $s, s_1 \in S$. The general formula for the differential, ∂T, of the billiard map [15] yields, for polygons:

$$\partial T = \begin{pmatrix} -\frac{\sin\theta}{\sin\theta_1} & \frac{h(s,s_1)}{\sin\theta_1} \\ 0 & -1 \end{pmatrix}. \tag{18}$$

Let $v \in V_{(s,\theta)}$ be a tangent vector to X at (s, θ). Then $\partial T(v) \in V_{(s_1,\theta_1)}$, and, by eq. 18

$$||\partial T(v)|| \leq ||v|| + h(s, s_1)|d\theta(v)| \leq ||v|| + \mathrm{diam}(S)|d\theta(v)|.$$

Iterating T, and using that, by eq. 18, $|d\theta(\partial T(v))| = |d\theta(v)|$, we obtain ($n$ arbitrary)

$$||\partial T^n(v)|| \leq ||v|| + |n|\mathrm{diam}(S)|d\theta(v)|. \tag{19}$$

If $C \geq \max(1, \mathrm{diam}(S))$, then, by equation above, $||\partial T^n(v)|| \leq (|n|+1)C||v||$. Thus, for any C^1 curve, $\gamma \subset X$, we have

$$|T^n(\gamma)| \leq (|n|+1)C|\gamma|. \tag{20}$$

By the 'false vertices' trick, and the general properties of the entropy (§2 **what theorem ?**), it suffices to show that $h(T, \mathcal{R}) = 0$ for the partition $\mathcal{R}$ above. Introduce coordinates $(s, t = -\cot\theta)$ in $\mathrm{int}(X)$. In these coordinates the curves $L(e, A)$ are linear segments, and the billiard map is piecewise projective [19]. Refining $\mathcal{R}$, if necessary, by additional straght segments, we can assume that the atoms of $\mathcal{R}$ are convex polygons, in the (s,t)-plane. Thus, for any $n \geq 1$, the atoms of $\mathcal{P}_n$ are convex, hence homeomorphic to the disc.

By Theorem 3.8, $h(T, \mathcal{R}) \leq \lambda$, where λ is the exponential growth rate of $|\partial\mathcal{P}_n|$. But $\partial\mathcal{P}_n = \gamma \cup T^{-1}\gamma \cup \cdots \cup T^{-n}\gamma$, for a certain curve γ, and, by eq. 20, $|\partial\mathcal{P}_n| \leq c_1 n^2$. Thus $\lambda = 0$. □

Elaborate later

Corollary 4.6 *Subexponential growth for geometric quantities defined by a polyhedral surface.*

Corollary 4.7 $h_\mu(T) = 0$ *for any invariant measure. Same for the flow entropy.*

4.3 Billiards in rational polytops

As we mentioned earlier, the billiard dynamics makes sense in any dimensions. In this subsection we study the billiards in polytops, $P \subset \mathbf{R}^3$. Let $L_i \subset \mathbf{R}^3$ be the hyperplanes containing the faces, $F_i, 1 \leq i \leq p$, of P, and let $s_i \in O(3)$ be the linear part of the Euclidean reflection about L_i. The group, $G = G_P \subset O(3)$, generated by $s_i, 1 \leq i \leq p$, is an important characteristic of P.

Definition 4.8 *A polytop P is* rational *if the group G_P is finite.*

Although the rationality condition in $\mathbf{R}^3$ (as opposed to $\mathbf{R}^2$) is very restrictive, there are interesting examples. Rational polygons have been well researched [23].

Let P be a rational polytop, and let $\Omega \subset S^2 \subset \mathbf{R}^3$ be a fundamental domain for the natural action of G. For any $\omega \in \Omega$ we define a submanifold, $Q_\omega \subset V$ in the phase space of the billiard flow. We have $V = \{(q, v) : q \in P, v \in S^2\}$. Set $Q_\omega = \{(q, v) : v \in G\omega\}$. By construction, Q_ω is invariant under the billiard flow, T^t, on V, and Q_ω is tiled by $|G\omega|$ copies of P.

Definition 4.9 *Let the notation be as above. The restriction,* $T^t|_{Q_\omega} = T^t_\omega$ *is the* billiard flow in direction ω.

Although, in general, Q_ω depends on ω, for $\omega \in \mathrm{int}(\Omega)$ we have $Q_\omega \simeq Q = Q(P)$, where Q is a polytop, tiled by $|G|$ copies of P. Note that Q is not necessarily a manifold. The family, T^t_ω, of directional flows on Q has a natural cross-section, S, the union of faces of Q. We view S as a union of faces of P, with multiplicities. More precisely, $S = \cup(F, \theta)$, where F runs through the faces of P, and for a $F \subset P$, the vector θ runs through the $|G|/2$ vectors in $G\omega$ that are directed inward. The first return map, $T_\omega : S \to S$, is the *directional billiard map.*

Theorem 4.10 *Let* $P \subset \mathbf{R}^3$ *be a rational polytop, and let* S *be the corresponding polyhedral surface. The directional billiard maps,* T_ω*, are affine GPEs, and* $h(T_\omega) = 0$*.*

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