EXPONENTIAL LAW FOR RANDOM MAPS ON COMPACT MANIFOLDS

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Abstract. We consider random dynamical systems on manifolds modeled by a skew product which have certain geometric properties and whose measures satisfy quenched decay of correlations at a sufficient rate. We prove that the limiting distribution for the hitting and return times to geometric balls are both exponential for almost every realisation. We then apply this result to random C^2 maps of the interval and random parabolic maps on the unit interval.

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Date: October 16, 2018.
This work was partially supported by FAPESB, CNPq and CAPES.
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1. Introduction

As a generalisation of deterministic dynamical systems but also as a better approximation of natural phenomena (e.g. existence of smalls perturbations), random dynamical systems have been extensively study in the last few decades. Unlike deterministic systems which only consider the iteration of one map, random systems allow the composition of different maps (for example by adding the presence of random noise or random perturbations) thus increasing the difficulty to analyze their statistical properties, in particular since these maps generally do not share a common invariant measure. We point the readers to the review paper by Kifer and Liu [15] for more details.

Among these statistical properties, we would like to focus on limit laws for rare events, more precisely Hitting Time Statistics (HTS) and Return Time Statistics (RTS). The study of hitting/return times for deterministic systems traces all the way back to the famous work of Poincaré [18] who prove that the orbit of almost every point come back as close as you want from its starting point and our main suject of interest will be the time needed for the orbit to come back. More precisely, if we denote by $\tau_A(x)$ the first time that the orbit of x enters the set A, then one can consider the functions

$$F_A^h(t) = \mu\left(x \in X : \tau_A(x) > \frac{t}{\mu(A)}\right)$$

and

$$F_A^r(t) = \frac{1}{\mu(A)}\mu\left(x \in A : \tau_A(x) > \frac{t}{\mu(A)}\right)$$

where μ is an invariant measure of the transformation. We can observe that the scaling factor $\frac{1}{\mu(A)}$ is suggested by the Kac's Lemma [14] which says that $\int_A \tau_A \, d\mu = 1$.

One is naturally interested whether the functions $F_{A_n}^h$ (respectively $F_{A_n}^r$) converge to a limiting function F^h (respectively F^r) as $n \to \infty$ when one choose a sequence of nested sets $\{A_n\}_{n=1}^{\infty}$. Indeed, if the sets A_n are taken to be cylinder sets with respect to a generating measurable partition, then the limit is known to be exponential for non-periodic points for mixing measures (see e.g. [8, 1]). In the case of Bowen balls the same result is known to be true [11]. For geometric balls $B_r(y)$ it has been proven that the limit F is exponential if μ has exponential decay of correlations (e.g. [19] and references therein). We refer to the reviews [9, 22] for more details on this subject.

Our goal in this paper is to extend some of these results for deterministic dynamical systems to the realm of random dynamical systems. We consider a family of maps $\{T_\omega\}_\omega$ with $T_\omega : X \to X$. The randomness came from a dynamical system (Ω, θ, ν) and the random orbit is given by

$$T_\omega^n(x) = T_{\theta^n\omega} \circ T_{\theta^{n-1}\omega} \circ \cdots \circ T_\omega(x).$$

Thus, one can define the (quenched) hitting time $\tau_A^\omega(x)$ as the first time the random orbit of x enters the set A. There are two ways to define the hitting times distribution, namely

$$F(t) = \mathbb{P}\left((\omega, x) : \tau_A^\omega(x) > \frac{t}{\mu(A)}\right)$$

and

$$F^\omega(t) = \mu^\omega\left(x : \tau_A^\omega(x) > \frac{t}{\mu(A)}\right).$$

The first is known as the annealed distribution, where the probability is taken w.r.t. the measure $\mathbb{P}$, which is invariant for the random dynamical system, i.e. invariant for the associated skew-product. The second is called the quenched distribution where the probability is taken with the measure μ^ω associated with the 'realisation' ω. In both cases, the scaling factor is $\frac{1}{\mu(A)}$, where μ is the marginal measure and is suggested by Kac's Lemma for the associated skew-product [17, 20].

In [4, 3, 19], it is proven that the annealed distribution for geometric balls converges to exponential for maps with (annealed) exponential decay of correlations. In the first two papers, their method exploits the relation between the hitting times statistics and the extreme value distribution, while in the third one the method of [13] is followed. In these paper, the convergence of the return time distribution to an exponential is also proven.

On the other hand, a quenched result is more interesting since it easily implies the annealed result by integrating over ω, but more difficult to get. The only known results are [20, 21] where random subshifts of finite type with fast decay of correlations are considered and an exponential law is proved for hitting times.

We emphasize that in both articles studying the quenched case, they did not prove the distribution for the return times and more importantly one can notice that the convergence of the return time distribution does not come immediately from the convergence of the hitting time distribution, as one could have hoped for from the deterministic case (e.g. [22]) or the annealed case ([19]).

In this paper, we extend the results of [20] to maps and prove that the quenched hitting time statistics converges, for almost every ω, to the exponential distribution for random maps which have certain geometric properties and with some rapidly mixing conditions. Moreover, this is the first paper where one also managed to obtain the convergence of the quenched return time statistics to the exponential distribution. The main theorem is stated in Section 2, and proven in Section 3, 4 and 5. The proof is based on the deterministic case [12] which in its turn was derived from the deterministic case on Young towers [10] but we emphasize that this is not just a simple adaptation of the deterministic case to the random case. In particular, one needs to be especially careful in the study of the very short returns (Section 4) and we encourage the reader to take a particular attention to Proposition 4.1 which is a complex adaptation of a lemma of [7] on the measure of the set of very short returns. In Section 7 we consider three examples, namely random C^2 expanding interval maps and random Pomeau-Manneville maps where we use the derivation of the fibered measures from [5] and the decay of sequential systems for parabolic maps [2]; the last example is the random perturbation of non-uniformly expanding maps with critical set, which can be modeled by a random Gibbs-Markov-Young structure [16].

2. Random Maps

Let $\theta : \Omega \to \Omega$ be the shift map on a full shift space Ω with θ-invariant probability measure ν. Let M be a compact manifold and for every $\omega \in \Omega$, let $T_\omega : M \to M$ be a measurable map. The skew product S on $\Omega \times M$ is then given by $S(\omega, x) = (\theta\omega, T_\omega x)$. For the iterates we obtain $S^n(\omega, x) = (\theta^n\omega, T^n_\omega x)$ where $T^n_\omega = T_{\theta^{n-1}\omega} \circ \cdots \circ T_{\theta\omega} \circ T_\omega$.

Assume that $\mathbb{P}$ is a measure on $\Omega \times M$ invariant under the skew action S and with marginal ν on Ω. There is a class of measures μ^ω for $\omega \in \Omega$ on M, such that $d\mathbb{P} = d\mu^\omega d\nu(\omega)$. These measures satisfy the invariance property $T^*_\omega\mu^\omega = \mu^{\theta\omega}$ for ν-almost every $\omega \in \Omega$. We denote by $\mu = \int_\Omega \mu^\omega \, d\nu(\omega)$ the marginal measure on M.

For every realisation $\omega \in \Omega$ let $\Gamma^u(\omega)$ be a collection of unstable leaves $\gamma^u(\omega)$ and $\Gamma^s(\omega)$ a collection of stable leaves $\gamma^s(\omega)$. We assume that $\gamma^u \cap \gamma^s$ consists of a single point for all $(\gamma^u, \gamma^s) \in \Gamma^u \times \Gamma^s$. The map T_ω contracts along the stable leaves and similarly T^{-1}_ω contracts along the unstable leaves.

For an unstable leaf $\gamma^u(\omega)$ denote by $\mu^\omega_{\gamma^u}$ the disintegration of μ^ω to the γ^u. We assume that μ^ω has a product like decomposition $d\mu^\omega = d\mu^\omega_{\gamma^u} d\upsilon^\omega(\gamma^u)$, where υ^ω is a transversal measure. That is, if f is a function on M then

$$\int f(x)\, d\mu^\omega(x) = \int_{\Gamma^u(\omega)} \int_{\gamma^u} f(x)\, d\mu^\omega_{\gamma^u}(x)\, d\upsilon^\omega(\gamma^u)$$

If $\gamma^u, \hat\gamma^u \in \Gamma^u(\omega)$ are two unstable leaves then the holonomy map $\mathcal{H} : \gamma^u \cap \Lambda \to \hat\gamma^u \cap \Lambda$ is defined by $\mathcal{H}(x) = \hat\gamma^u \cap \gamma^s(x)$ for $x \in \gamma^u \cap \Lambda$, where $\gamma^s(x)$ is the local stable leaf through x.

Let us denote by $J^\omega_n = \frac{dT^n_\omega \mu^\omega_{\gamma^u}}{d\mu^\omega_{\gamma^u}}$ the Jacobian of the map T^n_ω with respect to the measure μ^ω in the unstable direction.

Fix ω and let γ^u be a local unstable leaf. Assume there exists $R > 0$ and for every $n \in \mathbb{N}$ finitely many $y_k \in T^n_\omega\gamma^u$ so that $T^n_\omega\gamma^u \subset \bigcup_k B_{R,\gamma^u}(y_k)$, where $B_{R,\gamma^u}(y)$ is the embedded R-disk centered at y in the unstable leaf γ^u. Denote by $\zeta_{\varphi,k} = \varphi(B_{R,\gamma^u}(y_k))$ where $\varphi \in \mathscr{I}^\omega_n$ and $\mathscr{I}^\omega_n$ denotes the inverse branches of T^n_ω. We call ζ an n-cylinder. Then there exists a constant L so that the number of overlaps $N_{\varphi,k} = |\{\zeta_{\varphi',k'} : \zeta_{\varphi,k} \cap \zeta_{\varphi',k'} \neq \varnothing, \varphi' \in \mathscr{I}^\omega_n\}|$ is bounded by L for all $\varphi \in \mathscr{I}^\omega_n$ and for all k and n. This follows from the fact that $N_{\varphi,k}$ equals $|\{k' : B_{R,\gamma^u}(y_k) \cap B_{R,\gamma^u}(y_{k'}) \neq \varnothing\}|$ which is uniformly bounded by some constant L.

To obtain an exponential law for the distribution of hitting time and return time, we need a few assumptions. First of all, we need information on the annealed and quenched decay of correlations:

(I) There exists a decay function $\lambda(k)$ so that

$$\left| \int_\Omega \int_M G(H \circ T^k_\omega)\, d\mu^\omega\, d\nu(\omega) - \mu(G)\mu(H) \right| \leq \lambda(k) \|G\|_{Lip} \|H\|_\infty \qquad \forall k \in \mathbb{N}$$

for every $G \in Lip(M, \mathbb{R})$ and $H \in L^\infty(M, \mathbb{R})$.

(II) For ν-almost every ω, the individual measure μ^ω have the following decay of correlations

$$\left| \int_M G(H \circ T^k_\omega)\, d\mu^\omega - \mu^\omega(G)\mu^{\theta^k\omega}(H) \right| \leq \lambda(k) \|G\|_{Lip} \|H\|_\infty \qquad \forall k \in \mathbb{N},$$

for every $H \in L^\infty(M, \mathbb{R})$ which are constant on local stable leaves γ^s of T_ω and for every $G \in Lip(M, \mathbb{R})$.

Then, we need some geometric assumptions:

(III) (Distortion) We require that $\frac{J_n^\omega(x)}{J_n^\omega(y)}=\mathcal{O}(\Theta(n))$ for all $x,y\in\zeta$ and n, where ζ are n-cylinders in unstable leaves γ^u and Θ is a non-decreasing function which below we assume to be $\Theta(n)=\mathcal{O}(n^{\kappa'})$ for some $\kappa'\ge 0$. (For almost all ω.)
(IV) (Contraction) There exists a function $\delta(n)\to 0$ which decays at least summably polynomially, i. e. $\delta(n)=\mathcal{O}(n^{-\kappa})$ with $\kappa>1$, so that diam $\zeta\le\delta(n)$ for all n-cylinder ζ and all n and ω.
Finally, we need some informations on the measures:
(V) There exist $0<d_0<d_1$ and K such that $\rho^{d_0}\ge\mu(B_\rho)\ge\rho^{d_1}$ and

$$\frac{1}{K}\le\frac{\mu(B_\rho)}{\mu^\omega(B_\rho)}\le K$$

for all $\rho>0$ small enough and for ν-almost every ω.
(VI) (Annulus condition) Assume that for some $\xi\ge\beta>0$:

$$\sup_\omega\frac{\mu^\omega(B_{\rho+r}\setminus B_{\rho-r})}{\mu(B_\rho)}=\mathcal{O}(\frac{r^\xi}{\rho^\beta})$$

for every $r<\rho$.

Our main result is on the distribution of the first hitting and return times. For a set $B\subset M$ $\omega\in\Omega$ one defines the function

$$\tau_B^\omega(x)=\inf\{j\ge 1:\ T_\omega^j x\in B\}.$$

This is the *hitting time function* on M or the *return time function* when restricted to B itself. We can now state our main result, here μ_B^ω is the conditional measure of μ^ω restricted to the set $B\subset M$.

Theorem 2.1. *Let a random dynamical system satisfy the above requirements (I)–(VI) where δ and λ both decay super polynomially fast.*
Then

$$\mu^\omega\left(y\in M:\tau^\omega_{B_\rho(\mathsf{x})}(y)>\frac{t}{\mu(B_\rho(\mathsf{x}))}\right)\longrightarrow e^{-t}\qquad as\ \rho\to 0$$

and

$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y\in M:\tau^\omega_{B_\rho(\mathsf{x})}(y)>\frac{t}{\mu(B_\rho(\mathsf{x}))}\right)\longrightarrow e^{-t}\qquad as\ \rho\to 0$$

for all $t>0$ for μ^ω-almost every $\mathsf{x}\in M$ and ν-almost every $\omega\in\Omega$.

As can be observe in the next theorem, even if δ and λ do not decay super polynomially fast, one can still obtain an exponential distribution assuming some technical conditions on the constants present in the hypothesis (I)–(VI).

Let us denote u_0 the largest of the elements $\tilde{u}$ so that $\mu^\omega_{\gamma^u}(B_\rho(x))\le C_1\rho^{\tilde{u}}$ for all $\rho>0$ small enough and for almost all $x\in\gamma^u$, every unstable leaf γ^u and ν-almost all ω. By assumption (V), such element exists and is at least equal to d_0.

We have a version of Theorem 2.1 for polynomial decay:

Theorem 2.2. *Let a random dynamical system satisfy the above requirements (I)–(VI). Assume one of the following two conditions is satisfied.*
(A) $\delta(n)=\mathcal{O}(n^{-\kappa})$ and $\lambda(k)=\mathcal{O}(k^{-p})$ decay polynomially with the respective rates $\kappa>1$

and $p>1$ *satisfying* $\kappa\xi>1$, $\max\{\frac{d_1\beta}{\kappa\xi-1},\left(\frac{\beta}{\xi}+d_1\right)\frac{1}{p}\}<\min\{1,u_0\}$ *and* $\gamma=\kappa u_0-2-\kappa'>1$.
(B) δ *decays super polynomially,* $\lambda(k)=\mathcal{O}(k^{-p})$ *decays polynomially and* $\left(\frac{\beta}{\xi}+d_1\right)\frac{1}{p}<\min\{1,u_0\}$.

Then the conclusion of Theorem 2.1 holds.

Remark: If d is the dimension of the measure μ then $d_0<d<d_1$ can be chosen arbitrarily close to d. The assumptions in case (A) then simplify to $\max(\frac{d\beta}{k\xi-1},\frac{\beta/\xi+d}{p})<1\wedge u_0$.

The proof of the theorems is done in the next three sections. In Section 3 we prove that the limiting distribution is exponential (the convergence is realised for μ^ω-almost every point x) using a key proposition (Proposition 4.1). In Section 4, we prove the key proposition, i.e. we show the smallness of the measure of the set of points whose neighbourhoods return to themselves within a very small number of iterates. In Section 5 we then prove the limiting result for return times, an alternative proof is also given in Section 6. Finally, in Section 7 we look at interval maps as an example to apply our main result.

Throughout the paper $C_0,C_1,\dots$ and $\alpha,\beta,\dots$ denote global constants while $c_0,c_1,\dots$ are locally defined constants.

3. Entry times distribution

For a ball $B_\rho(\mathsf{x})\subset M$ we define the counting function

$$Z^\omega_{\mathsf{x},\rho,t}(y)=\sum_{n=0}^{\lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor-1}\mathbb{1}_{B_\rho(\mathsf{x})}\circ T^n_\omega(y)$$

which tracks the number of visits a trajectory of the point $y\in M$ makes to the ball $B_\rho(\mathsf{x})$ on an orbit segment of length $N=\lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor$, where t is a positive parameter. Clearly $\tau^\omega_{B_\rho(\mathsf{x})}(y)>N$ exactly if $Z^\omega_{\mathsf{x},\rho,t}(y)=0$.

Let us put $J=\mathfrak{a}|\log\rho|$ (with the number $\mathfrak{a}$ determined below) and define the following counting function for very short returns along the orbit segment:

$$Y^\omega_{\mathsf{x},\rho,t}(y)=\sum_{j=1}^{N}\mathbb{1}_{B_\rho(\mathsf{x})\cap\{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})}<J\}}\circ T^j_\omega(y).$$

For a positive parameter $\mathfrak{a}$ define the set

$$\mathcal{V}^\omega_\rho(\mathfrak{a})=\{\mathsf{x}\in M: B_\rho(\mathsf{x})\cap T^n_\omega B_\rho(\mathsf{x})\neq\varnothing\text{ for some }1\le n<\mathfrak{a}|\log\rho|\},\tag{1}$$

where $\rho>0$. The set $\mathcal{V}^\omega_\rho$ represents the points within M with very short return times with respect to the realisation ω.

Proposition 3.1. *Under the assumptions of Theorem 2.1 put* $u_1=u_0$ *if* δ *decays super-polynomially and* $u_1=u_0-\frac{1}{\kappa}$ *if* $\delta(n)=\mathcal{O}(n^{-\kappa})$.

Then there exist a positive ϵ *and a constant* C_2 *so that*

$$\left|\mu^\omega(\tau^\omega_{B_\rho}>N)-\prod_{j=1}^{N}(1-\mu^{\theta^j\omega}(B_\rho))\right|\le C_2\left(\rho^\epsilon+(\delta(J)^{u_1}+\rho^\epsilon)\sum_{j=1}^{N}\mu^{\theta^j\omega}(B_\rho)\right)+\mu^\omega(Y^\omega_{\mathsf{x},\rho,t})$$

for all balls B_ρ.

Proof of Theorem 2.1 for hitting times. According to [20] Lemma 14 the variance (as a function of ω) of

$$\mu^\omega(Z^\omega_{\mathsf{x},\rho,t}) = \sum_{j=1}^N \mu^{\theta^j\omega}(B_\rho)$$

is bounded by ρ^q for some $0 < q < \frac{(p-1)d_0\xi - d_1\xi - \beta}{p\xi+1}$ and for all x. Hence we obtain along any sequence ρ_i for which $\sum_{i=1}^\infty \rho_i^q < \infty$ by an application of Chebycheff's inequality and the Borel-Cantelli lemma that

$$\mu^\omega(Z^\omega_{\mathsf{x},\rho_i,t}) \to t \qquad \text{as } i \to \infty$$

for ν-almost every ω since $\mu(Z^\omega_{\mathsf{x},\rho,t}) = t$ for all $\rho > 0$. Since

$$\begin{aligned}
\prod_{j=1}^N (1 - \mu^{\theta^j\omega}(B_\rho)) &= \exp\sum_{j=1}^N \left(\mu^{\theta^j\omega}(B_\rho) + \mathcal{O}(\mu^{\theta^j\omega}(B_\rho))^2\right) \\
&= \exp\sum_{j=1}^N \left(\mu^{\theta^j\omega}(B_\rho) + \mathcal{O}(\max_j \mu^{\theta^j\omega}(B_\rho))\right) \\
&\to e^{-t}
\end{aligned}$$

as $\rho \to 0$ along a sequence ρ_i, since $\max_j \mu^{\theta^j\omega}(B_\rho) \le C_1\rho^{u_0} \to 0$.

Thus, our theorem is proven along a sequence ρ_i, if we prove that the right hand side of the inequality in Proposition 3.1 goes to zero as ρ_i goes to zero for almost every x which will follow immediately if $\mu^\omega(Y^\omega_{\mathsf{x},\rho_i})$ goes to zero for almost every x.

In order to estimate the term $\mu^\omega(Y^\omega_{\mathsf{x},\rho})$ let us put $N_\rho(\mathsf{x}) = \lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor$. Observe that if $\mathsf{x} \notin \mathcal{V}^{\theta^j\omega}_{2\rho}$ then $y \notin \mathcal{V}^{\theta^j\omega}_\rho$ for all $y \in B_\rho(\mathsf{x})$. That is $B_\rho(\mathsf{x}) \cap \mathcal{V}^{\theta^j\omega}_\rho = \varnothing$. Moreover, if $B_\rho(\mathsf{x}) \cap \{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})} < J\} \neq \varnothing$ then there exists $y \in B_\rho(\mathsf{x})$ and a $k < J$ so that $T^k_{\theta^j\omega}y \in B_\rho(\mathsf{x})$. Thus $d(y, T^k_{\theta^j\omega}y) < 2\rho$ and $y \in \mathcal{V}^{\theta^j\omega}_{2\rho}$. These two statements combined lead us to conclude that if $\mathsf{x} \notin \mathcal{V}^{\theta^j\omega}_{4\rho}$ then $B_\rho(\mathsf{x}) \cap \{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})} < J\} = \varnothing$.

Put $W^\omega_\rho(\mathsf{x}) = \sum_{j=1}^{N_{\rho/4}(\mathsf{x})} \mathbb{1}_{\mathcal{V}^{\theta^j\omega}_\rho}(\mathsf{x})$ and $q^\omega_\rho(\mathsf{x}) = \frac{W^\omega_\rho(\mathsf{x})}{N_{\rho/4}(\mathsf{x})}$. Let $M_k = \{\mathsf{x} \in M : N_{\rho/4}(\mathsf{x}) = k\}$ and put $a_{j,k} = \mu^\omega(\mathcal{V}^{\theta^j\omega}_\rho \cap M_k)$. Observe that, by Assumption (V) , $\sup_{\mathsf{x}} N^\omega_{\rho/4}(\mathsf{x})$ is bounded above by $\hat{N} = c_1 t\rho^{-d_1}$ for some constant c_1. Then

$$Q^\omega_\rho := \int_M q^\omega_\rho(\mathsf{x})\, d\mu^\omega(\mathsf{x}) = \sum_{k=1}^{\hat{N}} \frac{1}{k}\sum_{j=1}^k a_{j,k} \le \sum_{j=1}^{\hat{N}} \frac{1}{j}\sum_{k=1}^{\hat{N}} a_{j,k}.$$

Since by Proposition 4.1 $\mu^\omega(\mathcal{V}^{\theta^j\omega}_\rho) = \sum_{k=1}^{\hat{N}} a_{j,k} \lesssim |\log\rho|^{-\gamma-1}$, where $\gamma = \kappa u_0 - 2 - \kappa'$, we thus obtain

$$Q^\omega_\rho \lesssim |\log\rho|^{-\gamma-1}\sum_{j=1}^{\hat{N}} \frac{1}{j} \lesssim |\log\rho|^{-\gamma}.$$

Now define

$$\mathcal{B}^\omega_\rho = \left\{\mathsf{x} \in M : q^\omega_\rho(\mathsf{x}) > |\log\rho|^{-\gamma'}\right\}$$

for some $\gamma' \in (0,\gamma)$. By Markov's inequality:

$$\mu^\omega(\mathcal{B}^\omega_\rho) \le Q^\omega_\rho |\log\rho|^{\gamma'} \lesssim |\log\rho|^{-\gamma''},$$

where $\gamma'' = \gamma - \gamma'$.

If $\mathsf{x} \notin \mathcal{B}^\omega_{4\rho}$ then $q^\omega_{4\rho}(\mathsf{x}) \lesssim |\log\rho|^{-\gamma'}$ and $W^\omega_{4\rho}(\mathsf{x}) \lesssim \frac{N_\rho(\mathsf{x})}{|\log\rho|^{\gamma'}}$. Consequently there exists an index set $\mathcal{I}^\omega_\mathsf{x} \subset \{1,2,\dots,N_\rho(\mathsf{x})\}$ so that $|\mathcal{I}^\omega_\mathsf{x}| \lesssim N_\rho(\mathsf{x})|\log\rho|^{-\gamma'}$ and

$$\begin{cases} \mathsf{x} \in \mathcal{V}^\omega_{4\rho} & \forall j \in \mathcal{I}^\omega_\mathsf{x} \\ \mathsf{x} \notin \mathcal{V}^\omega_{4\rho} & \forall j \in \{1,\dots,N_\rho(\mathsf{x})\} \setminus \mathcal{I}^\omega_\mathsf{x}. \end{cases}$$

Since $B_\rho(\mathsf{x}) \cap \{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})} < J\} = \varnothing$ for all $j \in \{1,\dots,N_\rho(\mathsf{x})\} \setminus \mathcal{I}^\omega_\mathsf{x}$ we finally get by Assumption (V)

$$\begin{aligned} \mu^\omega(Y^\omega_{\mathsf{x},\rho}) &\le \mu^\omega\left(\sum_{j\in\mathcal{I}^\omega_\mathsf{x}} \mathbb{1}_{B_\rho(\mathsf{x})\cap\{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})}<J\}} \circ T^j_\omega\right) \\ &\le |\mathcal{I}^\omega_\mathsf{x}| K\mu(B_\rho(\mathsf{x})) \\ &\lesssim q^\omega_{4\rho}(\mathsf{x}) N_\rho(\mathsf{x}) K\mu(B_\rho(\mathsf{x})) \\ &\lesssim \frac{t}{|\log\rho|^{\gamma'}} \to 0 \end{aligned}$$

as ρ goes to zero.

In order to prove the statement in Theorem 2.1 for almost every $\mathsf{x} \in M$, we observe that $\mu^\omega(\mathcal{B}^\omega_\rho) \lesssim |\log\rho|^{-\gamma''}$ where $1 < \gamma'' < \gamma$ can be chosen arbitrarily close to γ. Let $\alpha \in (\frac{1}{\gamma''},1)$ and for $i \in \mathbb{N}$ large enough, we can choose $\rho_i = e^{-i^\alpha}$ (since this sequence satisfies $\sum_{i=1}^\infty \rho_i^q < \infty$). Thus,

$$\mu^\omega(\mathcal{B}^\omega_{\rho_i}) \le c_1 i^{-\alpha\gamma''}$$

and $\sum_i \mu^\omega(\mathcal{B}^\omega_{\rho_i}) < \infty$ since $\alpha\gamma'' > 1$. Using the Borel-Cantelli lemma we conclude $\mu^\omega(x \in \mathcal{B}^\omega_{\rho_i}$ i.o.$) = 0$ which proves the convergence in the theorem for almost every $\mathsf{x} \in M$ along the sequence ρ_i.

In order to get the convergence for arbitrary $\rho \to 0$ let $\rho > 0$ be sufficiently small and i so that $\rho_i \le \rho \le \rho_{i-1}$. We have $r_i = \rho_{i-1} - \rho_i = \rho_i.\mathcal{O}(i^{-(1-\alpha)})$, then

$$\begin{aligned}
&\left|\mu^\omega\left(\tau^\omega_{B_\rho} > \frac{t}{\mu(B_\rho)}\right) - \mu^\omega\left(\tau^\omega_{B_{\rho_i}} > \frac{t}{\mu(B_{\rho_i})}\right)\right| \\
\le\;& \mu^\omega\left(\tau^\omega_{B_\rho \setminus B_{\rho_i}} < \frac{t}{\mu(B_\rho)}\right) + \mu^\omega\left(\frac{t}{\mu(B_\rho)} < \tau^\omega_{B_{\rho_i}} < \frac{t}{\mu(B_{\rho_i})}\right) \\
\le\;& \sum_{j=1}^{\frac{t}{\mu(B_\rho)}} \mu^{\theta^j\omega}(B_\rho \setminus B_{\rho_i}) + \sum_{j=\frac{t}{\mu(B_\rho)}}^{\frac{t}{\mu(B_{\rho_i})}} \mu^{\theta^j\omega}(B_{\rho_i}) \\
\le\;& \mu(B_\rho) \sum_{j=1}^{\frac{t}{\mu(B_\rho)}} \frac{\mu^{\theta^j\omega}(B_{\rho+r_i} \setminus B_{\rho-r_i})}{\mu(B_\rho)} + K\mu(B_{\rho_i}) \left|\frac{t}{\mu(B_\rho)} - \frac{t}{\mu(B_{\rho_i})}\right| \\
\lesssim\;& t\frac{r_i^\xi}{\rho_i^\beta} + Kt\frac{\mu(B_\rho \setminus B_{\rho_i})}{\mu(B_\rho)} \\
\lesssim\;& t(1+K)\frac{r_i^\xi}{\rho_i^\beta} \\
\lesssim\;& \frac{\rho_i^{\xi-\beta}}{i^{\xi(1-\alpha)}}
\end{aligned}$$

using Assumption (VI) and Assumption (V). This difference goes to zero as $i \to \infty$ since $\xi \ge \beta$ and $1 - \alpha > 0$ which concludes the proof of the theorem. □

Proof of Proposition 3.1. We proceed as in [20] and note that

$$\left|\mu^\omega(\tau^\omega_{B_\rho} > N) - \prod_{j=1}^N (1 - \mu^{\theta^j\omega}(B_\rho))\right| \le \sum_{j=1}^N \epsilon_{\theta^j\omega}(B_\rho) \prod_{k=1}^{j-1} (1 - \mu^{\theta^k\omega}(B_\rho))$$

where

$$\epsilon_\omega(B_\rho) = \sup_{k\ge1} \left|\mu^\omega(\tau^\omega_{B_\rho} > k)\, \mu^\omega(B_\rho) - \mu^\omega(B_\rho \cap \{\tau^\omega_{B_\rho} > k\})\right|.$$

We now split the error term on the RHS into three parts using the fact that

$$\begin{aligned}
\epsilon_{\theta^j\omega} \;\le\; & \sup_{k\ge1} \left|\mu^{\theta^j\omega}(B_\rho \cap T^{-\Delta}_{\theta^j\omega}\{\tau^{\theta^{j+\Delta}\omega}_{B_\rho} \ge k\}) - \mu^{\theta^j\omega}(B_\rho)\, \mu^{\theta^{j+\Delta}\omega}(\{\tau^{\theta^{j+\Delta}\omega}_{B_\rho} \ge k\})\right| \\
& + \mu^{\theta^j\omega}(B_\rho \cap \{\tau^{\theta^j\omega}_{B_\rho} \le \Delta\}) + \mu^{\theta^j\omega}(B_\rho)\mu^{\theta^j\omega}(\tau^{\theta^j\omega}_{B_\rho} \le \Delta).
\end{aligned}$$

Thus

$$\left|\mu^\omega(\tau^\omega_{B_\rho} > N) - \prod_{j=1}^N (1 - \mu^{\theta^j\omega}(B_\rho))\right| \le \sum_{j=1}^N \epsilon_{\theta^j\omega}(B_\rho) = \mathcal{R} \le \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \tag{2}$$

where we now estimate the three terms on the RHS individually.

3.1. **Estimating $\mathcal{R}_1$.** We estimate the principal term by

$$\mathcal{R}_1 = N \sup_{\omega} \sup_{k\geq 1} \left| \mu^{\omega}(B_\rho \cap T_\omega^{-\Delta} S_k) - \mu^{\omega}(B_\rho)\, \mu^{\theta^\Delta \omega}(S_k) \right|.$$

where we put $S_k = S_k(\Delta) = \{y : \tau_{B_\rho}^{\theta^\Delta \omega}(y) \geq k\}$. We now use the decay of correlations from Assumption (II) to obtain an estimate. Approximate $\mathbb{1}_{B_\rho}$ by Lipschitz functions from above and below as follows:

$$\phi(x) = \begin{cases} 1 & \text{on } B_\rho \\ 0 & \text{outside } B_{\rho+\delta\rho} \end{cases} \qquad \text{and} \qquad \tilde{\phi}(x) = \begin{cases} 1 & \text{on } B_{\rho-\delta\rho} \\ 0 & \text{outside } B_\rho \end{cases}$$

with both functions linear within the annuli. The Lipschitz norms of both ϕ and $\tilde{\phi}$ are equal to $1/\delta\rho$ and $\tilde{\phi} \leq \mathbb{1}_{B_\rho} \leq \phi$. We obtain

$$\begin{aligned} &\mu^{\omega}(B_\rho \cap T_\omega^{-\Delta} S_k) - \mu^{\omega}(B_\rho)\, \mu^{\theta^\Delta \omega}(S_k) \\ &\leq \int_M \phi(\mathbb{1}_{S_k} \circ T_\omega^{\Delta})\, d\mu^{\omega} - \int_M \mathbb{1}_{B_\rho}\, d\mu^{\omega} \int_M \mathbb{1}_{S_k}\, d\mu^{\theta^\Delta \omega} \\ &= X + Y \end{aligned}$$

where

$$\begin{aligned} X &= \left(\int_M \phi\, d\mu^{\omega} - \int_M \mathbb{1}_{B_\rho}\, d\mu^{\omega} \right) \int_M \mathbb{1}_{S_k}\, d\mu^{\theta^\Delta \omega} \\ Y &= \int_M \phi\, (\mathbb{1}_{S_k} \circ T_\omega^{\Delta})\, d\mu^{\omega} - \int_M \phi\, d\mu^{\omega} \int_M \mathbb{1}_{S_k}\, d\mu^{\theta^\Delta \omega}. \end{aligned}$$

The two terms X and Y are estimated separately. The first term is estimated as follows:

$$X \leq \int_M \mathbb{1}_{S_k}\, d\mu^{\theta^\Delta \omega} \int_M (\phi - \mathbb{1}_{B_\rho})\, d\mu^{\omega} \leq \mu^{\omega}(B_{\rho+\delta\rho} \setminus B_\rho).$$

In order to estimate the second term Y we use the decay of correlations and have to approximate $\mathbb{1}_{S_k}$ by a function which is constant on local stable leaves. For that purpose put

$$\mathcal{S}_n = \bigcup_{\substack{\gamma^s \\ T_\omega^n \gamma^s \subset B_\rho}} T_\omega^n \gamma^s, \qquad \partial \mathcal{S}_n = \bigcup_{\substack{\gamma^s \\ T_\omega^n \gamma^s \cap B_\rho \neq \varnothing}} T_\omega^n \gamma^s$$

and

$$\mathscr{S}_\Delta^{N-j} = \bigcup_{n=\Delta}^{N-j} \mathcal{S}_n, \qquad \partial \mathscr{S}_\Delta^{n-j} = \bigcup_{n=\Delta}^{N-j} \partial \mathcal{S}_n.$$

The set

$$\mathscr{S}_\Delta^{N-j}(k) = S_k \cap \mathscr{S}_\Delta^{N-j}$$

is then a union of local stable leaves. This follows from the fact that by construction $T^n y \in B_\rho$ if and only if $T^n \gamma^s(y) \subset B_\rho$. We also have $S_k \subset \tilde{\mathscr{S}}_\Delta^{N-j}(k)$ where the set $\tilde{\mathscr{S}}_\Delta^{N-j}(k) = \mathscr{S}_\Delta^{N-j}(k) \cup \partial \mathscr{S}_\Delta^{N-j}$ is a union of local stable leaves.

Denote by ψ_Δ^{N-j} the characteristic function of $\mathscr{S}_\Delta^{N-j}(k)$ and by $\tilde\psi_\Delta^{N-j}$ the characteristic function of $\tilde{\mathscr{S}}_\Delta^{N-j}(k)$. Then ψ_Δ^{N-j} and $\tilde\psi_\Delta^{N-j}$ are constant on local stable leaves and satisfy

$$\psi_\Delta^{N-j} \leq \mathbb{1}_{S_k} \leq \tilde\psi_\Delta^{N-j}.$$

Since $\{y : \psi_\Delta^{N-j}(y) \neq \tilde\psi_\Delta^{N-j}(y)\} \subset \partial\mathscr{S}_\Delta^{N-j}$ we need to estimate the measure of $\partial\mathscr{S}_\Delta^{N-j}$.

By the contraction property $\operatorname{diam}(T_\omega^n\gamma^s(y)) \leq \delta(n)$ and consequently

$$\bigcup_{\substack{\gamma^s \\ T_\omega^n\gamma^s\subset B_\rho}} T_\omega^n\gamma^s \subset B_{\rho+\delta(n)} \setminus B_{\rho-\delta(n)}$$

and therefore

$$\mu^\omega(\partial\mathscr{S}_\Delta^{N-j}) \leq \mu^\omega\left(\bigcup_{n=\Delta}^{N-j} T_\omega^{-n}\left(B_{\rho+\delta(n)} \setminus B_{\rho-\delta(n)}\right)\right) \leq \sum_{n=\Delta}^{N-j} \mu^{\theta^n\omega}(B_{\rho+\delta(n)} \setminus B_{\rho-\delta(n)}).$$

Hence, by assumption (VI), using $r = 2\delta(n) = \mathcal{O}(n^{-\kappa})$ if δ decays polynomially with power κ:

$$\begin{aligned}\sum_{n=\Delta}^{N-j}\mu^\omega(\partial\mathscr{S}_\Delta^{N-j}) &= \mathcal{O}(1)\sum_{n=\Delta}^{\infty}\frac{n^{-\kappa\xi}}{\rho^{d_1\beta}}\mu(B_\rho)\\ &= \mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta}\mu(B_\rho))\end{aligned}$$

provided $\Delta \sim \rho^{-v}$ for some positive $v > \frac{\beta-d_0}{\kappa\xi-1}$ which is determined in Section 3.4 below. If we split $\Delta = \Delta' + \Delta''$ then we can estimate as follows:

$$\begin{aligned}Y &= \left|\int_M \phi\, T_\omega^{-\Delta'}(\mathbb{1}_{S_k(\Delta')})\,d\mu^\omega - \int_M \phi\,d\mu^\omega \int_M \mathbb{1}_{S_k(\Delta)}\,d\mu^{\theta^\Delta\omega}\right|\\ &\leq \lambda(\Delta')\|\phi\|_{Lip}\|\mathbb{1}_{\tilde{\mathscr{S}}_{\Delta''}^{N-j-p'}}\|_{\mathscr{L}^\infty} + 2\mu^\omega(\partial\mathscr{S}_{\Delta''}^{N-j}).\end{aligned}$$

Hence

$$\mu^\omega(B_\rho \cap T^{-\Delta}S_k) - \mu^\omega(B_\rho)\,\mu^{\theta^\Delta\omega}(S_k) \leq \frac{\lambda(\Delta/2)}{\delta\rho} + \mu^\omega(B_\rho \setminus B_{\rho-\delta\rho}) + \mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta}\mu(B_\rho)).$$

A similar estimate from below can be done using $\tilde\phi$. Hence

$$\mathcal{R}_1 \leq Nc_1\left(\frac{\lambda(\Delta/2)}{\delta\rho} + \sup_\omega \mu^\omega(B_{\rho+\delta\rho} \setminus B_{\rho-\delta\rho})\right) + \mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta}). \tag{3}$$

3.2. **Estimating the terms $\mathcal{R}_2$.** We will estimate the measure of each of the summands comprising $\mathcal{R}_2$ individually. We use the product form of the measures μ^ω. For that purpose fix j and and let γ^u be an unstable local leaf through B. Then we put

$$\mathscr{C}_j^\omega(B,\gamma^u) = \{\zeta_{\varphi,j} : \zeta_{\varphi,j} \cap B \neq \varnothing, \varphi \in \mathscr{I}_j^\omega\}$$

for the cluster of j-cylinders that covers the set B, where the sets $\zeta_{\varphi,k}$ are the images of imbedded R-balls in $T^j_\omega\gamma^u$. Then , using the distortion property (III),

$$\begin{aligned}\mu^\omega_{\gamma^u}(T^{-j}_\omega B_\rho\cap B_\rho) &\le \sum_{\zeta\in\mathscr{C}^\omega_j(B_\rho,\gamma^u)}\frac{\mu^\omega_{\gamma^u}(T^{-j}_\omega B_\rho\cap\zeta)}{\mu^\omega_{\gamma^u}(\zeta)}\mu^\omega_{\gamma^u}(\zeta)\\ &\le \sum_{\zeta\in\mathscr{C}^\omega_j(B_\rho,\gamma^u)}\Theta(j)\frac{\mu^{\theta^j\omega}_{T^j_\omega\gamma^u}(B_\rho\cap T^j_\omega\zeta)}{\mu^{\theta^j\omega}_{T^j_\omega\gamma^u}(T^j_\omega\zeta)}\mu^\omega_{\gamma^u}(\zeta)\end{aligned}$$

Since $\mu^{\theta^j\omega}_{T^j_\omega\gamma^u}(T^j_\omega\zeta)=\mu^{\theta^j\omega}_{T^j_\omega\gamma^u}(B_{R,\gamma^u}(y_k))$ (for some y_k) is uniformly bounded from below, we obtain

$$\begin{aligned}\mu^\omega_{\gamma^u}(T^{-j}_\omega B_\rho\cap B_\rho) &\le \Theta(j)\mu^{\theta^j\omega}_{T^j_\omega\gamma^u}(B_\rho)\sum_{\zeta\in\mathscr{C}^\omega_j(B_\rho,\gamma^u)}\mu^\omega_{\gamma^u}(\zeta)\\ &\le \Theta(j)\mu^{\theta^j\omega}_{T^j_\omega\gamma^u}(B_\rho)\,L\,\mu^\omega_{\gamma^u}\left(\bigcup_{\zeta\in\mathscr{C}^\omega_j(B_\rho,\gamma^u)}\zeta\right)\end{aligned}$$

Now, since diam $\bigcup_{\zeta\in\mathscr{C}^\omega_j(B_\rho,\gamma^u)}\zeta\le\delta(j)+\text{diam }B_\rho\le c_1\delta(j)$ (as we can assume that $\rho<\delta(j)$) we obtain

$$\mu^\omega_{\gamma^u}(T^{-j}_\omega B_\rho\cap B_\rho)\le c_3\Theta(j)\mu^{\theta^j\omega}_{T^j_\omega\gamma^u}(B_\rho)\delta(j)^{u_0}.$$

Since $d\mu^\omega=d\mu^\omega_{\gamma^u}dv^\omega(\gamma^u)$ we obtain

$$\mu^\omega(T^{-j}_\omega B_\rho\cap B_\rho)\le c_4\Theta(j)\mu^{\theta^j\omega}(B_\rho)\delta(j)^{u_0}.$$

Summing up the $\mu^\omega(T^{-j}_\omega B_\rho\cap B_\rho)$ over $j=J,\dots,\Delta-1$, we get

$$\begin{aligned}\mathcal{R}_2'(\omega) &= \mu^\omega(B_\rho\cap T^{-J}_\omega\{\tau^{\theta^J\omega}_{B_\rho}<\Delta-J\})+\mu^\omega(B_\rho\cap\{\tau^\omega_{B_\rho}<J\})\\ &\le \sum_{j=J}^{\Delta-1}\mu^\omega(T^{-j}_\omega B_\rho\cap B_\rho)+\mu^\omega(B_\rho\cap\{\tau^\omega_{B_\rho}<J\})\\ &\le c_4\sum_{j=J}^{\Delta-1}\Theta(j)\delta(j)^{u_0}\mu^{\theta^j\omega}(B_\rho)+\mu^\omega(B_\rho\cap\{\tau^\omega_{B_\rho}<J\}).\end{aligned}\tag{4}$$

For the entire error term we thus obtain

$$\begin{aligned}\mathcal{R}_2 &= \sum_{k=1}^N\mathcal{R}_2'(\theta^k\omega)\\ &\le c_4\sum_{j=J}^{\Delta}\Theta(j)\delta(j)^{u_0}\sum_{k=1}^N\mu^{\theta^{k+j}\omega}(B_\rho)+\sum_{k=1}^N\mu^{\theta^k\omega}(B_\rho\cap\{\tau^{\theta^k\omega}_{B_\rho}<J\})\\ &\le c_6t\delta(J)^{u_1}J^{\kappa'}\sum_{k=1}^N\mu^{\theta^k\omega}(B_\rho)+\mu^\omega(Y^\omega_{\mathsf{x},\rho,t})\end{aligned}$$

for some c_5, c_6 and almost every ω and ρ small enough (depending on ω). The exponent u_1 equals u_0 if $\delta(j)$ decays super polynomially and equals $u_0 - \frac{1}{\kappa}$ if $\delta(j)$ decays polynomially with power κ.

3.3. **Estimating the terms $\mathcal{R}_3$.** Assumption (V) yields

$$\mu^\omega(B_\rho) = \int \mu^\omega_{\gamma^u}(B_\rho)\, d\upsilon^\omega(\gamma^u) \le \int C_1\rho^{u_0}\, d\upsilon^\omega(\gamma^u) = C_1\rho^{u_0}$$

for every ω. Since $\mu^{\theta^j\omega}(\tau^{\theta^j\omega}_{B_\rho} \le \Delta) \le \sum_{k=1}^{\Delta} \mu^{\theta^{j+k}\omega}(B_\rho)$ we obtain by Assumption (V)

$$\begin{aligned}
\mathcal{R}_3 &= \sum_{j=1}^{N} \mu^{\theta^j\omega}(B_\rho)\mu^{\theta^j\omega}(\tau^{\theta^j\omega}_{B_\rho} \le \Delta)\\
&\le C_1\rho^{u_0} \sum_{j=1}^{N}\sum_{k=1}^{\Delta} \mu^{\theta^{j+k}\omega}(B_\rho)\\
&\le C_1\rho^{u_0}\Delta \sum_{j=1}^{N} \mu^{\theta^j\omega}(B_\rho) + C_1\rho^{u_0}\sum_{j=1}^{\Delta}(\Delta-j)\mu^{\theta^{N+j}\omega}(B_\rho)\\
&\le c_7\rho^{u_0}\Delta \sum_{j=1}^{N} \mu^{\theta^j\omega}(B_\rho) + c_7(\Delta\rho^{u_0})^2
\end{aligned}$$

for some c_7 for almost all ω and ρ small enough since the first sum converges ν-almost everywhere to t.

3.4. **The total error.** The total error is

$$\begin{aligned}
\mathcal{R} &= \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3\\
&\le Nc_1\left(\frac{\lambda(\Delta)}{\delta\rho} + \sup_\omega \mu^\omega(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho})\right) + \mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta})\\
&\qquad + \left(c_6\delta(J)^{u_1}J^{\kappa'} + C_1\rho^{u_0}\Delta\right)\sum_{j=1}^{N}\mu^{\theta^j\omega}(B_\rho) + c_7(\Delta\rho^{u_0})^2 + \mu^\omega(Y^\omega_{\mathsf{x},\rho,t}).
\end{aligned}$$

Let us consider the case when λ decays polynomially with power p, i.e. $\lambda(k) \sim k^{-p}$. We can choose $\Delta = \rho^{-v}$ so that $\lambda(\Delta) = \mathcal{O}(\rho^{-vp}) = \mathcal{O}(\rho^w\mu(B_\rho))\rho^{-(vp-w-d-1)}$ and so that then for some $\epsilon > 0$:

$$\Delta\rho^{u_0} \le c_8\rho^{u_0-\frac{w}{p}}\mu(B_\rho)^{-\frac{1}{p}} < c_9\rho^{u_0-\frac{w+d_1}{p}} = \mathcal{O}(\rho^\epsilon),$$

that is $\frac{d_1+w}{p} < v < \min\{1, u_0\}$. We then obtain with $\delta\rho = \rho^w$ that $N\frac{\lambda(\Delta)}{\delta\rho} \le c_{10}\frac{1}{\mu(B_\rho)}\frac{\Delta^{-p}}{\rho^w} = \mathcal{O}(\rho^{-(vp-w-d-1)})$. The second term is estimated by (maybe some smaller $\epsilon > 0$)

$$\sup_\omega \mu^\omega(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho}) = \mathcal{O}(\frac{\rho^{w\xi}}{\rho^\beta}) = \mathcal{O}(\rho^{w\xi-\beta}) = \mathcal{O}(\rho^\epsilon)$$

since $w\xi > \beta$. Hence we need constants $w, v > 0$ such that the following inequalities hold:

(1) $\frac{d_1+w}{p} < v < \min\{1, u_0\}$;

(2) $v(\kappa\xi - 1) - d_1\beta > 0$ from Section 3.1; and
(3) $w < \frac{\xi}{\beta}$ (w can be arbitrarily close to $\frac{\xi}{\beta}$).

These conditions hold if we require that $\max\{\frac{d_1\beta}{\kappa\xi-1}, \left(\frac{\beta}{\xi} + d_1\right)\frac{1}{p}\} < \min\{1, u_0\}$ which in particular implies $\rho^{v(\kappa\xi-1)-d_1\beta} = \mathcal{O}(\rho^{\epsilon})$. We therefore obtain

$$\mathcal{R} \leq \mathcal{O}(\rho^{\epsilon}) + \mathcal{O}(J^{\kappa'}\delta(J)^{u_1} + \rho^{\epsilon}) \sum_{j=1}^{N} \mu^{\theta^j\omega}(B_\rho) + \mu^{\omega}(Y^{\omega}_{\mathsf{x},\rho,t})$$

for all ρ small enough and every x. □

4. Very Short Returns

Let us recall that the set $\mathcal{V}_\rho \subset M$ is given by

$$\mathcal{V}^{\omega}_{\rho} = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^n_\omega B_\rho(\mathsf{x}) \neq \varnothing \text{ for some } 1 \leq n < J\},$$

where $J = \lfloor \mathfrak{a} \,|\log\rho\,| \rfloor$ and $\mathfrak{a} = (4\log A)^{-1}$ with

$$A = \sup_{\omega} \left(\| DT_\omega \|_{\mathscr{L}^\infty} + \| DT_\omega^{-1} \|_{\mathscr{L}^\infty}\right)$$

($A \geq 2$).

Now we can show that the set of centres where small balls have very short returns is small. Even if we follow the proof of Proposition 5.1 of [10] which modelled after Lemma 4.1 of [7], we emphasize that this is not a direct adaptation, in particular in view of Lemma 4.1 and its differences with the deterministic version.

Proposition 4.1. *There exist constants $C_3 > 0$ such that for all ρ small enough and all ω and $\hat{\omega}$:*

$$\mu^{\hat{\omega}}(\mathcal{V}^{\omega}_{\rho}) \leq C_3\Big(e^{-\mathfrak{c}|\log\rho|^{1/2}} + \delta(\mathfrak{a}\mathfrak{b}|\log\rho\,|)^{u_1}|\log\rho|^{\kappa'}\Big)$$

where $u_1 = u_0$ if δ decays superpolynomially and $u_1 = u_0 - \frac{1}{\kappa}$ if δ decays polynomially with power κ and $\mathfrak{b}, \mathfrak{c} > 0$ (recall that $\Theta(n) = \mathcal{O}(n^{-\kappa'})$).

Proof. Let us note that since T_ω is a diffeomorphism one has

$$B_\rho(\mathsf{x}) \cap T^n_\omega B_\rho(\mathsf{x}) \neq \varnothing \qquad \Longleftrightarrow \qquad B_\rho(\mathsf{x}) \cap T^{-n}_\omega B_\rho(\mathsf{x}) \neq \varnothing.$$

We partition $\mathcal{V}^{\omega}_{\rho}$ into level sets $\mathcal{N}^{\omega}_{\rho}(n)$ as follows

$$\mathcal{V}^{\omega}_{\rho} = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^{-n}_\omega B_\rho(\mathsf{x}) \neq \varnothing \text{ for some } 1 \leq n < J\} = \bigcup_{n=1}^{J-1} \mathcal{N}^{\omega}_{\rho}(n)$$

where

$$\mathcal{N}^{\omega}_{\rho}(n) = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^{-n}_\omega B_\rho(\mathsf{x}) \neq \varnothing\}.$$

The above union is split into two collections $\mathcal{V}^{\omega,1}_{\rho}$ and $\mathcal{V}^{\omega,2}_{\rho}$, where

$$\mathcal{V}^{\omega,1}_{\rho} = \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}^{\omega}_{\rho}(n) \quad \text{and} \quad \mathcal{V}^{\omega,2}_{\rho} = \bigcup_{n=\lceil \mathfrak{b}J \rceil}^{J} \mathcal{N}^{\omega}_{\rho}(n).$$

and where the constant $\mathfrak{b} \in (0,1)$ will be chosen below. In order to find the measure of the total set we will estimate the measures of the two parts separately.

(I) Estimate of $\mathcal{V}_\rho^{\omega,2}$

We will derive a uniform estimate for the measure of the level sets $\mathcal{N}_\rho^\omega(n)$ when $n > \mathfrak{b}J$. For this purpose define

$$\tilde{\omega} = \omega_0 \dots \omega_{n-1}\hat{\omega}_0\hat{\omega}_1 \dots = \omega_0 \dots \omega_{n-1}\hat{\omega}.$$

We have $T_{\tilde{\omega}}^n = T_{\omega_{n-1}} \circ \dots \circ T_{\omega_0} = T_\omega^n$. Also notice that $\theta^n\tilde{\omega} = \hat{\omega}$. Thus

$$\mu^{\hat{\omega}}(\mathcal{N}_\rho^\omega(n)) = \mu^{\tilde{\omega}}(T_{\tilde{\omega}}^{-n}\mathcal{N}_\rho^\omega(n)) \le \sum_\zeta \mu^{\tilde{\omega}}(T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta)$$

where the sum is over all n-cylinders ζ. We will consider each of the measures $\mu^{\tilde{\omega}}(T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta)$ separately by using the product form of the measures $\mu^{\hat{\omega}}$. By distortion of the Jacobian we obtain

$$\begin{aligned} \mu_{\gamma^u}^{\tilde{\omega}}(T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta) &= \frac{\mu_{\gamma^u}^{\tilde{\omega}}(T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta)}{\mu_{\gamma^u}^{\tilde{\omega}}(\zeta)}\,\mu_{\gamma^u}^{\tilde{\omega}}(\zeta) \\ &\le \Theta(n)\,\frac{\mu_{\hat{\gamma}^u}^{\hat{\omega}}(T_\omega^n(T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta))}{\mu_{\hat{\gamma}^u}^{\hat{\omega}}(T_\omega^n\zeta)}\,\mu_{\gamma^u}^{\tilde{\omega}}(\zeta), \end{aligned} \tag{5}$$

where, as before, $\hat{\gamma}^u = \gamma^u(T_\omega^n x)$ for $x \in \zeta \cap \gamma^u$. We estimate the numerator by finding a bound for the diameter of the set. Let the points x and z in $T_\omega^{-n}\mathcal{N}_\rho^\omega(n)$ be such that $x, z \in T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta \cap \gamma^u$ for an unstable leaf γ^u.

Note that $T_\omega^n x, T_\omega^n z \in \mathcal{N}_\rho^\omega(n)$, there exists $y \in B_\rho(T_\omega^n x)$ such that $T_\omega^n y \in B_\rho(T_\omega^n x)$ (as $B_\rho(T_\omega^n x) \cap T_\omega^{-n}B_\rho(x) \ne \varnothing$), thus

$$d(T_\omega^n x, x) \le d(T_\omega^n x, T_\omega^n y) + d(T_\omega^n y, y) + d(y,x) \le \rho + 2\rho + A^n d(T_\omega^n x, T_\omega^n y) \le (3 + A^n)\rho.$$

Hence as $y \in B_{A^n\rho}(x)$:

$$d(T_\omega^n x, T_\omega^n z) \le d(T_\omega^n x, x) + d(x,z) + d(z, T_\omega^n z) \le 6A^n\rho + d(x,z).$$

We have

$$d(x,z) \le \operatorname{diam}\zeta < \delta(n)$$

by assumption. Therefore

$$d(T_\omega^n x, T_\omega^n z) \le 6A^n\rho + d(x,z) \le 6\,A^n\rho + \delta(n)$$

If we choose $\mathfrak{a} > 0$ so that $\mathfrak{a} < \frac{1}{2\log A}$ then $A^n\rho < e^{-\frac{1}{2}|\log\rho|^{1/2}}$. If $n \ge \mathfrak{b}|\log\rho|$ for some $\mathfrak{b} \in (0,\mathfrak{a})$ then

$$d(T_\omega^n x, T_\omega^n z) \le c_1(e^{-\mathfrak{c}'|\log\rho|^{1/2}} + \delta(n))$$

for some constant c_1 where $\mathfrak{c}' = \min(\frac{1}{2}, \sqrt{\mathfrak{b}})$. Taking the supremum over all points x and z yields

$$|\,T_\omega^n(T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta \cap \gamma^u)\,| \le c_1(e^{-\mathfrak{c}'|\log\rho|^{1/2}} + \delta(n)).$$

By assumption (V) on the relationship between the measure and the metric

$$\mu_{\hat{\gamma}^u}^{\hat{\omega}}(T_\omega^n(T_\omega^{-n}\mathcal{N}_\rho^\omega(n) \cap \zeta)) \le c_2(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}} + \delta(n)^{u_0})$$

Incorporating the estimate into (5) yields

$$\mu^{\tilde\omega}_{\gamma^u}(T_\omega^{-n}\mathcal{N}^\omega_\rho(n)\cap\zeta)\le c_4\Theta(n)\,(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+\delta(n)^{u_0})\mu^{\tilde\omega}(\zeta),$$

for some c_4. Integrating over $d\upsilon(\gamma^u)$ and summing over ζ yields

$$\mu^{\hat\omega}(\mathcal{N}^\omega_\rho(n))\le c_4\Theta(n)\,(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+\delta(n)^{u_0})\sum_\zeta\mu^{\tilde\omega}(\zeta)\le c_5\Theta(n)\,(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+\delta(n)^{u_0})$$

as $\sum_\zeta\mu^{\tilde\omega}(\zeta)=\mathcal{O}(1)$. Consequently

$$\begin{aligned}
\mu^{\hat\omega}(\mathcal{V}^{\omega,2}_\rho) &\le \sum_{n=\lceil\,\mathfrak{b}J\,\rceil}^{J}\mu^{\hat\omega}(\mathcal{N}^\omega_\rho(n))\\
&\le c_5\Theta(J)\,Je^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+\sum_{n=\lceil\,\mathfrak{b}J\,\rceil}^{J}\Theta(n)\delta(n)^{u_0}\\
&\le c_6(e^{-\mathfrak{c}''|\log\rho|^{1/2}}+\delta(\mathfrak{a}\mathfrak{b}|\log\rho|)^{u_1}|\log\rho|^{\kappa'}) \qquad (6)
\end{aligned}$$

for some constant $\mathfrak{c}''>0$ (and ρ small enough) as $J=\lfloor\mathfrak{a}|\log\rho|\rfloor$. Here $u_1\le u_0$ is so that $\sum_{n=n_0}^\infty\delta(n)^{u_0}\le c_7\delta(n_0)^{u_1}$ for some constant c_7.

(II) Estimate of $\mathcal{V}^{\omega,1}_\rho$

We will need the following randomised version of Lemma B.3 from [7].

Lemma 4.1. *Put $s_p=2^p\,\frac{A^{n\,2^p}-1}{A^n-1}$. Then for every p,k integers, $\rho>0$ and ω there exists an $\tilde\omega$ so that*

$$\left\{\mathsf{x}\in M: B_\rho(\mathsf{x})\cap T^k_\omega B_\rho(\mathsf{x})\ne\varnothing\right\}\subset\left\{\mathsf{x}\in M: B_{s_p\rho}(\mathsf{x})\cap T^{k2^p}_{\tilde\omega}B_{s_p\rho}(\mathsf{x})\ne\varnothing\right\}.$$

Proof. Consider the case $p=1$. Let x such that $B_\rho(\mathsf{x})\cap T^k_\omega B_\rho(\mathsf{x})\ne\varnothing$. This implies that there exist $z\in B_\rho(\mathsf{x})\cap T^{-k}_\omega B_\rho(\mathsf{x})$. For any $u\in T^k_\omega B_\rho(\mathsf{x})$, there exist $v\in B_\rho(\mathsf{x})$ such that $T^k_\omega v=u$, thus

$$d(u,x)\le d(u,T^k_\omega z)+d(T^k_\omega z,x)\le d(T^k_\omega v,T^k_\omega z)+2\rho\le(2A^k+2)\rho.$$

Therefore, $T^k_\omega B_\rho(\mathsf{x})\subset B_{(2A^k+2)\rho}(\mathsf{x})$.

One can observe that if $B_\rho(\mathsf{x})\cap T^k_\omega B_\rho(\mathsf{x})\ne\varnothing$ then $T^k_\omega\left(B_\rho(\mathsf{x})\cap T^k_\omega B_\rho(\mathsf{x})\right)\ne\varnothing$ thus $T^k_\omega B_\rho(\mathsf{x})\cap T^k_\omega(T^k_\omega B_\rho(\mathsf{x}))\ne\varnothing$ and therefore $B_{(2A^k+2)\rho}(\mathsf{x})\cap T^k_\omega(T^k_\omega B_{(2A^k+2)\rho}(\mathsf{x}))\ne\varnothing$. Finally, this gives us

$$\{\mathsf{x}\in M: B_\rho(\mathsf{x})\cap T^k_\omega B_\rho(\mathsf{x})\ne\varnothing\}\subset\{\mathsf{x}\in M: B_{(2A^k+2)\rho}(\mathsf{x})\cap T^k_\omega(T^k_\omega B_{(2A^k+2)\rho}(\mathsf{x}))\ne\varnothing\}.$$

If we put $\tilde\omega=\omega_0\dots\omega_{k-1}\omega_0\omega_1\dots$ then $T^k_\omega\circ T^k_\omega=T^{2k}_{\tilde\omega}$. This proves the case $p=1$. The general case is shown similarly. □

Let us now consider the case $1\le n\le\lfloor\,\mathfrak{b}J\,\rfloor$ and let as in Lemma 4.1 $s_p=2^p\,\frac{A^{n\,2^p}-1}{A^n-1}$. Hence by Lemma 4.1 one has $\mathcal{N}^\omega_\rho(n)\subset\mathcal{N}^{\tilde\omega}_{s_p\rho}(2^pn)$, where $\tilde\omega=\tilde\omega(n)$ depends on n, for any $p\ge1$, and in particular for $p(n)=\lfloor\lg\mathfrak{b}J-\lg n\,\rfloor+1$. Therefore

$$\bigcup_{n=1}^{\lfloor\,\mathfrak{b}J\,\rfloor}\mathcal{N}^\omega_\rho(n)\subset\bigcup_{n=1}^{\lfloor\,\mathfrak{b}J\,\rfloor}\mathcal{N}^{\tilde\omega}_{s_{p(n)}\rho}(2^{p(n)}n).$$

Now define

$$n' = n2^{p(n)} \qquad \text{and} \qquad \rho' = s_{p(n)}\rho.$$

A direct computation shows that $1 \le n \le \lfloor \mathfrak{b}J \rfloor$ implies $\lceil \mathfrak{b}J \rceil \le n' \le 2\mathfrak{b}J$ and so

$$\mathcal{V}_\rho^{\omega,1} = \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_\rho^\omega(n) \subset \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_{s_{p(n)}\rho}^{\tilde\omega}(2^{p(n)}n) \subset \bigcup_{n'=\lceil \mathfrak{b}J \rceil}^{2\mathfrak{b}J} \mathcal{N}_{\rho'}^{\tilde\omega}(n').$$

Therefore to estimate the measure of $\mathcal{V}_\rho^1$ it suffices to find a bound for $\mu^{\hat\omega}(\mathcal{N}_{\rho'}^{\tilde\omega}(n'))$ when $n' \ge \mathfrak{b}J$. This is accomplished by using an argument analogous to the first part of the proof. Notice that since $\tilde\omega = (\omega_0 \dots \omega_{k-1})^{2^p-1}\omega$ we get that

$$T_{\tilde\omega}^{n'} = (T_\omega^n)^{2^p}.$$

Define $\omega' = \omega_0 \dots \omega_{k-1}\tilde\omega = (\omega_0 \dots \omega_{k-1})^{2^p}\omega$ and $\hat\omega' = (\omega_0 \dots \omega_{k-1})^{2^p}\hat\omega$. Notice that $\theta^{2^pn}\omega' = \theta^{(2^p-1)n}\tilde\omega = \omega$, we get

$$T_{\omega'}^{n'} = (T_\omega^n)^{2^p} = T_{\tilde\omega}^{n'}.$$

As a result,

$$\mathcal{N}_{\rho'}^{\tilde\omega}(n') = \{x : B_{\rho'} \cap T_{\tilde\omega}^{n'}B_\rho \ne \emptyset\} = \{x : B_{\rho'} \cap T_{\omega'}^{n'}B_\rho \ne \emptyset\} = \mathcal{N}_{\rho'}^{\omega'}(n').$$

Similar to the part (I),

$$\begin{aligned}
\mu^{\hat\omega}(\mathcal{N}_{\rho'}^{\tilde\omega}(n')) &= \mu^{\hat\omega}(\mathcal{N}_{\rho'}^{\omega'}(n')) \\
&= \mu_{\gamma^u}^{\hat\omega'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta) \\
&= \frac{\mu_{\gamma^u}^{\hat\omega'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta)}{\mu_{\gamma^u}^{\hat\omega'}(\zeta)}\, \mu_{\gamma^u}^{\hat\omega'}(\zeta) \\
&\le \Theta(n')\, \frac{\mu_{\hat\gamma^u}^{\hat\omega}(T_{\omega'}^{n'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta))}{\mu_{\hat\gamma^u}^{\hat\omega}(T_{\omega'}^{n'}\zeta)}\, \mu_{\gamma^u}^{\hat\omega'}(\zeta).
\end{aligned}$$

To estimate the measure of the numerator we follow the proof of Proposition 4.1 and replace all the n with n' and ρ with ρ'. We get for $\mathfrak{b} < 1/3$

$$\text{diam }(T_{\omega'}^{n'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta)) \le c_1(e^{-\mathfrak{c}'|\log\rho'|^{1/2}} + \delta(n')).$$

Therefore

$$\mu^{\hat\omega}(\mathcal{N}_{\rho'}^{\tilde\omega}(n')) \le c_5\Theta(n')(e^{-u_0|\log\rho'|^{1/2}} + \delta(n')^{u_0}).$$

Since $\rho' = s_p\rho$ and $\mathfrak{b} < \mathfrak{a} = \frac{1}{4\log A}$, we have

$$\begin{aligned}
\rho' &\le A^{2n2^p}\rho = A^{2n'}\rho \\
&\le A^{4\mathfrak{b}J} = A^{4\mathfrak{a}\mathfrak{b}|\log\rho|}\rho \\
&\le A^{\frac{4}{16\log A}|\log\rho|}\rho = \rho^{3/4}
\end{aligned}$$

which gives us

$$\mu^{\hat\omega}(\mathcal{N}_{\rho'}^{\tilde\omega}(n')) \le c_5\Theta(n')(e^{-u_0|\log\rho^{3/4}|^{1/2}} + \delta(n')^{u_0}).$$

Thus obtain an estimate similar to (6):

$$\mu^{\hat\omega}(\mathcal{V}_\rho^{\omega,1}) \le \sum_{n'=\lceil \mathfrak{b}J\rceil}^{2\mathfrak{b}J} \mu^{\hat\omega}(\mathcal{N}_{\rho'}^{\tilde\omega}(n')) \le c_8(e^{-\mathfrak{c}|\log\rho|^{1/2}} + \delta(\mathfrak{a}\mathfrak{b}|\log\rho|)^{u_1}|\log\rho|^{\kappa'}).$$

for some $\mathfrak{c}\in(0,\frac{3}{4}u_0)$.

(III) Final estimate

Overall we obtain for all ρ sufficiently small

$$\mu^{\hat\omega}(\mathcal{V}_\rho^{\omega}) \le \mu^{\hat\omega}(\mathcal{V}_\rho^{\omega,1}) + \mu^{\hat\omega}(\mathcal{V}_\rho^{\omega,2}) \le C_3\,(e^{-\mathfrak{c}|\log\rho|^{1/2}} + \delta(\mathfrak{a}\mathfrak{b}|\log\rho|)^{u_1}|\log\rho|^{\kappa'}),$$

for some C_3. □

5. Return times distribution

Proof of Theorem 2.1 for return times. Since we proved an exponential distribution for the hitting times, to get an exponential distribution for the return times we will estimate the difference between the hitting time statistics and the return time statistics:

$$\left|\mu_{B_\rho}^\omega(\tau_{B_\rho}^\omega > N) - \mu^\omega(\tau_{B_\rho}^\omega > N)\right| \le \frac{\epsilon_\omega(B_\rho)}{\mu^\omega(B_\rho)},$$

where we split the second term as follows:

$$\mu^\omega(B_\rho\cap\{\tau_{B_\rho}^\omega > N\}) = \mu^\omega(B_\rho)\mu^{\theta\omega}(\tau_{B_\rho}^{\theta\omega} > N-1) + \epsilon_\omega(B_\rho),$$

where ϵ_ω is as in the proof of Proposition 3.1. Observe that the convergence of the RTS does not come immediately from the first part of the theorem, as one could have hoped for from the deterministic case (e.g. [22]) or the annealed case ([19]).

In order to estimate the error term we split the RHS in three terms as in (2):

$$\left|\frac{\epsilon_\omega(B_\rho)}{\mu^\omega(B_\rho)}\right| = \tilde{\mathcal{R}} \le \tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 + \tilde{\mathcal{R}}_3.$$

We estimate the first term $\tilde{\mathcal{R}}_1$, the decay of correlations term (unlike the term $\mathcal{R}$ in Theorem 2.1 there is no factor N):

$$\tilde{\mathcal{R}}_1 = \frac{1}{\mu^\omega(B_\rho)}\sup_{k\ge1}\left|\mu^\omega(B_\rho\cap T_\omega^{-\Delta}S_k) - \mu^\omega(B_\rho)\,\mu^{\theta^\Delta\omega}(S_k)\right|.$$

As in Section 3.1, we get:

$$\tilde{\mathcal{R}}_1 \le \frac{c_1}{\mu^\omega(B_\rho)}\left(\frac{\lambda(\Delta/2)}{\delta\rho} + \mu^\omega(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho})\right) + \frac{\mu(B_\rho)}{\mu^\omega(B_\rho)}\mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta})$$

where we use Assumption (V) to estimate the term $\frac{\mu(B_\rho)}{\mu^\omega(B_\rho)}$ by K. The short hitting times term, $\tilde{\mathcal{R}}_3$, can be dealt with easily as in Section 3.3:

$$\tilde{\mathcal{R}}_3 = \mu^\omega(\{y:\tau_{B_\rho}^\omega(y) < \Delta\}) \le \sum_{k=1}^{\Delta}\mu^{\theta^k\omega}(B_\rho) \le c_2\Delta\rho^{u_0}.$$

We are left with the short return times term, $\tilde{\mathcal{R}}_2$ which we estimate similarly to Section 3.2:

$$\tilde{\mathcal{R}}_2 = \frac{\mathcal{R}_2'(\omega)}{\mu^\omega(B_\rho)} = \frac{1}{\mu^\omega(B_\rho)}\mu^\omega(B_\rho \cap \{y : \tau^\omega_{B_\rho}(y) < \Delta\}) \le c_3 \sum_{j=J}^{\Delta} \Theta(j)\delta(j)^{u_0} \frac{\mu^{\theta^j\omega}(B_\rho)}{\mu^\omega(B_\rho)}$$

for all $\mathsf{x} \notin \mathcal{V}^\omega_{4\rho}$ (as $B_\rho(\mathsf{x}) \cap \{\tau^\omega_{B_\rho(\mathsf{x})} < J\} = \varnothing$ for such x). This implies by Assumption (V) $\tilde{\mathcal{R}}_2 \le c_4\delta(J)^{u_1}J^{\kappa'}$.

Consequently, proceeding as in Section 3.4 we finally obtain

$$\tilde{\mathcal{R}} \le c_5\Big(\delta(J)^{u_0}J^{\kappa'} + \rho^\epsilon\Big)$$

for some $\epsilon >$ and for all $x \notin \mathcal{V}^\omega_{4\rho}$. □

6. Alternate argument for $\tilde{\mathcal{R}}_2 \to 0$ almost surely

First, we will estimate the measure of the set $\mathcal{W}^\omega_\rho = \{y : d(T^j_\omega y, y) < 2\rho \text{ for some } 1 < j < \Delta\}$. Let $J = \mathfrak{a}\mathfrak{b}|\log\rho|$ where $\mathfrak{a}, \mathfrak{b}$ be as before and put

$$\begin{aligned} \mathcal{W}^{\omega,1}_\rho &= \{y : d(T^j_\omega y, y) < 2\rho \text{ for some } 0 < j \le J\}, \\ \mathcal{W}^{\omega,2}_\rho &= \{y : d(T^j_\omega y, y) < 2\rho \text{ for some } J < j < \Delta\}. \end{aligned}$$

We will get estimates on the measure of $\mathcal{W}^{\omega,i}$, $i = 1, 2$.

6.1. Estimating $\mathcal{W}^{\omega,2}_\rho$.

We follow the proof of Proposition 4.1 and put $\mathcal{M}^\omega_\rho(n) = \{y : d(T^n_\omega y, y) < 2\rho\}$ the level set. As in (5) one has now

$$\mu^\omega(\mathcal{M}^\omega_\rho(n)) \le \sum_\zeta \Theta(n)\, \mu^\omega_{\hat{\gamma}^u}(T^n_\omega(T^{-n}_\omega \mathcal{M}^\omega_\rho(n) \cap \zeta))\, \mu^{\tilde{\omega}}_{\gamma^u}(\zeta).$$

To get a bound on the diameter of the set $T^n_\omega(T^{-n}_\omega \mathcal{M}^\omega_\rho(n) \cap \zeta)$ we take points $x, y \in T^{-n}_\omega \mathcal{M}^\omega_\rho(n) \cap \zeta$. Then

$$d(T^n_\omega x, T^n_\omega y) \le d(T^n_\omega x, x) + d(x, y) + d(y, T^n_\omega y) \le 4\rho + \delta(n).$$

As a result,

$$\mu^\omega(\mathcal{M}^\omega_\rho(n)) \le c_1\Theta(n)\,(\rho^{u_0} + \delta(n)^{u_0}) \sum_{\zeta:\, \zeta\cap T^{-n}_\omega \mathcal{M}^\omega_\rho \ne \varnothing} \mu^{\tilde{\omega}}_{\gamma^u}(\zeta).$$

Since the sum of ζ is bounded, we thus get with $\Theta(n) = \mathcal{O}(n^{-\kappa'})$

$$\mu^\omega(\mathcal{W}^{\omega,2}_\rho) \le \sum_{n=J}^{\Delta} \mu^\omega(\mathcal{M}^\omega_\rho(n)) \le c_2\left(\Delta^{\kappa'+1}\rho^{u_0} + \delta(J)^{u_1}J^{\kappa'}\right)$$

since $J = \mathfrak{a}\mathfrak{b}|\log\rho|$ where $u_1 = u_0$ if δ is superpolynomial and $u_1 = u_0 - \frac{1}{\kappa}$ if δ decays polynomially at rate κ.

6.2. Estimating $\mathcal{W}^{\omega,1}_\rho$.

Notice that $\mathcal{W}^{\omega,1}_\rho \subset \mathcal{V}^\omega_{2\rho}$ and by Proposition 4.1

$$\mu^\omega(\mathcal{W}^{\omega,1}_\rho) \le \mu^\omega(\mathcal{V}^\omega_{2\rho}) \le C_3\Big(e^{-\mathfrak{c}|\log 2\rho|^{1/2}} + \delta(\mathfrak{a}\mathfrak{b}|\log 2\rho|)^{u_1}|\log 2\rho|^{\kappa'}\Big).$$

6.3. **Almost sure limit of $\tilde{\mathcal{R}}_2$.** To estimate $\mathcal{W}^\omega_\rho = \mathcal{W}^{\omega,1}_\rho \cup \mathcal{W}^{\omega,2}_\rho$ we choose $\Delta = \rho^{-v'}$ and obtain

$$\mu^\omega(\{y : \tau^\omega_{2\rho}(y) < \Delta\}) = \mu^\omega(y : d(T^j_\omega y, y) < 2\rho \text{ for some } 0 < j < \Delta) = \mathcal{O}(|\log\rho|^{-b}),$$

where $b = u_1\kappa - \kappa'$ ($b > 0$) assuming δ decays polynomially with power κ and $0 < v' < \frac{u_0}{\kappa'+1}$.

In order to get a limit for $\rho \to 0$, we apply the Borel-Cantelli Lemma to the sequence $\rho_n = e^{-n^{2/b}}$ and obtain that for $\mu^\omega-$almost every y and all n large enough: $\tau^\omega_{2\rho_n}(y) \geq \Delta_n = \rho_n^{-v'}$ or

$$\tau^\omega_{2e^{-n^{2/b}}}(y) \geq e^{v' n^{2/b}}$$

which implies

$$\liminf_{n\to\infty} \frac{\log \tau^\omega_{2e^{-n^{2/b}}}(y)}{-\log 2e^{-n^{2/b}}} \geq \liminf_{n\to\infty} \frac{\log e^{v' n^{2/b}}}{-\log 2e^{-n^{2/b}}} = v'.$$

For every $\rho > 0$ small enough, there is an n so that $\rho_{n+1} \leq \rho < \rho_n$ and consequently

$$\frac{\log \tau^\omega_{2\rho_n}(y)}{-\log \rho_{n+1}} \leq \frac{\log \tau^\omega_{2\rho}(y)}{-\log \rho} \leq \frac{\log \tau^\omega_{2\rho_{n+1}}(y)}{-\log \rho_n}.$$

As $\frac{\log \rho_{n+1}}{\log \rho_n} = \frac{(n+1)^2}{n^2} \to 1$ as $n \to \infty$, we conclude that

$$\liminf_{\rho\to 0} \frac{\log \tau^\omega_{2\rho}(y)}{-\log \rho} = \liminf_{n\to\infty} \frac{\log \tau^\omega_{2e^{-n^{2/b}}}(y)}{-\log 2e^{-n^{2/b}}} \geq v'.$$

In other words for any $0 < v < v'$ one has $\mu^\omega(\mathcal{L}(\rho_0)) \to 1$ as $\rho_0 \to 0$ where

$$\mathcal{L}(\rho_0) = \{y : \tau^\omega_{2\rho}(y) > \rho^{-v} \ \forall \rho < \rho_0\}.$$

Finally, this implies by the Lebesgue density theorem (following the proof of Lemma 42 of [22]) that

$$\tilde{\mathcal{R}}_2 = \frac{1}{\mu^\omega(B_\rho)} \mu^\omega(B_\rho \cap \{y : \tau^\omega_{B_\rho}(y) < \Delta\}) \underset{\rho\to 0}{\longrightarrow} 0$$

for μ^ω-almost every x, where here $\Delta = \rho^{-v}$, $v < \frac{u_0}{\kappa'+1}$.

7. Examples

7.1. **Random C^2 interval maps.** As an example we consider random maps on the unit interval I. As above let $S : \Omega \times I \circlearrowleft$ be a skew action where the map θ is acting invertibly on Ω. For each ω the map $T_\omega : I \to I$ is a piecewise expanding map on the interval I. We assume that T_ω is piecewise C^2 with uniformly bounded C^2 norms. For $\varphi \in \mathscr{I}^\omega_n$ we denote by $\zeta_\varphi = \varphi(I)$ the n-cylinder associated with φ. As before, put

$$\delta(n) = \sup_\omega \max_{\varphi \in \mathscr{I}^\omega_n} |\zeta_\varphi|.$$

For a function $\psi : I \to \mathbb{R}$ be denote by $\operatorname{var}\psi$ its variation on the unit interval and let

$$\|f\| = \operatorname{var}\psi + \|f\|_{\mathscr{L}^1}$$

be its norm. This makes $X = \{f \in C(I, \mathbb{R}), \|f\| < \infty\}$ a Banach space which is equipped with the strong norm $\|\cdot\|$ and the weak norm $\|\cdot\|_{\mathscr{L}^1}$. Consider the transfer operator $\mathcal{L}$

on X which for each ω maps a function $\psi \in X$ on the interval to a function $\mathcal{L}_\omega\psi$ on the interval. It is given by

$$\mathcal{L}_\omega\psi(x) = \sum_{\varphi\in\mathscr{I}_1^\omega} \frac{\psi(\varphi x)}{|DT_\omega(\varphi x)|}.$$

The iterates of the transfer operator are $\mathcal{L}_\omega^n = \mathcal{L}_{\theta^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\theta\omega} \circ \mathcal{L}_\omega$. We shall next prove the Doeblin-Fortet inequality:

Lemma 7.1. *Assume that $\delta(k)$ decreases to zero as $k \to \infty$. Then there exist $\eta < 1$, $n \in \mathbb{N}$ and a constant C_4 so that for every $\omega \in \Omega$ and $\psi : I \to \mathbb{R}$ with $\|\psi\| < \infty$ one has*

$$var\, \mathcal{L}_\omega^n\psi \le \eta\, var\, \psi + C_4\|\psi\|_{\mathscr{L}^1}.$$

Proof. Let us fix ω. In order to estimate $\mathrm{var}\,|DT_\omega^n|^{-1}$ let $\ell \in \mathbb{N}$ and $n = p\ell$. Then, for $\varphi \in \mathscr{I}_n^\omega$, $|DT_\omega^{p\ell}|^{-1}\varphi = (|DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi)|DT_\omega^\ell|^{-1}\varphi$ and

$$\mathrm{var}\,|DT_\omega^{p\ell}|^{-1}\varphi \le ||DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi|_\infty \mathrm{var}\,|DT_\omega^\ell|^{-1}\varphi + ||DT_\omega^\ell|^{-1}\varphi|_\infty \mathrm{var}\,|DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi.$$

There exists a constant c_1 so that

$$||DT_\omega^\ell|^{-1}\varphi|_\infty \le c_1|\zeta_\varphi| \le c_1\delta(\ell)$$

and similarly $|DT_\omega^{(p-1)\ell}|^{-1}\varphi|_\infty \le c_1\delta((p-1)\ell)$. Hence since $T_\omega^\ell\varphi \in \mathscr{I}_{(p-1)\ell}^{\theta^\ell\omega}$ one has

$$\mathrm{var}\,|DT_\omega^{p\ell}|^{-1}\varphi \le c_1\delta(\ell)\mathrm{var}\,|DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi + c_1\delta((p-1)\ell)\mathrm{var}\,|DT_\omega^\ell|^{-1}\varphi.$$

Recursively one obtains for all $\varphi \in \mathscr{I}_{p\ell}^\omega$:

$$\begin{aligned}\mathrm{var}\,|DT_\omega^{p\ell}|^{-1}\varphi &\le \sum_{j=0}^{p-1}(c_1\delta(\ell))^j c_1\delta((p-j-1)\ell)\mathrm{var}\,|DT_\omega^\ell|^{-1}T_\omega^{j\ell}\varphi \\ &\le \sum_{j=0}^{p-1}(c_1\delta(\ell))^j c_1\delta((p-j-1)\ell)\mathrm{var}\,|DT_\omega^\ell|^{-1}.\end{aligned}$$

If we choose ℓ so that $\tilde\eta = c_1\delta(\ell) < 1$ then we obtain

$$\mathrm{var}\,|DT_\omega^n|^{-1}\varphi \le \Delta(n)\mathrm{var}\,|DT_\omega^\ell|^{-1}$$

for all $\varphi \in \mathscr{I}_n^\omega$, for all $n \in \mathbb{N}$ and for some function $\Delta(n)$ which decays to zero at the same rate as $\delta(n)$.

The variation of the transfer operator is then estimated as:

$$\mathrm{var}_I\, \mathcal{L}_\omega^n\psi = \sum_{\varphi\in\mathscr{I}_n^\omega}(|DT_\omega^n|^{-1}\psi)\varphi \le \sum_{\varphi\in\mathscr{I}_n^\omega}\left(\mathrm{var}\,\psi\varphi||DT_\omega^n|^{-1}\varphi|_\infty + |\psi\varphi|_\infty\mathrm{var}\,|DT_\omega^n|^{-1}\varphi\right).$$

Since $||DT_\omega^n|^{-1}|_\infty \le c_1|\zeta_\varphi| \le c_1\delta(n)$ one obtains

$$\mathrm{var}_I\, \mathcal{L}_\omega^n\psi \le c_1\delta(n)\sum_\varphi \mathrm{var}\,\psi\varphi + c_2\Delta(n)\sum_\varphi |\psi\varphi|_\infty$$

where $c_2 = \sup_\omega |DT_\omega^\ell|^{-1}$. With the estimate $|\psi\varphi|_\infty \le \int_I |\psi\varphi|\,d\lambda + \mathrm{var}\,\psi\varphi$ this leads to

$$\mathrm{var}_I\, \mathcal{L}_\omega^n\psi \le (c_1\delta(n) + c_2\Delta(n))\mathrm{var}\,\psi + c_2\Delta(n)\left|\frac{1}{|DT_\omega^n|^{-1}}\right|_\infty \int_I\sum_\varphi |\psi\varphi|_\infty|DT_\omega^n|^{-1}\varphi\,d\lambda$$

as $\sum_\varphi \operatorname{var}\psi\varphi = \operatorname{var}\psi$. Since the Lebesgue measure λ is a fixed point of the transfer operator we finally get

$$\operatorname{var}_I \mathcal{L}_\omega^n\psi \leq (c_1\delta(n) + c_2\Delta(n))\operatorname{var}\psi + c_2\Delta(n)|DT_\omega^n|_\infty \int_I |\psi|_\infty \, d\lambda.$$

Now, if we choose n so that $\eta = c_1\delta(n) + c_2\Delta(n) < 1$ we obtain

$$\operatorname{var}_I \mathcal{L}_\omega^n\psi \leq \eta \operatorname{var}\psi + c_3\|\psi\|_{\mathscr{L}^1},$$

where $c_3 \leq c_2\Delta(n)|DT_\omega^n|_\infty$. Put $C_4 = c_3$. Note that the constant $\eta < 1$ can be chosen arbitrarily small. $\square$

This proves the property (LY2) of [6]. The other two properties (LY0) and (LY1) are naturally satisfied as are the properties (V). To verify condition (RC) let $\psi \in \mathcal{C}_a$ where $\mathcal{C}_a = \{\psi > 0 : \operatorname{var}\psi \leq a\|\psi\|_{\mathscr{L}^1}\}$. Then by iterating Lemma 7.1 one obtains

$$\operatorname{var}\mathcal{L}_\omega^m\psi \leq \eta^{\frac{m}{n}}\operatorname{var}\psi + \frac{C_4}{1-\eta^{\frac{1}{n}}}\|\psi\|_{\mathscr{L}^1} \leq \left(\eta^{\frac{m}{n}} + \frac{C_4}{1-\eta^{\frac{1}{n}}}\right)\|\psi\|_{\mathscr{L}^1}$$

for all large m. If we choose $a \geq \frac{2C_4}{(1-\eta^{\frac{1}{n}})^2}$ this implies $\inf \mathcal{L}_\omega^m\psi \geq \|\mathcal{L}_\omega^m\psi\|_{\mathscr{L}^1} - \operatorname{var}\mathcal{L}_\omega^m\psi \geq \frac{a}{2}$. Hence the condition (RC) of [6] is satisfied with $\alpha_n = \frac{a}{2}$.

Therefore by the Main Theorem of [6] there exists a family of absolutely continuous measure μ^ω on the fibres $\{\omega\}\times I$ which satisfy the generalised invariance property $T_\omega^*\mu^\omega = \mu^{\theta\omega}$. In particular there is an S-invariant measure $\mathbb{P}$ on $\Omega\times I$ which is of the form $d\mathbb{P}(\omega,x) = d\mu^\omega(x)d\nu(\omega)$, where ν is a θ-invariant measure on Ω. Also note as a consequence of the lower bound $\inf\psi \geq \frac{a}{2}\|\psi\|_{\mathscr{L}^1}$ the densities h_ω of μ^ω have a uniform lower bound, that is there exists a constant $c_1 > 0$ so that $\inf h_\omega \geq c_1$ for all $\omega\in\Omega$.

Moreover one has decay of the annealed correlation function (I) and also the decay of the quenched correlation functions (II). In fact, the decay function $\lambda(n)$ decays exponentially fast to zero.

Since the measures μ^ω are absolutely continuous with respect to Lebesgue measure, condition (V) is satisfied with any values $d_0 < 1 < d_1$ arbitrarily close to 1 and from the uniform lower bound on the densities h_ω, i.e. we can take $K = 1/c_1$. Condition (III) follows from the uniform boundedness of second order derivatives. The annulus condition (VI) is satisfied with $\xi = \beta = 1$. We can therefore invoke Theorem 2.1 and obtain the following result:

Theorem 7.1. *Let $S : \Omega\times I \circlearrowleft$ be a skew system as described above, where the maps T_ω are piecewise C^2 with uniformly bounded C^2 derivatives. Let $\delta(n)$ be a summable sequence which monotonically decreases to zero so that $|\zeta_\varphi| \leq \delta(n)$ for all $\varphi \in \mathscr{I}_n^\omega$ for all n. Then*

$$\mu^\omega\left(y\in[0,1] : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho\to 0$$

and

$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y\in[0,1] : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho\to 0$$

for all $t > 0$ for Lebesgue almost every $\mathsf{x}\in[0,1]$ and ν-almost every $\omega\in\Omega$.

Clearly, if the maps T_ω are uniformly expanding then δ decays exponentially and satisfies the requirement of the theorem.

7.2. **Random parabolic interval maps.** We use the family of Pomeau-Manneville maps indexed by $\alpha > 0$ which is given by

$$T_\alpha(x) = \begin{cases} x + 2^{1+\alpha}x^{1+\alpha} & \text{if } x \in [0, \frac{1}{2}) \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

These maps have a neutral (parabolic) fixed point at $x = 0$ and are otherwise expanding. It is known that if $\alpha < 1$ then there exists an invariant absolutely continuous probability measure. Here we assume the setting of [5]. Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ be the 'driving space' with the left shift map $\theta : \Omega \circlearrowleft$. We equip Ω with the Bernoulli measure ν with weights $(\frac{1}{2}, \frac{1}{2})$. Let $0 < \alpha_0 < \alpha_1 < 1$ and define the function $\alpha : \Omega \to \{\alpha_0, \alpha_1\}$ by

$$\alpha(\omega) = \begin{cases} \alpha_0 & \text{if } \omega_0 = 0 \\ \alpha_1 & \text{if } \omega_0 = 1 \end{cases}.$$

Then we have a skew action $S : \Omega \times I$, with $I = [0, 1]$ defined by $S(\omega, x) = (\theta(\omega), T_\omega x)$, where we wrote $T_\omega = T_{\alpha(\omega)}$. Its iterates are $S(\omega, x) = (\theta^n\omega, T^n_\omega x)$, where $T^n_\omega = T_{\theta^{n-1}\omega} \circ \cdots \circ T_{\theta\omega} \circ T_\omega$. It is shown in [5] that there exists an S-invariant probability measure $\mu = h\nu \times \lambda$, where λ is the Lebesgue measure on I and where the density $h : \Omega \times I \to \mathbb{R}^+$ is Lipschitz continuous on compact subsets of $\Omega \times (0, 1]$. Notice that we here identify the shift space (Ω, θ) with the doubling interval map $p_1 = p_2 = \frac{1}{2}$ as in [5]. Let us note that in [5] Lemma 3.1 we can use the cone of functions

$$\mathcal{C}_2 = \left\{f \in C^0((0, 1]) \cap \mathscr{L}^1(\lambda) : f \geq 0, f \text{ decreasing}, x^{1+\alpha} \text{ increasing}, f(x) \leq a x^{-\alpha}\lambda(f)\right\}$$

where a according to [2] Lemma 1.2 is chosen large enough so that $\mathcal{C}_2$ is invariant under the transfer operators for T_{α_0} and T_{α_1}. Then lets us replace the cone $\mathcal{C}_a$ in [5] by $\mathcal{C}_2$ to obtain the invariant density h for the annealed measure μ. On the fibres $I_\omega = \{\omega\} \times I$ we then have the density h_ω given by $h_\omega(x) = h(\omega, x)$. This defines the fibred measures $\mu^\omega = h_\omega\lambda$ on I_ω which have the invariance property $T^*_\omega\mu^\omega = \mu^{\theta\omega}$. For the transfer operator $\mathcal{P}_\omega$ (adjoint to T_ω), one has $\mathcal{P}_\omega h_\omega = h_{\theta\omega}$ and $\mathcal{P}^*_\omega\lambda = \lambda$ and also by [2] Theorem 1.6:

$$\begin{aligned} \left|\int \psi(\phi \circ T^n_\omega)\, d\mu^\omega - \int \psi\, d\mu^\omega \int \phi\, d\mu^{\theta^n\omega}\right| &= \left|\int (\mathcal{P}^n_\omega \psi h_\omega)\phi\, d\lambda - \mu^\omega(\psi) \int \phi \mathcal{P}^n_\omega h_\omega\, d\lambda\right| \\ &\leq \int |\phi| \cdot |\mathcal{P}^n_\omega(\psi h_\omega - h_\omega\mu^\omega(\psi))|\, d\lambda \\ &\leq c_1|\phi|_\infty \left(\|\psi h_\omega\|_{\mathscr{L}^1(\lambda)} + \|h_\omega\mu^\omega(\psi)\|_{\mathscr{L}^1(\lambda)}\right) \frac{\log^{\frac{1}{\alpha_1}} n}{n^{\frac{1}{\alpha_1}-1}}, \end{aligned}$$

for some constant c_1 (which by [2] is equal to $\max\{C_{\alpha_0}, C_{\alpha_1}\}$). This is under the stated assumption that ϕh_ω and $h_\omega\mu^\omega(\psi)$ belong to the cone of functions $\mathcal{C}_2$. This in fact applies to the function $h_\omega\mu^\omega(\psi)$. A careful reading of the proof makes it apparent that the class of functions to which the contraction applies is far wider and in fact is only determined by the property that $\|\phi - \mathbb{A}_\epsilon\phi\|_{\mathscr{L}^1}$ is bounded by a multiple of $\epsilon^{1-\alpha}$. The smoothing operator $\mathbb{A}_\epsilon$ is given by $\mathbb{A}_\epsilon\phi(x) = \frac{1}{2\epsilon}\int_{B_\epsilon(x)} \phi(y)\, d\lambda(y)$. Since we want ψ to be the characteristic function of B_ρ, this requirement is clearly satisfied as $\|\phi - \mathbb{A}_\epsilon\phi\|_{\mathscr{L}^1} \lesssim \epsilon$. Consequently, for the purposes of Theorem 2.1, Assumption (I) is satisfied with $\lambda(n) = \mathcal{O}(n^{-p})$ for any $p < \frac{1}{\alpha_1} - 1$. Since one can integrate w.r.t. $d\nu(\omega)$ also Assumption (II) is satisfied with the same λ.

Clearly the dimension of μ is equal to one and Assumption (V) is satisfied with any $d_0 < 1 < d_1$ arbitrarily close to 1 from the fact that the density functions h_ω are uniformly bounded and bounded away from 0. Assumption (VI) is satisfied with $\xi = \beta = 1$. Also, if we denote by $\psi^n_{\theta^{-n}\omega}$ the (unique) inverse branch of $T^n_{\theta^{-n}\omega}$ which contains the parabolic point 0, then one has that $|\psi^n_{\theta^{-n}\omega}(I)| = \mathcal{O}(n^{-1/\alpha_1})$ for all ω. Hence $\delta(n) = \mathcal{O}(n^{-\kappa})$ with $\kappa = 1/\alpha_1$.

To estimate the distortion we again look at the 'worst case' which are the parabolic inverse branches $\psi^n_{\theta^{-n}\omega}$ of the map $T^n_{\theta^{-n}\omega}$. Put $a_n(\omega) = \psi^n_{\theta^{-n}\omega}(a_0)$, where $a_0 = \frac{1}{2}$. Then

$$DT^n_{\theta^{-n}\omega}(a_n) = \prod_{j=1}^{n} DT_{\theta^{-j}\omega}(a_j).$$

For the parabolic branch: $DT_{\theta^{-j}\omega}(x) = 1 + (1+\alpha(\theta^{-j}\omega))2^{1+\alpha(\theta^{-j}\omega)}x^{\alpha(\theta^{-j}\omega)})$. Also

$$a_{j-1} = T_{\theta^{-j}\omega}(a_j) = a_j\left(1 + 2^{1+\alpha(\theta^{-j}\omega)}a_j^{\alpha(\theta^{-j}\omega)}\right) = a_j(DT_{\theta^{-j}\omega}(a_j))^{1/(1+\alpha(\theta^{-j}\omega)} + \text{ h.o.t.}$$

and

$$DT_{\theta^{-j}\omega}(a_j) = \left(\frac{a_{j-1}}{a_j}\right)^{1+\alpha(\theta^{-j}\omega)} + \text{ h.o.t.}$$

which implies the estimate

$$DT^n_{\theta^{-n}\omega}(a_n) \le c_2 \prod_{j=0}^{n-1}\left(\frac{a_{j-1}}{a_j}\right)^{1+\alpha(\theta^{-j}\omega)} \le c_2\left(\prod_{j=0}^{n-1}\frac{a_{j-1}}{a_j}\right)^{1+\alpha_1} = c_2\, a_n^{-(1+\alpha_1)}.$$

Denote by $a'_k = a_k(\alpha_0)$ the inverse images of $a_0 = \frac{1}{2}$ in the deterministic case when all maps have the parameter value α_0. Then $a'_k \sim c_3 k^{-1/\alpha_0}$ for some $c_3 > 0$. Let us put $A_k = (a'_{k-1}, a_0]$. In order to estimate the distortion of the maps T^n_ω on the images of A_k and $A = (a_0, 1]$ under the inverse branches $\mathscr{I}^\omega_n$ of T^n_ω we look at the 'worst case' when the inverse branch is the parabolic branch ψ^n_ω. Then

$$\Theta(n) = \frac{DT^n_{\theta^n\omega}(a_n(\omega))}{DT^n_{\theta^{n+k'}\omega}(\tilde{a}_{n+k'}(\omega))}$$

where $\tilde{a}_{n+k'} = \psi^n_{\theta^{n+k'}\omega}(a'_k)$ and k' is so that $a'_k \in \psi^{k'}_{\theta^{k'}\omega}A$. We estimate the numerator from above by

$$DT^n_{\theta^n\omega}(a_n(\omega)) = \mathcal{O}(1)\prod_{j=0}^{n-1}\left(\frac{a_j}{a_{j+1}}\right)^{1+\alpha(\theta^{-j}\omega)} \lesssim \left(\frac{a_0}{a_n}\right)^{1+\alpha_1} \lesssim n^{\frac{1+\alpha_1}{\alpha_0}}$$

as $a_n \le a'_n = \mathcal{O}(n^{-1/\alpha_0})$. The denominator is estimated from below as follows:

$$DT^n_{\theta^n\omega}(\tilde{a}_{n+k'}(\omega)) = \mathcal{O}(1)\prod_{j=k'}^{n+k'-1}\left(\frac{\tilde{a}_j}{\tilde{a}_{j+1}}\right)^{1+\alpha(\theta^{-j}\omega)} \gtrsim \left(\frac{\tilde{a}_{k'}}{\tilde{a}_{n+k'}}\right)^{1+\alpha_0}.$$

Since $|\tilde a_{j+k'} - \tilde a_{j-1+k'}| = 2^{1+\alpha(\theta^{-j}\omega)}\tilde a_{j+k'}^{1+\alpha(\theta^{-j}\omega)} \le 2^{1+\alpha_1}\tilde a_{j+k'}^{1+\alpha_1}$ one obtains

$$\frac{\tilde a_{k'}}{\tilde a_{n+k'}} \gtrsim \frac{\tilde a_{k'}}{\left(\tilde a_{k'}^{-\alpha_1} + n\right)^{-1/\alpha_1}}$$

where $\tilde a_{k'} \sim c_3 n^{-\eta/\alpha_0}$ for some $\eta > 0$. Hence

$$DT^n_{\theta^n\omega}(\tilde a_{n+k'}(\omega)) \lesssim n^{\frac{1+\alpha_1}{\alpha_0}}\left(n^{\eta\frac{\alpha_1}{\alpha_0}} + n\right)^{-\frac{1+\alpha_0}{\alpha_1}} n^{\eta\frac{1+\alpha_0}{\alpha_0}}.$$

If we choose $\eta > 0$ so that $\eta\frac{\alpha_1}{\alpha_0} < 1$ then these two estimates combined yield

$$\Theta(n) \le c_4 \frac{1}{n^{\kappa'}},$$

where $\kappa' = \frac{1+\alpha_0}{\alpha_1} - \frac{1+\alpha_1}{\alpha_0} - \eta\frac{1+\alpha_0}{\alpha_0}$.

Assuming that $0 < \alpha_0 < \alpha_1 < \frac{1}{3}$, the condition $u_0\kappa - 2 - \kappa' > 1$ of Theorem 2.1 is satisfied since

$$\kappa - 2 - \kappa' = \frac{1}{\alpha_0\alpha_1}\left(\alpha_1^2 - \alpha_0^2 + \alpha_1(1 + \eta - 2\alpha_0 + \eta\alpha_0)\right) > \frac{(1/3+\eta)}{\alpha_0} > 1$$

is positive for any $\eta > 0$ and u_0 can be chosen arbitrarily close to 1.

For the inverse branches $\mathscr{I}^\omega_n$ of T^n_ω denote by $\hat\zeta_\varphi$ the 'n-cylinder' that is a pre-image of either $A_k \cup A = (a'_{k-1}, 1]$ under the inverse branches $\varphi \in \mathscr{I}^\omega_n$. We can now nearly use Theorem 2.1. Note that if $\mathsf{x} \in (0,1)$ then for all n large enough $\mathsf{x} \in A_{n^\theta} \cup A$. If we proceed as in the estimate of the term $\mathcal{R}_2$ in Section 3.2 we obtain

$$T^{-j}_\omega B_\rho \cap B_\rho \subset \bigcup_{\zeta:\zeta\cap B_\rho\neq\varnothing} T^{-j}B_\rho \cap \zeta = \mathscr{P}_1 \cup \mathscr{P}_2$$

where the union is over j-cylinders ζ and

$$\mathscr{P}_1 = \bigcup_{\zeta:\zeta\cap B_\rho\neq\varnothing} T^{-j}_\omega B_\rho \cap \hat\zeta, \qquad \mathscr{P}_2 = \bigcup_{\zeta:\zeta\cap B_\rho\neq\varnothing} T^{-j}_\omega B_\rho \cap \zeta \setminus \hat\zeta.$$

The first set is estimated as before in the main theorem. For the second term notice that

$$\mathscr{P}_2 = \bigcup_{\varphi\in\mathscr{I}^\omega_j:\varphi(I)\cap B_\rho\neq\varnothing} T^{-j}_\omega B_\rho \cap \varphi(D_{n^\eta})$$

where $D_k = (0, a'_{k-1}]$. Hence

$$\mathscr{P}_2 = \bigcup_{\varphi\in\mathscr{I}^\omega_j:\varphi(I)\cap B_\rho\neq\varnothing} \varphi(B_\rho \cap D_{n^\eta})$$

is empty for n large enough, i.e. so that $a_{n^\eta} < \mathsf{x}$.

Theorem 7.2. *Let $S : \{0,1\}^{\mathbb{Z}} \times I \circlearrowleft$ be the random system described above, where the maps T_ω are the parabolic maps T_{α_0} and T_{α_1}. Assume $0 < \alpha_0 < \alpha_1 < \frac{1}{3}$. Denote by μ the* *annealed invariant measure and by μ^ω the fibred measures. Then for all $t>0$:*

$$\mu^\omega\left(y\in[0,1]:\tau^\omega_{B_\rho(\mathsf{x})}(y)>\frac{t}{\mu(B_\rho(\mathsf{x}))}\right)\longrightarrow e^{-t}$$
$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y\in[0,1]:\tau^\omega_{B_\rho(\mathsf{x})}(y)>\frac{t}{\mu(B_\rho(\mathsf{x}))}\right)\longrightarrow e^{-t}$$

as $\rho\to 0$ for Lebesgue almost every $\mathsf{x}\in[0,1]$ and ν-almost every $\omega\in\{0,1\}^{\mathbb{Z}}$.

Proof. We verify the conditions of the theorem. Above we verified Assumptions (I)–(VI). Otherwise, as $\xi=1$ and $\alpha_1<\frac{1}{3}$ one clearly has $\kappa\xi>1$. Also, since u_0 and d_1 can be chosen arbitrarily close to one, $\beta=1$ and choosing $p=\frac{1}{\alpha_1'}-1$ with $\frac{1}{3}>\alpha_1'>\alpha_1$ arbitrarily close to α_1 we get $\max\left(\frac{d_1\beta}{\kappa\xi-1},(\frac{\beta}{\xi}+d_1)\frac{1}{p}\right)\le\max(\frac{d_1}{2},(1+d_1)\frac{\alpha_1'}{1-\alpha_1'})<\min(1,u_0)$. □

7.3. **Random perturbation of non-uniformly expanding maps with critical set.** Li and Vilarinho constructed in [16] a random Gibbs-Markov-Young structure for non-uniformly expanding maps with critical set. Examples of such maps include deformation of a uniformly expanding map by isotopy, and the Viana map in dimension 2. To be more precise, they consider a C^2 local diffeomorphism T on a compact Riemannian manifold M with possibly a set $\mathcal{C}\subset M$ consisting of critical points of T and ∂M. Then they take a skew product on $\Omega\times M$ where $\Omega=\{1,2,\dots,k\}^{\mathbb{Z}}$, so that T_ω only depends on the first symbol, and $T_\omega=T$ for some $\omega\in\Omega$.

In order to obtain the expanding property for almost every realization ω, they require that the system satisfies the following properties:

(1) there is $\alpha>0$ such that for $\nu\times m$ almost every (ω,x),
$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|DT_{\theta^j\omega}(T^j_\omega(x))^{-1}\|<-\alpha,$$
where m is the Lebesgue measure on M.

(2) given any $\gamma>0$ there is $\delta>0$, such that for $\nu\times m$ almost every (ω,x),
$$\limsup_{n\to+\infty}-\frac{1}{n}\sum_{j=0}^{n-1}\log\operatorname{dist}_\delta(T^j_\omega(x),\mathcal{C})<\gamma;$$
here $\operatorname{dist}_\delta$ is the truncated distance: $\operatorname{dist}_\delta(x,y)=d(x,y)$ if $d(x,y)<\delta$, and $\operatorname{dist}_\delta(x,y)=1$ otherwise.

Next, define the expansion time function:
$$\mathcal{E}_\omega(x)=\min\left\{N\ge 1:\frac{1}{n}\sum_{j=0}^{n-1}\log\|DT_{\theta^j\omega}(T^j_\omega(x))^{-1}\|<-\alpha,\ \text{for all } n\ge N\right\},$$
and the recurrence time function (to $\mathcal{C}$):
$$\mathcal{R}_\omega(x)=\min\left\{N\ge 1:\frac{1}{n}\sum_{j=0}^{n-1}\log\operatorname{dist}_\delta(T^j_\omega(x),\mathcal{C})<\gamma,\ \text{for all } n\ge N\right\}.$$

In the case when $\mathcal{C} = \emptyset$, condition (2) and $\mathcal{R}_\omega$ can be disregarded.

Consider the tail set at time n which is defined as

$$\Gamma_\omega^n = \{x : \mathcal{E}_\omega(x) > 0 \text{ or } \mathcal{R}_\omega(x) > n\}\,.$$

This is the set of point for the realization ω, such that the orbit does not exhibit enough hyperbolicity at time n, or get too close to the critical set $\mathcal{C}$. Then they prove the existence of an absolutely continuous probability measure, and quenched decay of correlations:

Theorem 7.3. [16, Theorem A] *Assume that there is $C, \gamma > 0$ and $0 < v \leq 1$ such that $m(\Gamma_\omega^n) < Ce^{-\gamma n^v}$ for ν almost every $\omega \in \Omega$. Then for some integer $q \geq 1$ we have:*

1. *for ν almost every ω there is an absolutely continuous probability μ_ω such that $(T_\omega^q)_*\mu_\omega = \mu_{\theta^q\omega}$;*
2. *there is $C_0, \gamma_0 > 0$, such that for ν almost every ω, we have stretched exponential decay of correlation for Lipschitz function and L^∞ functions:*
$$\left|\int_M G(H \circ T_\omega^k)\, d\mu^\omega - \mu^\omega(G)\mu^{\theta^k\omega}(H)\right| \leq C_1 e^{-\gamma_1 n^{v/2}} \|G\|_{Lip}\|H\|_\infty,$$
for every $H \in L^\infty(M, \mathbb{R})$ and for every $G \in Lip(M, \mathbb{R})$.

This verifies condition (II) for Theorem 2.1. Condition (I) follows by integrating over ω. Since μ_ω are absolutely continuous with respect to the Lebesgue measure for almost every ω, condition (V) and (VI) hold with d_0 and d_1 both close to $\dim M$, and $\xi = \beta = 1$. We have the following theorem:

Theorem 7.4. *Assume that the assumptions of Theorem 7.3 and condition (III) and (IV) in Theorem 2.1 hold. Then*

$$\mu^\omega\left(y \in M : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t}$$
$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y \in M : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t}$$

as $\rho \to 0$ for Lebesgue almost every $\mathsf{x} \in M$ and ν-almost every ω.

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