Statistical properties of equilibrium states for rational maps

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revised

Abstract

Equilibrium states of rational maps Hölder continuous potentials are not ϕ-mixing, mainly due to the presence of critical points. Here we prove that for disks the normalised return times of arbitrary orders are in the limit Poisson distributed as the radius of the disks go to zero. The return times are normalised by the measure of the disks. We also show that rational maps are weakly Bernoulli with respect to the partition given by Denker and Urbanski.

1 Introduction

We intend to investigate the effect of mixing on the distribution of return times. Let T be an expansive transformation on the space Ω and let μ be a probability measure on Ω. For a point x we denote by χ_ε the characteristic function of the ε-ball $B_\varepsilon(x)$. Then we can consider the 'random variable'

$$\xi_\varepsilon = \sum_{j=0}^{[t/\mu(\chi_\varepsilon)]} \chi_\varepsilon \circ T^j.$$

The value of ξ_ε measures the number of times a given point returns to the ε-neighbourhood of x within the normalised time t (the normalisation is with respect to the μ-measure of the set 'return-set' $B_\varepsilon(x)$). If μ is the measure of maximal entropy for the shift transformation on a subshift of finite type, then it was shown by Pitskel [10] that the return times are in the limit Poisson distributed for cylinder

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sets and μ-almost every x. For equilibrium states of Hölder continuous functions, Hirata ([6], [7]) has similar results for the zeroth return time $r = 0$ using the transfer operator restricted to the complement of ε-balls in the shiftspace (the argument for the higher order return times $r \geq 1$ seems to be incomplete).

Pitskel's proof relies on a general result by Sevast'yanov [12] which asserts that a process is Poisson distributed of all orders if one has sufficiently good mixing properties most of the time. In fact, Pitskel's result can easily be generalised to the case when μ is an equilibrium state for a Hölder continuous potential. Applied to cylinder sets, Galves and Schmitt [4] have succeeded in obtaining rates of convergence for the zeroth order return times ($r = 0$) although the return times have to suffer some additional rescaling at every step. Going one step further that result was subsequently used in [1] to show that repetition times are in the limit normal distributed.

The Central Limit Theorem for equilibrium measures for rational maps has been proven in [3] and requires a much weaker mixing property.

In fact, M Hirata, B Saussol and S Vaienti [8] are now able to show that certain family of interval maps with a parabolic point have Poisson distributed return times.

In this paper T is a rational map of degree at least 2 and J its Julia set. Assume that we executed appropriate branch cuts on the Riemann sphere so that we can define univalent inverse branches S_n of T^n on J for all $n \geq 1$ (see Lemma 2 for details). Put $\mathcal{A}^n = \{\varphi(J) : \varphi \in S_n\}$.

Let f be a Hölder continuous function on J so that $P(f) > \sup f$ ($P(f)$ is the pressure of f), let μ be its unique equilibrium state on J and define the 'random variable' ξ_ε to measures the number of times a given point returns to B_ε within the normalised time $t/\mu(B_\varepsilon)$. In our main result, Theorem 12, we then show that for almost every x there exists a sequence $\varepsilon_j \to 0$ so that

$$\mu(\mathcal{N}_{r,\varepsilon_j}) \to \frac{t^r}{r!}e^{-t}, \tag{1}$$

as $j \to \infty$, where $\mathcal{N}_{r,\varepsilon_j} = \{y \in \Omega : \xi_{\varepsilon_j}(y) = r\}$ are the r-levelsets of ξ_{ε_j}.

In Section 2 we recall prior results about rational maps and the inverse branches of their iterates and the convergence of the transfer operator. In the same section we also estimate the measures of neighbourhoods of critical values (Lemma 6). In Section 3 we show that one has uniform r-fold mixing properties for all r (Lemma 9). Section 4 is used to approximate metric balls by unions of cylinder sets. The main part of the proof of the main theorem in Section 5 is devoted to show that short return times occur rarely and can in the limit be neglected. This is done in Lemmas 14 to 16. In the last section we show that T is weakly Bernoulli.

Constants typed in capitals, $C_1, C_2, \ldots$, retain their meaning throughout the paper, while constants in lower case, $c_1, c_2, \ldots$ are used locally.

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2 The transfer operator

Let $T : \mathbf{C} \to \mathbf{C}$ be a rational map of degree $d \geq 2$, and denote by J its Julia set. For $f : J \to \mathbf{R}$, one defines the transfer operator $\mathcal{L}$ by

$$\mathcal{L}\phi(x) = \sum_{y \in T^{-1}x} e^{f(y)} \phi(y),$$

where ϕ are functions on J and $x \in J$. In order to use the euclidean metric on $\mathbf{C}$ (rather than the spherical metric on $\bar{\mathbf{C}}$) let us assume that $\infty \notin J$ (in particular we ask for simplicity's sake that $J \neq \mathbf{C}$), and denote by $C^\alpha(J), \alpha > 0$, the Hölder continuous functions on J with Hölder exponent α, that is, if $f \in C^\alpha(J)$ then there exists a smallest constant $|f|_\alpha$ so that $|f(x) - f(y)| \leq |f|_\alpha |x - y|^\alpha$, for all $x, y \in J$. If we denote by $|f|_\infty$ the supremum norm on J, then the natural norm on $C^\alpha(J)$ is given by $\| \cdot \|_\alpha = | \cdot |_\alpha + | \cdot |_\infty$.

If $f : J \to \mathbf{R}$ is a continuous function, then we would like to consider the action of the associated transfer operator $\mathcal{L}_f$. It is well known that for real f the operator $\mathcal{L}_f$ has a largest simple eigenvalue whose associated eigenfunction and eigenfunctional define an invariant measure μ on J which is conformal with respect to $P(f) - f$, where $P(f)$ is the pressure of f. If the function f is Hölder continuous and satisfies the condition $P(f) - f > 0$ ('supremum gap'), then it was shown [2] that μ is in fact the equilibrium state for f, which means that it realises the maximum in the variational principle

$$P(f) = \sup_\nu (h(\nu) + \mu(f)),$$

where the supremum is over all T-invariant probability measures ν on J, and $h(\nu)$ denotes the metric entropy of ν. Let us put $\rho = e^{\sup f - P(f)}$ which by assumption is less than one.

We shall need the following result.

Lemma 1 *[5] Let $0 < \lambda < 1$. Then there exist $\epsilon > 0$, $\eta \in (0,1)$, a sequence of simply connected regions $\Omega_n, n \in \mathbf{N}$, and a disjoint decomposition of the inverse branches of T^n on Ω_n into two subsets $\mathcal{S}'_n = \mathcal{S}'_n(\lambda)$ and $\mathcal{S}''_n = \mathcal{S}''_n(\lambda)$ so that*
(a) $|\mathcal{S}''_n| \leq c_1 \lambda^{-n}, n \in \mathbf{N}$, for some constant c_1.
(b) $|\varphi'(z)| \leq \frac{1}{2}\eta^n$ for $z \in \Omega_n$ and in particular $\mathrm{diam}(\varphi(\Omega_n)) \leq \frac{1}{2}\eta^n, \forall \varphi \in \mathcal{S}'_n, n \in \mathbf{N}$.
(c) $\mathrm{dist}(z, \Omega_n) \leq c_2 e^{-n\epsilon}$ for all $z \in J, n \in \mathbf{N}$, for some constant $c_2 > 0$.

In fact more can be said about the regions Ω_n on which the inverse branches are defined. Let us recall how the quasidisks Ω_n are constructed. In order to get uniform distortion estimates, the branch cuts had to be thickened by an exponentially small amount $e^{-\epsilon n}$. Moreover we grouped branch cuts that were too close and thus achieved that the thickened branch cuts were separated by at least a distance of order $e^{-\beta n}$, where $\beta > 0$ was smaller than ϵ. With the branch cuts all parallel to the imaginary axes we obtain contracting inverse branches $\tilde{S}'_n$ and non-contracting branches $\tilde{S}''_n$. If we use branch cuts which are at an angle with the imaginary axis, where the angle is chosen to be of the order $e^{-\beta n}$ (chosen so that the new fattened branchcut forms an angle with the previous one without intersection another nearby one), then we obtain a second set of contracting branches $\bar{S}'_n$ and non-contracting branches $\bar{S}''_n$ such that for every branch $\tilde{\varphi} \in \tilde{S}'_n$ we can find a (unique) branch $\bar{\varphi} \in \bar{S}_n$ which allows us to analytically continue $\tilde{\varphi}$ outside the region $\tilde{\Omega}_n$ to a quasi disk Ω_n which has ordinary branch cuts (no thickness) and deletes neighbourhoods of the critical values of T^n whose diameters are of the order $e^{-(\epsilon-\beta)n}$. In particular if $\tilde{\varphi}$ and $\bar{\varphi}$ are both contracting then the continuation φ is also contracting. We hence get the following improvement of the previous lemma ($C_2 \leq 2c_1$).

Lemma 2 *Let $0 < \lambda < 1$. Then there exist $\epsilon > 0$, $\eta \in (0,1)$, a sequence of quasidisks $\Omega_n, n \in \mathbf{N}$, which have regular branch cuts and delete $C_1 e^{-n\epsilon}$-neighbourhoods ($C_1 > 0$) of the critical values of T^n, and a disjoint decomposition of the inverse branches of T^n on Ω_n into two (disjoint) subsets $\mathcal{S}'_n = \mathcal{S}'_n(\lambda)$ and $\mathcal{S}''_n = \mathcal{S}''_n(\lambda)$ so that*
(a) $|\mathcal{S}''_n| \leq C_2 \lambda^{-n}, n \in \mathbf{N}$, for some C_2.
(b) $|\varphi'(z)| \leq \eta^n$ for $z \in \Omega_n$ and in particular $\mathrm{diam}(\varphi(\Omega_n)) \leq \eta^n, \forall \varphi \in \mathcal{S}'_n, n \in \mathbf{N}$.

In property (a) we can assume that $C_2 = 1$ for all large enough n.

In the following we shall indicate by $\tilde{\varphi}$ an extension of the inverse branch $\varphi \in S_n$ to a quasidisk $\tilde{\Omega}_n$ which has ordinary branch cuts (so that $\mu(\tilde{\Omega}_n \cap J) = 1$). We denote by $\tilde{S}_n$ those extended inverse branches, whose restrictions φ to Ω_n lie in S_n (similarly for $\tilde{S}'_n$ and $\tilde{S}'_n$). Put $\mathcal{A}^n = \{\varphi(J \cap \Omega_n) : \varphi \in S_n\}$ and $\tilde{\mathcal{A}}^n = \{\tilde{\varphi}(J \cap \tilde{\Omega}_n) : \tilde{\varphi} \in \tilde{S}_n\}$.

There are other ways to introduce inverse branches, notably by Denker and Urbanski in [2] whose inverse branches we shall use in the last section to prove the weak Bernoulli property. In the above setting (Lemma 2) we get better control on the number of non-contracting branches (property (a)) and in return have to pay for it in property (b), where the exponential contraction only applies to the image under the inverse branch $\varphi \in S'_n$ and nothing much can be said about $(T^k\varphi)'$ for $k = 1, 2, \ldots, n$. (We also cut out some holes around critical values.) On the other hand Denker and Urbanski's inverse branches (Lemma 23) don't provide such good control on the number on non-contracting branches but instead have uniform contraction estimates for $T^k\varphi$ for $k = 1, \ldots, n$.

By [3] the eigenfunction h to the largest eigenvalue e^P (which is a single eigenvalue) of the transfer operator $\mathcal{L}_f : C^\alpha \to C^\beta$, for real $f \in C^\alpha$, is Hölder continuous with some exponent β. Moreover h is bounded and strictly positive. Hence on can introduce a normalised transfer operator $\hat{\mathcal{L}} : C^\alpha \to C^\gamma$ by $\hat{\mathcal{L}} = e^{-P(f)}\mathcal{L}_{\hat{f}}$, where $\hat{f} = f + \log h - \log h \circ T$ is Hölder continuous with exponent β ($P(\hat{f}) = 0$). The principal eigenvalue of the normalised transfer operator is 1 and the associated eigenfunctions are the constants: $\hat{\mathcal{L}}\mathbf{1} = \mathbf{1}$. The normalised transfer operator is given by

$$\hat{\mathcal{L}}^n\phi = \sum_{\varphi\in\tilde{S}_n} (g_n\phi)\varphi,$$

where $g_n = e^{\hat{f}+\hat{f}T+\cdots+\hat{f}T^{n-1}}$. The equilibrium state μ now satisfies $\hat{\mathcal{L}}^*\mu = \mu$ and is $e^{-\hat{f}}$-conformal, where by assumption $e^{\hat{f}} \leq \rho < 1$.

Since T^n is one-to-one of the atoms $A_{\tilde{\varphi}} = \tilde{\varphi}(J \cap \tilde{\Omega}_n)$ we get $\mu(A_{\tilde{\varphi}}) \leq \rho^n$. Hence, with the choice $\lambda = \sqrt{\rho}$ (cf. Lemma 2) the collective μ-measure onS the atoms A_φ for non-contracting branches $\varphi \in \tilde{S}_n''$ is (for all large enough n) bounded by $(\rho/\lambda)^n = \rho^{n/2}$, since the number of non-contracting branches is bounded by λ^{-n}.

In Section 5 we shall repeatedly use the bound:

$$\sum_{\tilde{\varphi}\in\tilde{S}_n''} \mu(A_{\tilde{\varphi}}) \leq \left(\frac{\rho}{\lambda}\right)^n,$$

for all large enough n.

Proposition 3 *[5] Let $\psi \in C^\alpha$ and μ the equilibrium state for some potential $f \in C^\alpha$ which satisfies the supremum condition* $\sup f < P(f)$. *Then there exists a $\sigma < 1$ and a constant C_3, such that for all $k \geq 1$:*

$$|\hat{\mathcal{L}}^k\psi - \mu(\psi)|_\infty \leq C_3\sigma^k\|\psi\|_\alpha.$$

The following result will be needed below to prove the multiple mixing property of Lemma 7.

Lemma 4 *([3]) There exists a constant $C_4 > 1$ and $\gamma_0, \xi > 0$, so that* $\operatorname{diam}(\varphi(B_\gamma(x))) \leq$ *$C_4^n\gamma^\xi$ for all $x \in J$, inverse branches φ (on some quasidisk) of T^n, $n \geq 0$, $\gamma \leq \gamma_0$ and small enough δ.*

Let us denote by $\mathrm{HD}(\mu)$ the Hausdorff dimension of μ ($\mathrm{HD}(\mu) > 0$). The (lower) pointwise dimension of μ, $D(x) = \liminf_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon}$, is μ-almost everywhere equal to the Hausdorff dimension. The following lemma states that the pointwise dimension is always strictly positive. For completeness sake (and lack of a reference) we include a proof.

Lemma 5

$$\inf_{x\in J} D(x) > 0$$

Proof. According to [2] μ is an $e^{-\hat{f}}$-conformal measure, that is $\mu(TA) = \int_A e^{-\hat{f}(x)} d\mu(x)$ on (measurable) sets A on which T is one-to-one. This implies that $D(x) = D(Tx)$ for all $x \in J$ for which T is conformally one-to-one in a neighbourhood of x (i.e. $T'(x) \neq 0$). If x is a critical point of order k, then $T(y) = T(x) + (y-x)^k\psi(y)$, where $\psi(y)$ is analytic and non-zero in a neighbourhood of x. This implies that $D(x) = kD(Tx)$.

Let $x \in J$ be such that none of its preimages $T^{-k}x$, $k = 0, 1, 2, \ldots$, contains a critical point of T. Then D is a constant equal to $D(x)$ on $P = \bigcup_{k=0}^\infty T^{-k}x$. To show that $D(x) \geq \mathrm{HD}(\mu)$ let us assume that $D(x) < \mathrm{HD}(\mu)$ and choose $D(x) < \delta' < \mathrm{HD}(\mu)$. Then for every $\varepsilon > 0$ we can find numbers $0 < r(y) < \varepsilon/3$, so that

$$\mu(B_{r(y)}(y)) \geq r(y)^{\delta'}$$

for all $y \in P$. Since P is dense in J, $\{B_{r(y)}(y) : y \in P\}$ is an open cover of J, and by compactness there is a finite $P' \subset P$ so that $J \subset \bigcup_{y\in P'} B_{r(y)}(y)$. We now prune the set P' by successively removing points in the following manner. Let y be a point so that $r(y)$ is largest. Then remove all the y' for which $B_{r(y')}(y') \subset B_{3r(y)}(y)$. The remaining disks $B_{r(y')}(y')$ will be disjoint from $B_{r(y)}(y)$. Now pick among the remaining disks (other than $B_{r(y)}(y)$) the one with the largest r and remove in the same way (smaller) disks that are contained in the disk of triple radius. Successive application of this procedure yields after finitely many steps a set $P'' \subset P'$ which has the property that $B_{r(y)}(y) \cap B_{r(y')}(y') = \emptyset$ for $y, y' \in P''$, $y \neq y'$ and also satifies that $J \subset \bigcup_{y\in P''} B_{3r(y)}(y)$.

We therefore obtain

$$\sum_{y\in P''} (\,\mathrm{diam}\,(B_{3r(y)}(y)))^{\delta'} = 6^{\delta'} \sum_{y\in P''} r(y)^{\delta'} \leq 6^{\delta'} \sum_{y\in P''} \mu(B_{r(y)}(y)) \leq 6^{\delta'}.$$

Since ε was arbitrary, we found for every $\varepsilon > 0$ a (finite) cover $\mathcal{C}$ of J so that $\mathrm{diam}\,(A) \leq \varepsilon \;\; \forall\, A \in \mathcal{C}$ and

$$\sum_{A\in\mathcal{C}} (\,\mathrm{diam}\,(A))^{\delta'} \leq 6^{\delta'},$$

which implies that $\delta' \geq \mathrm{HD}(\mu)$. Hence $D(x) \geq \mathrm{HD}(\mu)$.

For an arbitrary point $x \in J$ we shall use the fact that every orbit in J contains at most c_1 critical points, where c_1 is some constant that depends only on the map T. Since the order of critical points is no more than the degree d of the map, one obtains for any $x \in J$ the bound:

$$D(x) \geq \frac{\mathrm{HD}(\mu)}{d^{c_1}}.$$

□

The complement Ω_n^c of the domain Ω_n consists of a union of $C_1 e^{-\epsilon n}$-disks (ϵ from Lemma 2) centred at the critical values of T^n (here we disregard the fact that in addition one has to do 'regular' branchcuts because their measure is zero). Note that the number of critical values of T^n grows linearly in n.

Lemma 6 *There exists an $\alpha \in (0,1)$ so that for all n large enough*

$$\mu(\Omega_n^c) \leq \alpha^n.$$

Proof. Since the radii at which one can estimate the measure of a ball using the local dimension are not of uniform size we have to do the following auxiliary argument.

The map T has finitely many critical points $z_1, z_2, \ldots$ of orders $k_1, k_2, \ldots$. For every j we thus can write $T(z) = T(z_j) + (z - z_j)^{k_j} \psi_j(z)$, where ψ_j is analytic and non-zero in a neighbourhood of z_j. Let $\varepsilon_1 > 0$ be so that $\log |\psi_j(z)/\psi_j(z_j)|$ is small (e.g. between the values $-\frac{1}{4}$ and $\frac{1}{4}$) for $z \in B_{\varepsilon_1}(z_j)$ ($j = 1, 2, \ldots$). Then there exists an $\varepsilon_2 > 0$ (assume $\varepsilon_2 < \varepsilon_1$) so that T is one-to-one on disks $B_{\varepsilon_2}(z)$ centred at $z \notin \bigcup_j B_{\varepsilon_1}(z_j)$. Within the disks $B_{\varepsilon_1}(z_j)$ we now use the fact that T behaves like a k_jth power of $z - z_j$. For $0 < \varepsilon_3 < \varepsilon_2$ we delete ε_3-neighbourhoods around the critical points. Then T is one-to-one on any set $A \subset B_{\varepsilon_1}(z_j)$ which satisfies $\mathrm{diam}\,(A) < \varepsilon_3$, $\mathrm{diam}\,(TA) < \varepsilon_3$ and so that A and TA both avoid the ε_3-neighbourhoods of the critical points.

If we choose ε_1 small enough then we can by Lemma 5 choose a positive $\delta < \inf_{x \in J} D(x)$ so that $\mu(B_\varepsilon(z_j)) \leq \varepsilon^\delta$ for all critical points z_j and all $\varepsilon \leq \varepsilon_1$.

Denote by $\mathrm{Crit}\,(T)$ the critical points of T, choose $u \in (e^{-\epsilon}, 1)$ (where ϵ is as in Lemma 2) and assume that n is large enough so that $C_1 e^{-n\epsilon} < u^{n/2} \leq \varepsilon_2/6$. Let $m = [n \frac{|\log u|}{2 \log |T'|_\infty}]$ and divide the critical values of T^n into two classes: P and Q. We say a critical value z of T^n lies in P if $\mathrm{dist}\,(T^k z, \mathrm{Crit}\,(T)) \leq 4u^{n/2}$ for some $k \leq m$ and $\mathrm{dist}\,(T^j z, \mathrm{Crit}\,(T)) > 4u^{n/2}$ for $j = 0, \ldots, k-1$. Otherwise we say z lies in Q. Note that by choice of m we have for $k \leq m$:

$$\mathrm{diam}\,(T^k B_{u^n}(z)) \leq 2u^n |T'|_\infty^k \leq 2u^{n/2}.$$

If $z \in Q$, then this implies the sets $T^k B_{u^n}(z)$ are disjoint from the $2u^{n/2}$-neighbourhood of $\mathrm{Crit}\,(T)$, and therefore (by the second paragraph of the proof) T^k is one-to-one on $B_{u^n}(z)$ for $k \le m$. In particular $T^m B_{u^n}(z) \subset \tilde{\Omega}_m$ (provided we arrange appropriately for the regular branchcuts) and there exists a unique inverse branch $\tilde{\varphi} \in \tilde{S}_m$ so that $B_{u^n}(z) = \tilde{\varphi}(T^m B_{u^n}(z))$. A rough estimate yields $\mu(B_{u^n}(z)) \le \mu(\tilde{\varphi}(J)) \le \rho^m$. Since $|Q| + |P| \le c_1 n$ for some constant c_1, one therefore obtains

$$\sum_{z \in Q} \mu(B_{u^n}(z)) \le c_1 n \rho^m \le \rho^{-1} c_1 n \left(\rho^{|\log u|/2 \log |T'|_\infty}\right)^n \to 0$$

exponentially fast as $n \to \infty$.

If $z \in P$ then, because of T-invariance of μ and the inclusion $B_{u^n}(z) \subset T^{-k}(T^k B_{u^n}(z))$, we have (since $\mathrm{dist}\,(T^k z, \mathrm{Crit}\,(T)) \le 4u^{n/2}$)

$$\mu(B_{u^n}(z)) \le \mu(T^k B_{u^n}(z)) \le \mu(B_{2u^{n/2}}(T^k z)) \le \mu(B_{6u^{n/2}}(z')),$$

where $z' \in \mathrm{Crit}\,(T)$ is such that $|T^k z - z'| \le 4u^{n/2}$. Hence, since $\mu(B_{6u^{n/2}}(z')) \le (6u^{n/2})^\delta$ for any $z' \in \mathrm{Crit}\,(T)$, and since $|P| \le c_1 n$ we conclude that

$$\sum_{z \in P} \mu(B_{u^n}(z)) \le c_2 n u^{n\delta/2}.$$

Since $\Omega_n^c \subset \bigcup_{z \in P \cup Q} B_{u^n}(z)$ the lemma now follows for any $\alpha > \max(e^{-\epsilon\delta/2}, \rho^{\epsilon/2 \log |T'|_\infty})$ $(\alpha < 1)$. $\square$

3 Mixing rates for rational maps

In this section we use the convergence properties of the transfer operator to deduce mixing properties for rational maps which will be sufficient to prove the main result, although, as one can see, it turns out that μ is not ϕ-mixing, since the right hand side in Lemma 8 is not independent of n but depends exponentially on n.

Lemma 7 *Let $\kappa > 1$. Then there exists a constant C_5 and $\sigma < 1$ so that*

$$\left|\mu(A_\varphi \cap T^{-k-n} Q) - \mu(A_\varphi)\mu(Q)\right| \le C_5 \sigma^k \kappa^n \mu(Q) |g_n \varphi|_\infty,$$

for any inverse branch $\varphi \in \tilde{S}_n$ ($A_\varphi = \varphi(J) \in \tilde{\mathcal{A}}^n$), $k, n > 0$ and Q measurable.

Proof. Let $\kappa > 1$, $\varphi \in S_n$ and note that $\hat{\mathcal{L}}^n \chi_{A_\varphi} = g_n \varphi$. To estimate the Hölder norm of $g_n \varphi$ let us use Lemma 4 according to which $|\varphi x - \varphi x'| \le C_4^n |x - x'|^\xi$ for all

$x, x' \in J \cap \tilde{\Omega}_n$ and some $\xi > 0$. Since $\hat{f}$ is β-Hölder continuous we obtain for every $\beta' \in (0, \beta]$ that

$$\begin{aligned}
|g_n(\varphi x) - g_n(\varphi x')| &= g_n(\varphi x)\left|1 - \frac{g_n(\varphi x')}{g_n(\varphi x)}\right| \\
&\leq |g_n\varphi|_\infty \left(C_4^n |x - x'|^\xi\right)^{\beta'} \\
&\leq |g_n\varphi|_\infty C_4^{\beta' n} |x - x'|^{\xi\beta'}.
\end{aligned}$$

Now let $\beta' > 0$ be so small that $C_4^{\beta'} = \kappa$ and then put $\gamma = \xi\beta'$. This gives $|g_n\varphi|_\gamma \leq |g_n\varphi|_\infty \kappa^n$ and consequently

$$\|g_n\varphi\|_\gamma \leq 2\kappa^n |g_n\varphi|_\infty.$$

Since

$$\mu(A_\varphi \cap T^{-k-n}Q) = \mu\left(\hat{\mathcal{L}}^n\left(\chi_{A_\varphi}(\chi_Q \circ T^{k+n})\right)\right) = \mu\left(\left(\chi_Q \circ T^k\right)(g_n\varphi)\right),$$

we get using Proposition 3 (note: $\mu(A_\varphi) = \mu(g_n\varphi)$)

$$\begin{aligned}
\left|\mu((\chi_Q \circ T^k)(g_n\varphi)) - \mu(Q)\mu(A_\varphi)\right| &\leq \mu\left(\chi_Q \left|\hat{\mathcal{L}}^k(g_n\varphi) - \mu(A_\varphi)\right|\right) \\
&\leq C_3\mu(Q)\sigma^k \|g_n\varphi\|_\gamma,
\end{aligned}$$

(where C_3 depends on the Hölder exponent γ). The lemma now follows with $C_5 = 2C_3$. □

Lemma 8 *There exist $\nu > 1$, $\sigma < 1$ and a constant C_6 so that for all $k, n \in \mathbf{N}$, Q measurable and finitely many (distinct) $A_1, \ldots, A_\ell \in \tilde{\mathcal{A}}^n$ one has*

$$\left|\mu(W \cap T^{-k-n}Q) - \mu(W)\mu(Q)\right| \leq C_6\sigma^k\nu^n\mu(W)\mu(Q),$$

where $W = \bigcup_{j=0}^{\ell} A_j$.

Proof. Let us first prove the lemma in the case when W consists of a single atom $A_\varphi = \varphi(J)$ for some $\varphi \in \tilde{S}_n$. Observe that there exists a constant $\nu' > 1$ (e.g. $\nu' = \exp(\sup \hat{f} - \inf \hat{f})$) so that for all $n \in \mathbf{N}$ and $\varphi \in \tilde{S}_n$ one has

$$|g_n|_\infty \leq \nu'^n \inf g_n\varphi,$$

which in turn implies

$$|g_n|_\infty \leq \nu'^n \inf g_n\varphi \leq \nu'^n\mu(g_n\varphi) = \nu'^n\mu(A_\varphi).$$

Now apply Lemma 7 and put $\nu = \kappa\nu'$.

For the general case we use the fact that the interiors of the atoms of $\tilde{\mathcal{A}}^n$ are disjoint and that the boundaries have zero measure (i.e. $\tilde{\mathcal{A}}^n$ is a partition for μ). Hence

$$\begin{aligned}\left|\mu(W \cap T^{-k-n}Q) - \mu(W)\mu(Q)\right| &\leq \sum_{j=0}^{\ell}\left|\mu(A_j \cap T^{-k-n}Q) - \mu(A_j)\mu(Q)\right| \\ &\leq \sum_{j=0}^{\ell} C_6\sigma^k\nu^n\mu(A_j)\mu(Q) \\ &\leq C_6\sigma^k\nu^n\mu(W)\mu(Q).\end{aligned}$$

□

For $r \geq 1$ and (large) N denote by $G_r(N)$ the r-vectors $\vec{v} = (v_1, \ldots, v_r)$ for which $0 \leq v_1 < v_2 < \cdots < v_r \leq N$. (Geometrically $G_r(N)$ is the portion of a cone in $\mathbf{N}^r$ which lies within distance N of the origin.) In our setting the numbers v_k are the return (or hitting) times. For $\vec{v} = (v_1, v_2, \ldots, v_r) \in G_r$ we put $d(\vec{v}) = \min_{1\leq s<r}(v_{s+1} - v_s)$.

Lemma 9 *Let $r > 1$ an integer. Then there exist $\varsigma \in (0,1)$, a constant C_7 and a $q > 0$ so that for any r numbers $n_1, \ldots, n_r$ and $\vec{v} = (v_1, v_2, \ldots, v_r) \in G_r$ for which $v_{s+1} - v_s > (1+q)n_s$ for $s = 1, \ldots, r-1$ one has*

$$\left|\mu(C_{\vec{v}}) - \prod_{s=1}^{r}\mu(W_s)\right| \leq C_7\varsigma^{d(\vec{v})}\prod_{s=1}^{r}\mu(W_s),$$

for any choice of W_s each of which is a union of atoms in $\tilde{\mathcal{A}}^{n_s}$, $s = 1, \ldots, r$, where $C_{\vec{v}} = \bigcap_{s=1}^{r} T^{-v_s}W_s$.

Proof. Put for $k = 1, 2, \ldots, r$:

$$D_k = \bigcap_{s=k}^{r} T^{-(v_s - v_k)}W_s.$$

In particular we have $C_{\vec{v}} = T^{-v_1}D_1$ and of course $\mu(C_{\vec{v}}) = \mu(D_1)$. Also note that

$$D_k = W_k \cap T^{-(v_{k+1}-v_k)}D_{k+1}$$

and $D_r = W_r$. Hence by Lemma 8 we obtain

$$|\mu(D_k) - \mu(W_k)\mu(D_{k+1})| \leq \mu(D_{k+1})\mu(W_k)\sigma^{v_{k+1}-v_k-n_k}\nu^{n_k}.$$

Repeated application of the triangle inequality yields

$$\left|\mu(C_{\vec{v}}) - \prod_{s=1}^{r} \mu(W_s)\right| \le \sum_{k=1}^{r-1} |\mu(D_k) - \mu(W_k)\mu(D_{k+1})| \prod_{s=1}^{k-1} \mu(W_s).$$

Now if we choose $q > 0$ so that $(\sigma^q \nu)^{1/(q+1)} = \varsigma < 1$, then we obtain:

$$\begin{aligned} \mu(D_k) &\le \mu(W_k)\mu(D_{k+1}) \left(1 + \sigma^{v_{k+1}-v_k-n_k} \nu^{n_k}\right) \\ &\le 2\mu(W_k)\mu(D_{k+1}) \end{aligned}$$

since by assumption that $v_{k+1} - v_k - n_k > qn_k$. Inductively then

$$\mu(D_k) \le 2^{r-k} \prod_{s=k}^{r} \mu(W_s),$$

for $k = r, r-1, r-2, r-3, \ldots, 1$. Thus (since $\sigma \le \varsigma$)

$$\begin{aligned} \left|\mu(C_{\vec{v}}) - \prod_{s=1}^{r} \mu(W_s)\right| &\le \sum_{k=1}^{r-1} \mu(D_{k+1})\mu(W_r)\sigma^{v_{k+1}-v_k-n_k} 2^{r-k} \nu^{n_k} \prod_{s=1}^{k-1} \mu(W_s) \\ &\le \sum_{k=1}^{r-1} \sigma^{v_{k+1}-v_k-(1+q)n_k} (\sigma^q \nu)^{n_k} 2^{r-k} \prod_{s=1}^{r} \mu(W_s) \\ &\le 2^{r+1} \varsigma^{d(\vec{v})} \prod_{s=1}^{r} \mu(W_s). \end{aligned}$$

Now put $C_7 = 2^{r+1}$. □

4 Approximability of balls by cylinder sets

Let us define for $\alpha > 0$ the set

$$\mathcal{O}_\alpha = \left\{ x \in J : \liminf_{\beta \to 0} \frac{\mu(B_\beta(x))}{\mu(B_{\beta-\beta^2}(x))} - 1 \ge \alpha \right\}.$$

Then for every $x \in \mathcal{O}_\alpha$ there exists a $\gamma(x)$ so that $\frac{\mu(B_\beta(x))}{\mu(B_{\beta-\beta^2}(x))} - 1 \ge \frac{\alpha}{2}$ for all $\beta < \gamma(x)$. Moreover let us define

$$\mathcal{O}_{\alpha,\beta} = \{ x \in \mathcal{O}_\alpha : \gamma(x) \ge \beta \}.$$

Clearly $\mathcal{O}_\alpha = \bigcup_{\beta > 0} \mathcal{O}_{\alpha,\beta}$ and therefore, if we can show that the measure of $\mathcal{O}_{\alpha,\beta}$ is zero for every positive β, then $\mathcal{O}_\alpha$ is a null set.

Lemma 10 *$\mu(\mathcal{O}_{\alpha,\beta}) = 0$ for every positive α and β.*

Proof. With $\alpha, \beta > 0$ given, we have by assumption $\mu(B_{\eta-\eta^2}(x)) \le \frac{1}{1+\alpha/2}\mu(B_\eta(x))$ for all $x \in \mathcal{O}_{\alpha,\beta}$ and $\eta \le \beta$. In particular

$$\mu(B_{(j+1)^{-1}}(x)) \le \frac{1}{1+\alpha/2}\mu(B_{j^{-1}}(x)),$$

for all $j \ge k = [1/\beta] + 1$. Iteration yields

$$\mu(B_{j^{-1}}(x)) \le \left(\frac{1}{1+\alpha/2}\right)^{j-k} \mu(B_{k^{-1}}(x)),$$

For simplicity's sake we can assume that the Julia set does not contain the point ∞ (a Möbius transform can achieve this) and therefore is contained in a compact subset of the open complex plane. There exists a constant c_1 (independent of $\eta > 0$) so that J can be covered by at most $c_1\eta^{-2}$ many balls of radius $\eta/2$. In particular, we can cover $\mathcal{O}_{\alpha,\beta}$ by at most $c_1\eta^{-2}$ balls of radius η and centres in $\mathcal{O}_{\alpha,\beta}$. Thus (as $\mu(B_{k^{-1}}(x)) \le 1$)

$$\mu(\mathcal{O}_{\alpha,\beta}) \le 4c_1 j^2 \left(\frac{1}{1+\alpha/2}\right)^{j-k},$$

and since j is arbitrary we conclude that $\mu(\mathcal{O}_{\alpha,\beta}) = 0$. □

As observed above, this implies that $\mu(\mathcal{O}_\alpha) = 0$ for every positive α. Taking a union over $\alpha > 0$ we put

$$\mathcal{O} = \bigcup_\alpha \mathcal{O}_\alpha = \left\{x \in J : \liminf_{\beta\to 0} \frac{\mu(B_\beta(x) \setminus B_{\beta-\beta^2}(x))}{\mu(B_{\beta-\beta^2}(x))} > 0\right\},$$

and obtain the following result which will be used in lemma 18.

Lemma 11

$$\mu(\mathcal{O}) = 0.$$

5 Distribution of return times

Let T be an expansive transformation on the space Ω and let μ be a probability measure on Ω. For $x \in J$ and $\varepsilon > 0$ let χ_ε be the characteristic function of $B_\varepsilon = B_\varepsilon(x)$. Then

$$\xi_\varepsilon = \sum_{k=0}^{[t/\mu(B_\varepsilon)]} \chi_\varepsilon \circ T^k. \tag{2}$$

measures the number of times x point returns to the set B_ε within the normalised time $t/\mu(B_\varepsilon)$. We have the following limiting behaviour.

Theorem 12 *For μ-almost every x, there exists a sequence of $\varepsilon_j \to 0$ so that*

$$\mu(\mathcal{N}_{r,\varepsilon_j}) \to \frac{t^r}{r!}e^{-t}, \tag{3}$$

as j tends to infinity, where $\mathcal{N}_{r,\varepsilon_j} = \{y \in J : \xi_{\varepsilon_j}(y) = r\}$ is the r-levelset of ξ_{ε_j}.

We shall need the following result which is due to Sevast'yanov [12] and was previously employed by Pitskel. It's usefulness mainly stems from relation (8) which is the mixing property of Lemma 9 and avoids estimating the measures of sets whose points return exactly r times (at times $\vec{v}$) within the time interval $[0, t/\mu(B_\varepsilon)]$. The mixing property (8) applies when the separation between the hitting times is large enough. The rare set, R_r below, consists of those hitting patterns that have short return times. Most of our effort will be directed to verify relation (6) (Lemma 16), that is to show that almost all returns are on a 'long timescale'.

Proposition 13 *Let $\{\eta_v^n : v = 1, \ldots, N(n)\}$ for $n \geq 1$ be an array of random $0,1$-valued variables and μ a probability measure. Put $\zeta_n = \sum_{v=1}^N \eta_v^n$, and for $\vec{v} \in G_r$ let $b_{\vec{v}}^n = \mu(\eta_{\vec{v}}^n)$, where $\eta_{\vec{v}}^n = \prod_{s=1}^r \eta_{v_s}^n$ (in particular $b_v^n = \mu(\eta_v^n)$). Assume that the following five assumptions are satisfied:*

$$\lim_{n\to\infty} \max_{1\leq v\leq N} b_v^n = 0, \tag{4}$$

$$\lim_{n\to\infty} \sum_{v=1}^N b_v^n = t > 0. \tag{5}$$

Moreover assume that there exist rare sets $R_r \subset G_r$ ($r \geq 1$)

$$\lim_{n\to\infty} \sum_{\vec{v}\in R_r} b_{\vec{v}}^n = 0, \tag{6}$$

$$\lim_{n\to\infty} \sum_{\vec{v}\in R_r} b_{v_1}^n \cdots b_{v_r}^n = 0, \tag{7}$$

$$\lim_{n\to\infty} \frac{b_{v_1}^n \cdots b_{v_r}^n}{b_{\vec{v}}^n} = 1, \tag{8}$$

uniformly in $\vec{v} \in G_r \setminus R_r$. Then

$$\lim_{n\to\infty} \mu(\mathcal{N}_r) = \frac{t^r e^{-t}}{r!},$$

where $\mathcal{N}_r = \{y : \zeta_n(y) = r\}$ is the r-levelset of ζ_n.

Put $N = [t/\mu(B_\varepsilon)] = [t/\mu(\chi_\varepsilon)]$. The random variable η is then given by $\eta_v^n = \chi_\varepsilon \circ T^v$. The sum ζ_n then equals ξ_ε of Theorem 12 and $\mu(\mathcal{N}_{r,\varepsilon}) = \sum_{\vec{v}\in G_r} b_{\vec{v}}^n$, where $b_{\vec{v}}^n = \mu(C_{\vec{v}})$, $C_{\vec{v}} = \bigcap_{k=1}^r T^{-v_k} B_\varepsilon$ (note that $\eta_{\vec{v}}^n$ is the characteristic function of $C_{\vec{v}}$).

The following lemma establishes that outside a zero measure set 'very short' returns occur only finitely many times.

Lemma 14 *There exists a monotone function $p(\varepsilon) \to \infty$ as $\varepsilon \searrow 0$ so that*

$$\lim_{\varepsilon\to 0} \mu\left(\left\{x \in J : B_\varepsilon(x) \cap T^{-m} B_\varepsilon(x) \neq \emptyset \ \text{ for some } \ 0 < m \le p(\varepsilon)\right\}\right) = 0$$

Proof. For $\varepsilon > 0$ and $m \in \mathbf{N}$ put

$$U_{m,\varepsilon} = \left\{x \in J : \ \ B_\varepsilon(x) \cap T^{-m} B_\varepsilon(x) \neq \emptyset\right\},$$

and denote by F_m the (finitely many) m-periodic points of T. For $j = 1, 2, \ldots$, put $\mathcal{F}_j = \bigcup_{m=1}^j F_m$ and let $\beta_j > 0$ be small enough so that $\mu(B_{2\beta_j}(\mathcal{F}_j)) \le 1/j$ (as μ is non-atomic), where $B_{2\beta_j}(\mathcal{F}_j) = \bigcup_{x\in\mathcal{F}_j} B_{2\beta_j}(x)$. Then

$$\gamma_j(k) = \inf_{x \notin B_{\beta_j}(\mathcal{F}_j)} d(x, T^{-k}x)$$

is positive for every $k \le j$, since $J \setminus B_{\beta_j}(\mathcal{F}_j)$ is closed and has no k-periodic points. Put $\gamma_j = \min_{k\le j} \gamma_j(k)$. If $\varepsilon_j \le \beta_j$ is small enough so that $\varepsilon_j \le \gamma_j/4$ and the components of $T^{-m} B_{\varepsilon_j}(x)$ have diameters less than $\gamma_j/2$, then

$$B_{\varepsilon_j}(x) \cap T^{-m} B_{\varepsilon_j}(x) = \emptyset,$$

$m = 1, \ldots, j$, for all $x \notin B_{2\beta_j}(\mathcal{F}_j)$. Consequently

$$\mu\left(\bigcup_{m=1}^j U_{m,\varepsilon_j}\right) \le \mu\left(B_{2\beta_j}(\mathcal{F}_j)\right) < \frac{1}{j}.$$

If we now define $p(\varepsilon) = \min\{j : \varepsilon_k \le \varepsilon \ \forall k \ge j\}$ we obtain that $\mu(\bigcup_{m=1}^{p(\varepsilon)} U_{m,\varepsilon})$ goes to zero as ε goes to zero. $\square$

Put differently, the lemma states that for μ-almost every $x \in J$ the intersections $B_\varepsilon(x) \cap T^{-m} B_\varepsilon(x)$ are empty for all $m = 1, \ldots, p(\varepsilon)$ for ε small enough. This implies that for almost every $x \in J$ and small enough ε the set

$$C_{\vec{v}} = \bigcap_{s=1}^r T^{-v_s} B_\varepsilon(x)$$

is empty if the hitting vector $\vec{v}$ lies in the set $I_r(N)$ (i.e. one of the repeat times $m_j = v_{j+1} - v_j$ is less or equal to $p(\varepsilon)$). We will need this fact in Lemma 15 below where we shall assume that for almost all points and small enough ε the rare set only consists of $K_r(N)$ and we don't encounter returns shorter than $p(\varepsilon)$.

As in Section 2 let $HD(\mu)$ denote the Hausdorff dimension of the measure μ. On a full measure set $\mathcal{M}$ one has $\lim_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon} = HD(\mu)$ for all $x \in \mathcal{M}$. We can assume that $\mathcal{M}$ lies in the complement of $\mathcal{O}$. Since $0 < HD(\mu)$ (or because of Lemma 5) one has for all $x \in \mathcal{M}$ that $\lim_{\varepsilon\to 0} \mu(B_\varepsilon(x)) \log \varepsilon = 0$. Pick some number δ' larger than $HD(\mu)$ and a positive $\delta < HD(\mu)$.

We shall assume that x is not a critical value of an iterate of T (of which there are only countably many).

Let ε_j be a decreasing sequence so that $\mu(B_{\varepsilon_j+\varepsilon_j^2/3})/\mu(B_{\varepsilon_j-\varepsilon_j^2/2}) \to 1$ as $j \to \infty$ (possible since $x \notin \mathcal{O}$ and $(\frac{1}{2}+\frac{1}{3})\varepsilon_j^2 \le (\varepsilon_j - \varepsilon_j^2/2)^2$).

We shall construct inner and outer approximations of B_ε by unions of elements in $\mathcal{A}^n$ for suitable $n = n(\varepsilon)$. First we determine n: Fix j, let $\ell = [\log \frac{\varepsilon_j}{3} / \log \eta] + 1$ (so that $\eta^\ell \le \frac{\varepsilon_j}{3}$) and find $n \ge 2\ell$ so that $(\rho/\lambda)^n \le \varepsilon_j^{1+(r-1)\delta'}$. In fact we can arrange that

$$n = [C_8 |\log \varepsilon_j|]$$

for some constant $C_8 > \max(r\delta'/\epsilon\delta, \delta'/|\log \alpha|)$ independent of j (where α is given by Lemma 6). With this choice of n we achieve that $\varepsilon_j \le e^{-n/C_8}$ and

$$\frac{\mu(\Omega_n^c)}{\mu(B_{\varepsilon_j})} \le c_1 \frac{\alpha^n}{\varepsilon_j^{\delta'}} \le c_2 \varepsilon_j^{C_8|\log \alpha| - \delta'} \to 0, \tag{9}$$

as $j \to \infty$, since $C_8 > \delta'/|\log \alpha|$ has been chosen large enough to get a positive exponent.

In equation (2) replace B_ε by $B_\varepsilon^\circ = B_\varepsilon \cap T^{-n}\Omega_n$ and define the modified process

$$\xi_\varepsilon^\circ = \sum_{k=0}^{N} \chi_{B_\varepsilon^\circ} \circ T^k$$

($\xi_\varepsilon^\circ \le \xi_\varepsilon$). The difference between the two processes can now be estimated using Lemma 6 (with some c_3):

$$\begin{aligned} \mu(|\xi_\varepsilon - \xi_\varepsilon^\circ|) &\le \sum_{k=0}^{N} \mu(T^{-k}(B_\varepsilon \setminus B_\varepsilon^\circ)) \\ &= \sum_{k=0}^{N} \mu(B_\varepsilon \setminus B_\varepsilon^\circ) \end{aligned}$$

$$
\begin{aligned}
&\leq (N+1)\mu(T^{-n}(\Omega_n^c)) \\
&\leq c_3 \frac{\mu(\Omega_n^c)}{\mu(B_{\varepsilon_j})} \to 0,
\end{aligned} \tag{10}
$$

by equation (9), as $j \to \infty$. Let us now define the following unions of preimages of Ω_n under the inverse branches of T^n:

$$
\begin{aligned}
W_j' &= \bigcup_{A_\varphi \subset B_{\varepsilon_j}, \varphi \in S_n'} A_\varphi, \\
W_j'' &= \bigcup_{A_\varphi \cap B_{\varepsilon_j} \neq \emptyset, \varphi \in S_n'} A_\varphi, \\
W_j''' &= \bigcup_{\varphi \in S_n''} A_\varphi,
\end{aligned}
$$

then by construction $B^\circ_{\varepsilon_j - \varepsilon_j^2/2} \subset W_j' \cup W_j'''$, $W_j' \subset B_{\varepsilon_j}$, $B^\circ_{\varepsilon_j} \subset W_j'' \cup W_j'''$ and $W_j'' \subset B_{\varepsilon_j + \varepsilon_j^2/3}$, since $\operatorname{diam}(A_\varphi) \leq \eta^n \leq \varepsilon_j^2/3$ for all $\varphi \in S_n'$.

We shall moreover use the sets $\tilde{W}_j', \tilde{W}_j'', \tilde{W}_j'''$ which are obtained from the above sets by applying the preimages to $\tilde{\Omega}_n$ (rather than Ω_n as above). We then have $B_{\varepsilon_j - \varepsilon_j^2/2} \subset \tilde{W}_j' \cup \tilde{W}_j'''$ and $B_{\varepsilon_j} \subset \tilde{W}_j'' \cup \tilde{W}_j'''$.

By equation (9) and choice of the values ε_j we have

$$
\frac{\mu(B_{\varepsilon_j} \setminus W_j')}{\mu(B_{\varepsilon_j})} \leq \frac{\mu(B_{\varepsilon_j} \setminus B_{\varepsilon_j - \varepsilon_j^2/2})}{\mu(B_{\varepsilon_j})} + \frac{\mu(\Omega_n^c)}{\mu(B_{\varepsilon_j})} \to 0 \tag{11}
$$

as $j \to \infty$, and, since $\mu(\tilde{W}_j''') \leq |S_n''|\rho^n \leq (\rho/\lambda)^n \leq \varepsilon_j^{1+(r-1)\delta'}$ $(x \in \mathcal{M})$, we also obtain

$$
\frac{\mu((W_j'' \cup W_j''') \setminus B_{\varepsilon_j})}{\mu(B_{\varepsilon_j})} \leq \frac{\mu(B_{\varepsilon_j + \varepsilon_j^2/3} \setminus B_{\varepsilon_j})}{\mu(B_{\varepsilon_j})} + \frac{\mu(W_j''')}{\mu(B_{\varepsilon_j})} \to 0 \tag{12}
$$

as $j \to \infty$.

We say that W_j' is the *inner approximation* of B_{ε_j} and $W_j'' \cup W_j'''$ is its *outer approximation*.

The 'rare sets' $R_r(N)$ consist of two disjoint subsets: $I_r(N)$ and $K_r(N)$. Let $p(\varepsilon)$ be the positive and integer valued function of Lemma 14 (which goes monotonically to infinity as $\varepsilon \searrow 0$), and let $q > 0$ be given by Lemma 9. Then we define the following disjoint subsets of $G_r(N)$ (where $n = n(\varepsilon)$ is as introduced above and, as always, $N = [t/\mu(B_\varepsilon)]$):

$$
\begin{aligned}
I_r(N) &= \{\vec{v} \in G_r(N) : \min(v_{s+1} - v_s) \leq p(\varepsilon)\}, \\
K_r(N) &= \{\vec{v} \in G_r(N) : p(\varepsilon) < \min(v_{s+1} - v_s) \leq (1+q)n\}.
\end{aligned}
$$

(If $p(\varepsilon) \geq n(\varepsilon)$ then $K_r(N)$ is empty.) The rare set is now $R_r(N) = I_r(N) \cup K_r(N)$. The next two lemmas together with Lemma 14 serve to verify condition (6) of Proposition 13.

Let us consider the following 'random variables' ξ_j', ξ_j'' associated with the inner and outer approximations of B_{ε_j} and which are defined by

$$\begin{aligned} \xi_j' &= \sum_{k=0}^{N} \chi_{W_j'} \circ T^k, \\ \xi_j'' &= \sum_{k=0}^{N} \chi_{W_j'' \cup W_j'''} \circ T^k, \end{aligned}$$

where χ_W is the characteristic function of the set W. We shall write * for $'$ (inner approximation) or $''$ (outer approximation). Let us introduce the sets $C_{\vec{v}}^*$ as follows

$$\begin{aligned} C_{\vec{v}}' &= \bigcap_{s=1}^{r} T^{-v_s} W_j' \\ C_{\vec{v}}'' &= \bigcap_{s=1}^{r} T^{-v_s} (W_j'' \cup W_j'''), \end{aligned}$$

where obviously $C_{\vec{v}}' \subset C_{\vec{v}}''$.

For a cylinder set $V = \psi(\Omega_m)$ for some $\psi \in S_m$ we denote by $\tilde{V}$ its associated extension $\tilde{\psi}(\tilde{\Omega}_m)$, and similarly for unions of cylinder sets. The following lemma serves to verify condition (7) of Theorem 13 outside the zero measure set of Lemma 14.

Lemma 15 *For almost all $x \in J$ the inner and outer approximations ($^* = ', ''$) both satisfy*

$$\lim_{n \to \infty} \sum_{\vec{v} \in K_r} b_{\vec{v}}^{*n} = 0,$$

provided ε is small enough.

Proof. We shall prove the lemma for the outer approximation ξ_j'' (and assume that $p(\varepsilon) < n(\varepsilon)$, for otherwise nothing has to be shown). Put K_r^s for those $\vec{v} \in K_r$ where $v_{i+1} - v_i \geq (1+q)n$ for exactly s indices $i_1, i_2, \ldots, i_s$ and note that always $s < r$.

Let $i_1, i_2, \ldots, i_s$ be the indices for which $v_{i_k+1} - v_{i_k} \geq n(1+q)$ for $k = 1, \ldots, s$. All the other differences we can by Lemma 14 assumed to be greater than $p_j = p(\varepsilon_j + \varepsilon_j^2/3)$ and are by assumption smaller than $(1+q)n$. Let $m_\ell = (v_{\ell+1} - v_\ell)$ be the return times. From now on we assume that $\ell \neq i_k$ ($k = 1, \ldots, s$). Hence we have

$m_\ell \in (p_j,(1+q)n)$. Put $m'_\ell = [m_\ell/(1+q)]$ and denote by $V''_{j,\ell}$ the union of those $A_{\psi_\ell} \in S'_{m'_\ell}$ which have non-empty intersection with W''_j. Clearly $W''_j \subset \tilde V''_{j,\ell} \cup \tilde V'''_{j,\ell}$, where $V'''_{j,\ell} = \bigcup_{\psi \in S''_{m'_\ell}} A_\psi$ and $\tilde V''_{j,\ell}, \tilde V'''_{j,\ell}$ are the associated extensions.

Using $\operatorname{diam} V''_{j,\ell} \leq \operatorname{diam} W''_j + \eta^{m'_\ell} \leq c_1\eta^{m'_\ell} + 2\varepsilon_j$ and Lemma 6 one obtains

$$\mu(\tilde V''_{j,\ell}) \leq (c_1\eta^{m'_\ell} + 2\varepsilon_j)^\delta + \mu(\Omega^c_{m'_\ell}) \leq 3c_1\eta^{m'_\ell\delta} + \alpha^{m'_\ell} \tag{13}$$

(where we assume without loss of generality that $\varepsilon_j < c_1\eta^{m'_\ell}$). Now we can use Lemma 9 (in the second inequality below), where we replace $d(\vec v)$ by 0 and use the identification: $W_{i_k} = \tilde W''_j$ for $k = 1, 2, \ldots, s-1$ and otherwise $W_\ell = \tilde V''_{j,\ell} \cup \tilde V'''_{j,\ell}$ if $\ell \neq i_k$ for $k = 1, \ldots, s-1$. Thus

$$\begin{aligned}
\mu\left(\bigcap_{i=1}^r T^{-v_i} W''_j\right) &\leq \mu\left(\bigcap_{k=1}^{s-1} T^{-v_{i_k}} \tilde W''_j \cap \bigcap_{v_\ell \neq v_{i_k},\, k=1,\ldots,s} T^{-v_\ell}(\tilde V''_{j,\ell} \cup \tilde V'''_{j,\ell})\right)\\
&\leq (1+C_7)\mu(\tilde W''_j)^s \prod_{v_\ell \neq v_{i_k},\, k=1,\ldots,s} (\mu(\tilde V''_{j,\ell}) + \mu(\tilde V'''_{j,\ell}))\\
&\leq c_2\mu(\tilde W''_j)^s \prod_{v_\ell \neq v_{i_k},\, k=1,\ldots,s} \left(\eta^{m'_\ell\delta} + \alpha^{m'_\ell} + \left(\frac{\rho}{\lambda}\right)^{m'_\ell}\right),
\end{aligned}$$

where in the last inequality we used (13) and the bound $\mu(\tilde V'''_{j,\ell}) \leq (\rho/\lambda)^{m'_\ell}$. Hence

$$\begin{aligned}
\mu(C''_{\vec v}) &\leq \mu\left(\bigcap_{i=1}^r T^{-v_i} \tilde W''_j\right) + r\mu(\tilde W'''_j)\\
&\leq c_3\mu(\tilde W''_j)^s \prod_{v_\ell \neq v_{i_k},\, k=1,\ldots,s} \varsigma'^{m_\ell} + r\mu(\tilde W'''_j),
\end{aligned}$$

where $\max(\eta^{\delta/(1+q)}, (\rho/\lambda)^{1/(1+q)}, \alpha^{1/(1+q)}) < \varsigma' < 1$.

For given values of the short return times m_ℓ ($\ell \neq i_k$, $k = 1, \ldots, s$), each of the s remaining entry times $\geq (1+q)n$ assumes no more than $t/\mu(B_{\varepsilon_j})$ values. Since the indices $i_1, \ldots, i_s$ can be picked in $\frac{r!}{s!(r-s)!}$ many ways, the number of possibilities is bounded by

$$\frac{r!}{s!(r-s)!}\left(\frac{t}{\mu(B_{\varepsilon_j})}\right)^s.$$

and therefore, since $\mu(\tilde W''_j) \leq \mu(B_{\varepsilon_j + \varepsilon_j^2/3}) + \mu(\Omega^c_n)$, we obtain

$$\sum_{\vec v \in K^s_r} \mu(C''_{\vec v}) \leq c_4 t^s \sum_{m_\ell} \left(\left(\frac{\mu(B_{\varepsilon_j+\varepsilon_j^2/3}) + \mu(\Omega^c_n)}{\mu(B_{\varepsilon_j})}\right)^s \prod_{v_\ell \neq v_{i_k},\, k=1,\ldots,s} \varsigma'^{m_\ell} + r\frac{\mu(\tilde W'''_j)}{\mu(B_{\varepsilon_j})^s}\right),$$

where the sum is over $m_{\ell_1}, \ldots, m_{\ell_{r-s}} \in [p_j, (1+q)n]$. To estimate this sum, let us note:

(i) The sum over the second term inside the brackets is crudely estimated using the fact that each of the short return times m_ℓ assumes no more than $(1+q)n$ values. Thus the summation has at most $((1+q)n)^{r-s} \le c_5 |\log \varepsilon_j|^{r-s}$ terms.

(ii) By (9) and choice of ε_j the ratio $\frac{\mu(B_{\varepsilon_j+\varepsilon_j^2/3})+\mu(\Omega_n^c)}{\mu(B_{\varepsilon_j})}$ (whose sth power appears within the first term inside the brackets) converges to 1 as $j \to \infty$.

(iii) The product which is part of the first term inside the brackets can be written as follows

$$\sum_{m_\ell} \prod_{v_\ell \neq v_{i_k},\, k=1,\ldots,s} \varsigma'^{m_\ell} = \sum_{m=(r-s)p_j}^{(r-s)(1+q)n} \varsigma'^m M_m,$$

where M_m is the number of integers $m_{\ell_1}, \ldots, m_{\ell_{r-s}} \in [p_j, (1+q)n]$ whose sum $m_{\ell_1} + \ldots + m_{\ell_{r-s}}$ equals m. For some constant $c_6 \sim 1/(r-s-1)!$ one has $M_m \le c_6 m^{r-s-1}$, which implies that

$$\sum_{m_\ell} \prod_{v_\ell \neq v_{i_k},\, k=1,\ldots,s} \varsigma'^{m_\ell} \le c_7 \varsigma'^{(r-s)p_j/2}$$

for some c_7.

We can thus estimate (note that $s \le r-1$)

$$\sum_{\vec{v} \in K_r^s} \mu(C''_{\vec{v}}) \le c_8 t^s \left(\frac{\mu(B_{\varepsilon_j+\varepsilon_j^2/3}) + \mu(\Omega_n^c)}{\mu(B_{\varepsilon_j})} \right)^s \varsigma'^{(r-s)p_j/2} + c_5 r |\log \varepsilon_j|^{r-s} \varepsilon_j^{1+(r-1)\delta'} \varepsilon_j^{-s\delta'}$$

which (for every value of $t > 0$) decays to zero. Therefore, as j goes to infinity,

$$\sum_{\vec{v} \in K_r} \mu(C''_{\vec{v}}) = \sum_{s=0}^{r-1} \sum_{\vec{v} \in K_r^s} \mu(C''_{\vec{v}}) \to 0.$$

The statement for the inner approximation is now evident because $\mu(C'_{\vec{v}}) \le \mu(C''_{\vec{v}})$ for all $\vec{v}$. $\square$

We can summarise the previous two lemmas as follows:

Lemma 16 *For almost every $x \in J$ and ε small enough one has (for both approximations)*

$$\lim_{n \to \infty} \sum_{\vec{v} \in R_r} b_{\vec{v}}^{*n} = 0.$$

The next lemmas serve to verify the remaining conditions (4), (5), (7) and (8) of Proposition 13 for both approximations, inner and outer.

Lemma 17 $\lim_{n\to\infty}\ \max_{1\le v\le N} b_v^{*n} = 0$.

Proof. Since (for both cases: $^* ='$ and $^* =''$)

$$b_v^{*n} \le \mu(\chi_{W_j''\cup W_j'''}\circ T^v) = \mu(W_j''\cup W_j''') \to 0,$$

as j, and therefore n, goes to infinity. Hence every b_v^n vanishes as n goes to infinity. □

Lemma 18 $\lim_{n\to\infty}\ \sum_{v=1}^{n} b_v^{*n} = t$.

Proof. Since $N = t/\mu(B_{\varepsilon_j})$ we obtain for the inner approximation

$$\sum_{v=1}^{N} b_v'^n = \sum_{v=1}^{N} \mu(\chi_{W_j'}\circ T^v) = N\mu(W_j') = t\left(1 - \frac{\mu(B_{\varepsilon_j}\setminus W_j')}{\mu(B_{\varepsilon_j})}\right) \to t$$

by construction of the sets W_j' and equation (11). Similarly for the outer approximation:

$$\sum_{v=1}^{N} b_v''^n = N\mu(W_j''\cup W_j''') = t\left(1 + \frac{\mu(W_j''\cup W_j'''\setminus B_{\varepsilon_j})}{\mu(B_{\varepsilon_j})}\right) \to t,$$

by construction of the sets W_j'', W_j''' and (12). □

Lemma 19

$$\lim_{n\to\infty} \sum_{\vec{v}\in R_r} b_{v_1}^* b_{v_2}^* \cdots b_{v_r}^* = 0$$

Proof. For the inner approximation: since $b_{v_s}' = \mu(W_j')$ for all s by invariance of the measure μ, we obtain

$$b_{v_1}' b_{v_2}' \cdots b_{v_r}' = \mu(W_j')^r \le \mu(B_{\varepsilon_j})^r.$$

To estimate the cardinality of $R_r(N)$ note $r-1$ indices each has at most N choices while the remaining one has at most $(r-1)(1+q)n$ choices. This implies

$$|R_r(N)| \le (r-1)(1+q)nN^{(r-1)}$$

and that ($c_1 > 0$)

$$\begin{aligned}\sum_{\vec{v}\in R_r} b'_{v_1}\cdots b'_{v_r} &\le |R_r(N)|\mu(B_{\varepsilon_j})^r \\ &\le (r-1)(1+q)nt^{r-1}\mu(B_{\varepsilon_j}) \\ &\le c_1\mu(B_{\varepsilon_j})|\log\varepsilon_j|\end{aligned}$$

converges to zero as $j\to\infty$.

For the outer approximation we get ($c_2, c_3 > 0$) by equation (12)

$$\begin{aligned}\sum_{\vec{v}\in R_r} b''_{v_1}\cdots b''_{v_r} &\le |R_r(N)|\mu(W''_j\cup W'''_j)^r \\ &\le |R_r(N)|\mu(B_{\varepsilon_j})^r\left(1+\frac{\mu(W''_j\cup W'''_j\setminus B_{\varepsilon_j})}{\mu(B_{\varepsilon_j})}\right)^r \\ &\le c_2 n\mu(B_{\varepsilon_j}) \\ &\le c_3\mu(B_{\varepsilon_j})|\log\varepsilon_j|\end{aligned}$$

converges to zero as $j\to\infty$. □

Lemma 20 *For both approximations:*

$$\lim_{n\to\infty}\frac{b^{*n}_{v_1}\cdots b^{*n}_{v_r}}{b^{*n}_{\vec{v}}} = 1,$$

uniformly for $\vec{v}\in G_r(N)\setminus R_r(N)$.

Proof. If $\vec{v}\notin R_r(N)$ then $v_{s+1}-v_s\ge(1+q)n$ for all $s=1,\dots,r-1$ and we obtain the desired result by lemma 9. (As $\varsigma<1$ the convergence is exponential.) □

It follows from Sevast'yanov's theorem (Proposition 13) that both processes ξ'_n, ξ''_n are in the limit Poisson distributed. Since

$$\xi'_j\le\xi_{\varepsilon_j},\quad \xi^\circ_{\varepsilon_j}\le\xi''_j,$$

where by equation (10)

$$\mu(\{y\in J:\xi_{\varepsilon_j}(y)\ne\xi^\circ_{\varepsilon_j}(y)\})\le\mu(|\xi_{\varepsilon_j}-\xi^\circ_{\varepsilon_j}|)\to 0.$$

Since

$$\begin{aligned}\mu(\{y\in J:\xi'_{\varepsilon_j}(y)=r\}) &\le \mu(\{y\in J:\xi_{\varepsilon_j}(y)=r\}) \\ &\le \mu(\{y\in J:\xi''_{\varepsilon_j}(y)=r\}) \\ &\quad +\mu(\{y\in J:\xi_{\varepsilon_j}(y)\ne\xi^\circ_{\varepsilon_j}(y)\}),\end{aligned}$$

we conclude that ξ_{ε_j} converges almost everywhere to a Poisson distribution as $j \to \infty$. This proves Theorem 12. $\square$

Remark 1: In the requirement that $\frac{\mu(B_{\varepsilon+\varepsilon^2})}{\mu(B_\varepsilon)} \to 1$ the square ε^2 can be replaced by any (finite) power of ε. The result will still hold true.

Remark 2: If in Lemma 11 one can replace "lim inf" by "lim sup" then the statement in Theorem 12 generalises to

$$\mu(\mathcal{N}_{r,\varepsilon}) \to \frac{t^r}{r!}e^{-t},$$

as $\varepsilon \to 0$ for almost every $x \in J$.

This is always true if the Hausdorff dimension of μ is larger than 1. To see this observe that the shell $B_{\varepsilon+\varepsilon^2} \setminus B_\varepsilon$ can be covered by at most $2\pi/\varepsilon$ balls with radii ε^2 each of which has μ-measure $\leq \varepsilon^{2\delta}$. On the other hand $\mu(B_\varepsilon) \geq \varepsilon^{\delta'}$. Hence for the measure of the shell we get the ratio:

$$\frac{\mu(B_{\varepsilon+\varepsilon^2} \setminus B_\varepsilon)}{\mu(B_\varepsilon)} \leq \frac{2\pi}{\varepsilon}\varepsilon^{2\delta}\varepsilon^{-\delta'} = 2\pi\varepsilon^{2\delta-1-\delta'},$$

where the exponent $2\delta - 1 - \delta'$ can be made positive for suitable choices of δ, δ', $1 < \delta < HD(\mu) < \delta'$. Thus the ratio goes to zero as $\varepsilon \to 0$ almost everywhere if $HD(\mu) > 1$.

Remark 3: Finally, instead of considering balls as return sets, let us consider the case when the return sets are 'cylinders'. For a point x and integer $n > 1$ we can find a $A_n(x) \in \tilde{\mathcal{A}}^n$ so that $x \in A_n(x)$. We denote by χ_n the characteristic function of $A_n(x)$. Then we can consider the 'random variable'

$$\zeta_n = \sum_{k=0}^{[t/\mu(A_n(x))]} \chi_n \circ T^k.$$

The value of ζ_n measures the number of times a point returns to the set $A_n(x)$ within the normalised time $t/\mu(A_n(x))$.

Corollary 21 *For μ-almost every x,*

$$\mu(\mathcal{N}_r) \to \frac{t^r}{r!}e^{-t},$$

as n tends to infinity, where $\mathcal{N}_r = \{y \in J : \zeta_n(y) = r\}$ is the r-levelset of ζ_n.

Remark 4: We can define the point process $P_{\varepsilon,x,y}$ on $\mathbf{R}^+$ by

$$P_{\varepsilon,x,y}[B] = \sum_{k=0}^{[t/\mu(A_\varepsilon]} \delta_{\tau_\varepsilon^k(y)/\mu(A_\varepsilon)}[B],$$

B a Borel set on $\mathbf{R}^+$, where δ is the unitmass and τ^k is the k-th return time, defined by $\tau^1 = \tau$ and inductively by $\tau^k = \tau^{k-1} + \tau \circ T^{\tau^{k-1}}$ for $k > 1$. A consequence of Theorem 12 is then the following convergence result ($|\cdot|$ denotes the Lebesgue measure on $\mathbf{R}$):

Corollary 22 *For almost every $x \in J$ there exists a sequence $\varepsilon_j \to 0$, so that for every Borel set $B \subset \mathbf{R}^+$*

$$\mu\left(\left\{y \in J : P_{\varepsilon_j,x,y}[B] \in B\right\}\right) \to \frac{|B|^r}{r!} e^{-|B|}$$

6 T is Weakly Bernoulli

Here we shall use the inverse branches for the iterates of T as introduced by Denker and Urbanski. Of particular interest is property (b) which will be needed in equation (15) below.

Lemma 23 *[2] Let $\epsilon > 0$ and $\lambda \in (0,1)$. Then there exists a quasidisk Ω, an integer m and inverse branches $\hat{S}_{jm}$ of T^{jm}, $j = 1, 2, \ldots$, on Ω so that*
(a) $\mu(\bigcup_{\varphi \in \hat{S}_{jm}} \varphi(J)) \geq 1 - \epsilon$ for all j.
(b) $|(T^{km}\varphi)'| \leq C_9 \lambda^{j-k}$ for $k = 0, 1, \ldots, j$ (some C_9), for $\varphi \in \hat{S}_{jm}$.
(c) there exists an integer s and a metric generator $\mathcal{G}^{sm}$ of T for μ so that $\hat{\mathcal{A}}^{sm} \subset \mathcal{G}^{sm}$, where $\hat{\mathcal{A}}^{sm} = \{\varphi(J) : \varphi \in \hat{S}_{sm}\}$

Let us note that Lemmas 7 to 9 remain valid if one uses the branches $\hat{S}_*$ instead of the contracting branches S'_*. Put $\hat{\mathcal{A}}^n = \{\varphi(J) : \varphi \in \hat{S}_n\}$, where we assume that n and p, q below are multiples of m, once m has been determined.

We shall show that the rational map T is weakly Bernoulli for the partitions $\hat{\mathcal{A}}^n$ for suitable but arbitrarily large n.

Theorem 24 *For every $\varepsilon > 0$ there exists a k_0 so that there are arbitrarily large u, v for which*

$$\sum_{U \in \mathcal{G}^u, V \in \mathcal{G}^v} \left|\mu(U \cap T^{-u-k}V) - \mu(U)\mu(V)\right| \leq \varepsilon,$$

for all $k \geq k_0$.

Proof. Let $\varepsilon > 0$, put in Lemma 23 $\epsilon = \varepsilon/6$ and determine m accordingly. Let $u \in \mathbf{N}$ be given and assume for simplicity's sake that v is a multiple of m. Let $r \leq u$ be a multiple of m and assume that the pressure of f is zero, that is, μ is e^f-conformal. Since U is a "u-cylinder", $T^r U$ is a "$(u-r)$-cylinder", i.e. $T^r U \in \hat{\mathcal{A}}^{u-r}$, and we obtain by Lemma 8

$$\left|\mu(T^r(U \cap T^{-k-u}V)) - \mu(T^r U)\mu(V)\right| \leq c_1 \nu^{u-r} \sigma^k \mu(T^r U)\mu(V). \tag{14}$$

If $P = \varphi(J)$ with $\varphi \in \hat{S}_q$ then we have by Lemma 23 (b):

$$|f^r(x) - f^r(x')| \leq c_2 \eta^{\alpha(u-r)}, \tag{15}$$

for some $c_2 > 0$ and all $x, x' \in U$, where f^r is the r-ergodic sum of f. Hence in (14) we can replace on both sides μ by $e^{-f^r}\mu$ (but not in the $\mu(V)$ term) without making an error larger than $c_2 \eta^{\alpha(u-r)} \mu(U)\mu(V)$ and use the fact that μ is e^f-conformal to obtain

$$\begin{aligned} \left|\mu(U \cap T^{-k-u}V) - \mu(U)\mu(V)\right| &\leq \left(c_1 \nu^{u-r}\sigma^k + c_2 \eta^{\alpha(u-r)}\right)\mu(U)\mu(V) \\ &\leq c_3 \sigma'^k \mu(U)\mu(V), \end{aligned}$$

for a suitable choice of r, and where $\sigma' = \max(\sqrt{\sigma}, \eta^{\frac{\alpha \log \sigma}{2 \log \nu}})$. Summing over $U \in \{\varphi(J) : \varphi \in \hat{S}_u\}$, $V \in \{\varphi(J) : \varphi \in \hat{S}_v\}$ yields

$$\sum_{U \in \hat{\mathcal{A}}^u, V \in \hat{\mathcal{A}}^v} \left|\mu(U \cap T^{-u-k}V) - \mu(U)\mu(V)\right| \leq \frac{\varepsilon}{3}$$

for all $k \geq k_0$ where k_0 chosen so that $c_3 \sigma'^{k_0} \leq \varepsilon/3$.

Since property (a) of Lemma 23 yields

$$\sum_{U \in \mathcal{G}^u \setminus \hat{\mathcal{A}}^u} \mu(U) \leq \frac{\varepsilon}{6}$$

we get

$$\sum_{U \in \mathcal{G}^u \setminus \hat{\mathcal{A}}^u, V \in \mathcal{G}^v} \left|\mu(U \cap T^{-u-k}V) - \mu(U)\mu(V)\right| \leq \frac{\varepsilon}{3}$$

(similarly for the sum over $U \in \mathcal{G}^v, V \in \mathcal{G}^u \setminus \hat{\mathcal{A}}^v$). This proves the theorem.

$\square$

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