PHASE TRANSITIONS IN ONE-DIMENSIONAL SUBSHIFTS

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Abstract. In this note we give simple examples of one-dimensional mixing subshift with positive topological entropy which have two distinct measures of maximal entropy. We also give examples of subshifts which have two mutually singular equilibrium states for Hölder continuous functions. We also indicate how the construction can be extended to yield examples with any number of equilibrium states.

1. **Introduction.** It is a well known fact that a wide class of uniformly hyperbolic transformations that satisfy some irreducibility condition have unique measures of maximal entropy. Irreducible subshifts have unique measures of maximal entropy if they are for instance of finite type or sofic (factors of shifts of finite type). More general hyperbolic systems which satisfy specification have unique measures of maximal entropy [1]. In either case the systems have a 'uniform' mixing property. Hofbauer had shown in [6] that even subshifts of finite type don't necessarily have unique equilibrium states. However the potentials in the provided examples lack of course regularity and one of the equilibrium states is supported on a single fixed point. On infinite alphabets there are examples of Markov shifts that have many Gibbs states although all are non-invariant [4] with only one exception. For a large class of Hölder continuous functions we get here multiple equilibrium states which are supported on embedded subshifts.

Here we produce a simple example which shows that even in the case of an irreducible subshift with positive entropy one cannot generally expect to have a unique measure of maximal entropy. Also Krieger [7] gave an example with a non-unique measure of maximal entropy—although not as simple and straightforward as this one. For two-dimensional subshifts of finite type, Burton and Steiff [2, 3] have given examples whose measures of maximal entropy form simplices that consist of more than single points.

In section 2 we give an example of a subshift which has two measures of maximal entropy. We then extend it to examples which can have any number of measures of

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maximal entropy including one with infinitely many. In the third section we construct an example that has two equilibrium states for Hölder continuous potentials which we then extend to examples with any number of equilibrium states and also one with infinitely many.

2. **Measures of maximal entropy.** We will first describe an example with two distinct ergodic measures of maximal entropy and then show how to construct an example with any number of distinct ergodic measures including one with infinitely many.

Let $\nu \geq 2$ be some positive integer and let us construct a subshift over the symbol set $\mathcal{A} = \{0, 1, 2, \ldots, 2\nu\}$. We shall refer to the symbols $\{1, 2, \ldots, \nu\}$ as the green symbols and denote them by $\mathcal{G}$. Similarly we call $\{\nu+1, \ldots, 2\nu\}$ the yellow symbols and write $\mathcal{Y}$ for them. Hence $\mathcal{A} = \{0\} \cup \mathcal{G} \cup \mathcal{Y}$. Let us denote by $\Sigma_{\mathcal{G}} = \mathcal{G}^{\mathbf{Z}}$ the (green) full ν-shift over the alphabet $\mathcal{G}$. Similarly we write $\Sigma_{\mathcal{Y}}$ for the (yellow) full ν-shift $\mathcal{Y}^{\mathbf{Z}}$.

Let $\tau > 0$. The subshift $\Sigma \subset \mathcal{A}^{\mathbf{Z}}$ will consist of the union of the two monochromatic shift spaces $\Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{Y}}$, to which we add bicoloured sequences in the following way: A sequence in Σ in which a word α of one colour (green or yellow) is followed by a word β of the other colour (yellow or green) will have a string of zeroes γ separating the two words α and β. The length $|\gamma|$ of that string of zeros is then at least, $\tau(a + b)$, where $a = |\alpha|, b = |\beta|$ are the lengths of the coloured words to the 'left' and 'right' of γ. If in particular the word α or β is infinitely long then it can only be followed by an infinite string of zeros and cannot 'transition' to a string of another colour.

More precisely:

Definition 2.1. The shift space $\Sigma \subset \mathcal{A}^{\mathbf{Z}}$ is defined as follows: We have $x \in \Sigma$ if
(i) either $x \in \Sigma_{\mathcal{G}} \cup \Sigma_{\mathcal{Y}}$, in which case we call x a *monochromatic* point.
(ii) or if bi-coloured blocks are of the form:

$$\ldots 0 g_1 g_2 \ldots g_{a-1} g_a 0^{\lambda} y_1 y_2 \ldots y_{b-s} y_b 0 \ldots$$

or

$$\ldots 0 y_1 y_2 \ldots y_{a-1} y_a 0^{\lambda} g_1 g_2 \ldots g_{b-s} g_b 0 \ldots$$

where $\lambda \geq \tau(a + b)$ and $g_i \in \mathcal{G}, y_i \in \mathcal{A}$ (0^{λ} is a string of zeros of length λ), $\tau > 0$.

If $\omega = \omega_m \omega_{m+1} \ldots \omega_{m+n-1}$ is an allowed word of some length n in Σ, then we denote by $U(\omega)$ the *cylinder set* $\{x \in \Sigma : x_i = \omega_i, m \leq i < m+n\}$. The shift transformation σ on Σ is defined in the usual way by $\sigma(x)_i = s_{i+1}, i \in \mathbf{Z}$, for $x \in \Sigma$. Notice that the transition rules are shift invariant. Thus Σ is indeed a subshift which we endow with the usual topology generated by the cylinder sets $U(\omega)$, where ω runs over all finite words in Σ.

If $\tau = 0$ then Σ is a sofic system in which every string of zeroes connects to coloured symbols of different colours, and whose topological entropy is $2 \log \nu$ (see Lemma 2.3).

Lemma 2.2. *The shift transformation σ on Σ is topologically mixing for every $\tau > 0$.*

Proof. We have to show that for any two finite words ω and η there exists a number N so that $U(\omega) \cap \sigma^n(U(\eta))$ is non-empty for all $n \geq N$. This is evidently true if ω and η are both monochromatic of the same colour. If the last symbol of η and the first symbol of ω are of different colour then consider the word $\pi = \eta 0^{\lambda} \omega$.

Clearly, π is an admissable word for every $\lambda > \tau(|\eta| + |\omega|)$, and the cylinder set $U(\pi) \subset U(\omega) \cap \sigma^n(U(\eta))$ is non-empty for all $n \geq N$, where N can then be chosen to be equal to $(1 + \tau)(|\eta| + |\omega|)$.

If the last symbol of η and the first symbol of ω are of the same colour, then let us consider the word $\pi = \eta 0^\kappa \varepsilon 0^\lambda \omega$, where ε is a symbol (or some other short word) of the other colour. Here we have $\kappa \geq \tau(|\eta| + |\varepsilon|)$ and $\lambda \geq \tau(|\omega| + |\varepsilon|)$. Clearly $U(\pi)$ is non-empty for those choices of λ and κ. Thus $U(\omega) \cap \sigma^n(U(\eta)) \neq \emptyset$ if $n \geq N = (\tau + 1)(|\eta| + 2|\varepsilon| + |\omega|)$. $\square$

Lemma 2.3. *If $\tau \geq \frac{\log 3}{\log \nu}$ then the topological entropy h of the shift $\sigma : \Sigma \to \Sigma$ is equal to $\log \nu$.*

Proof. Let us first get a lower bound on the topological entropy h of Σ. For that purpose let us observe that since Σ contains two full ν-shifts, $\Sigma_\mathcal{Y}$ and $\Sigma_\mathcal{G}$, the topological entropy of Σ must be at least $\log \nu$. Let us now estimate the number of the remaining words of length n from above. We have two cases to consider:

(i) Monochromatic words that might also contain zeroes. According to our rules such words begin or end with strings of zeroes. Thus we obtain purely yellow and green words of lengths $k = 0, \ldots, n$ which on at least one side are framed by strings of zeroes. Their number turns out to be $2 \sum_{k=0}^n (n - k) \nu^k \leq 2n\nu^n \sum_{k=0}^n \nu^{-k}$ which has exponential growth rate $\log \nu$.

(ii) To estimate the number of words of length n that genuinely contain symbols of both colours, let us observe that since any such word has at least one transition from yellow to green or vice versa, it therefore must also contain at least $n\frac{\tau}{\tau+1}$ zeroes, that is at most $n' = [\frac{n}{\tau+1}]$ ([ ] denotes integer part) coloured symbols (i.e. symbols in $\mathcal{A} \setminus \{0\} = \{1, 2, \ldots, 2\nu\}$). The coloured symbols come in monochromatic blocks of alternating colour. Denote by $P_{k,\ell}$ the number of possibilities in which one can arrange ℓ symbols in k blocks (separated by the appropriate number of zeroes), where $k = 1, 2, \ldots, \ell$. One finds that

$$P_{k,\ell} = \left(\begin{array}{c} \ell - 1 \\ k - 1 \end{array} \right),$$

which is the number of possibilities of picking the first element of every block but the very first one. The number of n-words in Σ which contain ℓ coloured symbols arranged in $k \leq \ell$ monochromatic blocks of alternating colour is $2P_{k,\ell}\nu^\ell$.

Distributing ℓ coloured symbols out of n' in k blocks can be done in $P_{k,\ell}$ many ways, where $1 \leq k \leq \ell \leq n'$. This leaves $m = n - (\tau + 1)\ell$ zeroes to be distributed on $k + 1$ intervals, namely the $k - 1$ gaps between the blocks of coloured symbols plus the two ends of the entire word. There are

$$Q_{k,m} = \left(\begin{array}{c} m + k \\ k \end{array} \right)$$

many possibilities. By Stirling's formula we have

$$
\begin{aligned}
Q_{k,m} &= \left(\begin{array}{c} n - (\tau + 1)\ell + k \\ k \end{array} \right) \\
&\leq \left(\begin{array}{c} n - (\tau + 1)\ell + k \\ \frac{1}{2}(n - (\tau + 1)\ell + k) \end{array} \right) \\
&\leq c_1 2^{n-(\tau+1)\ell+k} \sqrt{n - (\tau + 1)\ell + k} \\
&\leq c_1 2^{n-(\tau+1)\ell+k} \sqrt{n}.
\end{aligned}
$$

We can thus estimate the total number of bi-coloured strings of length n by

$$
\begin{aligned}
Q(n) &\leq 2\sum_{\ell=2}^{n'}\nu^\ell\sum_{k=2}^\ell\binom{\ell-1}{k-1}\binom{n-(\tau+1)\ell+k}{k} \\
&\leq 2c_1\sqrt{n}\sum_{\ell=2}^{n'}\nu^\ell 2^{n-(\tau+1)\ell}\sum_{k=2}^\ell\binom{\ell-1}{k-1}2^k \\
&\leq 2c_1\sqrt{n}\sum_{\ell=2}^{n'}\nu^\ell 2^{n-(\tau+1)\ell}3^\ell \\
&\leq c_2\sqrt{n}\begin{cases} 2^n & \text{if}\quad 3\nu 2^{-1-\tau}<1 \\ 2^n(3\nu 2^{-1-\tau})^{\frac{n}{\tau+1}} & \text{if}\quad 3\nu 2^{-1-\tau}\geq 1 \end{cases}
\end{aligned}
$$

Thus

$$
\lim_{n\to\infty}\frac{1}{n}\log Q(n)\leq\max\left(\log 2,\frac{\log 3\nu}{\tau+1}\right)
$$

which is $\leq\log\nu$ if $\tau\geq\frac{\log 3}{\log\nu}$.   □

Example 1. If $\tau\geq\frac{\log 3}{\log\nu}$, then there are two mutually singular measures of maximal entropy on Σ.

Proof. It is well known [8] that the measure of maximal entropy on the full ν-shift $\{1,\ldots,\nu\}^{\mathbf{Z}}$ is the Bernoulli measure with the probability vector $(\frac{1}{\nu},\ldots,\frac{1}{\nu})$. It's metric entropy is $\log\nu$. Let us define on Σ a Bernoulli measure $\mu_{\mathcal{Y}}$ which is given by the probability vector $(0,\frac{1}{\nu},\ldots,\frac{1}{\nu},0,\ldots,0)$ and the measure $\mu_{\mathcal{G}}$ with the probabilities $(0,0,\ldots,0,\frac{1}{\nu},\ldots,\frac{1}{\nu})$. Evidently both measures are shift invariant and have metric entropies $\log\nu$ which by Lemma 2.3 is the topological entropy of Σ. Hence $\mu_{\mathcal{Y}}$ and $\mu_{\mathcal{G}}$ (as well as all their linear combinations) are distinct ergodic measures of maximal entropy for the subshift Σ.   □

2.1. Example with $L\geq 2$ distinct ergodic measures of maximal entropy.
Let $\nu\geq 2$ be an integer and consider the alphabet $\mathcal{A}=\{0,1,2,\ldots,L\nu\}$. We label the embedded full ν-shift spaces by $\Sigma_j=\{(j-1)\nu+1,(j-1)\nu+2,\ldots,j\nu\}^{\mathbb{Z}}$, $j=1,2,\ldots,L$. A transition from a word α in Σ_j happens only to a word β of a 'different colour' if it is separated by a string of zeros of length $\geq\tau(|\alpha|+|\beta|)$ and β lies either in Σ_{j-1} or Σ_{j+1}. One sees, as was proven in Lemma 2.2 for $L=2$, that the resulting shift space $\Sigma\subset\{0,1,2,\ldots,L\nu\}^{\mathbb{Z}}$ is topologically mixing.

Example 2. If $\nu\geq 2$ and $\tau\geq\frac{\log 5}{\log\nu}$ then there are $L\geq 2$ distinct ergodic measures of maximal entropy on Σ.

Proof. The topological entropy of Σ is at least $\log\nu$ as this is the topological entropy of the embedded subshifts Σ_j. To get an upper bound on the topological entropy of Σ we estimate the number $Q(n)$ of words of length n that are not 'monochromatic', i.e. don't belong to a single Σ_j. We proceed as before and obtain:

$$
Q(n)\leq L\sum_{\ell=2}^{n'}\nu^\ell\sum_{k=2}^\ell\binom{\ell-1}{k-1}2^{k-1}\binom{m+k}{k},
$$

where $m=n-[(\tau+1)\ell]$, ℓ counts the number of 'coloured' symbols (i.e. symbols in $\mathcal{A}\setminus\{0\}$), $k-1$ is the number of changes of colour that is where there is a transition

from a word in some Σ_j to $\Sigma_{j\pm 1}$ (the factor 2^{k-1} accounts for number of possible switches in colour). As before we thus get (with $n' = [n/(\tau + 1)]$)

$$
\begin{aligned}
Q(n) &\leq L\sqrt{n}c_1 \sum_{\ell=2}^{n'} \nu^\ell 2^{n-(\tau+1)\ell} \sum_{k=2}^{\ell} \binom{\ell-1}{k-1} 2^k 2^{k-1} \\
&\leq 2L\sqrt{n}c_1 \sum_{\ell=2}^{n'} 2^n (5\nu 2^{-\tau-1})^\ell \\
&\leq c_2\sqrt{n} \begin{cases} 2^n & \text{if } 5\nu 2^{-\tau-1} < 1 \\ 2^n (5\nu 2^{-\tau-1})^{\frac{n}{\tau+1}} & \text{if } 5\nu 2^{-\tau-1} \geq 1 \end{cases}.
\end{aligned}
$$

Consequently we get that

$$
Q(n) \leq c_2\sqrt{n}\, 2^n \left(1 + (5\nu 2^{-\tau-1})^{\frac{n}{\tau+1}}\right)
$$

and thus

$$
\lim_{n\to\infty} \frac{1}{n} \log Q(n) \leq \max\left(\log 2, \frac{\log 5\nu}{1+\tau}\right).
$$

Hence, if $\tau \geq \frac{\log 5}{\log \nu}$ then $h_{\text{top}}(\Sigma) = \log \nu$. It is now clear how we get the L distinct ergodic measures of maximal entropy: they are the L Bernoulli measures μ_j with weights $p_k^j = \frac{1}{\nu}$ for $k = (j-1)\nu + 1, (j-1)\nu + 2, \ldots, j\nu$ and $p_k^j = 0$ for all other k ($j = 1, 2, \ldots, L$). These measures all mutually singular and have metric entropy $\log \nu$ i.e. they are measures of maximal entropy. $\qquad\square$

2.2. **Example with infinitely many distinct ergodic measures of maximal entropy.** Let $\nu \geq 2$. We now let L go to infinity and obtain a subshift Σ over the alphabet $\mathcal{A} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. There are infinitely many embedded full ν-shift

$$
\Sigma_j = \{(j-1)\nu + 1, (j-1)\nu + 2, \ldots, j\nu\}^{\mathbb{Z}},
$$

for $j = 1, 2, \ldots$. As before we only transition from a word α in Σ_j a word β in $\Sigma_{j\pm 1}$ by keeping them separated by a string of zeros of length $\geq \tau(|\alpha| + |\beta|)$. Again the resulting shift space Σ is topologically mixing but it is not a compact space. Here we use the Gurevich entropy [5] which is defined by $h_{\text{top}} = \lim_{n\to\infty} \frac{1}{n} \log N(n)$, where $N(n)$ is the number of n-words $\alpha_1\alpha_2\cdots\alpha_n$ which begin with a fixed symbol, that is $\alpha_1 = a$ for a fixed $a \in \mathcal{A}$. The entropy h_{top} is independent of the choice of a. One can also consider the one-point-compactification $\bar{\Sigma} = \Sigma \cup \{\infty\}$ and a suitable metric which in particular implies that neighbourhoods of ∞ contain the embedded subshifts Σ_j for large enough j. The topological entropy can then be defined by (ε, n)-separated sets[1] $E_{\varepsilon,n}$ (see e.g. [8]) and is given by $h_{\text{top}}(\Sigma) = \lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \frac{\log |E_{\varepsilon,n}|}{n}$.

Example 3. If $\nu \geq 2$ and $\tau \geq \frac{\log 5}{\log \nu}$ then there are infinitely many distinct ergodic measures of maximal entropy on Σ.

Proof. Evidently $h_{\text{top}}(\Sigma) \geq \log \nu$. Moreover if we denote by $R_L(n)$ the number of words of length n in Σ whose first coordinate lie in $\mathcal{A}_L = \{0, 1, \ldots, L\nu\}$ then we get that $h_{\text{top}}(\Sigma) = \lim_{L\to\infty} \lim_{n\to\infty} \frac{\log R_L(n)}{n}$. We have that

$$
R_L(n) \leq L\nu^n + Lc_3\sqrt{n} \begin{cases} 2^n & \text{if } 5\nu 2^{-\tau-1} < 1 \\ 2^n (5\nu 2^{-\tau-1})^{\frac{n}{\tau+1}} & \text{if } 5\nu 2^{-\tau-1} \geq 1 \end{cases}.
$$

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[1] The metric on a shift space with infinite alphabet uses the distance between symbols $d_s(a, b) = \left|\frac{1}{a+1} - \frac{1}{b+1}\right|$ for $a, b \in \mathcal{A} = \mathbb{N}_0$. Then $d(\vec{x}, \vec{y}) = \sum_j \vartheta^{|j|} d_s(x_j, y_j)$ for some $\vartheta \in (0, 1)$.

and consequently

$$\lim_{n \to \infty} \frac{1}{n} \log R_L(n) \leq \max\left(\log \nu, \frac{\log 5\nu}{1+\tau}\right).$$

Now, letting L go to infinity we obtain $h_{\text{top}}(\Sigma) \leq \max\left(\log \nu, \frac{\log 5\nu}{1+\tau}\right)$ which is equal to $\log \nu$ if $\tau \geq \frac{\log 5}{\log \nu}$ then $h_{\text{top}}(\Sigma) = \log \nu$. The equal weight Bernoulli measures on the spaces Σ_j are all measures of maximal entropy and are mutually singular.   □

3. Equilibrium states.

Let us construct now subshifts with non unique equilibrium states for Hölder continuous potentials. We say a function $f : \Sigma \to \mathbb{R}$ is Hölder continuous on the shift space Σ if there exists a $\vartheta \in (0,1)$ and a constant C so that for every $n > 0$ one has

$$\sup |f(\vec{x}) - f(\vec{y})| \leq C\vartheta^n$$

where the supremum is over all pairs of sequences $\vec{x} = \cdots x_{-1}x_0x_1x_2\ldots, \vec{y} = \cdots y_{-1}y_0x_1y_2 \cdots \in \Sigma$ for which $x_i = y_i \forall |i| < n$ and either $x_{-n} \neq y_{-n}$ or $x_n \neq y_n$.

3.1. Example with two distinct equilibrium states.

Let $\tau > 0$ and $\nu \geq 3$ and integer. Then we put as above $\mathcal{A} = \{0\} \cup \mathcal{G} \cup \mathcal{Y} = \{0, 1, \ldots, 2\nu\}$ and define as above the subshift $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ which has the two embedded full ν-shifts Σ_*, $* = \mathcal{G}, \mathcal{Y}$ and where the transitions are strings of zeros of lengths $\geq \tau(|\alpha| + |\beta|)$ with α, β being the adjacent monochromatic words of different colours.

Let f be a real valued Hölder continuous function on the full shift $\Sigma_{\mathcal{G}}$ so that $P_{\mathcal{G}}(f) > \sup f + \log 2$, where $P_{\mathcal{G}}(f)$ is the pressure of f on the full shift $\Sigma_{\mathcal{G}}$ (see [8]). For $\nu \geq 3$ this is possible since $P_{\mathcal{G}}(f) - \sup f$ is bounded above by $\log \nu$ (and equal to $\log \nu$ if f is cohomologous to a constant, i.e. $f = u \circ \sigma - u + c$ for a Hölder continuous u) and the pressure function is continuous.

Let us define a Hölder continuous function g on Σ in two steps. We first extend the function to $\Sigma_{\mathcal{Y}}$ by identification: that is $g(\vec{y}) = f(\vec{x})$ for $\vec{x} \in \Sigma_{\mathcal{G}}, \vec{y} \in \Sigma_{\mathcal{Y}}$ for which $y_i = x_i + \nu \forall i$. Now we extend g to bi-coloured words in the following way: If $x_0 = 0$ then we put $g(\vec{x}) = \sup f$. If $x_0 \neq 0$ and $\vec{x} \in \Sigma$ is a sequence which contains symbols of both colours then let $x_{-n} \cdots x_n$ be the longest monochromatic string of either green or yellow symbols (that is $x_{-n} \cdots x_n$ is a word in either $\Sigma_{\mathcal{G}}$ or $\Sigma_{\mathcal{Y}}$ and either $x_{-n-1} = 0$ or $x_{n+1} = 0$). We then pick a monochromatic point $\vec{y} \in U(x_{-n} \cdots x_n) \subset \Sigma_*$ $(* = \mathcal{G}, \mathcal{Y})$ and define $g(\vec{x}) = f(\vec{y})$. In this way g is a Hölder continuous function on Σ. Let us now prove that the pressure of g (on Σ) is in fact equal to $P_{\mathcal{G}}(f)$ (which equals $P_{\mathcal{Y}}(g)$) for a suitable choice of τ. Let us note that if f is not cohomologous to a constant on $\Sigma_{\mathcal{G}}$ then the extension g will also not be cohomologous to a constant on Σ.

Lemma 3.1. *If $\tau \geq \frac{\log 4\nu}{P_{\mathcal{G}}(f) - \sup f} - 1$, then the pressure $P(g)$ of g on Σ is equal to $P_{\mathcal{G}}(f)$.*

Proof. Denote by W_n the collection of Σ-words $\alpha = \alpha_1 \cdots \alpha_n$ of lengths n. The pressure of g is then given by the exponential growth rate of the partition function

$$Z_n = \sum_{\alpha \in W_n} e^{g^n(x(\alpha))}.$$

The point $x(\alpha) \in \Sigma$ is an arbitrary point in the cylinder set $U(\alpha) \subset \Sigma$, and $g^n = g + g\sigma + g\sigma^2 + \cdots + g\sigma^{n-1}$ is the n-th ergodic sum of g. If in the above sum we restrict to monochromatic words α of either colour, then we obtain the green or

yellow partition function $Z_n^* = \sum_{\alpha \in W_n^*} e^{g^n(x(\alpha))}$ ($* = \mathcal{G}, \mathcal{Y}$) whose growth rates are exactly $P_{\mathcal{G}}(f)$. We denoted by W_n^* the set of monochromatic words $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ in Σ_*.

Let us note that the growth rate of $\sum_{k=1}^n Z_k^*$ is also $P_{\mathcal{G}}(f)$. Hence, if we denote by W_n' those words in W_n that contain symbols of exactly one colour and otherwise at least one zero, then it follows that the growth rate of $Z_n = \sum_{\alpha \in W_n'} e^{g^n(x(\alpha))}$ is also given by $P_{\mathcal{G}}(f)$.

It thus remains to show that the exponential growth rate of $\sum_{\alpha \in W_n''} e^{g^n(x(\alpha))}$ is $\leq P_{\mathcal{G}}(f)$, where W_n'' denotes the genuinely bi-coloured words in W_n (i.e. those words that contain green and yellow symbols). In Lemma 2.2 we estimated

$$|W_n''| = Q(n) \leq c_2 \sqrt{n} 2^n \left(1 + (3\nu 2^{-1-\tau})^{n/(\tau+1)}\right),$$

and since $\sup g = \sup f$, we obtain

$$Z_n'' = \sum_{\alpha \in W_n''} e^{g^n(x(\alpha))} \leq |W_n''| e^{n \sup g} \leq c_2 e^{n \sup f} \sqrt{n} 2^n \left(1 + (3\nu 2^{-1-\tau})^{n/(\tau+1)}\right).$$

Thus

$$\lim_{n \to \infty} \frac{\log Z_n''}{n} \leq \sup f + \max \left(\log 2, \frac{\log 3\nu}{\tau + 1}\right)$$

which is $\leq P_{\mathcal{G}}(f)$ if $\tau + 1 \geq \frac{\log 3\nu}{P_{\mathcal{G}}(f) - \sup f}$. $\qquad\square$

Example 4. Let $\nu \geq 3$ and $\tau \geq \frac{\log 3\nu}{P_{\mathcal{G}}(f) - \sup f} - 1$. Then there are two mutually singular equilibrium states for g on Σ.

Proof. For $\nu \geq 3$ there are many potentials $f : \Sigma_{\mathcal{G}} \to \mathbb{R}$ for which $P_{\mathcal{G}}(f) \geq \sup f + \log 2$. Then for the extension $g : \Sigma \to \mathbb{R}$ there is an equilibrium state on each of the embedded full ν-shifts $\Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{Y}}$ both of which, by Lemma 3.1, assume the pressure $P(g) = P_{\mathcal{G}}(f)$ in the variational principle. These two measures are both invariant and mutually singular. $\qquad\square$

3.2. **Example with $L \geq 2$ distinct equilibrium states.** For $\nu \geq 3$, an integer, and construct as above a subshift Σ over the alphabet $\mathcal{A} = \{0, 1, 2, \ldots, L\nu\}$ so that it contains the embedded full ν-shift spaces $\Sigma_j = \{(j-1)\nu + 1, \ldots, j\nu\}^{\mathbb{Z}}$, $j = 1, 2, \ldots, L$. Again we allows for transitions from words α in some Σ_j to a word β in $\Sigma_{j\pm1}$ that have to separated by a string of zeros of length $\geq \tau(|\alpha| + |\beta|)$.

Let f be a real valued Hölder continuous function on the full two-shift Σ_1 so that $P_1(f) > \sup f + \log 2$, where $P_1(f)$ is the pressure of f on the full shift Σ_1.

Let us define a Hölder continuous function g on Σ in two steps. We first extend the function from Σ_1 to Σ_j for $j = 2, \ldots, L$ by identification: If $\vec{x} \in \Sigma_1, \vec{y} \in \Sigma_j$ so that $y_i = x_i + (j-1)\nu \forall i$, then we put $g(\vec{y}) = f(\vec{x})$. Then one extends g to bi-coloured words as before, namely we put $g(\vec{x}) = \sup f$ if $x_0 = 0$ and if $x_0 \neq 0$ for a non-monochromatic $\vec{x} \in \Sigma$ then we put $g(\vec{x}) = f(\vec{y})$ where $\vec{y}$ is a monochromatic point that realises the longest symmetric monochromatic word of $\vec{x}$.

Example 5. If $\tau \geq \frac{\log 4\nu}{P_{\mathcal{G}}(f) - \sup f} - 1$, then the pressure $P(g)$ of g on Σ is equal to $P_{\mathcal{G}}(f)$.

Proof. We have to find the exponential growth rate $P(g)$ of the partition function $Z_n = \sum_{\alpha \in W_n} e^{g^n(x(\alpha))}$, where W_n are the Σ-words of lengths n. Clearly $P(g) \geq P_1(f)$.

In order to estimate the exponential growth rate of $Z_n'' = \sum_{\alpha \in W_n''} e^{g^n(x(\alpha))}$ where W_n'' are the genuinely multi-coloured words in W_n we get as above that

$$|W_n''| = Q(n) \leq c_2 \sqrt{n} 2^n \left(1 + (5\nu 2^{-1-\tau})^{n/(\tau+1)}\right),$$

and therefore

$$Z_n'' = \sum_{\alpha \in W_n''} e^{g^n(x(\alpha))} \leq |W_n''| e^{n \sup g} \leq c_2 e^{n \sup f} \sqrt{n} 2^n \left(1 + (5\nu 2^{-1-\tau}))^{n/(\tau+1)}\right)$$

which yields

$$\lim_{n \to \infty} \frac{\log Z_n''}{n} \leq \sup f + \max\left(\log 2, \frac{\log 5\nu}{\tau + 1}\right).$$

This is $\leq P_\mathcal{G}(f)$ if $\tau + 1 \geq \frac{\log 5\nu}{P_\mathcal{G}(f) - \sup f}$. $\qquad\square$

Lemma 3.2. *Let $\nu \geq 3$ and $\tau \geq \frac{\log 5\nu}{P_1(f) - \sup f} - 1$. Then there are $L \geq 2$ mutually singular equilibrium states for g on Σ.*

Proof. For $\nu \geq 3$ there are many potentials $f : \Sigma_\mathcal{G} \to \mathbb{R}$ for which $P_\mathcal{G}(f) \geq \sup f + \log 2$. Then for the extension $g : \Sigma \to \mathbb{R}$ there is an equilibrium state on each of the embedded full ν-shifts $\Sigma_\mathcal{G}$ and $\Sigma_\mathcal{Y}$ both of which, by Lemma 3.1, assume the pressure $P(g) = P_\mathcal{G}(f)$ in the variational principle. These two measures are both invariant and mutually singular. $\qquad\square$

3.3. Example with infinitely many distinct equilibrium states.

Let $\nu \geq 3$ and construct the subshift $\Sigma \subset \mathbb{N}_0^{\mathbb{Z}}$ as was done above. A Hölder continuous function f on Σ_1 which satisfies $P_1(f) \geq \sup f + \log 2$ is then as above extended to a Hölder function g on Σ so that it is a 'copy' on each of the embedded full shifts Σ_j, $j = 1, 2, \ldots$. Moreover we can extend g to the compact space $\bar{\Sigma}$ by defining $g(\infty) = \sup f$. If we denote by μ_j the equilibrium state on Σ_j for the potential $g|_{\Sigma_j}$ then $P_j(g|_{\Sigma_j}) = P_1(f) = P(g)$ for all j. On the infinite symbol subshift Σ we use the Gurevich pressure which is given by

$$P(g) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(g),$$

where the sum in the partition function

$$Z_n(g) = \sum_\alpha e^{g^n(x(\alpha))}$$

is over all n-words $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ so that $\alpha_1 = a$ for a given $a \in \mathcal{A}$. The pressure does not depend the choice of $a \in \mathcal{A}$. We thus obtain:

Example 6. If $\nu \geq 3$ and $\tau \geq \frac{\log 5}{\log \nu}$ and $\tau \geq \frac{\log 5\nu}{P_1(f) - \sup f} - 1$ then there are infinitely many distinct equilibrium states μ_j on Σ for g which are mutually singular.

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