Hitting and return times in ergodic dynamical systems

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Abstract

Given an ergodic dynamical system (X, T, μ), and $U \subset X$ measurable with $\mu(U) > 0$, let $\mu(U)\tau_U(x)$ denote the normalized hitting time of $x \in X$ to U. We prove that given a sequence (U_n) with $\mu(U_n) \to 0$, the distribution function of the normalized hitting time to U_n converges weakly to some pseudo-distribution F if and only if the distribution function of the normalized return time converges weakly to some distribution function $\tilde{F}$, and that in the converging case,

$$F(t) = \int_0^t (1 - \tilde{F}(s))ds, \ t \geq 0.$$

This in particular characterizes asymptotics for hitting times, and shows that the asymptotic for return times is exponential if and only if the one for hitting times is too.

1. Introduction

Throughout $(X, \mathcal{B}, \mu)$ *is a probability space,* $T: X \to X$ *is measurable and preserves* μ*, i.e.* $T\mu = \mu$. We also assume the *dynamical system* (X, T, μ) to be *ergodic*.

For $U \subset X$ with $\mu(U) > 0$, *Poincaré's recurrence theorem* [K, Theorem 1'] states that the variable

$$\tau_U(x) = \inf\{k \geq 1 : T^k x \in U\}$$

is μ-a.s. well defined. If $x \in U$, $\tau_U(x)$ denotes the *return time* of x to U, while when dropping the requirement that x be in U, $\tau_U(x)$ is the *hitting time* of x to U (also called entrance time). The *return time theorem* [K, Theorem 2'] reads

$$\mathbb{E}(\mu(U)\tau_U) = \sum_{t \geq 1} t\mu(U \cap \{\tau_u = t\}) = 1,$$

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where the expectation is computed with respect to the induced probability measure on U, $\mu_U := \frac{\mu}{\mu(U)}$.

Finer statistical properties of the variable $\mu(U)\tau_U$ have been investigated, in a rather large number of recent papers, where particular attention was given to the study of weak convergence of the variable $\mu(U_n)\tau_{U_n}$ as $\mu(U_n) \to 0$. See [A-G] for a recent survey in the mixing case.

We say a sequence of distribution functions (F_n) converges weakly to a function F (which might not be a distribution function itself) if F is increasing and at any point of continuity of F, say t_0, $F_n(t_0) \to F(t_0)$. Notice that **we assume F increasing a priori**. We will write $F_n \Rightarrow F$ if (F_n) converges weakly to F.

Given a $U \subset X$ measurable with $\mu(U) > 0$, we define

$$\tilde{F}_U(t) := \frac{1}{\mu(U)}\mu(U \cap \{\tau_U \mu(U) \leq t\}) \text{ and } F_U(t) = \mu(\{\mu(U)\tau_U \leq t\}).$$

Define

$$\begin{cases} \mathcal{F} = \{F : \mathbb{R} \to [0,1],\ F \equiv 0 \text{ on }]-\infty, 0],\ F \text{ increasing, continuous,} \\ \qquad\qquad\qquad\qquad \text{concave on } [0, +\infty[,\ F(t) \leq t \text{ for } t \geq 0\}\,; \\ \tilde{\mathcal{F}} = \left\{\tilde{F} : \mathbb{R} \to [0,1],\ \tilde{F} \text{ increasing},\ \tilde{F} \equiv 0 \text{ on }]-\infty, 0],\ \int_0^{+\infty}(1 - \tilde{F}(s))ds \leq 1\right\}. \end{cases}$$

These functional classes appear in the following :

Theorem [L], [K-L].

[L] *: given (X, T, μ) ergodic aperiodic, given any $\tilde{F} \in \tilde{\mathcal{F}}$, there exists (U_n) in X such that $\tilde{F}_{U_n} \Rightarrow \tilde{F}$ and $\mu(U_n) \to 0$.*

[K-L] *: given (X, T, μ) ergodic aperiodic, given any $F \in \mathcal{F}$, there exists (U_n) in X such that $F_{U_n} \Rightarrow F$ and $\mu(U_n) \to 0$.*

No connection between one kind of asymptotic and the other is known, except in [HSV, Theorem 2.1] where it is shown that if $\tilde{F}_{U_n} \to \tilde{F}$ and $\tilde{F}(t) = 1 - e^{-t}$ for $t \geq 0$, then $F_{U_n} \to F$ and $F(t) = \tilde{F}(t)$ for $t \geq 0$.

In this note we prove obtain the following rather unexpected, and surprisingly unknown :

Main Theorem. *Given (X, T, μ) ergodic, and a sequence of positive measure measurable subsets $(U_n)_{n\geq 1}$ in X, $(\tilde{F}_{U_n})_{n\geq 1}$ converges weakly if and only if $(F_{U_n})_{n\geq 0}$ converges weakly.*

Moreover, if the convergence holds, and if $\tilde{F}$ and F are the corresponding limiting distributions, then

$$(\diamondsuit) \qquad F(t) = \int_0^t (1 - \tilde{F}(s))ds,\ t \geq 0.$$

Obvious consequences are :

Corollary.

The asymptotic for hitting times, if exists, is positive exponential with parameter 1 *if and only if the one for return times is, too.*

[L, Theorem 1]$\Longleftrightarrow$[K-L, Theorem 1].

The proof of the Corollary is left to the reader. We insist how strange it is that the map $t \geq 0 \mapsto 1 - e^{-t}$ is the only fixed point of $(\diamondsuit)$.

2. Proof of the Main Theorem

We will need two lemmas :

Lemma 1. *Given $U \subset X$ with $\mu(U) > 0$, if $\bar{F}_U(t)$ denotes the smallest piecewise linear map, continuous, concave on $[0, +\infty[$, and greater than F_U, then, letting ${\bar{F}'_U}{}^+$ denote its right-hand side derivative, one has*

$$(\star) \qquad {\bar{F}'_U}{}^+(t) = 1 - \tilde{F}_U(t),\ t \geq 0.$$

Notice that

$$(\star\star) \qquad \| F_U - \bar{F}_U \|_\infty \leq \mu(U).$$

Proof of Lemma 1. The reader will be immediately convinced once he plots a self made hand made example. See [L] and [K-L] for further details on the construction of F_U and $\tilde{F}_U$. □

Lemma 2. *If $(f_n)_{n\geq 0}$ is a sequence of concave functions defined on a non-empty open interval $]a, b[$, and converges pointwise to f, then off an at most countable subset of I, the sequence of derivatives (f'_n) converges pointwise to the derivative f' of f.*

Proof of Lemma 2. This is a straightforward adaptation of [R, Theorem 25.7].

Indeed, by [R, Theorem 25.3], off an at most countable subset of I, the functions f_n, and f, are differentiable, as concave functions.

Next, using the argument for the proof of [R, Theorem 25.7], but for a fixed $x \in I$ rather than along a sequence of point x_i or points x_i in a closed bounded subset of I, the convergence of the derivatives, when all defined, follows at once. □

$\underline{\Rightarrow}$ **in the Theorem** : we assume $\mu(U_n) \to 0$ and that $\tilde{F}_{U_n} \Rightarrow \tilde{F}$ for some $\tilde{F} \in \tilde{\mathcal{F}}$. Since $\tilde{F}$ is increasing, this implies that $\tilde{F}_{U_n} \to \tilde{F}$ Lebesgue almost surely on $[0, +\infty[$. Whence, for given $t \geq 0$, by the Lebesgue dominated convergence theorem on $[0, t]$ ($\tilde{F} \in [0, 1]$), combining with $(\star)$ in Lemma 1, one has

$$\bar{F}_{U_n}(t) = \int_0^t (1 - \tilde{F}_{U_n}(s))ds \to \int_0^t (1 - \tilde{F}(s))ds =: F(t).$$

We define $F(t) = 0$ for $t < 0$. Then because $\tilde{F} \in \tilde{\mathcal{F}}$, it is clear that $F \in \mathcal{F}$.

And by $(\star\star)$, $F_{U_n}(t) \to F(t)$ for any $t \in \mathbb{R}$ (the convergence is in fact uniform on compact subsets of $\mathbb{R}$ by [R, Theorem 10.8]).

Remark 1. *Given $U \subset X$ measurable with $\mu(U) > 0$, $\bar{F}_U$ is concave. Moreover if $F_{U_n} \Rightarrow F$, then $F_{U_n} \to F$ Lebesgue almost surely, whence on a dense subset of $[0, +\infty[$. By $(\star\star)$ and* [R, Theorem 10.8], *it follows that if $F_{U_n} \Rightarrow F$, then $F \in \mathcal{F}$.*

$\Leftarrow$ **in the Theorem** : we assume $F_{U_n} \Rightarrow F$. Then either we assume $F \in \mathcal{F}$ by [K-L], or not and use Remark 1 above to deduce it.

Whence by $(\star)$ and $(\star\star)$, we have, for $t \geq 0$,

$$\bar{F}_{U_n}(t) = \int_0^t \bar{F}'_{U_n}{}^{+}(s)ds = \int_0^t (1 - \tilde{F}_{U_n}(s))ds \to F(t) \ (= \int_0^t F'^{+}(s)ds).$$

By Lemma 2, we deduce that off an at most countable subset Ω of $]0, +\infty[$, $1 - \tilde{F}_{U_n}(s) \to F'(s)$. We put $\tilde{F}(s) := 1 - F'^{+}(s)$ for $s \in \mathbb{R}$.

It remains to show that if F'^{+} is continuous at s, then $\tilde{F}_{U_n}(s) \to \tilde{F}(s)$. Clearly if $s \notin \Omega$ or $s < 0$ there is nothing to do. Else, for any $s_1 < s < s_2$ not in Ω, we have

$$\tilde{F}(s_1) \leq \liminf_n \tilde{F}_{U_n}(s) \leq \limsup_n \tilde{F}_{U_n}(s) \leq \tilde{F}(s_2),$$

and since Ω is dense in $[0, +\infty[$, this ends the proof. □

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