A DERIVATION OF THE POISSON LAW FOR RETURNS OF SMOOTH MAPS WITH CERTAIN GEOMETRICAL PROPERTIES

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Abstract. We consider invariant measures of maps on manifolds whose correlations decay at a sufficient rate and which satisfy a geometric contraction property. We then prove the that the limiting distribution of returns to geometric balls is Poissonian. This does not assume a tower construction or any partition. The decay of correlations is used to show that the independence generated results in the Poisson distribution for returns that are sufficiently separated. A geometric contraction property is then used to show that short return times have a vanishing contribution to the return times distribution. We then also show that the set of very short returns which are of a small linear order of the logarithm of the radius of the balls has a vanishing measure. We obtain error terms which decay polynomially in the logarithm of the radius. We also obtain a extreme value law for such systems.

Contents

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1. Introduction

The limiting distribution of higher order return times in dynamics goes back to Doeblin [10] who establised the Poisson distribution in the the limit for the Gauss map at the origin. In more recent times there has been a large number of results for returns to cylinder sets where certain mixing properties are assumed. Pitskel [23] proved for Axiom A maps and equlibrium measures for Hölder continuous potentials that unions of cylinder sets have in the limit Poisson distributed return times. An approximation argument allowed him then to also deduce the same result for balls in the case of an Axiom A map on the two dimensional torus. He showed that the moments converge and then invoked a theorem of Sevast'yanov to conclude that the return times are Poissonian in the limit. Other subsequent results like by Denker [8] which is along similar lines and in [19] which considers parabolic maps on the interval extended those results to more general settings. For rational maps this was done in [14] which allowed approximations for balls if the dimension of the measure was not too large. More recently in [1, 2] results were obtain for ϕ-mixing and α-mixing systems along a nested sequence of cylinder sets. Using the method of Chen and Stein the Poisson distribution was established in [17] along sequences of unions of cylinders for ϕ-mixing systems over countable alphabets. This also allowed for approximations for balls under some favourable conditions. In [16] this was extended to α-mixing systems and applied to the returns to Bowen balls. In [9] the Chen-Stein method was used for toral automorphisms and ball-like sets where it was shown that for non-periodic points one obtains the Poisson distribution in the limit. For a review see e.g. [15].

For returns to geometric balls apart from Pitskel's result [23] from 1990 all results are quite recent except for the one by Pitskel and on intervals where approximations by cylinder sets can easily be used. For systems that can be modelled by Young towers, Chazottes and Collet [7] proved the limiting distribution to be Poissonian if the decay of correlations are exponential and the unstable manifold is one dimensional. This was generalised to polynomially decaying correlations and arbitrary dimensions in [18]. In this case the speed of convergence is polynomial in the logarithm of the radius of the return ball. A similar result without speed of convergence was obtained in [22] by using the Lebesgue density theorem and in [13] it was shown that for Billiard maps the limiting return times distribution is Poissonian. In this paper we provide a limiting result for maps on manifolds whose correlations decay and which satisfy a certain uniform contraction property (Assumption (IV) below).

2. Assumptions and main results

Let M be a manifold and $T : M \to M$ a C^2 diffeomorphism with the properties described below in the assumptions. Let μ be a T-invariant probability measure on M.

For a ball $B_\rho(\mathsf{x}) \subset M$ we define the counting function

$$\xi^t_{\rho,\mathsf{x}}(x) = \sum_{n=0}^{\lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor - 1} \mathbb{1}_{B_\rho(\mathsf{x})} \circ T^n(x).$$

which tracks the number of visits a trajectory of the point $x \in M$ makes to the ball $B_\rho(\mathsf{x})$ on an orbit segment of length $N = \lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor$, where t is a positive parameter. (We often omit the sub- and superscripts and simply use $\xi(x)$.)

Let Γ^u be a collection of unstable leaves γ^u and Γ^s a collection of stable leaves γ^s. We assume that $\gamma^u \cap \gamma^s$ consists of a single point for all $(\gamma^u, \gamma^s) \in \Gamma^u \times \Gamma^s$. The map T contracts along the stable leaves (need not to be uniform) and similarly T^{-1} contracts along the unstable leaves.

For an unstable leaf γ^u denote by μ_{γ^u} the disintegration of μ to the γ^u. We assume that μ has a product like decomposition $d\mu = d\mu_{\gamma^u} d\upsilon(\gamma^u)$, where υ is a transversal measure. That is, if f is a function on M then

$$\int f(x)\, d\mu(x) = \int_{\Gamma^u} \int_{\gamma^u} f(x)\, d\mu_{\gamma^u}(x)\, d\upsilon(\gamma^u)$$

If $\gamma^u, \hat\gamma^u \in \Gamma^u$ are two unstable leaves then the holonomy map $\Theta : \gamma^u \to \hat\gamma^u$ is defined by $\Theta(x) = \hat\gamma^u \cap \gamma^s(x)$ for $x \in \gamma^u$, where $\gamma^u(x)$ be the local unstable leaf through x.

Let us denote by $J_n = \frac{dT^n \mu_{\gamma^u}}{d\mu_{\gamma^u}}$ the Jacobian of the map T^n with respect to the measure μ in the unstable direction.

Let γ^u be a local unstable leaf. Assume there exists $R > 0$ and for every $n \in \mathbb{N}$ finitely many $y_k \in T^n\gamma^u$ so that $T^n\gamma^u \subset \bigcup_k B_{R,\gamma^u}(y_k)$, where $B_{R,\gamma^u}(y)$ is the embedded R-disk centered at y in the unstable leaf γ^u. Denote by $\zeta_{\varphi,k} = \varphi(B_{R,\gamma^u}(y_k))$ where $\varphi \in \mathscr{I}_n$ and $\mathscr{I}_n$ denotes the inverse branches of T^n. We call ζ an n-cylinder. Then there exists a constant L so that the number of overlaps $N_{\varphi,k} = |\{\zeta_{\varphi',k'} : \zeta_{\varphi,k} \cap \zeta_{\varphi',k'} \neq \varnothing, \varphi' \in \mathscr{I}_n\}|$ is bounded by L for all $\varphi \in \mathscr{I}_n$ and for all k and n. This follows from the fact that $N_{\varphi,k}$ equals $|\{k' : B_{R,\gamma^u}(y_k) \cap B_{R,\gamma^u}(y_{k'}) \neq \varnothing\}|$ which is uniformly bounded by some constant L.

We make the following assumptions:

(I) *Decay of correlations:* There exists a decay function $\lambda(k)$ so that

$$\left|\int_M G(H \circ T^k)\, d\mu - \mu(G)\mu(H)\right| \leq \lambda(k)\|G\|_{Lip}\|H\|_\infty \qquad \forall k \in \mathbb{N},$$

for functions H which are constant on local stable leaves γ^s of T.

(II) *Dimension:* There exist $0 < d_0 < d_1$ such that $\rho^{d_0} \geq \mu(B_\rho) \geq \rho^{d_1}$.

(III) *Unstable dimension:* There exists a u_0 so that $\mu_{\gamma^u}(B_\rho(x)) \leq C_1\rho^{u_0}$ for all $\rho > 0$ small enough and for almost all $x \in \gamma^u$, every unstable leaf γ^u.

(IV) Assume there are sets $\mathcal{G}_n$ so that

(i) *Non-uniform setsize:* $\mu(\mathcal{G}_n^c) = \mathcal{O}(n^{-q})$ for some positive q.

(ii) *Distortion:* $\frac{J_n(x)}{J_n(y)} = \mathcal{O}(\omega(n))$ for all $x, y \in \zeta$, $\zeta \cap \mathcal{G}_n^c = \varnothing$ for $n \in \mathcal{N}$, where ζ are n-cylinders in unstable leaves γ^u and $\omega(n)$ is a non-decreasing sequence.

(iii) *Contraction:* There exists a $\kappa > 1$, so that diam $\zeta \leq n^{-\kappa}$ for all n-cylinders ζ for which $\zeta \cap \mathcal{G}_n^c = \varnothing$ and all n.

(V) *Annulus condition:* Assume that for some $\eta, \beta > 0$:

$$\frac{\mu(B_{\rho+r} \setminus B_{\rho-r})}{\mu(B_\rho)} = \mathcal{O}(r^\eta \rho^{-\beta})$$

for every $r < \rho_0$ for some $\rho_0 < \rho$ (see remark below).

For a positive parameter $\mathfrak{a}$ define the set

$$\mathcal{V}_\rho(\mathfrak{a}) = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^n B_\rho(\mathsf{x}) \neq \varnothing \text{ for some } 1 \leq n < \mathfrak{a}|\log\rho|\}, \tag{1}$$

where $\rho > 0$. The set $\mathcal{V}_\rho$ represents the points within M with very short return times.

2.1. **Return times are Poisson distributed.**

Theorem 1. *Assume that the map $T : M \to M$ satisfies the assumptions (I)–(V) where $\lambda(k)$ decays at least polynomially with power $p > \frac{\frac{\beta}{\eta}+d_1}{d_0}$. Moreover we assume that $d_0 > \max\{\frac{d_1}{q-1}, \frac{\beta}{\kappa\eta-1}\}$ and $\kappa u_0 > 1$ Assume $\omega(j) \sim j^{\kappa'}$ for some $\kappa' \in [0, \kappa u_0 - 1)$.*

Then

$$\mathbb{P}(\xi_{\rho,\mathsf{x}} = r) = e^{-t}\frac{t^r}{r!} + \mathcal{O}(|\log\rho|^{-\sigma})$$

for all $\mathsf{x} \notin \mathcal{V}_\rho(\mathfrak{a})$ for some positive $\mathfrak{a}$, where $\sigma = \min\{\kappa u_0, p\} - \kappa' - 1$ is positive. Moreover, there exists an $\mathfrak{a} > 0$ so that

$$\mu(\mathcal{V}_\rho(\mathfrak{a})) = \mathcal{O}(|\log\rho|^{-\sigma}).$$

If $\delta(n) = \mathcal{O}(\vartheta^n), \vartheta < 1$ is exponential then

$$\mathbb{P}(\xi_{\rho,\mathsf{x}} = r) = e^{-t}\frac{t^r}{r!} + \mathcal{O}(t\rho^{u_0\mathfrak{a}|\log\vartheta|}).$$

The proof of the first part of the theorem is given in the next section. The bound on the size of the very short return set is given in Section 4.2.

Remark 1: The standard case of bounded distortion corresponds to the value $\kappa' = 0$. Then the rate of convergence is $\sigma = \kappa u_0 - 1$.

Remark 2: If μ has dimension d then the condition on the decay of correlation is $p > 1 + \frac{\beta}{d\eta}$.

Remark 3: For an absolutely continuous measure β and β can be chosen arbitrarily close to 1, d_0 and d_1 are arbitrarily close to $D = \dim M$; thus the requirement for p is to be larger than $\frac{1}{D} + 1$ and $\kappa > \frac{1}{D} + 1$ as u_0 is arbitrarily close to D.

Remark 4: In the annulus condition (V) we require that $r < \rho_0$ where according to Section 3.4 $\rho_0 = \mathcal{O}(\rho^{v\kappa})$ and $v < d_0$ can be arbitrarily close to d_0.

Remark 5: Condition (IV) does not imply uniform contraction. In fact the intermittent on the unit interval satisfies this condition. For more details see Section 7.

2.2. **Extremal Values Distribution.**

We take a point $z \in M$ and define

$$\varphi(x) = g(\mu(B_{d(x,z)}(z)))$$

where g is a function from M to $\mathbb{R} \cup \{+\infty\}$ with the following properties: g is strictly decreasing in a neighborhood of 0; 0 is a global maximum for g; g satisfies one of the following three properties:

Type 1. There exist some strictly positive function p such that

$$g^{-1}\left(g(\frac{1}{n})+yp(g(\frac{1}{n}))\right)=(1+\varepsilon_n)\frac{e^{-y}}{n}.$$

with $\epsilon_n\to 0$ as $n\to\infty$ for all $y\in\mathbb{R}$.
Type 2. $g(0)=+\infty$ and there exist $\beta>0$ such that

$$g^{-1}\left(g(\frac{1}{n})y\right)=(1+\varepsilon_n)\frac{y^{-\beta}}{n}$$

for all $y>0$.
Type 3. $g(0)=D<+\infty$ and there exist $\gamma>0$ such that

$$g^{-1}\left(D-g(\frac{1}{n})y\right)=(1+\varepsilon_n)g^{-1}\left(D-g(\frac{1}{n})\right)\frac{y^{\gamma}}{n}$$

for all $y>0$.
Examples of functions satisfying the three types are $g_1(x)=-\log x$, $g_2(x)=x^{-\frac{1}{\beta}}$ and $g_3(x)=D-x^{\frac{1}{\gamma}}$.

We put $X_n=\varphi\circ T^n$ and $M_n=\max\{X_k:0\le k\le n-1\}$. We also write

$$M_{j,n}=\max\{X_j,\cdots,X_{j+n-1}\}.$$

Let $\{\hat{X}_n\}$ be the a stationary, independent process such that $\hat{X}_0$ has the same distribution as X_0. Denote by $\hat{M}_n$ the corresponding maxima of $\{\hat{X}_n\}$. From the extreme value theory of stationary, independent processes we know that under proper linear normalization, $a_n(\hat{M}_n-b_n)$ converges to one of the following three limits:
Type 1.

$$G(x)=e^{-e^{-x}},\qquad -\infty<x<\infty.$$

Type 2.

$$G(x)=\begin{cases}0 & \text{if } x\le 0\\ e^{-x^{-\beta}} & \text{if } x>0, \text{ for some } \beta>0.\end{cases}$$

Type 3.

$$G(x)=\begin{cases}e^{-(-x)^{\gamma}} & \text{if } x\le 0 \text{ for some} \gamma>0\\ 1 & \text{if } x>0.\end{cases}$$

Now we state the Theorem.

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EVL **Theorem 2.** *Assume that the map $T:M\to M$ satisfies the assumptions (I)–(V) where $\lambda(k)$ decays at least polynomially with power $p>\frac{\beta}{\eta d_1}+1$. Moreover we assume that $\kappa\eta-1>\beta$, $\frac{d_1}{q-1}<1$ and $\kappa u_0>2$ if $\omega(j)\sim j^{\kappa'}$ for some $\kappa'\in[0,\kappa u_0-3)$ (where diam $\zeta=\mathcal{O}(n^{-\kappa})$ for n-cylinders ζ).*

Then we have Type i extreme value law for observables g_i with type i, $i=1,2,3$.

3. Proof of Theorem 1

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3.1. **Poisson approximation of the return times distribution.** To prove Theorem 1 we will employ the Poisson approximation theorem from Section 5. Let x be a point in the phase space and $B_\rho := B_\rho(\mathsf{x})$ for $\rho > 0$. Let $X_n = \mathbb{1}_{B_\rho} \circ T^{n-1}$, then we put $N = \lfloor t/\mu(B_\rho) \rfloor$, where t is a positive parameter. We write $S_a^b = \sum_{n=a}^b X_n$ (and $S = S_1^N$). Then for any $2 \leq \Delta \leq N$ (C_3 from Section 5)

errorUSE

$$\left| \mathbb{P}(S = k) - \frac{t^k}{k!} e^{-t} \right| \leq C_3(N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta\, \mu(B_\rho)), \tag{2}$$

where

$$\mathcal{R}_1 = \sup_{\substack{0<j<N-\Delta \\ 0<q<N-\Delta-j}} \left| \mathbb{E}(\mathbb{1}_{B_\rho} \mathbb{1}_{S_{\Delta+1}^{N-j}=q}) - \mu(B_\rho)\, \mathbb{E}(\mathbb{1}_{S_{\Delta+1}^{N-j}=q}) \right|$$

$$\mathcal{R}_2 = \sum_{n=1}^{\Delta-1} \mathbb{E}(\mathbb{1}_{B_\rho}\, \mathbb{1}_{B_\rho} \circ T^n).$$

Since we restrict to the complement of the set $\mathcal{V}_\rho$ (cf. (1)) we have from now on

$$\mathcal{R}_2 = \sum_{n=J}^{\Delta-1} \mu(B_\rho \cap T^{-n} B_\rho),$$

where $J = \lfloor \mathfrak{a} \,|\log \rho\,| \rfloor$. Since $\mathbb{P}(S = k) = 0$ for $k > N$ we obtain

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$$\left| \mathbb{P}(S = k) - \frac{t^k}{k!} e^{-t} \right| = \frac{t^k}{k!} e^{-t} \leq c_1 |\log \rho\,|^{-\kappa u_0} \qquad \forall k > N \tag{3}$$

using the fact that $\mu(B_\rho) \lesssim \rho^{d_0}$ and for ρ sufficiently small.

We now proceed to estimate the error between the distribution of S and a Poissonian for $k \leq N$ based on Theorem 3.

{R1_section}

3.2. **Estimating $\mathcal{R}_1$.** By invariance of the measure μ we can also write

$$\mathcal{R}_1 = \sup_{\substack{0<j<N-\Delta \\ 0<q<N-\Delta-j}} \left| \mu(B_\rho \cap T^{-\Delta}\{S_1^{N-j-\Delta} = q\}) - \mu(B_\rho)\, \mu(\{S_1^{N-j-\Delta} = q\}) \right|.$$

We now use the decay of correlations (I) to obtain an estimate for $\mathcal{R}_1$. Approximate $\mathbb{1}_{B_\rho}$ by Lipschitz functions from above and below as follows:

$$\phi(x) = \begin{cases} 1 & \text{on } B_\rho \\ 0 & \text{outside } B_{\rho+\delta\rho} \end{cases} \qquad \text{and} \qquad \tilde{\phi}(x) = \begin{cases} 1 & \text{on } B_{\rho-\delta\rho} \\ 0 & \text{outside } B_\rho \end{cases}$$

with both functions linear within the annuli. The Lipschitz norms of both ϕ and $\tilde{\phi}$ are equal to $1/\delta\rho$ and $\tilde{\phi} \leq \mathbb{1}_{B_\rho} \leq \phi$.

We obtain

$$\begin{aligned}\mu(B_\rho\cap\{S_\Delta^{N-j}=q\})-\mu(B_\rho)\,\mu(\{S_1^{N-j-\Delta}=q\})\\ \le\int_M\phi\;\cdot\mathbb{1}_{S_\Delta^{N-j}=q}\,d\mu-\int_M\mathbb{1}_{B_\rho}\,d\mu\int_M\mathbb{1}_{S_1^{N-j-\Delta}=q}\,d\mu\\ =X+Y\end{aligned}$$

where

$$\begin{aligned}X&=\left(\int_M\phi\,d\mu-\int_M\mathbb{1}_{B_\rho}\,d\mu\right)\int_M\mathbb{1}_{S_1^{N-j-\Delta}=q}\,d\mu\\ Y&=\int_M\phi\,(\mathbb{1}_{S_\Delta^{N-j}=q})\,d\mu-\int_M\phi\,d\mu\int_M\mathbb{1}_{S_1^{N-j-\Delta}=q}\,d\mu.\end{aligned}$$

The two terms X and Y are estimated separately. The first term is estimated by:

$$X\le\int_M\mathbb{1}_{S_1^{N-j-\Delta}=q}\,d\mu\int_M(\phi-\mathbb{1}_{B_\rho})\,d\mu\le\mu(B_{\rho+\delta\rho}\setminus B_\rho).$$

In order to estimate the second term Y we use the decay of correlations and have to approximate $\mathbb{1}_{S_1^{N-j-\Delta}=q}$ by a function which is constant on local stable leaves. For that purpose put

$$\mathcal{S}_n=\bigcup_{\substack{\gamma^s\\ T^n\gamma^s\subset B_\rho}}T^n\gamma^s,\qquad\partial\mathcal{S}_n=\bigcup_{\substack{\gamma^s\\ T^n\gamma^s\cap B_\rho\neq\varnothing}}T^n\gamma^s$$

and

$$\mathscr{S}_\Delta^{N-j}=\bigcup_{n=\Delta}^{N-j}\mathcal{S}_n,\qquad\partial\mathscr{S}_\Delta^{n-j}=\bigcup_{n=\Delta}^{N-j}\partial\mathcal{S}_n.$$

The set

$$\mathscr{S}_\Delta^{N-j}(q)=\{S_1^{N-j-\Delta}=q\}\cap\mathscr{S}_\Delta^{N-j}$$

is then a union of local stable leaves. This follows from the fact that by construction $T^ny\in B_\rho$ if and only if $T^n\gamma^s(y)\subset B_\rho$. We also have $\{S_p^{N-j}=q\}\subset\tilde{\mathscr{S}}_\Delta^{N-j}(q)$ where the set $\tilde{\mathscr{S}}_\Delta^{N-j}(k)=\mathscr{S}_\Delta^{N-j}(k)\cup\partial\mathscr{S}_\Delta^{N-j}$ is a union of local stable leaves.

Denote by ψ_Δ^{N-j} the characteristic function of $\mathscr{S}_\Delta^{N-j}(k)$ and by $\tilde\psi_\Delta^{N-j}$ the characteristic function of $\tilde{\mathscr{S}}_\Delta^{N-j}(k)$. Then ψ_Δ^{N-j} and $\tilde\psi_\Delta^{N-j}$ are constant on local stable leaves and satisfy

$$\psi_\Delta^{N-j}\le\mathbb{1}_{S_1^{N-j-\Delta}=q}\le\tilde\psi_\Delta^{N-j}.$$

Since $\{y:\psi_\Delta^{N-j}(y)\neq\tilde\psi_\Delta^{N-j}(y)\}\subset\partial\mathscr{S}_\Delta^{N-j}$ we need to estimate the measure of $\partial\mathscr{S}_\Delta^{N-j}$.

By the contraction property $\operatorname{diam}(T^n\gamma^s(y))\le\delta(n)=n^{-\kappa}$ outside the set $\mathcal{G}_n^c$ and consequently

$$\bigcup_{\substack{\gamma^s\\ T^n\gamma^s\subset B_\rho\\ \gamma^s\subset\mathcal{G}_n}}T^n\gamma^s\subset B_{\rho+\delta(n)}\setminus B_{\rho-\delta(n)}$$

and therefore

$$\begin{aligned}\mu(\partial\mathscr{S}_\Delta^{N-j}) &\le \mu\left(\bigcup_{n=\Delta}^{N-j} T^{-n}\left(B_{\rho+\delta(n)}\setminus B_{\rho-\delta(n)}\right)\right)+\sum_{n=\Delta}^\infty \mu(\mathcal{G}_{N-j}^c)\\ &\le \sum_{n=\Delta}^{N-j}\mu(B_{\rho+\delta(n)}\setminus B_{\rho-\delta(n)})+\sum_{n=\Delta}^\infty\mu(\mathcal{G}_n^c).\end{aligned}$$

The last term is estimated by

$$\sum_{n=\Delta}^\infty \mu(\mathcal{G}_n^c)=\mathcal{O}(1)\sum_{n=\Delta}^\infty n^{-q}=\mathcal{O}(\Delta^{-q+1})=\mathcal{O}\rho^{v(q-1)}\le\rho^\epsilon\mu(B_\rho),$$

if we put $\Delta\sim\rho^{-v}$ assume that q is large enough so that $v(q-1)>d_1$ ($\epsilon=v(q-1)-d_1>0$). Hence, by assumption (IV),

$$\mu(\partial\mathscr{S}_\Delta^{N-j})=\mathcal{O}(1)\sum_{n=\Delta}^\infty\frac{n^{-\kappa\eta}}{\rho^\beta}\mu(B_\rho)+\rho^\epsilon\mu(B_\rho)=\mathcal{O}(\rho^{v(\kappa\eta-1)-\beta}+\rho^\epsilon)\mu(B_\rho)$$

with $\delta(n)=\mathcal{O}(n^{-\kappa})$ and $\Delta\sim\rho^{-v}$ where this time we also need that $v>\frac{\beta}{\kappa\eta-1}$ which is determined in Section total_error 3.4 below. Both constraints imply that we must have $v>\max\{\frac{d_1}{q-1},\frac{\beta}{\kappa\eta-1}\}$. If we split $\Delta=\Delta'+\Delta''$ then we can estimate as follows:

$$\begin{aligned}Y&=\left|\int_M\phi\,T^{-\Delta'}(\mathbb{1}_{S_{\Delta''}^{N-j-\Delta'}=q})\,d\mu-\int_M\phi\,d\mu\int_M\mathbb{1}_{S_1^{N-j-\Delta}=q}\,d\mu\right|\\ &\le\lambda(\Delta')\|\phi\|_{Lip}\|\mathbb{1}_{\mathscr{S}_{\Delta''}^{N-j-\Delta'}}\|_{\mathscr{L}^\infty}+2\mu(\partial\mathscr{S}_{\Delta''}^{N-j}).\end{aligned}$$

Hence

$$\begin{aligned}&\mu(B_\rho\cap T^{-\Delta}\{S_1^{N-j-\Delta}=q\})-\mu(B_\rho)\,\mu(\{S_1^{N-j-\Delta}=q\})\\ &\qquad\le\frac{\lambda(\Delta/2)}{\delta\rho}+\mu(B_{\rho+\delta\rho\setminus B_\rho})+\mathcal{O}(\rho^{v(\kappa\eta-1)-\beta}+\rho^\epsilon)\mu(B_\rho)\end{aligned}$$

by taking $\Delta'=\Delta''=\frac{\Delta}{2}$. A similar estimate from below can be done using $\tilde\phi$. Hence

R1est (4) $$\mathcal{R}_1\le c_2\left(\frac{\lambda(\Delta/2)}{\delta\rho}+\mu(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho})\right)+\mathcal{O}(\rho^{v(\kappa\eta-1)-\beta}+\rho^\epsilon)\mu(B_\rho).$$

In the exponential case when $\delta(n)=\mathcal{O}(\vartheta^n)$ we choose $\Delta=s|\log\rho|$ for some $s>0$ and obtain the estimate

$$\mathcal{R}_1\le c_2\left(\frac{\lambda(\Delta/2)}{\delta\rho}+\mu(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho})\right)+\mathcal{O}(\rho^{s|\log\vartheta|-\beta}+\rho^\epsilon)\mu(B_\rho).$$

3.3. **Estimating the terms $\mathcal{R}_2$.** We will estimate the measure of each of the summands comprising $\mathcal{R}_2$ individually. We use the product form of the measures μ. For that purpose fix j and and let γ^u be an unstable local leaf through B. Then we put

$$\mathscr{C}_j(B,\gamma^u)=\{\zeta_{\varphi,j}:\zeta_{\varphi,j}\cap B\ne\varnothing,\varphi\in\mathscr{I}_j\}$$

for the cluster of j-cylinders that covers the set B, where the sets $\zeta_{\varphi,k}$ are the pre-images of embedded R-balls in $T^j\gamma^u$. Then

$$\begin{aligned}\mu_{\gamma^u}(T^{-j}B_\rho\cap B_\rho) &\le \sum_{\zeta\in\mathscr{C}_j(B_\rho,\gamma^u)}\frac{\mu_{\gamma^u}(T^{-j}B_\rho\cap\zeta)}{\mu_{\gamma^u}(\zeta)}\mu_{\gamma^u}(\zeta)\\ &\le \sum_{\zeta\in\mathscr{C}_j(B_\rho,\gamma^u)} c_3\omega(j)\frac{\mu_{T^j\gamma^u}(B_\rho\cap T^j\zeta)}{\mu_{T^j\gamma^u}(T^j\zeta)}\mu_{\gamma^u}(\zeta)\end{aligned}$$

Since $\mu_{T^j\gamma^u}(T^j\zeta)=\mu_{T^j\gamma^u}(B_{R,\gamma^u}(y_k))$ (for some y_k) is uniformly bounded from below, we obtain

$$\begin{aligned}\mu_{\gamma^u}(T^{-j}B_\rho\cap B_\rho) &\le c_3\omega(j)\mu_{T^j\gamma^u}(B_\rho)\sum_{\zeta\in\mathscr{C}_j(B_\rho,\gamma^u)}\mu_{\gamma^u}(\zeta)\\ &\le c_3\omega(j)\mu_{T^j\gamma^u}(B_\rho)\,L\,\mu_{\gamma^u}\left(\bigcup_{\zeta\in\mathscr{C}_j(B_\rho,\gamma^u)}\zeta\right)\end{aligned}$$

Now, since outside the set $\mathcal{G}_n^c$ one has diam $\bigcup_{\zeta\in\mathscr{C}_j(B_\rho,\gamma^u)}\zeta\le j^{-\kappa}+\text{diam }B_\rho\le c_1j^{-\kappa}$ (as we can assume that $\rho<j^{-\kappa}$) we obtain

$$\mu_{\gamma^u}\left(\bigcup_{\substack{\zeta\in\mathcal{C}_j(B_\rho,\gamma^u)\\ \zeta\subset\mathcal{G}_j}}\zeta\right)=\mathcal{O}(j^{-\kappa u_0})$$

and therefore

$$\mu_{\gamma^u}\left(\bigcup_{\zeta\in\mathcal{C}_j(B_\rho,\gamma^u)}\zeta\right)\le\mathcal{O}(j^{-\kappa u_0})+\mu(\mathcal{G}_j^c)=\mathcal{O}(j^{-\kappa u_0}+j^{-q})$$

and consequently

$$\mu_{\gamma^u}(T^{-j}B_\rho\cap B_\rho)\le c_4\omega(j)\mu_{T^j\gamma^u}(B_\rho)(j^{-\kappa u_0}+j^{-q}).$$

Since $d\mu=d\mu_{\gamma^u}d\upsilon(\gamma^u)$ we obtain

$$\mu(T^{-j}B_\rho\cap B_\rho)\le c_5\omega(j)\mu(B_\rho)(j^{-\kappa u_0}+j^{-q}).$$

Summing up the $\mu(T^{-j}B_\rho\cap B_\rho)$ over $j=J,\dots,\Delta-1$, we see that outside the set of forbidden ball centers $\mathcal{V}_\rho$ we get

R_2 (5) {R_2} $$\mathcal{R}_2=\sum_{j=J}^{\Delta-1}\mu(T^{-j}B_\rho\cap B_\rho)\le c_5\sum_{j=J}^{\Delta-1}\omega(j)(j^{-\kappa u_0}+j^{-q})\mu(B_\rho)$$

for some c_5. If we assume that $\omega(j)\sim j^{\kappa'}$ and $\delta(j)$ decays polynomially with power κ then

$$\mathcal{R}_2\le c_6\mu(B_\rho)\sum_{j=J}^{\Delta-1}j^{\kappa'}(j^{-\kappa u_0}+j^{-q})\le c_7J^{-\sigma}\mu(B_\rho)$$

for ρ small enough, where $\sigma = \min\{\kappa u_0, q\} - \kappa' - 1$ is positive by assumption. If diam ζ (ζ n-cylinders) and $\mu(\mathcal{G}_n^c)$ decay super polynomially then

$$\mathcal{R}_2 \leq c_7 J^{\kappa'}(\delta(J)^{u_0} + \delta'(n))\mu(B_\rho),$$

where diam $\zeta \leq \delta(n)$, $\mu(\mathcal{G}_n^c) \leq \delta'(n)$ are super polynomial.

In the exponential case ($\delta(n) = \mathcal{O}(\vartheta^n)$) we obtain

$$\mathcal{R}_2 = \mathcal{O}(\rho^{u_0 \mathfrak{a} |\log \vartheta|})\mu(B_\rho)$$

as $J = \mathfrak{a}|\log \rho|$.

3.4. **The total error.** For the total error we now put $\delta\rho = \rho^w$ and $\Delta = \rho^{-v}$ where $v < d_0$ since $\Delta \ll N$ and $N \geq \rho^{-d_0}$. Then $\lambda(\Delta) = \mathcal{O}(\Delta^{-p}) = \mathcal{O}(\rho^{pv})$ and thus (in the polynomial case)

$$\begin{aligned} \left|\mathbb{P}(S=k) - \frac{t^k}{k!}e^{-t}\right| &\leq N c_1 \left(\frac{\lambda(\Delta)}{\delta\rho} + \mu(B_{\rho+\delta\rho} \setminus B_{\rho-\delta\rho})\right) + c_7 J^{-\sigma} + \mathcal{O}(\rho^{v(\kappa\eta-1)-\beta} + \rho^\epsilon) \\ &\leq c_8 \left(\rho^{vp-w-d_1} + \rho^{w\eta-\beta} + tJ^{-\sigma} + \rho^{v(\kappa\eta-1)-\beta} + \rho^\epsilon\right). \end{aligned}$$

We can choose $v < d_0$ arbitrarily close to d_0 and then require $d_0 p - w - d_1 > 0$, $w\eta - \beta > 0$ and $d_0(\kappa\eta - 1) - \beta > 0$. We can choose $w > \frac{\beta}{\eta}$ arbitrarily close to $\frac{\beta}{\eta}$ and can satisfy all requirements if $p > \frac{\frac{\beta}{\eta}+d_1}{d_0}$ in the case when λ decays polynomially with power p, i.e. $\lambda(k) \sim k^{-p}$. Hence

$$\left|\mathbb{P}(S=k) - \frac{t^k}{k!}e^{-t}\right| \leq C_1 J^{-\sigma}$$

for some C_1.

In the exponential case (diam $\zeta = \mathcal{O}(\vartheta^n)$ for n cylinders ζ) we obtain

$$\mathbb{P}(S=k) - \frac{t^k}{k!}e^{-t} = \mathcal{O}(1)\left(\rho^{s|\log\vartheta| - w - d_1} + \rho^{w\eta-\beta} + t\rho^{u_0\mathfrak{a}|\log\vartheta|} + \rho^{s|\log\vartheta|-\beta} + \rho^\epsilon\right).$$

Choosing s and w large enough, we obtain that the RHS is of order $\mathcal{O}(t\rho^{u_0\mathfrak{a}|\log\vartheta|})$. □

4. Very Short Returns

4.1. **Assumptions.** Let (M, T) be a dynamical system equipped with a metric d. Assume that the map $T: M \to M$ is a C^2-diffeomorphism. As at the start of the paper the set $\mathcal{V}_\rho \subset M$ is given by

$$\mathcal{V}_\rho = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^n B_\rho(\mathsf{x}) \neq \varnothing \text{ for some } 1 \leq n < J\},$$

where $J = \lfloor \mathfrak{a} |\log\rho| \rfloor$ and $\mathfrak{a} = (4 \log A)^{-1}$ with

$$A = \sup_\omega \left(\| DT \|_{\mathscr{L}^\infty} + \| DT^{-1} \|_{\mathscr{L}^\infty}\right)$$

($A \geq 2$). We will need the following assumptions:

(V1) Assume there are sets $\mathcal{G}_n$ so that

(i) *Non-uniform setsize:* $\mu(\mathcal{G}_n^c) = \mathcal{O}(n^{-q})$ for some positive q.

(ii) *Distortion:* $\frac{J_n(x)}{J_n(y)} = \mathcal{O}(\omega(n))$ for all $x, y \in \zeta$, $\zeta \cap \mathcal{G}_n^c = \varnothing$ for $n \in \mathcal{N}$, where ζ are n-cylinders in unstable leaves γ^u and $\omega(n)$ is a non-decreasing sequence.

(iii) *Contraction:* There exists a $\kappa > 1$, so that $\operatorname{diam}\zeta \le n^{-\kappa}$ for all n-cylinders ζ for which $\zeta \cap \mathcal{G}_n^c = \varnothing$ and all n.

(V2) *Geometric regularity of the measure on the unstable leaves:* Assume there exists $u_0 > 0$ such that

$$\mu_{\gamma^u}(B_\rho(\mathsf{x})) \le \rho^{u_0}$$

for all x, unstable leaves γ^u and ρ small enough.

4.2. **Estimate on the measure of $\mathcal{V}_\rho$.** Now we can show that the set of centres where small balls have very short returns is small. To be precise we have the following result:

Proposition 1. *Assume that the map $T : M \to M$ satisfies the assumptions (V1) and (V2). Then there exist constants $C_2 > 0$ such that for all ρ small enough*

$$\mu(\mathcal{V}_\rho) \le \frac{C_2}{|\log\rho|^\sigma}$$

where $\sigma = \kappa u_0 - \kappa' - 1$ if δ decays polynomially with power $\kappa > 1$ and ω grows polynomially with power $\kappa' \ge 0$ assuming $\sigma > 0$.

If δ decays exponentially and $\limsup_{n\to\infty} \frac{\log\log\omega(n)}{\log n} < \frac{1}{2}$ then the error term on the RHS is $\mathcal{O}(\delta(|\log\rho|)^{u_0})$.

Proof. We follow the proof of Proposition 5.1 of [18] which modelled after Lemma 4.1 of [7]. Let us note that since T is a diffeomorphism one has

$$B_\rho(\mathsf{x}) \cap T^n B_\rho(\mathsf{x}) \ne \varnothing \qquad \Longleftrightarrow \qquad B_\rho(\mathsf{x}) \cap T^{-n} B_\rho(\mathsf{x}) \ne \varnothing.$$

We partition $\mathcal{V}_\rho$ into level sets $\mathcal{N}_\rho(n)$ as follows

$$\mathcal{V}_\rho = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^{-n} B_\rho(\mathsf{x}) \ne \varnothing \text{ for some } 1 \le n < J\} = \bigcup_{n=1}^{J-1} \mathcal{N}_\rho(n)$$

where

$$\mathcal{N}_\rho(n) = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^{-n} B_\rho(\mathsf{x}) \ne \varnothing\}.$$

The above union is split into two collections $\mathcal{V}_\rho^1$ and $\mathcal{V}_\rho^2$, where

$$\mathcal{V}_\rho^1 = \bigcup_{n=1}^{\lfloor \mathfrak{b} J \rfloor} \mathcal{N}_\rho(n) \quad \text{and} \quad \mathcal{V}_\rho^2 = \bigcup_{n=\lceil \mathfrak{b} J \rceil}^{J} \mathcal{N}_\rho(n).$$

and where the constant $\mathfrak{b} \in (0,1)$ will be chosen below. In order to find the measure of the total set we will estimate the measures of the two parts separately.

(I) Estimate of $\mathcal{V}_\rho^2$

We will restrict to the set $\mathcal{G}_n$ and derive a uniform estimate for the measure of the level sets $\tilde{\mathcal{N}}_\rho(n) = \mathcal{N}_\rho(n) \cap \mathcal{G}_n$ when $n > \mathfrak{b} J$. Then

$$\mu(\tilde{\mathcal{N}}_\rho(n)) = \mu(T^{-n}\tilde{\mathcal{N}}_\rho(n)) \le \sum_\zeta \mu(T^{-n}\tilde{\mathcal{N}}_\rho(n) \cap \zeta)$$

We will consider each of the measures $\mu(T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta)$ separately by using the product form of the measures μ. By distortion of the Jacobian we obtain

$$\begin{aligned}\mu_{\gamma^u}(T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta) &= \frac{\mu_{\gamma^u}(T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta)}{\mu_{\gamma^u}(\zeta)}\,\mu_{\gamma^u}(\zeta)\\ &\le c_1\omega(n)\,\frac{\mu_{\hat\gamma^u}(T^n(T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta))}{\mu_{\hat\gamma^u}(T^n\zeta)}\,\mu_{\gamma^u}(\zeta), \qquad (6)\ \{\texttt{level\_summand}\}\end{aligned}$$

el_summand

where, as before, $\hat\gamma^u=\gamma^u(T^nx)$ for $x\in\zeta\cap\gamma^u$. We estimate the numerator by finding a bound for the diameter of the set. Let the points x and z in $T^{-n}\tilde{\mathcal{N}}_\rho(n)$ be such that $x,z\in T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta\cap\gamma^u$ for an unstable leaf γ^u.

Note that $T^nx, T^nz\in\tilde{\mathcal{N}}_\rho(n)$, there exists $y\in B_\rho(T^nx)$ such that $T^ny\in B_\rho(T^nx)$, thus

$$d(T^nx,x)\le d(T^nx,T^ny)+d(T^ny,y)+d(y,x)\le\rho+\rho+A^nd(T^nx,T^ny)\le(2+A^n)\rho.$$

Hence

$$d(T^nx,T^nz)\le d(T^nx,x)+d(x,z)+d(z,T^nz)\le 4A^n\rho+d(x,z).$$

We have

$$d(x,z)\le\operatorname{diam}\zeta<n^{-\kappa}$$

by assumption. Therefore

$$d(T^nx,T^nz)\le 4A^n\rho+d(x,z)\le 4\,A^n\rho+n^{-\kappa}$$

If we choose $\mathfrak{a}>0$ so that $\mathfrak{a}<\frac{1}{2\log A}$ then $A^n\rho<e^{-\frac12|\log\rho|^{1/2}}$. If $n\ge\mathfrak{b}|\log\rho|$ for some $\mathfrak{b}\in(0,\mathfrak{a})$ then

$$d(T^nx,T^nz)\le c_2(e^{-\mathfrak{c}'|\log\rho|^{1/2}}+n^{-\kappa})$$

for some constant c_2 where $\mathfrak{c}'=\min(\frac12,\sqrt{\mathfrak{b}})$. Taking the supremum over all points x and z yields

$$|\,T^n(T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta\cap\gamma^u)\,|\le c_2(e^{-\mathfrak{c}'|\log\rho|^{1/2}}+n^{-\kappa}).$$

By assumption (V2) on the relationship between the measure and the metric

$$\mu_{\hat\gamma^u}(T^n(T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta))\le c_3(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+n^{-\kappa u_0})$$

Incorporating the estimate into (6) yields level_summand

$$\mu_{\gamma^u}(T^{-n}\tilde{\mathcal{N}}_\rho(n)\cap\zeta)\le c_4\omega(n)(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+n^{-\kappa u_0})\mu(\zeta),$$

for some c_4. Integrating over $dv(\gamma^u)$ and summing over ζ yields

$$\mu(\tilde{\mathcal{N}}_\rho(n))\le c_4\omega(n)(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+n^{-\kappa u_0})\sum_\zeta\mu(\zeta)\le c_5\omega(n)(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+n^{-\kappa u_0})$$

as $\sum_\zeta \mu(\zeta)=\mathcal{O}(1)$. Consequently, if $\omega(n)$ is so that $\limsup_{n\to\infty}\frac{\log\log\omega(n)}{\log n}<\frac12$ (as can be seen from the estimates above, the value $\frac12$ can be replaced by 1) then

Th2refPt2 (7)
$$\begin{aligned}\mu(\mathcal{V}_\rho^2) &\le \sum_{n=\lceil \mathfrak{b}J\rceil}^{J}\Big(\mu(\tilde{\mathcal{N}}_\rho(n))+\mu(\mathcal{G}_n^c)\Big)\\ &\le c_5 e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}\sum_{n=\lceil \mathfrak{b}J\rceil}^{J}\omega(n)+c_6\sum_{n=\lceil \mathfrak{b}J\rceil}^{J}\left(\omega(n)n^{-\kappa u_0}+n^{-q}\right)\\ &\le c_7(e^{-\mathfrak{c}''|\log\rho|^{1/2}}+(\mathfrak{a}\mathfrak{b}|\log\rho|)^{-\sigma})\end{aligned}\tag{7}$$

for some constant $\mathfrak{c}''>0$ (and ρ small enough) as $J=\lfloor \mathfrak{a}|\log\rho|\rfloor$. As before, $\sigma=\min\{\kappa u_0,q\}-\kappa'-1$ as $\omega(n)\sim n^{\kappa'}$.

(II) Estimate of $\mathcal{V}_\rho^1$

We will need the following version of Lemma B.3 from CC13 [7].

{B.3}

B.3 **Lemma 1.** *Put $s_p=2^p\frac{A^{n\,2^p}-1}{A^n-1}$. Then for every p,k integers, $\rho>0$*

$$\left\{\mathsf{x}\in M: B_\rho(\mathsf{x})\cap T^kB_\rho(\mathsf{x})\neq\varnothing\right\}\subset\left\{\mathsf{x}\in M: B_{s_p\rho}(\mathsf{x})\cap T^{k2^p}B_{s_p\rho}(\mathsf{x})\neq\varnothing\right\}.$$

Proof. Consider the case $p=1$. Let x such that $B_\rho(\mathsf{x})\cap T^kB_\rho(\mathsf{x})\neq\varnothing$. This implies that there exist $z\in B_\rho(\mathsf{x})\cap T^{-k}B_\rho(\mathsf{x})$. For any $u\in T^kB_\rho(\mathsf{x})$, there exist $v\in B_\rho(\mathsf{x})$ such that $T^kv=u$, thus

$$d(u,x)\le d(u,T^kz)+d(T^kz,x)\le d(T^kv,T^kz)+2\rho\le(2A^k+2)\rho.$$

Therefore, $T^kB_\rho(\mathsf{x})\subset B_{(2A^k+2)\rho}(\mathsf{x})$.

One can observe that if $B_\rho(\mathsf{x})\cap T^kB_\rho(\mathsf{x})\neq\varnothing$ then $T^k\left(B_\rho(\mathsf{x})\cap T^kB_\rho(\mathsf{x})\right)\neq\varnothing$ thus $T^kB_\rho(\mathsf{x})\cap T^k(T^kB_\rho(\mathsf{x}))\neq\varnothing$ and therefore $B_{(2A^k+2)\rho}(\mathsf{x})\cap T^k(T^kB_{(2A^k+2)\rho}(\mathsf{x}))\neq\varnothing$. Finally, this gives us

$$\{\mathsf{x}\in M: B_\rho(\mathsf{x})\cap T^kB_\rho(\mathsf{x})\neq\varnothing\}\subset\{\mathsf{x}\in M: B_{(2A^k+2)\rho}(\mathsf{x})\cap T^k(T^kB_{(2A^k+2)\rho}(\mathsf{x}))\neq\varnothing\}.$$

The general case is shown similarly. □

The lemma thus shows that $\mathcal{N}_\rho(n)\subset\mathcal{N}_{s_p\rho}(2^pn)$ and consequently we only need to estimate $\mu(\mathcal{N}_{s_p\rho}(2^pn))$.

Let us now consider the case $1\le n\le\lfloor \mathfrak{b}J\rfloor$ and let as in Lemma B.3 1 $s_p=2^p\frac{A^{n\,2^p}-1}{A^n-1}$. Hence by Lemma B.3 1 one has $\mathcal{N}_\rho(n)\subset\mathcal{N}_{s_p\rho}(2^pn)$. for any $p\ge1$, and in particular for $p(n)=\lfloor \lg\mathfrak{b}J-\lg n\rfloor+1$. Therefore

$$\bigcup_{n=1}^{\lfloor \mathfrak{b}J\rfloor}\mathcal{N}_\rho(n)\subset\bigcup_{n=1}^{\lfloor \mathfrak{b}J\rfloor}\mathcal{N}_{s_{p(n)}\rho}(2^{p(n)}n).$$

Now define

$$n'=n2^{p(n)}\qquad\text{and}\qquad\rho'=s_{p(n)}\rho.$$

A direct computation shows that $1 \le n \le \lfloor \mathfrak{b}J \rfloor$ implies $\lceil \mathfrak{b}J \rceil \le n' \le 2\mathfrak{b}J$ and so

$$\mathcal{V}_\rho^1 = \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_\rho(n) \subset \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_{s_{p(n)}\rho}(2^{p(n)}n) \subset \bigcup_{n'=\lceil \mathfrak{b}J \rceil}^{2\mathfrak{b}J} \mathcal{N}_{\rho'}(n').$$

Therefore to estimate the measure of $\mathcal{V}_\rho^1$ it suffices to find a bound for $\mathcal{N}_{\rho'}(n')$ when $n' \ge \mathfrak{b}J$. This is accomplished by using an argument analogous to the first part of the proof. We replace all the n with n' and ρ with ρ'. We get for $\mathfrak{b} < 1/3$

$$\mu(\mathcal{N}_{\rho'}(n')) \le c_6 \xi(n')(e^{-u_0|\log \rho'|^{1/2}} + n'^{-\kappa u_0} + n'^{-q})$$

and thus obtain an estimate similar to (7):

$$\mu(\mathcal{V}_\rho^1) \le \sum_{n'=\lceil \mathfrak{b}J \rceil}^{2\mathfrak{b}J} \mu(\mathcal{N}_{\rho'}(n')) \le c_8(e^{-\mathfrak{c}|\log \rho'|^{1/2}} + (\mathfrak{a}\mathfrak{b}|\log \rho|)^{-\sigma}).$$

for some $\mathfrak{c} \in (0, u_0)$.

(III) Final estimate

Overall we obtain for all ρ sufficiently small

$$\mu(\mathcal{V}_\rho) \le \mu(\mathcal{V}_\rho^1) + \mu(\mathcal{V}_\rho^2) \le c_9(e^{-\mathfrak{c}|\log \rho'|^{1/2}} + (\mathfrak{a}\mathfrak{b}|\log \rho|)^{-\sigma}) \le C_2 |\log \rho|^{-\sigma},$$

for some C_2. □

5. Poisson Approximation Theorem

This section contains the abstract Poisson approximation theorem which establishes the distance between sums of $\{0,1\}$-valued dependent random variables X_n and a random variable that is Poisson distributed. It is used in Section 3.1 in the proof of Theorem 1 and compares the number of occurrences in a finite time interval with the number of occurrences in the same interval for a Bernoulli process $\{\tilde{X}_n : n\}$.

Theorem 3. [7] *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\{0,1\}$-valued process and t a positive parameter. Let $S_a^b = \sum_{n=a}^b X_n$ and define $S := S_1^N$ for convenience's sake where $N = \lfloor t/\epsilon \rfloor$ and $\epsilon = \mathbb{P}(X_1 = 1)$. Additionally, let ν be the Poisson distribution measure with mean $t > 0$. Finally, assume that $\epsilon < \frac{t}{2}$. Then there exists a constant C_3 such that for any $E \subset \mathbb{N}_0$, and $2 \le \Delta < N$ we have*

$$|\,\mathbb{P}(S \in E) - \nu(E)\,| \le C_3 \#\{E \cap [0,N]\}\ (N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta\epsilon)$$

where,

$$\mathcal{R}_1 = \sup_{\substack{0<j<N-\Delta \\ 0<q<N-\Delta-j}} \{|\,\mathbb{P}(X_1 = 1 \wedge S_{\Delta+1}^{N-j} = q) - \epsilon\,\mathbb{P}(S_{\Delta+1}^{N-j} = q)\,|\}$$

$$\mathcal{R}_2 = \sum_{n=2}^{\Delta} \mathbb{P}(X_1 = 1 \wedge X_n = 1).$$

Proof. Let $(\tilde{X}_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables taking values in $\{0,1\}$, constructed so that $\mathbb{P}(\tilde{X}_1 = 1) = \epsilon$. Further assume that the $\tilde{X}_n$'s are independent of the X_n's. Let $\tilde{S} = \sum_{n=1}^N \tilde{X}_n$. Then

$$\begin{aligned} |\,\mathbb{P}(S\in E) - \nu(E)\,| &\le |\,\mathbb{P}(S\in E) - \mathbb{P}(\tilde{S}\in E)\,| + |\,\mathbb{P}(\tilde{S}\in E) - \nu(E)\,| \\ &\le \sum_{k\in E\cap[0,N]} |\,\mathbb{P}(S=k) - \mathbb{P}(\tilde{S}=k)\,| + \sum_{k=0}^{\infty}\left|\mathbb{P}(\tilde{S}=k) - \frac{t^k}{k!}e^{-t}\right| \end{aligned}$$

Thanks to [3] we can bound the second sum using the estimate

$$\sum_{k=0}^{\infty}\left|\mathbb{P}(\tilde{S}=k) - \frac{t^k}{k!}e^{-t}\right| \le \frac{2t^2}{N}. \tag{8}$$

For summands of the remaining term we utilize the proof of Theorem 2.1 from [7] according to which for every $k\le N$,

$$|\,\mathbb{P}(S=k) - \mathbb{P}(\tilde{S}=k)\,| \le 2N(\mathcal{R}_1 + \mathcal{R}_2 + \Delta\epsilon^2) + 4\Delta\epsilon.$$

As $N \le t/\epsilon$ this becomes

$$|\,\mathbb{P}(S=k) - \mathbb{P}(\tilde{S}=k)\,| \le 6\,t\,(N(\mathcal{R}_1+\mathcal{R}_2) + \Delta\epsilon). \tag{9}$$

Combining (8) and (9) yields

$$\begin{aligned} |\,\mathbb{P}(S\in E) - \nu(E)\,| &\le \sum_{k\in E\cap[0,N]} |\,\mathbb{P}(S=k) - \mathbb{P}(\tilde{S}=k)\,| \;+\; \frac{2t^2}{N} \\ &\le \sum_{k\in E\cap[0,N]} 6\,t\,(N(\mathcal{R}_1+\mathcal{R}_2)+\Delta\epsilon) \;+\; \frac{2t^2}{t/\epsilon - 1} \\ &\le 6\,t\,\#\{E\cap[0,N]\}\,(N(\mathcal{R}_1+\mathcal{R}_2)+\Delta\epsilon) + 4\,t\epsilon \\ &\le C_3\#\{E\cap[0,N]\}\,(N(\mathcal{R}_1+\mathcal{R}_2)+\Delta\epsilon) \end{aligned}$$

for some $C_3 < \infty$. □

6. Proof of Theorem 2

In [21] Leadbetter et al gave two conditions called D and D', under which $a_n(M_n - b_n) \to G$ is equivalent to $a_n(\hat{M}_n - b_n) \to G$. Recall that $\hat{M}_n$ is the maxima of the independent, stationary process $\{\hat{X}_n\}$. Later Freitas et al [12] replaced condition D by D_2 and obtained the same result. To state the conditions we put $u_n = v/a_n + b_n$ for $v\in\mathbb{R}$ and sequences a_n, b_n.

Condition. $D_2(u_n)$ [12] *We say condition $D_2(u_n)$ holds if for any integers l, t and n*

$$|\mu(X_0 > u_n, M_{t,l} < u_n) - \mu(X_0 > u_n)\mu(M_l < u_n)| \le \gamma(n,t)$$

where $\gamma(n,t)$ is a non-increasing sequence in t for every n and satisfies $\gamma(n,t_n) = o(\frac{1}{n})$ for some sequence $t_n = o(n)$, $t_n \to \infty$.

Condition. $D'(u_n)$ FF08 [12] *We say condition* $D'(u_n)$ *holds if*

$$\lim_{k\to\infty}\limsup_n n\cdot\sum_{j=1}^{\lfloor n/k\rfloor}\mu(X_0>u_n,X_j>u_n)=0$$

Below we will verify both conditions for Type I observable, i.e. g with $g^{-1}\left(g(\frac{1}{n})+yp(g(\frac{1}{n}))\right)=(1+\varepsilon_n)\frac{e^{-y}}{n}$. The other two cases follow similarly.

6.1. **Condition** $D_2(u_n)$**.** First we show $D_2(u_n)$. Put $u_n(y)=g\left(\frac{1}{n}\right)+yp\left(g\left(\frac{1}{n}\right)\right)$, then

$$\{X_0>u_n\}=B_{l(g^{-1}(u_n))}(z);$$

here $l(y)=\inf\{r>0:\mu(B_r(z))\geq y\}$. Since $\mu(B_{l(y)}(z))=y$, by Assumption (II) we get

$$Cy^{1/d_0}\leq l(y)\leq C'y^{1/d_1}$$

for some constant C and C'. In particular we have

$$Cn^{-1/d_0}\leq l(g^{-1}(u_n))\leq C'n^{-1/d_1}. \tag{10}$$

Here both constants depend on y.

To simplify notations we write $r_n=l(g^{-1}(u_n))$ and omit z. We approximate the indicator function of $\{Y_0>u_n\}=B_{r_n}$ by Lipschitz functions $\phi(x)$ and $\tilde{\phi}(x)$ as in the proof of $\mathcal{R}_1$. The same estimate as in Section est_R1_section 3.2 yields ($\epsilon=v(q-1)-d_1$)

$$\begin{aligned}
&|\mu(Y_0>u_n,M_{t,l}<u_n)-\mu(Y_0>u_n)\mu(M_l<u_n)|\\
&\qquad = \left|\int \mathbb{1}_{B_{r_n}}\mathbb{1}_{\{M_{t,l}<u_n\}}\,d\mu-\int\mathbb{1}_{B_{r_n}}\,d\mu\int\mathbb{1}_{\{M_{t,l}<u_n\}}\,d\mu\right|\\
&\qquad \leq c_2\left(\frac{\lambda(t/2)}{\delta r_n}+\mu(B_{r_n+\delta r_n}\setminus B_{r_n-\delta r_n})\right)+\mathcal{O}(r_n^{v(\kappa\eta-1)-\beta}+r_n^{\epsilon})\mu(B_{r_n}).
\end{aligned}$$

Putting $\delta r_n=r_n^w$, $w>1$, and $t=n^v$ with $0<v<1$ gives

$$\begin{aligned}
\gamma(n,t) &= |\mu(Y_0>u_n,M_{t,l}<u_n)-\mu(Y_0>u_n)\mu(M_l<u_n)|\\
&\leq \mathcal{O}(1)\left(n^{-vp+w/d_1}+n^{-1-(w\eta-\beta)/d_1}\right)+\mathcal{O}(r_n^{v(\kappa\eta-1)-\beta}+r_n^{\epsilon})\frac{1}{n}
\end{aligned}$$

as $\mu(Y_0>u_n)=\mu(B_{r_n})=\mathcal{O}(1/n)$. In order that $n\gamma(n,t)\to 0$ we need $1-vp+w/d_1<0$, $-(w\eta-\beta)/d_1<0$ and, as before, $v(\kappa\eta-1)-\beta$ and $\epsilon=v(q-1)-d_1$ positive. We choose $w>\beta/\eta$ and v close to 1 which can be satisfied since by the assumptions $\frac{d_1}{q-1}<1$. In the case when λ decays polynomially with power p, $1-vp+w/d_1<0$ is satisfied if $p>\frac{\beta}{\eta d_1}+1$.

6.2. **Condition** $D'(u_n)$**.** Notice that

$$\sum_{j=1}^{\lfloor n/k\rfloor}\mu(X_0>u_n,X_j>u_n)=\sum_{j=1}^{\lfloor n/k\rfloor}\mu(B_{r_n}\cap T^{-j}B_{r_n}).$$

This is exactly $\mathcal{R}_2$ in Section 3.1 with $\Delta = \lfloor n/k \rfloor$. We split the sum into two parts as follows:

$$\sum_{j=1}^{\lfloor n/k \rfloor} \mu(B_{r_n} \cap T^{-j} B_{r_n}) = \sum_{j=J}^{\lfloor n/k \rfloor} \mu(B_{r_n} \cap T^{-j} B_{r_n}) + \sum_{j=1}^{J} \mu(B_{r_n} \cap T^{-j} B_{r_n})$$

with $J = \lfloor \mathfrak{a} |\log r_n| \rfloor$. By (5) we get

$$\sum_{j=J}^{\Delta} \mu(B_{r_n} \cap T^{-j} B_{r_n}) \leq c_5 \sum_{j=J}^{\Delta-1} \omega(j)\delta(j)^{u_0} \mu(B_{r_n}),$$

where $\mu(B_{r_n}) = \mathcal{O}(\frac{1}{n})$. For the second term we restrict to points $z \notin \mathcal{V}_{r_n}$ which implies that $B_{r_n} \cap T^{-j} B_{r_n} = \varnothing$ for $j = 1, \dots, J-1$, where by Section 4

$$\mu(\mathcal{V}_{r_n}) \leq C_2 |\log r_n|^{-\sigma},$$

with $\sigma = \min\{\kappa u_0, p\} - \kappa' - 1 > 0$.

To finish the proof we use the maximal function technique by Collet in [6]. For this purpose we fix some $0 < \xi < \theta < 1$ and define the set

$$F_k = \left\{ \mu(B_{r_{\exp(k^\xi)}} \cap \mathcal{V}_{r_{\exp(k^\xi)}}) \geq \mu(B_{r_{\exp(k^\xi)}}) \cdot k^{-\theta} \right\}$$

and

$$M_r(x) = \sup_{s>0} \frac{1}{\mu(B_s(x))} \int_{B_s(x)} \mathbb{1}_{\mathcal{V}_r}(y)\, d\mu(y)$$

where $\mathcal{V}_r = \{\mathsf{x} \in M : B_r \cap T^n B_r \neq \varnothing \text{ for some } 1 \leq n < J\}$ as before. Since

$$F_k \subset \{M_{r_{\exp(k^\xi)}} \geq k^{-\theta}\},$$

we conclude that

$$\mu(F_k) \leq \mu(M_{r_{\exp(k^\xi)}} \geq k^{-\theta}) \leq \frac{\mu(\mathbb{1}_{\mathcal{V}_{r_{\exp(k^\xi)}}(y)})}{k^{-\theta}} \leq k^{-(\xi\sigma - \theta)}.$$

If $\xi\sigma - \theta > 1$ we get $\sum_k \mu(F_k) < \infty$ and thus by Borel-Cantelli there exist $N(x)$ for almost every x such that $x \notin F_k$ for all $k > N(x)$. For every n, we choose k such that

$$\exp(k^\xi) \leq n < \exp((k+1)^\xi),$$

we have $r_{\exp((k+1)^\xi)} \leq r_n \leq r_{\exp(k^\xi)}$. As a result

$$B_{r_n} \cap T^{-j} B_{r_n} \subset B_{r_{\exp(k^\xi)}} \cap T^{-j} B_{r_{\exp(k^\xi)}} \subset B_{r_{\exp(k^\xi)}} \cap \mathcal{V}_{r_{\exp(k^\xi)}}$$

for every $j < J$. Therefore

$$\begin{aligned} n \cdot \sum_{j=1}^{J} \mu(B_{r_n} \cap T^{-j} B_{r_n}) &\leq n \cdot \sum_{j=1}^{J} \mu(B_{r_{\exp(k^\xi)}} \cap \mathcal{V}_{r_{\exp(k^\xi)}}) \\ &\leq C \exp((k+1)^\xi) \cdot (k+1)^\xi \mu(B_{r_{\exp(k^\xi)}}) \cdot k^{-\theta} \\ &\leq C \frac{\exp((k+1)^\xi)}{\exp(k^\xi)} \cdot k^{\xi-\theta} \to 0 \end{aligned}$$

since $\xi - \theta < 0$. If $\sigma = \kappa u_0 - \kappa' - 1 > 2$ then we can choose $0 < \xi < \theta < 1$ both close to 1 so that $\sigma\xi - \theta > 1$.

7. Example

7.1. **Mostly contracting diffeomorphisms.** As an example we consider a C^2 diffeomorphisms f on a three dimensional manifold. We assume that it is partially hyperbolic, dynamically coherent and mostly contracting as in [11]. We have the following theorem:

Theorem 4. *Let f be a C^2 partially hyperbolic diffeomorphism on a three dimensional manifold whose center bundle is dynamically coherent, u-convergent and mostly contracting. Let ν be its SRB-measure.*

Then for almost every x we have

$$\lim_{\rho\to 0} \nu(\xi_{\rho,\mathrm{x}} = r) = e^{-t}\frac{t^r}{r!}.$$

Condition (IV) is easy to verify since f is C^2 and uniformly expanding on the unstable direction. Dolgopyat proved that such map has a unique SRB measure ν which has exponential decay of correlation with respect to Hölder continuous function. Notice that Corollary 6.2 and 6.3 in [11] still hold true if one assumes that A is L^∞ which is constant on stable leaves. Furthermore ν has no zero Lyapunov exponents. As a result the pointwise dimension of ν coincide with its Hausdorff dimension $d_H(\mu)$ by [4]. Thus we can take d_0 and d_1 both close to $d_H(\mu)$ and such that $d_0 < d_H(\mu) < d_1$ in Condition (II). Condition (III) is also satisfied since μ is absolutely continuous w.r.t. the Lebesgue measure on the unstable direction.

The annulus condition (V) is proven by the following easy lemma:

Lemma 2. *Let f and μ be as above. Then there exist $\beta > 0$ such that*

$$\frac{\mu(B_{\rho+r} \setminus B_{\rho-r})}{\mu(B_\rho)} = \mathcal{O}(r^{1/2}\rho^{-\beta})$$

for $0 < r \ll \rho$.

Proof. The disintegration of ν along unstable leaves is absolutely continuous w.r.t. Lebesgue measure with bounded density. Let γ be an unstable leaf, we have

$$\nu_\gamma((B_{\rho+r} \setminus B_{\rho-r}) \cap \gamma) \le Cm((B_{\rho+r} \setminus B_{\rho-r}) \cap \gamma) \le C'(r\rho)^{1/2}.$$

The lemma follows by integrating along the transversal direction. □

7.2. **Manville-Pommeau map.** As another example we consider the Manville-Pommeau map on the unit interval. It is given by

$$Tx = \begin{cases} x + 2^{1+\alpha}x^{1+\alpha} & \text{for } 0 \le x \le \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} < x \le 1 \end{cases},$$

where $\alpha \in (0,1)$ is a parameter. In this case T has an absolutely continuous invariant measure μ whose density is $h(x) \sim x^{-\alpha}$. The return times distribution has previous been shown to be Poissonian in [19]. Also, an inducing argument was used in [5] to show that the first return time is almost surely exponentially distributed. Here we apply our main theorem to give a short argument to deduce the Poisson distribution of entry times. For this we also rely on a result of Hu [20] which proves that the transfer operator converges at a polynomial rate and thus that the decay of correlations (as in Assumption (I)) is polynomial.

There is a sequence of points $a_n, n = 0, 1, \dots$ which decreases to 0 so that $T_0 = \frac{1}{2}$ and $Ta_{n+1} = a_n$ for all n. If we put $I_n = (a_{n+1}, a_n]$, then all the intervals I_n are pairwise disjoint and satisfy $TI_{n+1} = I_n$ for all n and $\bigcup_n I_n = (0, \frac{1}{2})$. Moreover $a_n \sim n^{-\gamma}$, where $\gamma = \frac{1}{\alpha}$ is larger than 1. Since $h(x) \sim x^{-\alpha}$ one has $\mu(I_n) \sim n^{-\gamma}$ and $\mu(J_n) = n^{1-\gamma}$ where $J_n = \bigcup_{j=n}^{\infty} I_j$ is a punctured neighbourhood of 0.

The two elements $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ cover the entire unit interval and denote by $\mathscr{I}_n$ the inverse branches of T^n.

Put $A_k = (a_{k+1}, a_0] = \bigcup_{j=0}^{k} I_k$ and let $\mathscr{I}_n$ be the inverse branches of T^n. If $\hat{\zeta}_\varphi$ is an n-cylinder, that is a preimage of either A_k or $(\frac{1}{2}, 1]$ under the inverse branch $\varphi \in \mathscr{I}_n$ then the distortion of DT^n on $\hat{\zeta}_\varphi$ is bounded by $c_1\left(\frac{1}{k} + \frac{1}{n}\right)^{-\gamma(\gamma+1)}$ for some constant c_1. In particular if we choose $k = n^\theta$ for some $\theta \in [0, 1)$ then we can put $\omega(n) = c_2 n^{\theta\gamma(\gamma+1)}$ for some c_2.

If we want to use Theorem 1 then we would put $\mathcal{G}_n^c = J_{n^\theta}$ which would give us $\mu(\mathcal{G}_n^c) = \mu(J_{n^\theta}) \sim n^{-\theta(\gamma-1)}$ and thus $q = \theta(\gamma - 1)$. We can thus directly apply Theorem 1 if $\gamma > 2$, i.e. if $\alpha \in (0, \frac{1}{2})$ because then we can choose θ close enough to 1 in order to satisfy Assumptions (I)–(V). In this case we can do a little better by tailoring the estimates in the proof to the situation at hand.

We now can nearly use the theorem for higher order returns, but let us remark that if $\mathsf{x} \in (0, 1)$ then for n large enough we have that $\mathsf{x} \in A_{n^\theta} \cup (\frac{1}{2}, 1]$. If we proceed as in the estimate of the term $\mathcal{R}_2$ we obtain

$$T^{-j}B_\rho \cap B_\rho \subset \bigcup_{\zeta:\zeta\cap B_\rho \neq \varnothing} \zeta = \mathscr{P}_1 \cup \mathscr{P}_2$$

where the union is over j-cylinders ζ and

$$\mathscr{P}_1 = \bigcup_{\zeta:\zeta\cap B_\rho \neq \varnothing} T^{-j}B_\rho \cap \hat{\zeta}, \qquad \mathscr{P}_2 = \bigcup_{\zeta:\zeta\cap B_\rho \neq \varnothing} T^{-j}B_\rho \cap \zeta \setminus \hat{\zeta}.$$

The first set is estimated as before in the main theorem. For the second term notice that

$$\mathscr{P}_2 = \bigcup_{A\in\mathcal{A}} \bigcup_{\varphi\in\mathscr{I}_j:\varphi(A)\cap B_\rho\neq\varnothing} T^{-j}B_\rho \cap \varphi(A \setminus \hat{A}) = \bigcup_{\varphi\in\mathscr{I}_j:\varphi(A_0)\cap B_\rho\neq\varnothing} T^{-j}B_\rho \cap \varphi(J_{n^\theta})$$

where $\mathcal{A} = \{(0, \frac{1}{2}], (\frac{1}{2}, 1]\}$ and $A = A_{j^\theta}$ if $A = (0, \frac{1}{2}]$ and otherwise $\hat{A} = A$. Hence

$$\mathscr{P}_2 = \bigcup_{\varphi\in\mathscr{I}_j:\varphi(A_0)\cap B_\rho\neq\varnothing} \varphi(B_\rho \cap J_{n^\theta})$$

which is empty for n large enough, i.e. so that $a_{n^\theta} < \mathsf{x}$.

By [20] Proposition 5.2 one has that the correlations decay polynomially at the rate of $\gamma - 1$, that is $\lambda(k) = c_3 k^{1-\gamma}$ ($p = \gamma - 1$) for some c_3. The dimensions here are $d_0 = d_1 = 1$ and the annulus condition is satisfied with $\eta = 1$ and $\beta = 0$. Similarly, $u_0 = 1$. In order to get the contraction rate consider the 'worst' case for the contraction, when the partition element $(0, \frac{1}{2}]$ is n times mapped by the inverse branch that contains the parabolic branch. Its image is then $(0, a_n)$ and therefore diam $\zeta = a_n \sim n^{-\gamma}$ for n-cylinders ζ. Hence $\kappa = \gamma$. Since $\gamma > 1$ the conditions of the theorem are satisfied since we can choose $\theta > 0$ arbitrarily close to 0. For any $\sigma < \gamma$ one can choose $\theta > 0$ so that $\sigma \leq \gamma - \theta\gamma(\gamma - 1)$ and therefore we obtain the following result.

Corollary. *Let T be the Manneville-Pommeau map for the parameter $\alpha \in (0, 1)$. Let μ be the invariant absolutely continuous probability measure. Then for any $\sigma < \gamma = \frac{1}{\alpha}$ one has*

$$\mathbb{P}(\xi_{\rho,\mathsf{x}} = r) = e^{-t}\frac{t^r}{r!} + \mathcal{O}(|\log\rho|^{-\sigma})$$

for all $\mathsf{x} \notin \mathcal{V}_\rho(\mathfrak{a})$ for some positive $\mathfrak{a}$. Moreover $\mu(\mathcal{V}_\rho(\mathfrak{a})) = \mathcal{O}(|\log\rho|^{-\sigma})$.

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