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Convergence of rare event point processes to the Poisson process for planar billiards

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Abstract
We show that for planar dispersing billiards the distribution of return times is, in the limit, Poisson for metric balls almost everywhere w.r.t. the SRB (Sinai–Ruelle–Bowen) measure. Since the Poincaré return map is piecewise smooth but becomes singular at the boundaries of the partition elements, recent results on the limiting distribution of return times cannot be applied, as they require the maps to have bounded second derivatives everywhere. We first prove the Poisson limiting distribution assuming exponentially decaying correlations. For the case where the correlations decay polynomially, we induce on a subset on which the induced map has exponentially decaying correlations. We then prove a general theorem according to which the limiting return times statistics of the original map and the induced map are the same.

Keywords: rare events, point processes, billiards
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1. Introduction

The purpose of this work is to study the statistical laws governing the occurrence of rare events for billiards. The starting point is the analysis of stationary stochastic processes $X_0, X_1, \ldots$ generated by the dynamics of the billiards considered. A billiard map (T, X, μ) is a measure preserving transformation $T : X \to X$ of a phase space X which preserves a volume measure μ. The stationary stochastic processes that we consider will be generated by the time series $\{\phi \circ T^j\}$ generated by an observable $\phi : X \to \mathbb{R}$ which is maximized at a unique point $\zeta \in X$.

The rare events will be the exceedances of a high threshold u, meaning the occurrences of the event $X_j > u$, for some $j \in \mathbb{N}_0$, which correspond to the entrance of the orbit, at time $j \in \mathbb{N}_0$, into a small region of the phase space, namely into a small neighbourhood of the point ζ.

We will consider rare event point processes (REPPs), which keep a record of the number of exceedances (or entrances into certain small balls around ζ) in a certain normalized time interval and show that for certain planar billiards, these REPP converge typically to a Poisson process. We postpone the formal definition of the REPP to section 2.1, but in order to illustrate our main results in a more intuitive way, we introduce the random variable

$$N_n(t) = \#\left\{i = 0, \ldots, \left\lfloor \frac{t}{\mathbb{P}(X_0 > u_n)} \right\rfloor : \; X_i > u_n \right\}, \tag{1}$$

which counts the number of exceedances among the first $\lfloor t/\mathbb{P}(X_0 > u_n) \rfloor$ random variables of the process, where $\mathbb{P}$ is a probability on the space of realizations of the stochastic process that makes it stationary and $(u_n)_{n\in\mathbb{N}}$ is a sequence chosen such that $u_n \to u_F :=$ ess sup $|X_0|$ and hence $\mathbb{P}(X_0 > u_n) \to 0$. The probability $\mathbb{P}$ is given by the invariant measure μ and we will use these interchangeably.

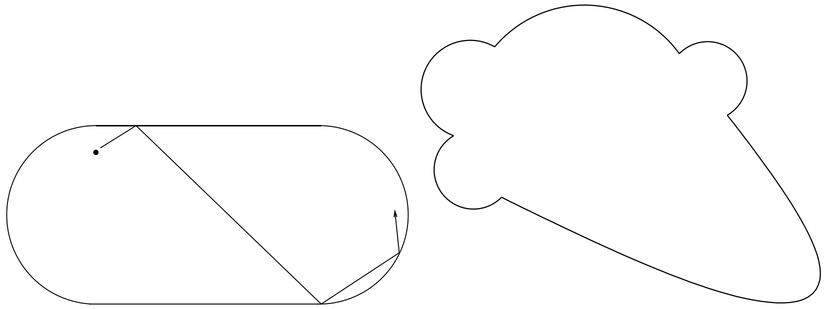
Recently Chazottes and Collet [6] showed that for any two-dimensional dynamical system (T, X, μ) modelled by a Young tower which has bounded derivative and exponential tails (and hence exponential decay of correlations for Hölder observations), then for μ-a.e. point $\zeta \in X$, if $B_r(\zeta)$ is a ball of radius r about ζ, then

$$\mu\left\{x \in X : \sum_{j=0}^{\left\lfloor \frac{t}{\mu B_r(\zeta)} \right\rfloor} 1_{B_r(\zeta)}(T^j x) = k\right\} \to \mathrm{e}^{-t}\frac{t^k}{k!}$$

as $r \to 0$. Our result implies this Poisson law for shrinking balls about generic points for a broad class of billiard systems. This was then extended to polynomial decay for tails and correlations in [17,25]. Both results also gave rates of convergence: the error is a positive power of the diameter in the exponential case and a negative power of the logarithm of the diameter in the polynomial case. Unfortunately, those results rely on the boundedness of the derivative of T (i.e. $|DT|_\infty < C < \infty$) and therefore do not apply to exponentially or polynomially mixing Sinai dispersing billiards since those have unbounded derivatives.

Our goal is to show that for planar Sinai dispersing billiards (with finite or infinite horizon) and also for certain billiard systems with polynomial decay of correlations, the REPP, typically, converges in distribution to a standard Poisson process, where the thresholds u_n converge to the maximum value attainable (and the corresponding neighbourhoods shrink to ζ) in a scaled way. This means that, with such a scaling, the REPP convergence to a standard Poisson occurs for a.e. point ζ chosen in the phase space, with respect to the invariant measure, which, in the setting of billiards, is equivalent to the Lebesgue measure.

Note that there are two perspectives from which to look at rare events in a dynamical setting: one consists in looking at the exceedances as extreme values for the random variables X_j, for $j \in \mathbb{N}_0$, in which case one uses tools of extreme value theory; the other consists in looking at rare events as hits on or returns to small sets for the orbits in the phase space, a phenomenon which is tied to that of recurrence. Motivated by the work of Collet [9], in [11, 12] the authors established formally a connection between the existence of extreme value laws (EVL) for $X_0, X_1, \ldots$, i.e., the existence of a distributional limit for the maximum of the first n variables of the process, and the existence of hitting times statistics (HTS), i.e., the existence of a distributional limit for the normalized hitting time for shrinking neighbourhoods of the point ζ. In this way, these two perspectives were shown to be linked and essentially one can look at them just as two sides of the same coin.



(a) A Bunimovich stadium. (b) Flower like stadia.

Figure 1. Some polynomially mixing billiards.

Our proofs are based upon extreme value theory and some remarkable ideas of Collet [9]. These techniques are especially powerful in the setting of billiards, which have strong hyperbolic properties and preserve a volume measure. We first give proofs for Sinai dispersing billiards, and then show how recent work of Chernov and Zhang [8] and Markarian [23] allows us to extend these results to billiards with polynomial decay by inducing on a subset for which the return map has good hyperbolic properties.

To illustrate the kinds of billiards to which we can apply our results we refer the reader to Chernov and Zhang [8], where examples of polynomially mixing billiards can be found. In particular, we mention: semi-dispersing billiards in rectangles with internal scatters, Bunimovich stadia, Bunimovich flower-like regions and skewed stadia (see figure 1). As a consequence of the convergence of the REPP to the Poisson process, stated in theorems 2.4 and 2.5 below, we can assert in more general terms that for stationary stochastic processes $X_0, X_1, \ldots$, as mentioned above, arising from the dynamics of Sinai dispersing billiards (with finite or infinite horizon), of the Bunimovich stadia or of Bunimovich flower-like billiards, then for a.e. ζ chosen in the phase space, we have that, for all $t > 0$ and $k \in \mathbb{N}_0$,

$$\lim_{n\to\infty} \mathbb{P}(N_n(t) = k) = \mathrm{e}^{-t}\frac{t}{k!}. \tag{2}$$

We mention that after this paper was submitted, in [24], Pène and Saussol, using recurrence rates to cope with the short returns, generalized the main result of Chazottes and Collet, given in [6], which, in particular, also allowed them to obtain the same limit in (2) for billiards with polynomial decay of correlations such as the Bunimovich stadia. For another generalization of the Chazottes–Collet result, see also [17, 25]. Also, see [26] for related results under exponential decay of correlations.

2. The setting and a statement of results

Let (T, X, μ) be an ergodic transformation of a probability space. We suppose that X is embedded in a Riemannian manifold of dimension d. Suppose that the time series $X_0, X_1, \ldots$ arises from such a system simply through evaluation of a given observable $\varphi : X \to \mathbb{R}\cup\{\pm\infty\}$ along the orbits of the system, or, in other words, the time evolution given by successive iterations by T:

$$X_n = \varphi \circ T^n, \quad \text{for each } n \in \mathbb{N}. \tag{3}$$

Clearly, $X_0, X_1, \ldots$ defined in this way is not an independent sequence. However, the T-invariance of μ guarantees that this stochastic process is stationary.

We suppose that the r.v. $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ achieves a global maximum at $\zeta \in X$ (we allow $\varphi(\zeta) = +\infty$). We assume that φ and μ are sufficiently regular that, for u sufficiently close to $u_F := \varphi(\zeta)$, the event

$$U(u) := \{x \in X : \varphi(x) > u\} = \{X_0 > u\}$$

corresponds to a topological ball centred at ζ. Moreover, the quantity $\mu(U(u))$, as a function of u, varies continuously on a neighbourhood of u_F.

We are interested in studying the extremal behaviour of the stochastic process $X_0, X_1, \ldots$ which is tied to the occurrence of exceedances of high levels u. The occurrence of an exceedance of level u at time $j \in \mathbb{N}_0$ means that the event $\{X_j > u\}$ occurs, where u is close to u_F. Observe that a realization of the stochastic process $X_0, X_1, \ldots$ is achieved if we pick, at random and according to the measure μ, a point $x \in X$, compute its orbit and evaluate φ along it. Then saying that an exceedance occurs at time j means that the orbit of the point x hits the ball $U(u)$ at time j, i.e., $T^j(x) \in U(u)$.

For more details on the choice of the observables such that the above properties hold and the link between extreme values and hitting/returns to small sets endures, we suggest that the readers look at [14, section 4.1]. However, for definiteness we mention that a possible choice for φ in this setting, where the invariant measure μ will be equivalent to the Lebesgue measure, is the following: consider some point $\zeta \in X$ and take

$$\varphi(x) = -\log(\mathrm{dist}(x, \zeta)), \tag{4}$$

where $\mathrm{dist}(\cdot, \cdot)$ denotes the usual Euclidean metric in X.

For technical reasons, due to the techniques prevailing in extreme value theory, we will consider sequences $(u_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} n\mu(X_0 > u_n) = \tau, \tag{5}$$

for some $\tau > 0$. The motivation for using such normalizing sequences comes from the case when $X_0, X_1, \ldots$ are independent and identically distributed (i.i.d.). Let $M_n = \max\{X_0, \ldots, X_{n-1}\}$. In this i.i.d. setting, it is clear that $\mathbb{P}(M_n \leqslant u) = (F(u))^n$, where F is the d.f. of X_0, i.e., $F(x) := \mathbb{P}(X_0 \leqslant x)$. Hence, condition (5) implies that

$$\mathbb{P}(M_n \leqslant u_n) = (1 - \mathbb{P}(X_0 > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \to \mathrm{e}^{-\tau},$$

as $n \to \infty$. This means that the waiting times between exceedances of u_n are approximately exponentially distributed.

For example, if φ is given as in (4) and if μ has a density with respect to the Lebesgue measure m where $\rho(\zeta) := \frac{\mathrm{d}\mu}{\mathrm{d}m}(\zeta)$, then the scaling constants can be chosen as $u_n = (1/d)\log n + \rho(\zeta)$.

2.1. Rare event point processes

Before we give the formal definition for REPP, we introduce some formalism. Let $\mathcal{S}$ denote the semi-ring of subsets of $\mathbb{R}_0^+$ whose elements are intervals of the type $[a, b)$, for $a, b \in \mathbb{R}_0^+$. Let $\mathcal{R}$ denote the ring generated by $\mathcal{S}$. Recall that if $J \in \mathcal{R}$, there are $k \in \mathbb{N}$ and k intervals $I_1, \ldots, I_k \in \mathcal{S}$ such that $J = \cup_{i=1}^k I_j$. In order to fix notation, let $a_j, b_j \in \mathbb{R}_0^+$ be such that $I_j = [a_j, b_j) \in \mathcal{S}$. For $I = [a, b) \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, we define $\alpha I := [\alpha a, \alpha b)$ and $I + \alpha := [a + \alpha, b + \alpha)$. Similarly, for $J \in \mathcal{R}$ define $\alpha J := \alpha I_1 \cup \cdots \cup \alpha I_k$ and $J + \alpha := (I_1 + \alpha) \cup \cdots \cup (I_k + \alpha)$.

Definition 2.1. For stationary stochastic processes $X_0, X_1, \ldots$ and sequences $(u_n)_{n\in\mathbb{N}}$ satisfying (5), we define the REPP by counting the number of exceedances (or hits of $U(u_n)$)

during the (rescaled) time period $v_n J \in \mathcal{R}$, where $J \in \mathcal{R}$, and $v_n := 1/\mu(X_0 > u_n)$ is, according to Kac's theorem, the expected waiting time before the occurrence of one exceedance. To be more precise, for every $J \in \mathcal{R}$, set

$$N_n(J) := \sum_{j \in v_n J \cap \mathbb{N}_0} 1_{\{X_j > u_n\}}.$$

Our main result states that the REPP N_n converges in distribution to a standard Poisson process. For the sake of completeness, we give next the meaning of convergence in distribution of point processes and also the definition of a standard Poisson process. (See [20] for more details.)

Definition 2.2. Suppose that $(N_n)_{n\in\mathbb{N}}$ is a sequence of point processes defined on $\mathcal{S}$ and N is another point process defined on $\mathcal{S}$. Then, we say that N_n *converges in distribution to* N if the sequence of vector random variables $(N_n(J_1), \ldots, N_n(J_k))$ converges in distribution to $(N(J_1), \ldots, N(J_k))$, for every $k \in \mathbb{N}$ and all $J_1, \ldots, J_k \in \mathcal{S}$ such that $N(\partial J_i) = 0$ a.s., for $i = 1, \ldots, k$.

Definition 2.3. Let $T_1, T_2, \ldots$ be an i.i.d. sequence of random variables with common exponential distribution of mean 1. Given this sequence of r.v., for $J \in \mathcal{R}$, set

$$N(J) = \#\left\{ i \in \mathbb{N} : \sum_{j=1}^{i} T_j \in J \right\}.$$

We say that N defined this way is a standard Poisson process.

To simplify the notation, whenever $J = [0, t)$ for some $t > 0$ then we will write

$$N_n(t) := N_n([0, t))) \qquad \text{and} \qquad N(t) := N([0, t)).$$

Note that $N_n(t)$ just defined is consistent with (1).

Remark 2.3.1. The random variable $N(J)$ has distribution

$$\mathbb{P}(N(J) = k) = \mathrm{e}^{-m(J)} \frac{m(J)^k}{k!}.$$

where $m(J)$ is the Lebesgue measure of J.

Remark 2.3.2. In the literature, the study of rare events is often tied to the existence of EVLs or the existence of HTS and return times statistics (RTS). The existence of EVL has to do with the existence of distributional limits for $M_n = \max\{X_0, \ldots, X_{n-1}\}$. On the other hand, the existence of exponential HTS means the existence of a distributional limit for the elapsed time until the orbit hits certain balls around ζ, when properly normalized. When the orbit starts in the target ball around ζ and consequently we look at the first return (rather than hit) and its limit distribution, then we say that we have RTS, instead. Since no exceedances of u_n up to time n means that there are no entrances into a certain ball around ζ, the existence of EVLs is equivalent to the existence of HTS (see [11, 12]). Moreover, in [16] it was proved that an integral formula relates the distributions of HTS and RTS, which in particular yields the standard exponential distribution as its unique fixed point. We note also that certain extreme value statistics lift from base transformations to suspension flows [19].

Remark 2.3.3. Observe that the definition that we give here of the REPP is related to the EVL approach. In fact, the REPP defined here can be identified as the exceedance point process defined in [11, section 3]. One can define germane point processes that were referred to as hitting times point processes, again in [11, section 3]. We note that by [11, theorems 3 and 4], the convergence of the REPP to a Poisson process can be reformulated in terms of the convergence of the hitting times point processes to the Poisson process, as well.

Remark 2.3.4. The convergence of the REPP to the Poisson process is stronger than the existence of an EVL for $X_0, X_1, \ldots$. In particular, not only can we recover the distributional limit for the maxima by observing that $\{M_n \leqslant u_n\} = \{N_n(n/v_n) = 0\}$, but also we can obtain the distributional limit of the order statistics. Namely, if $X_{1,n} \leqslant X_{2,n} \leqslant \ldots \leqslant X_{n,n}$ denote the order statistics of the first n random variables of the process, then $\{X_{n-k,n} \leqslant u_n\} = \{N_n(n/v_n) \leqslant k\}$.

Leadbetter [21] introduced some conditions on the dependence structure of general stationary stochastic processes, called $D(u_n)$ and $D'(u_n)$, which can be used to prove the convergence of REPP to the Poisson process (see [22, section 5]). However, condition $D(u_n)$, which imposes some sort of uniform mixing, is often too strong to be verified in a dynamical setting. Recently, Freitas *et al* [11] gave an alternative condition, named $D_3(u_n)$, which together with the original $D'(u_n)$ was enough to prove the convergence of the REPP N_n in distribution to the standard Poisson process that we denote by N. This is precisely the statement of [11, theorem 5]. The great advantage of this weaker condition $D_3(u_n)$ is that it is much easier to check in a dynamical setting.

We will show that the stochastic processes arising from the billiard systems considered satisfy both the conditions $D_3(u_n)$ and $D'(u_n)$. Hence, we give next the precise formulation of the two conditions.

For every $A \in \mathcal{R}$ we define

$$M(A) := \max\{X_i : i \in A \cap \mathbb{Z}\}.$$

In the particular case where $A = [0, n)$ we simply write, as before, $M_n = M([0, n))$. Also note that $\{M(A) \leqslant u_n\} = \{N_n(v_n^{-1}A) = 0\}$.

Condition ($D_3(u_n)$). We say that $D_3(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if there exists $\gamma(n, t)$ nonincreasing in t for each n, and $n\gamma(n, t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$ (which means that $t_n/n \to 0$ as $n \to \infty$) such that

$$|\mathbb{P}(\{X_0 > u_n\} \cap \{M(A + t) \leqslant u_n\}) - \mathbb{P}(X_0 > u_n)\mathbb{P}(M(A) \leqslant u_n)| \leqslant \gamma(n, t),$$

for all $A \in \mathcal{R}$ and $t \in \mathbb{N}$.

This last condition is reminiscent of the α-mixing property that is often used for describing rates of mixing with respect to partitions. In some sense the partition here is given by a neighbourhood ($\{X > u_n\}$ of ζ and its complement. The increment t here plays the role of the 'gap' which is needed to obtain a speed of mixing, where the events $\{X_0 > u_n\}$ and $\{M(A + t) \leqslant u_n\}$ become increasingly independent as $t \to \infty$. Here however, the rate of mixing γ is not uniform in n for a given value of t, but its dependence on n is sufficiently weak as to still allow for good limiting statistics. It is specially adapted to the problem of counting exceedances. Using the decay of correlations of the billiard systems considered, we will verify it for the stochastic processes arising from such systems.

Condition ($D'(u_n)$). We say that $D'(u_n)$ holds for the sequence $X_0, X_1, X_2, \ldots$ if

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=1}^{\lfloor n/k \rfloor} \mathbb{P}(X_0 > u_n, X_j > u_n) = 0. \tag{6}$$

While $D_3(u_n)$ is a condition on the long range dependence structure of the stochastic process $X_0, X_1, \ldots$, $D'(u_n)$ is instead a condition on the short range dependence structure which inhibits the appearance of clusters of exceedances. In other words, if we break the first n random variables into blocks of size $\lfloor n/k \rfloor$, then $D'(u_n)$ restricts the existence of more than one exceedance in each block, which means that the exceedances should appear scattered through the time line.

2.2. Planar dispersing billiards.

Let $\Gamma = \{\Gamma_i, i = 1, \ldots, k\}$ be a family of pairwise disjoint, simply connected C^3 curves with strictly positive curvature on the two-dimensional torus $\mathbb{T}^2$. The billiard flow B_t is the dynamical system generated by the motion of a point particle in $Q = \mathbb{T}^2/(\cup_{i=1}^k(\text{ interior } \Gamma_i))$ with constant unit velocity inside Q and with elastic reflections at $\partial Q = \cup_{i=1}^k \Gamma_i$, where elastic means 'angle of incidence equals angle of reflection'. If each Γ_i is a circle, then this system is called a periodic Lorentz gas, a well-studied model in physics [3]. The billiard flow is Hamiltonian and preserves a probability measure (which is the Liouville measure) $\tilde{\mu}$ given by $\mathrm{d}\tilde{\mu} = C_Q\, \mathrm{d}q\, \mathrm{d}t$ where C_Q is a normalizing constant and the $q \in Q$, $t \in \mathbb{R}$, are Euclidean coordinates.

We first consider the billiard map $T : \partial Q \to \partial Q$. Let r be a one-dimensional coordinatization of Γ corresponding to length and let $n(r)$ be the outward normal to Γ at the point r. For each $r \in \Gamma$ we consider the tangent space at r consisting of unit vectors v such that $(n(r), v) \geqslant 0$. We identify each such unit vector v with an angle $\theta \in [-\pi/2, \pi/2]$. The boundary M is then parametrized by $M := \partial Q = \Gamma \times [-\pi/2, \pi/2]$, so M consists of the points (r, θ). $T : M \to M$ is the Poincaré map that gives the position and angle $T(r, \theta) = (r_1, \theta_1)$ after a point (r, θ) flows under B_t and collides again with M, according to the rule 'angle of incidence equals angle of reflection'. Thus if (r, θ) is the time of flight before collision, then $T(r, \theta) = B_{h(r,\theta)}(r, \theta)$. The billiard map preserves a probability measure $\mathrm{d}\mu = c_M \cos\theta\, \mathrm{d}r\, \mathrm{d}\theta$ equivalent to the two-dimensional Lebesgue measure $\mathrm{d}m = \mathrm{d}r\, \mathrm{d}\theta$ with density $\rho(x) = c_M \cos\theta$ where $x = (r, \theta)$ and c_M is a normalizing constant, to ensure that the invariant measure is a probability measure, i.e. has total mass 1.

We define the time of flight, $h : \partial Q \to \mathbb{R}$, by $h(x, r) = \min\{t > 0 : B_t(x, r) \in \partial Q\}$, i.e. the flow time that it takes for a point on the boundary of Q to return to the boundary. Under the assumption of a finite horizon condition, namely, that the time of flight $h(r, \theta)$ is uniformly bounded above, Young [27] proved that the billiard map has exponential decay of correlations for Hölder observations. The strategy relied on building a Gibbs–Markov structure, that is now usually called a *Young tower*, with a corresponding induced map bearing nice hyperbolic properties. Then the idea was to pass the good statistical properties of the induced map to the original system, in which the tail of the inducing time ended up playing a prominent role—in particular, in the determination of the system's mixing rates. This settled a long-standing question about the rate of decay of correlations in such systems. Chernov [7] extended this result to planar dispersing billiards with piecewise C^3 smooth boundaries and where the flight time $h(x, r)$ can become singular along countable numbers of smooth curves. Chernov also proved exponential decay for dispersing billiards with corner points (a class of billiards that we do not discuss in this paper). Good references for background results for this section are the papers [4, 5, 7, 27].

Our first theorem is as follows.

Theorem 2.4. *Let $T : M \to M$ be a planar dispersing billiard map. Consider that the stochastic process $X_0, X_1, \ldots$ is given as in (3) for observables φ of the type considered above. Then for μ-a.e. ζ, conditions $D_3(u_n)$ and $D'(u_n)$ hold for $X_0, X_1, \ldots$ and sequences $(u_n)_{n\in\mathbb{N}}$ satisfying (5). Consequently, the REPP N_n given in definition 2.1 converges in distribution to the standard Poisson process.*

Remark 2.4.1. Observe that the convergence of the REPP N_n to the Poisson process N implies that $N_n(t)$ converges in distribution to $N(t)$ for all $t > 0$. In particular, for each $t > 0$ and each integer $k \in \mathbb{N}_0$,

$$\lim_{n\to\infty} \mu(N_n(t) = k) = \mu(N(t) = k) = \mathrm{e}^{-t}\frac{t^k}{k!},$$

which is exactly the statement of (2).

The strategy for proving theorem 2.4 is to show the validity of conditions $D_3(u_n)$ and $D'(u_n)$ for various dynamical systems modelled by Young towers, in particular dispersing planar billiards. The proof of $D'(u_n)$ has been given in Gupta *et al* [15] but we reproduce it for completeness in section 3.1. The proof of $D_3(u_n)$ is similar to the proof for a related condition $D_2(u_n)$ (useful in establishing the existence of EVL) given in [15].

2.3. Billiards with polynomial mixing rates

In [27], Young introduced a Gibbs–Markov structure (which became known as a Young tower) which she used to study dispersing billiards with exponential decay of correlations. Later on, Markarian [23] developed an elegant technique for using inducing to establish polynomial upper bounds for rates of decay of correlation in certain billiard systems. Young [28] had used coupling to establish polynomial decay for certain non-uniformly expanding maps and Markarian's ideas built upon this work.

Markarian's idea was to find a subset $M \subset X$ on which the first-return map $F : M \to M$ has strong hyperbolic behaviour, and in particular admits a Young tower with exponential tails. His approach was subsequently extended by Chernov and Zhang [8] to many billiard systems exhibiting polynomial decay.

Notation: Given a finite measure μ on X and a measurable set $A \subset X$ ($\mu(A) > 0$), we denote by μ_A the corresponding conditional measure on A, i.e. $\mu_A(B) = \mu(A \cap B)/\mu(A)$ for $B \subset X$ measurable.

The first-hitting-time function going to M is given by

$$r_M(x) := \min\{j \geqslant 1 : T^j(x) \in M\} \tag{7}$$

and measures the time until the orbit of a point $x \in X$ enters M. The induced map $F : M \circlearrowleft$ is then given by $F = T^{r_M}$ and its invariant measure is the normalized measure μ_M. If the tails for the return time decay polynomially, that is if $\mu(x \in X : r_M(x) > n) = \mathcal{O}(n^{-a})$ for some constant $a > 0$, then, as shown by Markarian [23],

$$\left| \int \phi\, \psi \circ T^n \,\mathrm{d}\mu - \int \phi \,\mathrm{d}\mu \int \psi \,\mathrm{d}\mu \right| \leqslant C n^{-a} \|\phi\|_{\mathrm{Lip}} \|\psi\|_{\mathrm{Lip}} \tag{8}$$

for some constant C. This allows us to extend our results above on Poisson limit laws to the setting of billiards with polynomial mixing rates, by first inducing on M and then realizing $T : X \to X$ as a first-return-time tower over (F, M, μ_M).

Theorem 2.5. *Suppose that (T, X, μ) is a billiard system with SRB measure μ and $M \subset X$ is a subset such that the first-return map $F : M \to M$ admits the structure of a Young tower with exponential tails. Suppose further that the function r_M, defined in (7), is integrable with respect to μ. Consider now that the stochastic process $X_0, X_1, \ldots$ is given as in (3) for observables φ of the type considered above. Then for μ-a.e. ζ, the REPP N_n given in definition 2.1 converges in distribution to the standard Poisson process.*

The idea for proving theorem 2.5 is to use the same strategy as was used for dispersing billiards to show that for the first-return-time map $F : M \to M$ and for the stochastic processes that it gives rise to, we have convergence of the point processes N_n to the standard Poisson process, μ-a.e. Then we use an idea introduced in [2], which essentially says that the original system T shares the same property of the first-return-time map F, meaning that for stochastic processes arising from the dynamics of T we also have that the point processes N_n converge to the standard Poisson process, for μ-a.e. ζ. Unfortunately, the original statement of [2] only allows to conclude that if the first-return-time map F has exponential HTS/RTS for balls around μ-a.e. ζ, then the original system T also has the same property. However, as remarked in [2], a small adjustment to the argument used there allows one to prove the stronger statement that the same holds for the convergence of point processes to the standard Poisson process. For completeness, we state here such a result and prove it in section 4.

In order to distinguish objects of the induced system F from the corresponding objects of the original system, we will use the symbol $\hat{}$ over these objects. In particular we will write $\hat{\mu} := \mu_M$. Let $\zeta \in M$ and φ be an observable as above, which achieves a global maximum at ζ.

This new induced system gives rise to a new set of random variables

$$\hat{X}_n = \varphi \circ F^n.$$

We can thus consider $\hat{N}_n(J)$ for $J \in \mathcal{S}$ and $\hat{v} = 1/\hat{\mu}(\hat{X}_0 > u_n)$ defined analogously to $N_n(J)$ in definition 2.1 for the original system.

Proposition 2.6. *Suppose that (T, X, μ) is a dynamical system with μ absolutely continuous with respect to the Lebesgue measure, and that $M \subset X$ is a measurable set with $\mu(M) > 0$, and let $F : M \to M$ denote the first-return induced map. Assume that $\hat{N}_n$ converges in distribution (w.r.t. $\hat{\mu}$) to a standard Poisson process N, for $\hat{\mu}$-a.e. $\zeta \in M$. Then for the original map (T, X, μ) we can say that N_n converges in distribution (w.r.t. the measure μ) to a standard Poisson process for μ-a.e. point $\zeta \in M$.*

We remark that the statement of [2], which said that the limit distribution for HTS/RTS for the induced map F was equal, at μ-a.e. point ζ, to the respective HTS/RTS distributional limit for the original system T, was extended in [18] by removing the μ-a.e. point ζ restriction. In an ongoing work concerning an extremal dichotomy for intermittent maps, the first-named author with A C M Freitas, M Todd and S Vaienti have proved an extension of the [18] result to include the convergence of point processes, which implies proposition 2.6.

3. Condition $D_3(u_n)$ for Young towers with exponential tails

We will make an assumption on the invariant measure μ, which is automatically satisfied for planar billiard maps. We assume the following.

Assumption A : For μ-a.e. $\zeta \in M$ there exists $\bar{q} := \bar{q}(\zeta) > 0$ such that if $A_{r,\epsilon}(\zeta) = \{y \in M : r \leqslant d(\zeta, y) \leqslant r + \epsilon\}$ is a shell of inner radius r and outer radius $r + \epsilon$ about the point ζ and if r is sufficiently small, $0 < \epsilon \ll r < 1$, then $\mu(A_{r,\epsilon}(\zeta)) \leqslant \epsilon^{\bar{q}}$.

Assumption A is satisfied by planar dispersing billiards with finite and infinite horizons as the invariant measure is equivalent to the Lebesgue one. This is proved in [5, appendix 2] where it is shown that $\bar{q}$ may be taken as 1 in the case of a finite horizon and 4/5 in the case of an infinite horizon.

The Young tower assumption implies that there exists a subset $\Lambda \subset M$ such that Λ has a hyperbolic product structure and that (P1)–(P4) of [27] hold. We refer the reader to Young's paper [27] and the book by Baladi [1] for details. A similar axiomatic construction of a tower is given by Chernov [7], which is a good reference for background on dispersing billiard maps and flows.

By taking T to be a local diffeomorphism, we allow the map T or its derivative to have discontinuities or singularities.

Next we describe briefly the structure of a Young tower with exponential return time tails for a local diffeomorphism $T : M \to M$ of a Riemannian manifold M equipped with Lebesgue measure m. There is a set Λ with a hyperbolic product structure as in Young [27] and we assume that there is an $\mathscr{L}^1(m)$ return time function $R : \Delta_0 \to \mathbb{N}$. Moreover assume that there is a countable partition $\Lambda_{0,i}$ of Δ_0 such that R is constant on each partition element $\Lambda_{0,i}$. We put $R_i := R|_{\Lambda_{0,i}}$. Now the Young tower is defined by

$$\Delta = \bigcup_{i\in\mathbb{N},0\leqslant l\leqslant R_i-1} \{(x,l) : x \in \Lambda_{0,i}\}$$

and the tower map $F : \Delta \to \Delta$ by

$$F(x,l) = \begin{cases} (x, l+1) & \text{if } x \in \Lambda_{0,i}, l < R_i - 1 \\ (T^{R_i}x, 0) & \text{if } x \in \Lambda_{0,i}, l = R_i - 1 \end{cases}.$$

We will refer to $\Delta_0 := \cup_i(\Lambda_{0,i}, 0)$ as the base of the tower Δ and define $\Lambda_i := \Lambda_{0,i}$. Similarly we call $\Delta_l = \{(x,l) : l < R(x)\}$ the lth level of the tower. Define the return map $f = T^R : \Delta_0 \to \Delta_0$ by $f(x) = T^{R(x)}(x)$. We may form a quotiented tower (see [27] for details) by introducing an equivalence relation for points on the same stable manifold. We now list the features of the tower that we will use.

There exists an invariant measure m_0 for $f : \Delta_0 \to \Delta_0$ which has absolutely continuous conditional measures on local unstable manifolds in Δ_0, with density bounded uniformly from above and below.

There exists an F-invariant measure ν on Δ which is given by $\nu(B) = \frac{m_0(F^{-l}B)}{\int_{\Delta_0} R\,\mathrm{d}m_0}$ for measurable $B \subset \Lambda_l$, and extended to the entire tower Δ in the obvious way. There is a projection $\pi : \Delta \to M$ given by $\pi(x,l) = T^l(x)$ which semi-conjugates F and T, that is it satisfies $\pi \circ F = T \circ \pi$. The invariant measure μ, which is an SRB measure for $T : M \to M$, is then given by $\mu = \pi_*\nu$. Denote by $W^s_\varepsilon(x)$ the local stable manifold through x, i.e. there exist $\varepsilon(x) > 0$, $C > 0$, $0 < \alpha < 1$ such that

$$W^s_\varepsilon(x) = \{y : d(x,y) < \varepsilon, d(T^n y, T^n x) < C\alpha^n \ \forall n \geqslant 0\}.$$

We use the notation $W^s_{\mathrm{loc}}(x)$ rather than $W^s_\varepsilon(x)$ in contexts where the length of the local stable manifold is not important. Analogously one defines the local unstable manifold $W^u_{\mathrm{loc}}(x)$. Let $B(x,r)$ denote the ball of radius r centred at the point x. We lift a function $\phi : M \to \mathbb{R}$ to Δ by defining, with abuse of notation, $\phi(x,l) = \phi(T^l x)$.

Under the assumption of exponential tails, that is if $m(R > n) = \mathcal{O}(\theta^n)$ for some $0 < \theta < 1$, then from the computations in [27] one can deduce that there exists $0 < \theta_1 < 1$ such that for all Lipschitz ϕ, ψ we have

$$\left|\int \phi\psi \circ T^n \,\mathrm{d}\mu - \int \phi \,\mathrm{d}\mu \int \psi \,\mathrm{d}\mu\right| \leqslant C\theta_1^n \|\phi\|_{\mathrm{Lip}} \|\psi\|_{\mathrm{Lip}} \tag{9}$$

for some constant C. Moreover, if the lift of ψ is constant on local stable leaves of the Young tower, then

$$\left|\int \phi\psi \circ T^n \,\mathrm{d}\mu - \int \phi \,\mathrm{d}\mu \int \psi \,\mathrm{d}\mu\right| \leqslant C\theta_1^n \|\phi\|_{\mathrm{Lip}} \|\psi\|_{\infty}. \tag{10}$$

As before, let ζ be in the support of μ and define a stochastic process X_n given by $X_n(x) = -\log d(T^n x, \zeta)$. In the remainder of this section we establish condition $D_3(u_n)$ for maps modelled by a Young tower with exponential tails satisfying assumption A. Our main theorem for this section is as follows.

Theorem 3.1. *Let $T : (M,\mu) \to (M,\mu)$ be a dynamical system modelled by a Young tower with exponential tails satisfying assumption A. Then the stochastic process $X_0, X_1, \ldots$ defined as in (3) satisfies the condition $D_3(u_n)$.*

Proof. For $\zeta \in M$, we first define

$$B_{r,k}(\zeta) = \left\{x : T^k(W^s_\varepsilon(x)) \cap \partial B(\zeta, r) \neq \emptyset\right\},$$

and obtain as an immediate consequence of assumption A the following:

Proposition 3.2. *Under assumption A there exist constants $C > 0$ and $0 < \tau_1 < 1$ such that for any r, k,*

$$\mu(B_{r,k}(\zeta)) \leqslant C\tau_1^k. \tag{11}$$

Proof. As a consequence of the uniform contraction of local stable manifolds [27, (P2)], there exist $\alpha \in (0, 1)$ and $c_1 > 0$ such that $d(T^n(x), T^n(y)) \leqslant c_1\alpha^n$ for all $y \in W^s_\varepsilon(x)$. In particular, this implies that $|T^k(W^s_\varepsilon(x))| \leqslant c_1\alpha^k$ where $|\cdot|$ denotes the length with respect to the Lebesgue measure. Therefore, for every $x \in B_{r,k}(\zeta)$ the leaf $T^k(W^s_\varepsilon(x))$ lies in an annulus of width $2c_1\alpha^k$ around $\partial B(\zeta, r)$. By assumption A and the invariance of μ, the result follows, with $C = (2c_1)^{\bar{q}}$ and $\tau_1 = \alpha^{\bar{q}}$. □

We now continue the proof of theorem 3.1. The constant τ_1 below is from proposition 3.2. Let $A \in S$ such that $A = \cup_{j=1}^{l}[a_j, b_j)$, and define $I_A = [a_1, b_l]$.

Lemma 3.1. *Suppose that $\Phi : M \to \mathbb{R}$ is Lipschitz and Ψ_A is the indicator function*

$$\Psi_A := 1_{\{M(A)\leqslant u_n\}}.$$

Then, for all $j \geqslant 0$,

$$\left|\int \Phi\Psi_A \circ T^j \, \mathrm{d}\mu - \int \Phi \, \mathrm{d}\mu \int \Psi_A \, \mathrm{d}\mu\right| \leqslant \mathcal{O}(1)\left(\|\Phi\|_\infty\tau_1^{\lfloor j/2\rfloor} + \|\Phi\|_{\mathrm{Lip}}\theta^{\lfloor j/2\rfloor}\right). \tag{12}$$

Proof. Define the function $\tilde{\Phi} : \Delta \to \mathbb{R}$ by $\tilde{\Phi}(x, r) = \Phi(T^r(x))$ and define the function $\tilde{\Psi}_A(x, r) = \Psi_A(T^r(x))$. We choose a reference unstable manifold $\tilde{\gamma}^u \subset \Delta_0$, and by the hyperbolic product structure each local stable manifold $W^s_\varepsilon(x)$ will intersect $\tilde{\gamma}^u$ at a unique point $\hat{x}$. Here x denotes a point in the base of the tower Δ_0 and we therefore have $x \in W^s_\varepsilon(\hat{x})$.

We define the function $\overline{\Psi}_A(x, r) := \Psi_A(\hat{x}, r)$. We note that $\overline{\Psi}_A$ is constant along stable manifolds in Δ and that the set of points where $\overline{\Psi}_A \neq \tilde{\Psi}_A$ is, by definition, the set of (x, r) which project to points $T^r(x)$ for which there exist x_1, x_2 on the same local stable manifold as $T^r(x)$ for which

$$x_1 \in \{M(A) \leqslant u_n\}$$

but

$$x_2 \notin \{M(A) \leqslant u_n\}.$$

This set is contained inside $\cup_{k=a_1}^{a_1+b_l} T^{-k}B_{u_n,k}(\zeta)$. If we let $a_1 \geqslant \lfloor j/2\rfloor$, then by proposition 3.2 we have

$$\nu\left\{\tilde{\Psi}_{\lfloor j/2\rfloor,\lfloor j/2\rfloor+b_l} \neq \overline{\Psi}_{\lfloor j/2\rfloor,\lfloor j/2\rfloor+b_l}\right\} \leqslant \sum_{k=\lfloor j/2\rfloor}^{\infty} \mu(B_{u_n,k}) \leqslant \mathcal{O}(1)\tau_1^{\lfloor j/2\rfloor}.$$

By the decay of correlations as proved in [27] under the assumption of exponential tails, we have

$$\left|\int \tilde{\Phi}\overline{\Psi}_{A+\lfloor j/2\rfloor} \circ F^{j-\lfloor j/2\rfloor} \, \mathrm{d}\nu - \int \tilde{\Phi} \, \mathrm{d}\nu \int \overline{\Psi}_{A+\lfloor j/2\rfloor} \, \mathrm{d}\nu\right| \leqslant \mathcal{O}(1)\|\Phi\|_{\mathrm{Lip}}\|\Psi\|_\infty\theta^{\lfloor j/2\rfloor}.$$

Recall that

$$\left|\int \Phi\Psi_{A+\lfloor j/2\rfloor}\circ T^{j-\lfloor j/2\rfloor}\,\mathrm{d}\mu-\int\Phi\,\mathrm{d}\nu\int\Psi_{A+\lfloor j/2\rfloor}\,\mathrm{d}\mu\right|$$
$$=\left|\int\tilde\Phi\tilde\Psi_{A+\lfloor j/2\rfloor}\circ F^{j-\lfloor j/2\rfloor}\,\mathrm{d}\nu-\int\tilde\Phi\,\mathrm{d}\nu\int\tilde\Psi_{A+\lfloor j/2\rfloor}\,\mathrm{d}\nu\right|.$$

We will use the identity $\int\tilde\phi\tilde\psi\circ F-\int\tilde\phi\int\tilde\psi=\int\tilde\phi(\tilde\psi\circ F-\bar\psi\circ F)+\int\tilde\phi\bar\psi\circ F-\int\tilde\phi\int\bar\psi+\int\tilde\phi\int\bar\psi-\int\tilde\phi\int\tilde\psi$. Thus

$$\left|\int \Phi\Psi_{A+\lfloor j/2\rfloor}\circ T^{j-\lfloor j/2\rfloor}\,\mathrm{d}\mu-\int\Phi\,\mathrm{d}\nu\int\Psi_{A+\lfloor j/2\rfloor}\,\mathrm{d}\mu\right|$$
$$=\left|\int\tilde\Phi\tilde\Psi_{A+\lfloor j/2\rfloor}\circ F^{j-\lfloor j/2\rfloor}\,\mathrm{d}\nu-\int\tilde\Phi\,\mathrm{d}\nu\int\tilde\Psi_{A+\lfloor j/2\rfloor}\,\mathrm{d}\nu\right|$$
$$\leqslant\left|\int\tilde\Phi\left(\tilde\Psi_{A+\lfloor j/2\rfloor}-\overline{\Psi}_{A+\lfloor j/2\rfloor}\right)\circ F^{j-\lfloor j/2\rfloor}\,\mathrm{d}\nu\right|+\mathcal{O}(1)\|\Phi\|_{\mathrm{Lip}}\theta^{\lfloor j/2\rfloor}$$
$$+\left|\int\tilde\Phi\,\mathrm{d}\nu\int\left(\overline{\Psi}_{A+\lfloor j/2\rfloor}-\tilde\Psi_{A+\lfloor j/2\rfloor}\right)\mathrm{d}\nu\right|$$
$$\leqslant\mathcal{O}(1)\left(2\|\Phi\|_\infty\nu\left\{\overline{\Psi}_{A+\lfloor j/2\rfloor}\neq\tilde\Psi_{A+\lfloor j/2\rfloor}\right\}+\|\Phi\|_{\mathrm{Lip}}\theta^{\lfloor j/2\rfloor}\right)$$
$$\leqslant\mathcal{O}(1)\left(\|\Phi\|_\infty\tau_1^{\lfloor j/2\rfloor}+\|\Phi\|_{\mathrm{Lip}}\theta^{\lfloor j/2\rfloor}\right).\tag{13}$$

We complete the proof by observing that $\int\Psi_A\,\mathrm{d}\mu=\int\Psi_{A+\lfloor j/2\rfloor}\,\mathrm{d}\mu$ by the μ-invariance of T and that $\Psi_{A+\lfloor j/2\rfloor}\circ T^{j-\lfloor j/2\rfloor}=\Psi_{A+j}=\Psi_A\circ T^j$. □

To prove condition $D_3(u_n)$, we will approximate the characteristic function of the set $U_n=\{X_0>u_n\}$ by a suitable Lipschitz function. This approximation will decrease sharply to zero near the boundary of the set U_n. The bound in lemma 3.1 involves the Lipschitz norm; therefore, we need to be able to bound the increase in this norm.

We approximate the indicator function 1_{U_n} by a Lipschitz continuous function Φ_n as follows. Since U_n is a ball of some radius $r_n\sim\frac{1}{\sqrt{n}}$ centred at the point ζ, we define Φ_n to be 1 inside a ball centred at ζ of radius $r_n-n^{-\frac{2}{\bar q}}$, where $\bar q$ comes from assumption A and decaying to 0, so on the boundary of U_n, Φ_n vanishes. The Lipschitz norm of Φ_n is seen to be bounded by $n^{\frac{2}{\bar q}}$ and $\|1_{U_n}-\Phi_n\|_1\leqslant\frac{1}{n^2}$. Therefore

$$\left|\int 1_{U_n}\Psi_{A+\lfloor j/2\rfloor}\circ T^{j-\lfloor j/2\rfloor}\,\mathrm{d}\mu-\mu(U_n)\int\Psi_{A+l}\,\mathrm{d}\mu\right|$$
$$\leqslant\left|\int\left(1_{U_n}-\Phi_n\right)\Psi_{A+\lfloor j/2\rfloor}\,\mathrm{d}\mu\right|+\mathcal{O}(1)\left(\|\Phi_n\|_\infty j^2\tau_1^{\lfloor j/4\rfloor}+\|\Phi_n\|_{\mathrm{Lip}}\theta^{\lfloor j/2\rfloor}\right)$$
$$+\left|\int\left(1_{U_n}-\Phi_n\right)\mathrm{d}\mu\int\Psi_{A+\lfloor j/2\rfloor}\,\mathrm{d}\mu\right|,\tag{14}$$

and consequently

$$|\mu(U_n\cap\{M(A+l)\leqslant u_n\})-\mu(U_n)\mu(\{M(A)\leqslant u_n\})|\leqslant\gamma(n,j),$$

where

$$\gamma(n,j)=\mathcal{O}(1)\left(n^{-2}+n^{\frac{2}{\bar q}}\theta_1^{\lfloor j/2\rfloor}\right)$$

where $\theta_1=\max\{\tau_1,\theta\}$. Let $j=t_n=(\log n)^5$. Then $n\gamma(n,t_n)\to0$ as $n\to\infty$. Note that we had considerable freedom of choice as regards t_n; anticipating our applications, we choose $t_n=(\log n)^5$. □

3.1. Property $D'(u_n)$ for planar dispersing billiard maps

We have shown that $D_3(u_n)$ is immediate in the case of dispersing billiard maps with finite horizon, as they are modelled by a Young tower in [27] and have exponentially decaying correlations. Chernov [7, section 5] (see also [5, section 5]) constructs a Young tower for billiards with infinite horizon to prove exponential decay of correlations, so condition $D_3(u_n)$ is satisfied by this class of billiard map as well. Hence to prove a Poisson limit law, we need only prove condition $D'(u_n)$, which we do in this section.

It is known (see [7, lemma 7.1] for finite horizon and [7, section 8] for infinite horizon) that dispersing billiard maps expand in the unstable direction in the Euclidean metric $|\cdot| = \sqrt{(\mathrm{d}r)^2 + (\mathrm{d}\phi)^2}$,[4] in that $|\mathrm{D}T_u^n v| \geqslant C\tilde{\lambda}^n |v|$ for some constant $C > 0$ and $\tilde{\lambda} > 1$ independently of v.

If we choose N_0 such that $\lambda := C\tilde{\lambda}^{N_0} > 1$, then T^{N_0} (or $\mathrm{D}T^{N_0}$) expands unstable manifolds (tangent vectors to unstable manifolds) uniformly in the Euclidean metric.

It is common to use the p-metric in proving ergodic properties of billiards. Recall that for any curve γ, the p-norm of a vector tangent to γ is given as $|v|_p = \cos\phi(r)|\,\mathrm{d}r|$ where γ is parametrized in the (r,ϕ) plane as $(r,\phi(r))$. Since the Euclidean metric in the (r,ϕ) plane is given by $\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}\phi^2$, this implies that $|v|_p \leqslant \cos\phi(r)\,\mathrm{d}s \leqslant \mathrm{d}s = |v|$. We will use $l_p(C)$ to denote the length of a curve in the p-metric and $l(C)$ to denote length in the Euclidean metric. If γ is a local unstable manifold or local stable manifold, then $C_1 l(\gamma)_p \leqslant l(\gamma) \leqslant C_2\sqrt{l_p(\gamma)}$.

For planar dispersing billiards, there exists an invariant measure μ (which is equivalent to the two-dimensional Lebesgue measure), and through μ-a.e. point x, there exist a local stable manifold $W^s_{\mathrm{loc}}(x)$ and a local unstable manifold $W^u_{\mathrm{loc}}(x)$. The SRB measure μ has absolutely continuous (with respect to the Lebesgue measure) conditional measures μ_x on each $W^u_{\mathrm{loc}}(x)$. The expansion by $\mathrm{D}T$ is unbounded, however, in the p-metric at $\cos\theta = 0$ and this may lead to quite different expansion rates at different points on $W^u_{\mathrm{loc}}(x)$. To overcome this effect and obtain uniform estimates on the densities of conditional SRB measure, it is common to definite homogeneous local unstable and local stable manifolds. This approach was adopted in [4, 5, 7, 27]. Fix a large k_0 and define for $k > k_0$

$$I_k = \left\{(r,\theta) : \frac{\pi}{2} - k^{-2} < \theta < \frac{\pi}{2} - (k+1)^{-2}\right\}$$

$$I_{-k} = \left\{(r,\theta) : -\frac{\pi}{2} + (k+1)^{-2} < \theta < -\frac{\pi}{2} + k^{-2}\right\}$$

and

$$I_{k_0} = \left\{(r,\theta) : -\frac{\pi}{2} + k_0^{-2} < \theta < \frac{\pi}{2} - k_0^{-2}\right\}.$$

We call a local unstable (stable) manifold $W^u_{\mathrm{loc}}(x)$ ($W^s_{\mathrm{loc}}(x)$) *homogeneous* if $T^n W^u_{\mathrm{loc}}(x)$ ($T^{-n} W^s_{\mathrm{loc}}(x)$) does not intersect any of the line segments in $\cup_{k>k_0}(I_k \cup I_{-k}) \cup I_{k_0}$ for all $n \geqslant 0$. Homogeneous $W^u_{\mathrm{loc}}(x)$ have almost constant conditional SRB densities $\frac{\mathrm{d}\mu_x}{\mathrm{d}m_x}$ in the sense that there exists $C > 0$ such that $\frac{1}{C} \leqslant \frac{\mathrm{d}\mu_x}{\mathrm{d}m_x}(z_1)/\frac{\mathrm{d}\mu_x}{\mathrm{d}m_x}(z_2) \leqslant C$ for all $z_1,\ z_2 \in W^u_{\mathrm{loc}}(x)$ (see [7, section 2] and the remarks following theorem 3.1).

From this point on, all the local unstable (stable) manifolds that we consider will be homogeneous. Bunimovich *et al* [5, appendix 2, equation A2.1] give quantitative estimates for the length of homogeneous $W^u_{\mathrm{loc}}(x)$. They show that there exist $C,\ \tau > 0$ such that $\mu\{x : l(W^s_{\mathrm{loc}}(x)) < \epsilon \text{ or } l(W^u_{\mathrm{loc}}(x)) < \epsilon\} \leqslant C\epsilon^\tau$, where $l(C)$ denotes the one-dimensional Lebesgue measure or length of a rectifiable curve C. In our setting, τ could be taken to be $\frac{2}{9}$;

[4] Note that before we used the notation $|\cdot|$ to denote the length of a set with respect to the Lebesgue measure. The argument will be used to distinguish between the Euclidean metric (applied to vectors) and the length (applied to sets).

its exact value will play no role, but for simplicity in the forthcoming estimates we assume that $0 < \tau < \frac{1}{2}$.

The natural measure μ has absolutely continuous conditional measures μ_x on local unstable manifolds $W^u_{\text{loc}}(x)$, which have almost uniform densities with respect to the Lebesgue measure on $W^u_{\text{loc}}(x)$, by [7, equation 2.4].

3.1.1. Controlling the measure of the set of rapidly returning points. Let $A_{\sqrt{\epsilon}} = \{x : |W^u_{\text{loc}}(x)| > \sqrt{\epsilon}\}$; then $\mu(A^c_{\sqrt{\epsilon}}) < c_1\epsilon^{\tau/2}$ by Bunimovich's result. Let $x \in A_{\sqrt{\epsilon}}$ and consider $W^u_{\text{loc}}(x)$. Since $|T^{-k}W^u_{\text{loc}}(x)| < \lambda^{-1}|W^u_{\text{loc}}(x)|$ for $k > N_0$ we obtain, by the triangle inequality, for $y, y' \in W^u_{\text{loc}}(x)$,

$$d(y, y') \leqslant d(T^{-k}y', y') + d(T^{-k}y, T^{-k}y') + d(T^{-k}y, y) \leqslant 2\epsilon + \frac{1}{\lambda}d(y, y')$$

which implies that $d(y, y') \leqslant 2(1 - \frac{1}{\lambda})\epsilon$. Thus

$$l\{y \in W^u_{\text{loc}}(x) : d(y, T^{-k}y) < \epsilon\} \leqslant 2(1 - \lambda^{-1})\epsilon \leqslant c_2\sqrt{\epsilon}\, l\{y \in W^u_{\text{loc}}(x)\}.$$

Since the density of the conditional SRB measure μ_x is bounded above and below with respect to the one-dimensional Lebesgue measure, we obtain $\mu_x(y \in W^u_{\text{loc}}(x) : d(y, T^{-k}y) < \epsilon) < c_3\sqrt{\epsilon}$. Integrating over all unstable manifolds in $A_{\sqrt{\epsilon}}$ (discarding the set $\mu(A^c_{\sqrt{\epsilon}})$), we obtain $\mu\{x : d(T^{-k}x, x) < \epsilon) < c_4\epsilon^{\tau/2}$ ($c_4 \leqslant c_1 + c_3$). Since μ is T-invariant, we get

$$\mathcal{E}_k(\epsilon) := \mu\{x : d(T^k x, x) < \epsilon\} < c_4\epsilon^{\tau/2}$$

for $k > N_0$. Consequently,

$$E_k := \{x : d(T^j x, x) \leqslant \frac{2}{\sqrt{k}} \text{ for some } 1 \leqslant j \leqslant \log^5 k\}$$

obeys the upper bound $\mu(E_k) \leqslant c_5 k^{-\sigma}$ for any $\sigma > \frac{\tau}{4}$. Let us note that a similar result has been shown in [6], lemma 4.1.

3.1.2. Controlling the measure of the set of points whose neighbourhoods have large overlaps with the sets E_k. As in [9], we define the Hardy–Littlewood maximal function $\mathcal{M}_l$ for $\phi(x) = 1_{E_l}(x)\rho(x)$, where $\rho(x) = \frac{d\mu}{dm}(x)$, such that

$$\mathcal{M}_l(x) := \sup_{a>0} \frac{1}{m(B(x, a))} \int_{B(x,a)} 1_{E_l}(y)\rho(y)\, dm(y).$$

Hence (see [10, page 96]),

$$m(|\mathcal{M}_l| > C) \leqslant \frac{\|1_{E_l}\rho\|_1}{C}$$

where $\|\cdot\|_1$ is the $\mathscr{L}^1$ norm with respect to m. Let

$$F_k := \{x : \mu(B(x, k^{-\gamma/2}) \cap E_{k^{\gamma/2}}) \geqslant (k^{-\gamma\beta/2})k^{-\gamma}.$$

Then $F_k \subset \{\mathcal{M}_{k^{\gamma/2}} > k^{-\gamma\beta/2}\}$ and hence

$$m(F_k) \leqslant \mu(E_{k^{\gamma/2}})k^{\gamma\beta/2} \leqslant Ck^{-\gamma\sigma/2}k^{\gamma\beta/2}.$$

If we take $0 < \beta < \sigma/2$ and $\gamma > \sigma/4$, then for some $\delta > 0$, we have $k^{-\gamma\sigma/2}k^{\gamma\beta/2} < k^{-1-\delta}$, and hence

$$\sum_k m(F_k) < \infty.$$

Thus by the Borel–Cantelli lemma, for m-a.e. (and hence μ-a.e.) $\zeta \in X$, there exists $N(\zeta)$ such that $\zeta \notin F_k$ for all $k > N(\zeta)$. Thus along the subsequence $n_k = k^{-\gamma/2}$, we have $\mu(U_{n_k} \cap T^{-j}U_{n_k}) \leqslant n_k^{-1-\delta}$ for $k > N(\zeta)$, where, as before, $U_n = \{X_0 > u_n\}$ (and thus $T^{-j}U_n = \{X_0 \circ T^j > u_n\}$). This is sufficient for obtaining an estimate for all u_n. For if $k^{\gamma/2} \leqslant n \leqslant (k+1)^{\gamma/2}$, then $\mu(U_n \cap T^{-j}U_n) \leqslant \mu(U_{n_k} \cap T^{-j}U_{n_k}) \leqslant n_k^{-1-\delta} \leqslant 2n^{-1-\delta}$ for all n large enough as $(\frac{k+1}{k})^{\gamma/2} \to 1$.

We now control the iterates $1 \leqslant j \leqslant N_0$. If ζ is not periodic, then $\min_{1\leqslant i<j\leqslant N_0} d(T^i\zeta, T^j\zeta) \geqslant s(\zeta) > 0$ and hence $\mu(U_n \cap T^{-j}U_n) = 0$ for all $1 \leqslant j \leqslant N_0$ and n large enough.

Since u_n was chosen such that $n\mu(U_n) \to 1$, we get

$$\mu(U_n \cap T^{-j}U_n) \leqslant 2n^{-1-\delta}$$

for any $1 \leqslant j \leqslant \log^5 n$, and consequently

$$\lim_{n\to\infty} n \sum_{j=1}^{\log^5 n} \mu(U_n \cap T^{-j}U_n) = 0.$$

3.1.3. *Accounting for exceedances between* $\log^5 n$ *and* $\sqrt{n}$.

We use exponential decay of correlations to show that

$$\lim_{n\to\infty} n \sum_{j=\log^5 n}^{p=\sqrt{n}} \mu(U_n \cap T^{-j}U_n) = 0. \tag{15}$$

As before, we approximate the indicator function 1_{U_n} of the set U_n by a suitable Lipschitz function. Recall that U_n is a ball of some radius $r_n \sim \frac{1}{\sqrt{n}}$ centred at the point ζ. We define Φ_n to be 1 inside $B(\zeta, r_n - n^{-\frac{2}{\bar{q}}})$, where $\bar{q}$ comes from assumption A, and to decay to $\Phi_n = 0$ on $X \setminus U_n$. The Lipschitz norm of Φ_n is then bounded by $n^{\frac{2}{\bar{q}}}$. Thus

$$\begin{aligned}\left|\int 1_{U_n}\left(1_{U_n}\circ T^j\right)\mathrm{d}\mu - \left(\int 1_{U_n}\mathrm{d}\mu\right)^2\right| \leqslant{}& \left|\int \Phi_n\left(\Phi_n\circ T^j\right)\mathrm{d}\mu - \left(\int \Phi_n\,\mathrm{d}\mu\right)^2\right| \\ &+\left|\left(\int \Phi_n\,\mathrm{d}\mu\right)^2 - \left(\int 1_{U_n}\mathrm{d}\mu\right)^2\right| \\ &+\left|\int 1_{U_n}\left(1_{U_n}\circ T^j\right)\mathrm{d}\mu - \int \Phi_n\left(\Phi_n\circ T^j\right)\mathrm{d}\mu\right|.\end{aligned}$$

If $(\log n)^5 \leqslant j \leqslant p = \sqrt{n}$, then we obtain, by decay of correlations, for the first term,

$$\left|\int \Phi_n\left(\Phi_n\circ T^j\right)\mathrm{d}\mu - \left(\int \Phi_n\,\mathrm{d}\mu\right)^2\right| \leqslant Cn^{\frac{4}{\bar{q}}}\theta^j \leqslant \frac{C}{n^2}$$

if n is sufficiently large. For the second term we obtain, for n large enough,

$$\left|\left(\int \Phi_n\,\mathrm{d}\mu\right)^2 - \left(\int 1_{U_n}\mathrm{d}\mu\right)^2\right| \leqslant \mu\left(A_{r_n,n^{-2/\bar{q}}}\right) \leqslant \left(n^{-2/\bar{q}}\right)^{\bar{q}} < Cn^{-2}.$$

Similarly we estimate the third term as follows:

$$\left|\int \Phi_n\left(\Phi_n\circ T^j\right)\mathrm{d}\mu - \int 1_{U_n}\left(1_{U_n}\circ T^j\right)\mathrm{d}\mu\right| \leqslant 2\mu\left(A_{r_n,n^{-2/\bar{q}}}\right) \leqslant \frac{C}{n^2}.$$

Hence equation (15) is satisfied, which concludes the proof of theorem 2.4.

4. Billiards with polynomial mixing rates

Proof of theorem 2.5. First suppose that ζ is a generic point in M. We may establish a Poisson limit law for nested balls about ζ by proving $D_3(u_n)$ and $D'(u_n)$ as in the case of Sinai dispersing billiards for the map $F : M \to M$ with respect to the measure μ_M. To prove $D_3(u_n)$, note that local stable manifolds contract exponentially, assumption A holds (as the measure $\mu_M(\cdot) = \frac{1}{\mu(M)}(\cdot \cap M)$) and the exponential decay of equation (10) in the Lipschitz norm versus $\mathscr{L}^\infty(m)$ holds because we have the structure of a Young tower for $F : M \to M$. Hence $D_3(u_n)$ holds for generic points ζ in M. These are all of the ingredients of the proof for $D_3(u_n)$.

The proof of $D'(u_n)$ also proceeds in the same way as for Sinai dispersing billiards; as the local unstable manifolds contract uniformly under F^{-1}, the measure μ_M decomposes into a conditional measure on the local unstable manifolds which is absolutely continuous with respect to the Lebesgue measure. These are all of the ingredients of the proof of $D'(u_n)$ for Sinai dispersing billiards.

Finally we use proposition 2.6 to extend this result to generic points in phase space. □

Proof of proposition 2.6. The argument below is built on adjustments of the proofs of [2, theorem 2.1] and [13, theorem 5]. Since N is a simple point process, without multiple events, we may use a criterion proposed by Kallenberg [20, theorem 4.7] to show the stated convergence. Namely we need to verify that:

(1) $\mathbb{E}(N_n(I)) \xrightarrow{n\to\infty} \mathbb{E}(N(I))$, for all $I \in \mathcal{S}$;
(2) $\mu(N_n(J) = 0) \xrightarrow{n\to\infty} \mu(N(J) = 0)$, for all $J \in \mathcal{R}$,

where $\mathbb{E}(\cdot)$ denotes the expectation with respect to μ. As before, let us put $U_n = \{X_0 > u_n\}$.

The first condition follows trivially by definition of the point process N_n. In fact, let $a, b \in \mathbb{R}^+$ be such that $I = [a, b)$; then, recalling that $v_n = 1/\mu(U_n)$, we have

$$\begin{aligned}\mathbb{E}(N_n(I)) &= \mathbb{E}\left(\sum_{j=\lfloor v_n a\rfloor+1}^{\lfloor v_n b\rfloor} 1_{T^{-j}U_n}\right) = \sum_{j=\lfloor v_n a\rfloor+1}^{\lfloor v_n b\rfloor} \mathbb{E}(1_{T^{-j}U_n})\\ &= (\lfloor v_n b\rfloor - (\lfloor v_n a\rfloor + 1))\,\mu(U_n)\\ &\sim (b-a)v_n\mu(U_n)n \to \infty(b-a) = \mathbb{E}(N(I)).\end{aligned}$$

To prove (2), note that by [29, corollary 6] we only need to show that

$$\mu_M(N_n(J) = 0) \xrightarrow{n\to\infty} \mathbb{P}(N(J) = 0), \quad \text{for all } J \in \mathcal{R}.$$

Let

$$E_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} r_M \circ F^i(x).$$

Then by the ergodic theorem, we get, for μ-a.e. $x \in M$,

$$E_n(x) \to c := \int_M r_M\, \mathrm{d}\mu_M = \frac{1}{\mu(M)},$$

where the final equality follows from Kac's theorem. Moreover $c = v_n/\hat{v}_n$.

For μ-a.e. $x \in M$, there exists a finite number $j(x, \varepsilon)$ such that $|E_n(x) - c| < \varepsilon$ for all $n \geqslant j(x, \varepsilon)$. Let $\tilde{G}_n^\varepsilon := \{x \in M : j(x, \varepsilon) < n\}$. Moreover, we define $N = N(\varepsilon)$ to be such that

$$\hat{\mu}(\tilde{G}_N^\varepsilon) > 1 - \varepsilon. \tag{16}$$

Since

$$\left|\sum_{i=0}^{n-1} r_M(F^i(x)) - cn\right| < \varepsilon n \quad \text{for } x \in \tilde{G}_N^\varepsilon \quad \text{and } n \geqslant N,$$

for all such n, there exists $s = s(x)$ with $|s| < \varepsilon n$ such that $F^n(x) = T^{cn+s}(x)$. Since $r_{U_n} = \sum_{i=0}^{\hat{r}_{U_n}-1} r_M \circ F^i$, we obtain

$$r_{U_n}(x) = c\hat{r}_{U_n}(x) + s$$

for some $|s| < \varepsilon \hat{r}_{U_n}(x)$ whenever $\hat{r}_{U_n}(x) \geqslant N$ and $x \in \tilde{G}_N^\varepsilon$, where we used that $c = \upsilon_n/\hat{\upsilon}_n$.

Note that since $U_{n+1} \subset U_n \forall n$, the sets $L_{N,n}^\varepsilon := \{\hat{r}_{U_n} > N\}$ are nested, i.e. $L_{N,n}^\varepsilon \subset L_{N,n+1}^\varepsilon \forall n$. Hence, as $\mu_M(\hat{r}_{U_n} \leqslant j) \leqslant j\mu_M(U_n) \to 0$ as $n \to \infty$ there exists $N' = N'(\varepsilon)$ sufficiently large such that

$$\mu_M((L_{N,n}^\varepsilon)^c) < \varepsilon \tag{17}$$

for all $n > N'$.

Let $J_{\sup} = \sup J + 1$. Observe that

$$\mu_M\left(N_n([0, J_{\sup})) > \kappa\right) \leqslant \mu_M\left(\hat{N}_n(\upsilon_n/\hat{\upsilon}_n[0, J_{\sup})) > \kappa\right)$$
$$= \mu_M\left(\hat{N}_n(c[0, J_{\sup})) > \kappa\right) \xrightarrow{u \to u_F} \mathbb{P}(N([0, cJ_{\sup}) > \kappa) \xrightarrow{\kappa \to \infty} 0.$$

This implies that we can choose $K(J)$ independent of ε such that $\mu_M(N_n(J) > K(J)) < \varepsilon$.

Also, for any $x \in M$ and $i = 2, \ldots$, let $r_{U_n}^{(i)}(x) := r_{U(n)}(T^{r_{U_n}^{(i-1)}})(x)$ where $r_{U_n}^{(1)} := r_{U_n}$ and put $\tau_{U_n}^i = \tau_{U_n}^{i-1} + r_{U_n}^{(i)}$, with $\tau_{U_n}^1 = r_{U_n}$ for the time of the ith return to U_n under the map T. Similarly we define $\hat{r}_{U_n}^{(i)}(x) := \hat{r}_{U_n}(F^{\hat{r}_{U_n}^{(i-1)}})(x)$ and $\hat{\tau}_{U_n}^i = \hat{\tau}_{U_n}^{i-1} + \hat{r}_{U_n}^{(i)}$ for the time of the ith return to U_n under F. We will use the ergodic theorem to approximate $\tau_{U_n}^i(x)$ by $c\hat{\tau}_{U_n}^i(x)$ on a large set.

For that purpose put

$$E(u_n, J, \varepsilon) := \{N_n(J) = 0\} \cap \{N_n([0, J_{\sup})) > K\} \cap \left(\bigcap_{j=1}^K T^{-\tau_{U_n}^j}\left(\tilde{G}_N^{\varepsilon/K} \cap L_{N,N'}^{\varepsilon/K}\right)\right).$$

By stationarity, (16) and (17), for K, N and n sufficiently large we have

$$\left|\mu_M(N_n(J) = 0) - \mu_M(E(u_n, J, \varepsilon))\right|$$
$$\leqslant \mu_M(N_n([0, J_{\sup})) > K) + K\mu_M\left(\left(\tilde{G}_N^{\varepsilon/K}\right)^c\right) + K\mu_M\left(\left(L_{N,N'}^{\varepsilon/K}\right)^c\right) \leqslant 3\epsilon. \tag{18}$$

By the definition of $\tilde{G}_N^{\varepsilon/K}$ we now conclude that for $x \in E(u_n, J, \varepsilon)$ and $j = 1, \ldots, K$, there exist $|s_j| < \varepsilon \hat{r}_{U_n}^{(j)}(x)$ such that

$$r_{U_n}^{(j)}(x) = c\hat{r}_{U_n}^{(j)}(x) + s_j.$$

Hence

$$\left|\tau_{U_n}^j - c\hat{\tau}_{U_n}^j\right| \leqslant K\varepsilon \tag{19}$$

on $E(u_n, J, \varepsilon)$ for $j = 1, \ldots, K$. Since $\hat{\upsilon}_n = \upsilon_n/c$, from (19) we get that, for $x \in E(u_n, J, \varepsilon)$ and every $j = 1, \ldots, K$,

$$\tau_{U_n}^j(x) \in \upsilon_n J \quad \Rightarrow \quad \hat{\tau}_{U_n}^j(x) \in \hat{\upsilon}_n(1 + B(0, K\varepsilon/c))J \tag{20}$$

and also

$$\hat{\tau}^j_{U_n}(x) \in \hat{v}_n J \;\Rightarrow\; \tau^j_{U_n}(x) \in v_n(1 + B(0, K\varepsilon/c))J, \tag{21}$$

where we used $(1 + B(0, \delta))J = \{x = (1 + y)z : |y| < \delta, z \in J\}$. Hence,

$$\mu_M(\hat{N}_n(J) = 0) \leqslant \mu_M(E(u_n, (1 + B(0, K\varepsilon/c))J, \varepsilon)) \leqslant \mu_M(\hat{N}_n((1 + B(0, 2K\varepsilon/c))J) = 0).$$

Taking limits as $n \to \infty$, by hypothesis, we get that

$$\mathbb{P}(N(J) = 0) \leqslant \mu_M(E(u_n, (1 + B(0, K\varepsilon/c))J, \varepsilon)) \leqslant \mathbb{P}(N((1 + B(0, 2K\varepsilon/c))J) = 0).$$

Finally, using (18) and that $\lim_{\delta\to 0} \mathbb{P}(N((1 + B(0, \delta))J) = 0) = \mathbb{P}(N(J) = 0)$ (as J is a finite union of disjoint intervals), we get

$$\lim_{n\to\infty} \mu_M(N_n(J) = 0) = \mathbb{P}(N(J) = 0). \qquad \square$$

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