Generalised Gibbs states for expanding maps

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Abstract

We show that for expanding maps there is a one-to-one correspondence between equilibrium states and Gibbs' states (measures which are equivalent to their pullbacks by local homeomorphism which preserve the 'hyperbolic structure'). In particular we show that there is a one-one relationship between families of multipliers potentials (up to constants).

1 Introduction

In this paper we consider the equivalence of Gibbs' and equilibrium states for expanding maps on compact metric spaces. A Gibbs' state is determined by a family of locally defined multipliers that satisfy a (natural) cocycle relation (see definition 2 below), where a multiplier is the rescaling factor that is needed to have the Gibbs' measure agree with its pullback by a conjugating homeomorphism. Ordinarily, an equilibrium state for a potential satisfies a variational principle and a (sufficiently regular) potential determines a family of multipliers (see equation (1) below) which then implies that equilibirium states satisfy the Gibbs' property with respect to that family of multipliers. Here we show that the reverse implication applies, which means that a measure that satisfies the local comparisons prescribed by the multiplier maximise the variational principle (on the entire space). This is largely a consequence of expansiveness and the mixing property. Combined with previous results [4] this establishes a one-one relationship between multipliers and potentials (up to additive constants).

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We show that a family of (local) multipliers gives rise to a potential and the Gibbs measure for these multipliers will prove to be the equilibrium state for that potential. In particular for a Hölder continuous potential the cohomology class of potentials that differ by coboundaries yields a class of equivalent measures exactly one of which is invariant under the map. This implies that a modification of the multipliers by 'multiplicative coboundary' like terms yields one which is invariant under the map (see the remark below).

Assuming the relation (1) applies, then the papers [2, 4] established the equivalence of Gibbs and equilibrium states in various settings. Moreover, if for some f the multipiers are given by equation (1), then Ruelle showed [4] that for expanding maps the associated Gibbs' state is the equilibrium state for the potential f (which is known to be characterised by the eigenfunctional and eigenvector to the largest eigenvalue of a suitable transfer operator).

In the special case of a subshift of finite type, we showed in [2] that a given family of Hölder continuous multipliers (with sufficiently large Hölder exponents) gives rise to a representation by an expression similar to equation (1) with a Hölder continuous potential, provided the family of multipliers is shift invariant. The construction takes advantage of the explicit hyperbolic structure and local coordinates. The potential is determined up to additive constants and coboudaries.

2 Definitions

A map T on a compact metric space Ω is *expanding* if there exists a (expanding) constant $\lambda > 1$ such that $d(Tx, Ty) \geq \lambda d(x, y)$ for $d(x, y) \leq \zeta_0$, for some positive ζ_0. For every $x \in \Omega$, the set $T^{-1}x = \{y \in \Omega : Ty = x\}$ is at most countable, and, moreover, there exists a positive $\zeta < \zeta_0$ such that for every $x \in \Omega$, the set $T^{-1}(B_\zeta(x))$ is the finite union of disjoint open subsets $A_1, A_2, \ldots$, of Ω on which T is one-to-one ($B_\zeta(x)$ denotes the ball of radius ζ centered at x). The sets $A_1, A_2, \ldots$ are the components of $T^{-1}(B_\zeta(x))$ to each of which T restricted is a homeomorphism. For every positive integer n and $x \in \Omega$ the set $T^{-n}(B_\zeta(x))$ is the disjoint union of components $A_1, A_2, \ldots$ in Ω. If z lies in some A_j and satisfies $T^n z = x$, then for every $y \in B_\zeta(x)$ there exists a unique $z' \in A_j$ satisfying $T^n z' = y$.

Mixing Assumption: We shall require that or every positive ε and $x \in \Omega$, the set $T^{-n}(x)$ is ε-dense in Ω for all large enough n.

Definition 1 *[1, 3] (i) Two points $x, y \in \Omega$ are called* conjugate *or* n-conjugate *if $T^n x = T^n y$ for some positive integer n (and consequently also $T^m x = T^m y$ for all $m \geq n$.)*
(ii) A (local) homeomorphism $\varphi : U_\varphi \to \Omega$, $U_\varphi \subset \Omega$ open, is called conjugating, *if every $x \in U_\varphi$ is conjugate to $\varphi(x)$. In fact $T^j x = T^j \varphi(x)$ for all $x \in U_\varphi$ and $j \geq n$ for some positive integer n.*

The composition of conjugating homeomophisms is again conjugating: $\varphi = \varphi' \circ \varphi'$ is conjugating on the open set $U_\varphi = \varphi^{-1}(U_{\varphi''} \cap \varphi'(U_{\varphi'}))$ (if U_φ is non-empty). In particular, if φ' and φ'' are both n-conjugating then also φ is n-conjugating.

Definition 2 *A family of positive and continuous functions $\{r_\varphi : U_\varphi \to (0, \infty) : \varphi \quad \text{conjugating}\}$ is said to be a* family of multipliers *if for any two conjugating homeomorphisms φ' and φ'' the following cocyle equation is satisfied:*

$$(r_{\varphi''} \circ \varphi')\, r_{\varphi'} = r_{\varphi'' \circ \varphi'},$$

on $U'' = \varphi^{-1}(U_{\varphi''} \cap \varphi'(U_{\varphi'}))$ (provided U'' is non-empty).

Let us write $f^n = f + fT + fT^2 + \cdots + fT^{n-1}$ for the nth ergodic sum of f and introduce the function space $V(\Omega)$ [5] as follows. We say a function $f : \Omega \to \mathbf{R}$, belongs to $V(\Omega)$ if f satisfies the following two conditions:
(i) For every positive $\delta < \zeta$ the norm $\|f\|_\delta = \sup_{d(x,x')<\delta} C(x, x')$ is finite, where $C(x, x')$ is the smallest number for which

$$\sup_{n \geq 1} \sup_{(y,y') \in T^{-n}x \times T^{-n}x'} |f^n(y) - f^n(y')| \leq C(x, x'),$$

where the supremum is over all pairs $(y, y') \in \Omega \times \Omega$ for which both points lie in the same component A_j of $T^{-n}(B_\zeta(x))$, that is, $y, y' \in A_j$, $j = 1, 2, \ldots$.
(ii) The constant $C(x, x')$ goes to zero as $\delta \to 0$.

The prime example of a family of multipliers is given as follows. Let f be a function in $V(\Omega)$ and φ an n-conjugating homeomorphism. Then put

$$r_\varphi = \exp\left(f^n \circ \varphi - f^n\right) = \exp \sum_{k=0}^{\infty} (f \circ T^k \circ \varphi - f \circ T^k). \tag{1}$$

The function r_φ is defined on U_φ and one sees that, as φ runs through the entire set of conjugating homeomorphism, one obtains a family of multipliers which satisfies the cocyle equations of Definition 2. In this paper we show that for expanding maps all families of multipliers are given by equation (1) for suitably chosen f.

Definition 3 *[1] Let* $\{r_\varphi : \varphi \text{ conjugating}\}$ *be a family of multipliers. A probability measure* μ *on* Ω *is called* Gibbs' *for the family* $\{r_\varphi : \varphi\}$ *if for every conjugating homeomorphism* $\varphi : U_\varphi \in \Omega$ *the following holds true:*
(i) $\varphi^*\mu$ *restricted to* U_φ *is absolutely continuous with respect to* μ*.*
(ii) $\frac{d\varphi^*\mu}{d\mu} = r_\varphi$*, that is*

$$\int \chi \circ \varphi \, r_\varphi \, d\mu = \int \chi \, d\mu$$

for measurable (test) functions χ *which are supported in* $\varphi(U_\varphi)$*.*

For $x \in \Omega$ let us write T_x for the homeomorphism which is given by the restriction of T to the ball $B_\zeta(x)$. Its inverse T_x^{-1} is then a homeomorphism as well. If μ is a measure on Ω such that $T_x^*\mu$ is absolutely continuous with respect to μ we write h for the Radon-Nikodym derivative $\frac{dT_x^*\mu}{d\mu}$ (for which we shall also write $\frac{d\mu(x)}{d\mu(Tx)} = \frac{d\mu}{d\mu T}(x)$).

3 Results

The families of multipliers we consider shall satisfy the following two conditions:

(I) There exists a summable sequence of constants $C_k \searrow 0$, as $k \to \infty$, such that

$$\left| \log \frac{r_{\varphi^1}}{r_\varphi \circ T} \right| \leq C_k,$$

for φ for which $d(T^j\varphi x, T^j x) \leq \varepsilon_0, j = 0, 1, \ldots, k$, where $\varphi^1 = T \circ \varphi \circ T^{-1}$ and T^{-1} is the local inverse of T whose range contains the domain of φ.

(II) For every n:

$$\sum_{\varphi \in \Phi(n)} r_\varphi(x) < \infty,$$

for all $x \in \Omega$, where $\Phi(n)$ is the collection of all n-conjugating homeomorphism in Ω ($r_\varphi(x) = 0$ if $x \notin U_\varphi$).

Theorem 4 *Let μ be a Gibbs' measure on Ω for a family of multipliers $\{r_\varphi : \varphi \text{ conjugating}\}$ which satisfies the conditions (I) and (II). Then there exists a continuous function $f \in V(\Omega)$ so that the multipliers r_φ are of the form given above in (1).*

Let us note that we don't require the individual multipliers to satisfy any particular regularity condition. It is the quasi-invariant condition (I) which is used to show that the potential f lies in the space $V(\Omega)$. Also, if T is a finite to one map then the condition (II) is naturally satisfied.

Let us note that the potential f is not unique. Any change by a constant will yield the same equilibirium state (and multipliers), and if a coboundary is added to f the Gibbs state will change to an equivalent measure.

Theorem 4 shows in particular that μT and μ are equivalent and that $\frac{d\mu}{d\mu T} = e^f$.

Remark: If the potential f is Hölder continuous, adding a coboundary $u - u \circ T$ yields an equivalent but T-invariant measure [6]. If $\tilde{r}_\varphi$ are the multipliers given by equation (1) for the modified potential $\tilde{f} = f + u - u \circ T$, then $\tilde{r}_\varphi = e^{u - u\circ\varphi} r_\varphi$, i.e. the multipliers change by a 'multiplicative coboundary' $e^{u-u\circ\varphi}$ (conjugating homeomorphisms don't form a group because compositions not always exist). As a consequence, the multipliers $\tilde{r}_\varphi$ are T-invariant, which means

$$\tilde{r}_{\varphi^1} = \tilde{r}_\varphi \circ T,$$

where $\varphi^1 = T^{-1}\varphi T^{-1}$, and the invariant measure $\tilde{\mu}$ has density $\frac{d\tilde{\mu}}{d\mu} = e^u$, i.e. $\tilde{\mu}(\chi) = \mu(e^u \chi)$ for integrable χ.

4 Proof of Theorem 4

We shall construct a sequence of approximating potentials $f_n \in V(\Omega)$ which will allow us to express the multipliers r_φ through equation (1). Define the following 'partition function':

$$\omega_n(x) = \sum_{\varphi \in \Phi(n)} r_\varphi(x)$$

($\omega_n > 0$), where the sum is over all n-conjugating φ so that $x \in U_\varphi$. Note that the partition functions are well defined by assumption (II). For any

n-conjugating ψ one has by the 'cocycle property' for multipliers

$$\omega_n(x) = \sum_{\varphi' \in \Phi(n)} r_\varphi(\psi x)\, r_\psi(x) = r_\psi(x)\, \omega_n \circ \psi(x)$$

($x \in U_\psi$), where the sum is over all $\varphi' = \varphi \circ \psi^{-1}$ (φ' is n-conjugating). Let us define:

$$f_n(x) = \log \frac{\omega_{n-1} T(x)}{\omega_n(x)},$$

for $n = 2, 3, \ldots$.

Lemma 5 *If φ is n-conjugating then $r_\varphi = \exp\left(f_n^n \circ \varphi - f_n^n\right)$.*

Proof. Using the definition of f_n we get on U_φ:

$$\begin{aligned}
\exp\left(f_n^n \circ \varphi - f_n^n\right) &= \exp \sum_{0 \le k < n} \left(\log \frac{\omega_{n-1} T}{\omega_n} T^k \varphi - \log \frac{\omega_{n-1} T}{\omega_n} T^k \right) \\
&= \prod_{0 \le k < n} \frac{\omega_{n-1} T^{k+1} \varphi}{\omega_n T^{k+1} \varphi} \, \frac{\omega_n T^k}{\omega_{n-1} T^{k+1}} \\
&= \prod_{0 \le k < n} \frac{\omega_{n-1} T^{k+1} \varphi}{\omega_{n-1} T^{k+1}} \, \frac{\omega_n T^k}{\omega_n T^{k+1} \varphi}.
\end{aligned}$$

For $x \in U_\varphi$, the two points $T^k x$ and $T^k \varphi(x)$ are $(n-k)$-conjugate (and thus also n-conjugate). Let us put φ^k for the conjugating homeomorphism defined on $T^k(U_\varphi)$ by the requirement that it maps $T^k x$ to $T^k \varphi(x)$ for all $x \in U_\varphi$. We obtain $\varphi^k T^k = T^k \varphi$ and

$$\frac{\omega_n T^k}{\omega_n T^k \varphi} = \frac{\omega_n}{\omega_n \varphi^k} \circ T^k = r_{\varphi^k} \circ T^k,$$

and also

$$\frac{\omega_{n-1} T^{k+1} \varphi}{\omega_{n-1} T^{k+1}} = \frac{\omega_{n-1} \varphi^{k+1}}{\omega_{n-1}} \circ T^{k+1} = \frac{1}{r_{\varphi^{k+1}}} \circ T^{k+1}.$$

Since $\varphi^0 = \varphi$ ($k = 0$) and $\varphi^n = \mathrm{id}$ (as $T^n x = T^n \varphi(x)$), this yields

$$\exp(f_n^n \circ \varphi - f_n^n) = \prod_{0 \le k < n} \frac{r_{\varphi^k} T^k \varphi}{r_{\varphi^{k+1}} T^{k+1}} = r_\varphi.$$

□

Proof of Theorem 4. We first have to show that the family of functions $\{f_n : n \in \mathbf{N}\}$ is equicontinuous. Assume that y and y' are two points in X that are close so that there exists a $k \geq 0$ such that $d(T^j y, T^j y') < \zeta, j = 0, 1, \ldots, k$. For $m > k$ we can find $z \in \Omega$ so that $T^m z = T^m y$ (i.e. z and y are m-conjugate) and so that z lies in the same component of $T^{-m}(B_\zeta(T^m y'))$. Since we can choose m arbitrarily large we can achieve that

$$|f_n(y') - f_n(z)|$$

is arbitrarily small. On the other hand, since $d(T^j z, T^j y) \leq \zeta,\ \forall j \leq k$, we obtain

$$|f_n(z) - f_n(y)| = \left|\log \frac{r_{\varphi^1}}{r_\varphi \circ T}(y)\right| \leq C_k,$$

where $\varphi^1 = T \circ \varphi \circ T^{-1}$ (T^{-1} is the local inverse of T which maps to the domain of φ) and $C_k \to 0$ as $k \to \infty$ by property (I) of the family of multiplers. Hence

$$|f_n(y') - f_n(y)| \leq |f_n(y') - f_n(z)| + C_k,$$

and if we let m go to infinity we obtain $|f_n(y') - f_n(y)| \leq C_k$, uniformly in $y, y' \in \Omega$ for which $d(T^j y, T^j y') < \zeta, 0 \leq j \leq k$. This proves that the family $\{f_n : n\}$ is equicontinuous and thus has a subsequence that converges to a limit f.

To show that f lies in the space $V(\Omega)$, we have to show that for $x, x' \in \Omega$ there exists $C(x, x') \to 0$ as $d(x, x') \to 0$ so that for all N

$$|f^N(y) - f^N(y')| \leq C(x, x'),$$

for all y, y' in the same component of $T^{-N}x, T^{-N}x'$. For simplicity's sake we shall assume that f_n converges to f (rather than a subsequence). Let $k \geq 0$ be the largest integer for which $d(T^j x, T^j x') \leq \zeta$ for $j = 0, 1, \ldots, k$. Note that $k \to \infty$ in a uniform way as $d(x, x') \to 0$. By choosing n large enough we can make the term

$$|f^N(y) - f_n^N(y)| + |f^N(y') - f_n^N(y')|$$

arbitrarily small. Hence let n be (very) large and consider

$$|f_n^N(y) - f_n^N(y')| \leq |f_n^N(y') - f_n^N(z(m))| + |f_n^N(z(m)) - f_n^N(y)|, \quad (2)$$

where $z(m)$ and y are m-conjugate ($m >> N, k$). Since T is mixing we can choose $z(m)$ so that $d(y', z(m)) \to 0$ as $m \to \infty$ (in particular we can of course assume that $z(m)$ lies in the same component of $T^{-N-k}B_\zeta(T^k x)$ as the point y'). Hence

$$|f_n^N(y') - f_n^N(z(m))| \to 0,$$

as $m \to \infty$. To estimate the second term on the right hand side of equation (2) let φ be an m-conjugating homeomorphism in a neighbourhood of y which is determined by the relation $\varphi(y) = z(m)$. By Lemma 5 we have in the domain U_φ of φ:

$$f_n \circ \varphi - f_n = \log \frac{r_{\varphi^1}}{r_\varphi \circ T},$$

and consequently (by assumption (I)):

$$\begin{aligned} |f_n^N(y) - f_n^N(z(m))| &\leq \sum_{j=0}^{N-1} \left| \log \frac{r_{\varphi^{j+1}}}{r_{\varphi^j} \circ T}(T^j y) \right| \\ &\leq \sum_{j=0}^{N-1} C_{k+N-j}, \end{aligned}$$

where $\varphi^j = T^j \circ \varphi \circ T^{-j}$ (T^{-j} is the local inverse which maps to a neighbourhood of y) is $(n-j)$-conjugating. It now follows that uniformly in n, m and N:

$$|f_n^N(y) - f_n^N(z(m))| \leq \sum_{j=k+1}^{\infty} C_j.$$

Using inequality (2) and taking limits $m, n \to \infty$ we thus obtain for all N

$$|f^N(y) - f^N(y')| \leq \sum_{j=k+1}^{\infty} C_j.$$

Since $d(T^j x, T^j x') \leq \zeta$ for $j = 0, 1, \ldots, k$, we can now put

$$C(x, x') = \sum_{j=k+1}^{\infty} C_j.$$

If $d(x, x') \to 0$ then $k \to \infty$ and thus $C(x, x') \to 0$ as the C_j are summable. Hence f lies in $V(\Omega)$, and this concludes the proof of the theorem. □

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