# EXTREME VALUE THEORY WITH SPECTRAL TECHNIQUES:APPLICATION TO A SIMPLE ATTRACTOR.

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Abstract. We give a brief account of application of extreme value theory in dynamical systems by using perturbation techniques associated to the transfer operator. We will apply it to the baker's map and we will get a precise formula for the extremal index. We will also show that the statistics of the number of visits in small sets is compound Poisson distributed.

## 1. Introduction

Extreme value theory (EVT) has been widely studied in the last years in application to dynamical systems both deterministic and random. A review of the recent results with an exhaustive bibliography is given in our collective work [17]. As we will see, there is a close connection between EVT and the statistics of recurrence and both could be worked out simultaneously by using perturbations theories of the transfer operator. This powerful approach is limited to systems with quasi-compact transfer operators and exponential decay of correlations; nevertheless it can be applied to situations where more standard techniques meet obstructions and difficulties, in particular to:

- non-stationary systems
- observable with non-trivial extremal sets
- higher-dimensional systems.

Another big advantage of this technique is the possibility of defining in a precise and universal way the extremal index (EI). We defer to our recent paper [3] for a critical discussion of this issue with several explicit computations of the EI in new situations. The germ of the perturbative technique of the transfer operator applied to EVT is in the fundamental paper [16] by G. Keller and C. Liverani; the explicit connection with recurrence and extreme value theory has been done by G. Keller in the article [15], which contains also a list of suggestions for further investigations. We successively applied this method to i.i.d. random transformations in [1, 3], to coupled maps on finite lattices in [10], and to open systems with targets and holes [11].

The object of this note is to illustrate this technique by presenting a new application to a bi-dimensional invertible system. We will firstly show that the main steps are independent of the nature of the system provided one could find the good functional spaces where the transfer operator exhibits quasi-compactness. Let us point out that we already gave a presentation of that technique in [17], chapter 7, when applied to random systems.

We will find a few limitations to a complete application of the theory. We will point out those issues with bold items; moreover we will address the question of the generalization

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to wider class of maps in higher dimensions.

The spectral technique discussed above does not allow us, for the moment, to get another property related to limiting return and hitting times distribution in small sets, namely the limiting statistics of the number of visits. The latter takes usually the form of a compound Poisson distribution. In the recent paper by two of us [14], a new technique was developed to get such a distribution for invertible maps in higher dimension and arbitrary small sets. Moreover a relation between the limiting return times distribution and the probability of the cluster sizes was established, where clusters consist of the portion of points that have finite return times in the limit where random return times go to infinity. This lead to another definition of the extremal index as the reciprocal of the expected length of the clusters [3], and it will coincide with that given by the spectral technique.

## 2. A pedagogical example: the baker's map

We now treat an example for which there are not apparently established results for the extreme value distributions. This example, the baker's map, is a prototype for uniformly hyperbolic transformations in more than one dimension, two in our case, and in order to study it with the transfer operator, we will introduce suitable anisotropic Banach spaces. Our original goal was to investigate directly larger classes of uniformly hyperbolic maps, including Anosov ones, but, as we said above, the generalizations do not seem straightforward; we will explain the reason later on. With the usual probabilistic approaches extreme value distributions have been obtained for the linear automorphisms of the torus in [4].

We will refer to the baker transformation studied in Section 2.1 in [6], but we will write it in a particular case in order to make the exposition more accessible. The baker's transformation $T(x_n, y_n)$ is defined on the unit square $X = [0,1]^2 \subset \mathbb{R}^2$ into itself by:

$$x_{n+1} = \begin{cases} \gamma_a x_n & \text{if } y_n < \alpha \\ (1-\gamma_b) + \gamma_b x_n & \text{if } y_n > \alpha \end{cases}$$

$$y_{n+1} = \begin{cases} \frac{1}{\alpha} y_n & \text{if } y_n < \alpha \\ \frac{1}{\beta}(y_n - \alpha) & \text{if } y_n > \alpha, \end{cases}$$

with $\beta = 1 - \alpha$, $\gamma_a + \gamma_b \leq 1$, see Fig. 1. To simplify a few of next formulae, we will take $\alpha = \beta \leq 0.5$ and $\gamma_a = \gamma_b$.

The map $T$ is discontinuous on the horizonal line $\Gamma : \{y = \alpha\}$. The singularity curves for $T^l, l > 1$ are given by $T^{-l}\Gamma$ and they are constructed in this way: take the preimages $T_Y^{-l}(\alpha)$ of $y = \alpha$ on the $y$-axis according to the map $T_Y(y) = \alpha y, y < \alpha$; $T_Y(y) = \frac{1}{\beta} y - \frac{\alpha}{\beta}, y \geq \alpha$. Then $T^{-l}\Gamma = \{y = T_Y^{-l}(\alpha)\}$. Any other horizontal line will be a stable manifold of $T$. The singularity set for $T^{-1}$ will be the intersection of $T(X)$ with the image of $\Gamma$, which gives the two boundaries of $T(X)$ respectively on the top of rectangle on the right and on the bottom of the rectangle on the left. The invariant non-wandering set $\Lambda$ will be at the end an attractor foliated by vertical lines which are all unstable manifolds. We denote by $\mathcal{W}^s(\mathcal{W}^u)$ the set of full horizontal (vertical) stable (unstable) manifolds of length 1 just constructed. We point out that a stable horizontal manifold $W_s$ will originate two disjoint full stable manifold when iterate backward by $T^{-1}$ not for the presence of singularity, but because the map $T^{-1}$ will only be defined on the two images of $T(X)$ as illustrated in Fig. 1.

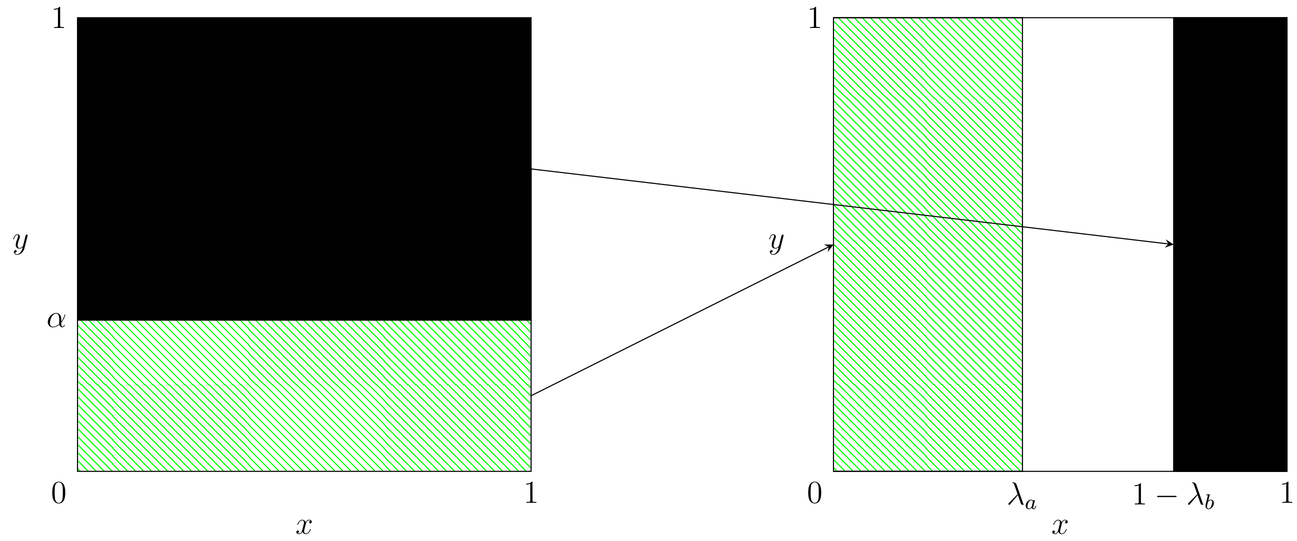


FIGURE 1. Action of baker map on the unit square. The lower part of the square is mapped in the left part and the upper part in the right part.

In order to obtain useful spectral information from the transfer operator $\mathcal{L}$, its action is restricted to a Banach space $\mathcal{B}$. We now give the construction of the norms on $\mathcal{B}$ and an associated "weak" space $\mathcal{B}_w$ in the case of the baker's map, following partly the exposition in [6]. In this case, those spaces are easier to define and the norms will be constructed directly on the horizontal stable manifolds instead of admissible leaves, which are smooth curves in approximately the stable direction, see [7]. As we anticipated above, we follow [6], but we slightly change the definition of the stable norms by adapting ourselves to that originally introduced in [7].

Given a full stable manifold $W_s$, we consider the collection $\Sigma$ of all the connected interval $W \in W_s$ of length less than one. Then we denote $C^\kappa(W, \mathbb{C})$ the set of continuous complex-valued functions on $W$ with Hölder exponent $\kappa$ and define the norm

$$|\varphi|_{W,\kappa} := |W|^\kappa \cdot |\varphi|_{C^\kappa(W,\mathbb{C})}, \tag{1}$$

where $|W|$ denotes the length of $W$. For $h \in C^1(X, \mathbb{C})$ we define the *weak norm* of $h$ by

$$|h|_w = \sup_{W \in \Sigma} \sup_{\substack{\varphi \in C^1(W,\mathbb{C}) \\ |\varphi|_{C^1(W,\mathbb{C})} \leq 1}} \left| \int_W h\varphi \, dm \right|$$

where $dm$ is unnormalized Lebesgue measure along $W$, instead with $m_L$ we will denote the Lebesgue measure over $X$.

The strong stable norm is defined as:

$$\|h\|_s = \sup_{W \in \Sigma} \sup_{\substack{\varphi \in C^\kappa(W,\mathbb{C}) \\ |\varphi|_{W,\kappa} \leq 1}} \left| \int_W h\varphi \, dm \right|. \tag{2}$$

We then need to define the strong unstable norm which allows us to compare expectations along different stable manifolds. For completeness sake we now give such a definition, even if we will not use it for the next considerations. Following the suggestion of the footnote 6 in [5], we define the strong unstable norm of $f$ as

$$\|h\|_u = \sup_{W\in\mathcal{W}_s} \sup_{\varphi\in C^1(W,\mathbb{C})|\varphi|_{C^1(W,\mathbb{C})}\leq 1} \int_W (d^u f)\varphi dm, \tag{3}$$

where $d^u f$ is the derivative of $f$ in the unstable direction. Finally we can define the *strong norm* of $h$ by

$$\|h\| = \|h\|_s + b\|h\|_u.$$

We put $\mathcal{B}$ the completion of $C^1(X,\mathbb{C})$ with respect to the norm $\|\cdot\|$, and, similarly, we define $\mathcal{B}_w$ to the completion of $C^1(X,\mathbb{C})$ with respect to the norm $|\cdot|_w$. Let us note that $\mathcal{B}$ lies in the dual of $C^1(X,\mathbb{C})$ and its elements are distributions.

**Remark 2.1.** *These inclusions and all the properties on the transfer operator which we are going to use, requires that $\kappa < 1$. We will see that the fact that $\kappa$ is not bounded from below by a positive number will be of substantial help for establishing the extreme value distribution.*

More precisely, there exists $C > 0$ such that any $h \in \mathcal{B}$ induces a linear functional $\varphi \to h(\varphi)$ with the property that

$$|h(\varphi)| \leq C|h|_w|\varphi|_{C^1}, \quad \text{for } \varphi \in C^1(X,\mathbb{C}), \tag{4}$$

see [7, Remark 3.4] for details. In particular, for $h \in C^1(X,\mathbb{C})$ we have that (see [7, Remark 2.5])

$$h(\varphi) = \int_X h\varphi\, dm_L, \quad \text{for } \varphi \in C^1(X,\mathbb{C}). \tag{5}$$

The transfer operator $\mathcal{L}$ is defined as, for $h \in L^1(X,\mathbb{C})$, the space of $m_L$ summable functions with complex values, see [7, Section 2.1]:

$$\mathcal{L}h = \left(\frac{h}{|\det DT|}\right) \circ T^{-1} = \frac{h \circ T^{-1}}{\alpha^{-1}\gamma}, \tag{6}$$

where the last equality on the r.h.s. uses the particular choices for the parameters defining the map $T$. Whenever $h \in C^1(X,\mathbb{C})$, the use of (5) and (6) gives

$$(\mathcal{L}h)(\varphi) = h(\varphi \circ T), \quad \text{for } h \in \mathcal{B} \text{ and } \varphi \in C^1(X,\mathbb{C}),$$

which, by completeness, can be extended to any $h \in \mathcal{B}$.

## 3. The spectral approach for EVT

3.1. **Formulation of the problem.** We now take a ball $B(z,r)$ of center $z \in \Lambda$ and radius $r$ and denote with $B(z,r)^c$ its complement, where $d(\cdot,\cdot)$ is the Euclidean metric.

Let us consider for $x \in X$ the observable

$$\phi(x) = -\log d(x,z) \tag{7}$$

and the function

$$M_n(x) := \max\{\phi(x), \cdots, \phi(T^{n-1}x)\}. \tag{8}$$

For $u \in \mathbb{R}_+$, we are interested in the distribution of $M_n \leq u$, where $M_n$ is now seen as a random variable on the probability space $(X,\mu)$, being $\mu$ the Sinai-Bowen-Ruelle (SRB) measure. Notice that the event $\{M_n \leq u\}$ is equivalent to the set $\{\phi \leq u, \ldots, \phi\circ T^{n-1} \leq u\}$ which in turn coincides with the set

$$E_n := B(z,e^{-u})^c \cap T^{-1}B(z,e^{-u})^c \cap \cdots \cap T^{-(n-1)}B(z,e^{-u})^c.$$

We are therefore following points which will enter the ball $B(z,e^{-u})$ for the first time after at least $n$ steps, and this could be seen as the distribution of the maximum of the observable $\phi \circ T^j, j = 0, \dots, n-1$. It is well known from elementary probability that the distribution of the maximum of a sequence of i.i.d. random variable is degenerate. One way to overcame it, is to make the *boundary level* $u$ depending upon the time $n$ in such a way the sequence $u_n$ grows to infinity and gives, hopefully, a non-degenerate limit for $\mu(M_n \le u_n)$.

**Question I** It would be useful if we could take as the underlying probability, the normalized (non-stationary) Lebesgue measure instead of the SRB measure. Would we get the same distribution for the extreme values? Let us recall that the SRB measure is associated with a positive Lebesgue measure invariant set $B_\mu$, the basin of attraction, such that for every continuous function $g$ we have: $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} g(T^i x) = \int g d\mu, \ \forall x \in B_\mu$. If we construct the absolutely continuous measures $m_L^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} T_*^i m_L$, and since the boundary of the set $E_n$ (see above) is of zero SRB-measure, being composed of the preimages of smooth curves, by the Portmanteau Theorem, we have for any $x \in B_\mu$ :

$$\lim_{n\to\infty} \mu(M_n \le u_n) = \lim_{n\to\infty} \lim_{m\to\infty} m_L^{(m)}\{x; \phi(T^m(x) \le u_n, \dots \phi(T^{m+n-1}(x) \le u_n\}.$$

From now on we set: $B_n = B(z, e^{-u_n})$ and $B_n^c$ the complement of $B_n$.
We easily have

$$\mu(M_n \le u_n) = \int \mathbf{1}_{B_n^c}(x)\mathbf{1}_{B_n^c}(Tx)\cdots \mathbf{1}_{B_n^c}(T^{n-1}x)\, d\mu. \tag{9}$$

By introducing the perturbed operator, for $h \in \mathcal{B}$:

$$\mathcal{L}_n h := \mathcal{L}(\mathbf{1}_{B_n^c} h), \tag{10}$$

we can write (9) as

$$\mu(M_n \le u_n) = \mathcal{L}_n^n \mu(1). \tag{11}$$

We explicitly used here two facts which deserve justification.

- $\mathbf{1}_{B_n^c}$ and $\mathbf{1}_{B_n^c} h$ are in the Banach space. This is included in [8], Lemma 5.2 or Lemma 3.5 in [9], which show that piecewise Holder functions are in the Banach space as long as the discontinuities have some weak transversality property with the stable cone, which is clearly satisfied for our balls.
- $\mathbf{1}_A h(\phi) = h(\mathbf{1}_A \phi)$, when $h$ is a Borel measure; this was used above in the representation in terms of $\mathcal{L}_n^n$. This follows from the construction of the space $\mathcal{B}$ whenever $h$ is a Borel measure in $\mathcal{B}$.

It has been proved in [6] that the operator $\mathcal{L}$ is quasi-compact, in the sense that it can be written as

$$\mathcal{L} = \mu Z + Q, \tag{12}$$

where $\mu = \mathcal{L}\mu$ is the SRB measure normalized in such a way that $\mu(1) = 1$ and spanning the one-dimensional eigenspace corresponding to the eigenvalue 1; $Z$ is the generator of the one-dimensional eigenspace of the dual space $\mathcal{B}^*$ and corresponding to the eigenvalue 1 and normalized in such a way that $Z(1) = 1$; finally $Q$ is a linear operator on $\mathcal{B}$ with spectral radius $sp(Q)$ strictly less than one.

3.2. **The perturbative approach.** We now introduce the assumptions which allow us to apply the perturbative technique of Keller and Liverani [16]: we adapt them to our current situation and we defer to the aforementioned paper for details and equivalent hypothesis. We begin with a list of four assumptions, postponing the last fifth assumption to the next section.

- **A1** The unperturbed operator $\mathcal{L}$ is quasi-compact in the sense expressed by (12).
- **A2** There are constants $\beta, D > 0$, such that $\forall n$ sufficiently large, $\forall h \in \mathcal{B}$ and $\forall k \in \mathbb{N}$ we have

$$|\mathcal{L}_n^k h|_w \leq D|h|_w, \tag{13}$$
$$||\mathcal{L}_n^k h|| \leq D\beta^k ||h|| + D|h|_w.$$

- **A3** We can bound the weak norm of $(\mathcal{L} - \mathcal{L}_n)h$, with $h \in \mathcal{B}$, in terms of the norm of $h$ as:

$$|(\mathcal{L} - \mathcal{L}_n)h|_w \leq \chi_n ||h||$$

  where $\chi_n$ is a monotone upper semi-continuous sequence converging to zero. This is achieved in lemma 6.5 in [7], where the contracting factor is given by $\text{diam}(B_n)^\kappa$, for $n$ large enough; this is called the *triple norm* estimate.
- **A4** If we define

$$\Delta_n = Z(\mathcal{L} - \mathcal{L}_n)(\mu), \tag{15}$$

  and for $h \in \mathcal{B}$

$$\eta_n := \sup_{||h|| \leq 1} |Z(\mathcal{L}(h\mathbf{1}_{B_n}))|, \tag{16}$$

  we must show that

$$\lim_{n \to \infty} \eta_n = 0, \tag{17}$$

$$\eta_n ||\mathcal{L}(\mathbf{1}_{B_n}\mu)|| \leq \text{const } \Delta_n. \tag{18}$$

With these assumptions, and as a consequence of the theory in [16], the decomposition (12) holds for $n$ large enough, namely

$$\lambda_n^{-1}\mathcal{L}_n = \mu_n Z_n + Q_n \tag{19}$$
$$\mathcal{L}_n \mu_n = \lambda_n \mu_n \tag{20}$$
$$Z_n \mathcal{L}_n = \lambda_n Z_n \tag{21}$$
$$Q_n(\mu_n) = 0, \quad Z_n Q_n = 0, \tag{22}$$

where $\lambda_n \in \mathbb{C}; \mu_n \in \mathcal{B}; Z_n \in \mathcal{B}^*; Q_n \in \mathcal{B}$, and $\sup_n sp(Q_n) < sp(Q)$. We observe that the previous assumptions (19)–(22) imply that $Z_n(\mu_n) = 1, \forall n$; moreover $\mu_n$ can be normalized in such a way that $Z_n(\mu) = 1$. It remains to prove **A2** and **A4**.

Let us start with the former, **A2**. The proof is basically the same as the proof of Proposition 4.2 in [6], with the difference that we allow subsets of the stable manifolds of length less than one . By density of $C^1(X, \mathbb{C})$ in both $\mathcal{B}$ and $\mathcal{B}_w$, it will be enough to take $h \in C^1(X, \mathbb{C})$. We have to control integral of type: $\int_W \mathcal{L}_n h\varphi \, dm$, where $W \in \Sigma$ and $\varphi \in C^1(W, \mathbb{C})$ (resp. $C^\kappa(W, \mathbb{C})$), according to the estimate of the weak (resp. strong) norm. Let us give the argument for the weak norm, analogous considerations will apply to the other two norms. Let us consider for instance $\mathcal{L}_n^2$, we have

$$\int_W \mathcal{L}_n^2 h\varphi dm = \int_W \frac{\mathbf{1}_{B_n^c}(T^{-1}x)\mathcal{L}(\mathbf{1}_{B_n^c}h)(T^{-1}x)\varphi(x)}{\alpha^{-1}\gamma_a}\, dm(x) = \tag{23}$$

$$\sum_{i=1,2}\int_{W_i} \frac{\mathbf{1}_{B_n^c}(y)\mathcal{L}(\mathbf{1}_{B_n^c}h)(y)\varphi(Ty)}{\alpha^{-1}}\, dm(y), \tag{24}$$

where $W_i, i = 1, 2$ are the two preimages of $W$ and we performed a change of variable along the stable manifold with jacobian $\gamma_a$. The measure $m$ along $W_i$ is again the unnormalized Lebesgue measure. Iterating one more time we will produce at most two new pieces of stable manifolds, and we get:

$$\sum_{j=1,\cdots,4}\int_{W_j} \alpha^2 h(y)\varphi(T^2y)\mathbf{1}_{B_n^c}(y)\mathbf{1}_{B_n^c}(Ty)\, dm(y), \tag{25}$$

In the integral we replace the $W_j$ with $(W_j \cap B_n^c \cap T^{-1}B_n^c)$ getting again small pieces $W_j^{(n)}$ of stable manifolds. In order to compute the weak norm of $\mathcal{L}_n^2$ we must take the test function $\varphi$ verifying $|\varphi|_{C^1(W,\mathbb{C})} \leq 1$. If we now take two points $y_1, y_2 \in W_j^{(n)}$ we have

$$|\varphi(T^2(y_1)) - \varphi(T^2(y_2))| \leq |T^2(y_1) - T^2(y_2)| \leq \gamma_a^2|y_1 - y_2|,$$

and therefore $|\varphi \circ T^2|_{C^1(W_j^{(n)},\mathbb{C})} \leq 1$. By multiplying and dividing (25) by $|\varphi \circ T^2|_{C^1(W_j^{(n)},\mathbb{C})}$ we finally get

$$(25) \leq \sum_{j=1,\cdots,4} \alpha^2|h|_w \leq |h|_w,$$

where the last bound comes from our choice of $\alpha \leq \frac{1}{2}$. The proof generalizes immediately to any power $\mathcal{L}_n^k, k \geq 2$.

We now pass to justify **A4.** First of all we show how $Z$ is defined: it is the unique solution of the eigenvalue equation $\mathcal{L}^*Z = Z$, where $\mathcal{L}^*$ is the dual of the transfer operator. We define

$$Z(h) := h(1),\ \ h \in \mathcal{B}. \tag{26}$$

We in fact have for $h \in \mathcal{B}$:

$$\mathcal{L}^*Z(h) = Z(\mathcal{L}h) = (\mathcal{L}h)(1) = h(1 \circ T) = h(1) = Z(h).$$

Coming back to $\Delta_n$ we see immediately that

$$\Delta_n = Z(\mathcal{L})(\mathbf{1}_{B_n}\mu) = \mathcal{L}(\mathbf{1}_{B_n}\mu)(1) = \int \mathbf{1}_{B_n}\, d\mu = \mu(B_n). \tag{27}$$

The term $||\mathcal{L}(\mathbf{1}_{B_n}\mu)||$ can be handled very easily using the Lasota-Yorke inequality. Without any restrictions, suppose the latter is established without higher iterations of the operator. Then by Proposition 2.7 in [7] we have

$$||\mathcal{L}(\mathbf{1}_{B_n}\mu)|| \leq C_1||\mathbf{1}_{B_n}\mu|| + C_2|\mathbf{1}_{B_n}\mu|_w,$$

where $C_1, C_2$ are two constants depending only on the map. By Lemma 4.3 in [5] we have that there exists a third constant $C_3$ such that

$$||\mathbf{1}_{B_n}\mu|| \leq C_3||\mu|| \quad \text{and} \quad |\mathbf{1}_{B_n}\mu|_w \leq C_3|\mu|_w.$$

We notice that the constant $C_3$ depend on two parameters, called $B_0, B_1$ in [5], which in our case are respectively 2 and $2\pi$ for the particular shape of $B_n$, and therefore $C_3$ is independent of $n$. By setting

$$C_4 := C_1C_3||\mu|| + C_2C_3|\mu|_w,$$

we are led to prove that (see (18)), $\eta_n C_4 \leq$ const $\Delta_n$, namely

$$\eta_n \leq \text{ const } \Delta_n = \text{const } \mu(B_n). \tag{28}$$

Before continuing, we have to focus on $\mu(B_n) = \mu(B(z, e^{-u_n}))$. It is well known that for $\mu$-almost all $z$ and by taking the radius sufficiently small, depending on the value $\iota$, $e^{-u_n(d+\iota)} \leq \mu(B(z, e^{-u_n}) \leq e^{-u_n(d-\iota)}$, where $\iota > 0$ is arbitrarily small. The quantity $d$ is the Hausdorff dimension of the measure $\mu$ and in our case reads [18]:

$$d = 1 + d_s, \text{ where } d_s := \frac{\alpha \log \alpha^{-1} + (1-\alpha)\log(1-\alpha)^{-1}}{\log \gamma^{-1}}.$$

Notice that $d_s$ is strictly smaller than 1; for instance, with the choices $\alpha = 0.5, \gamma = 0.25$, we get $d = 0.5$. We now have:

**Lemma 3.1.** *Assume $\kappa > d_s$.*
*Then*

$$\eta_n \leq 2\mu(B_n).$$

*Proof.* We have

$$Z(\mathcal{L}(h\ \mathbf{1}_{B_n})) = \int h\ \mathbf{1}_{B_n} dm.$$

Put $\tilde{W}_\xi = W_\xi \cap B_n$; by disintegrating along the stable partition $\mathcal{W}^s$ we get:

$$\begin{aligned} \int h\ \mathbf{1}_{B_n}\, dm_L &= \int_\xi d\lambda(\xi) \left[\int_{W_\xi} (\mathbf{1}_{B_n} h)(x)\, dm(x)\right] \\ &\leq \int_\xi d\lambda(\xi) \left[|\tilde{W}_\xi|^\kappa \|h\|_s\right] \\ &\leq e^{-u_n\kappa}||h||_s \lambda(\xi; B_n \cap W_\xi \neq \emptyset), \end{aligned}$$

where $\lambda$ is a counting measure on the stable fibers $W_\xi$ belonging to $\mathcal{W}^s$; and indexed by $\xi$. By definition of disintegration we have that

$$\lambda(\xi; B_\epsilon \cap W_\xi \neq \emptyset) = m_L(\bigcup W_\xi, B_n \cap W_\xi \neq \emptyset) = 2e^{-u_n},$$

and therefore

$$\eta_n \leq 2e^{-u_n(\kappa+1)}.$$

We finally have

$$\eta_n \leq 2e^{-u_n(\kappa+1)} \leq 2e^{-u_n(d+\iota)} \leq 2\mu(B_n),$$

provided we choose

$$\kappa > d + \iota - 1$$

which can be satisfied by assumption. □

**Remark 3.2.** *The local comparison between the Lebesgue and the SRB measure of a ball of center $z$ obliged us to choose $z$ $\mu$-almost everywhere because in this way we have a precise value for the locally constant dimension $d$. We are therefore discarding several points, eventually periodic, where the limiting distribution for the Gumbel's law (see next section) could exhibit extremel indices different from* 1.

**Remark 3.3.** *For piecewise hyperbolic diffeomorphisms in dimension 2, the construction of the Banach space imposes strong constraints on the value of the Hölder exponent $\kappa$; in particular $\kappa \leq 1/3$. However the relationship between $\kappa$ and $d$ persists as $\kappa > d$. This is surely violated if $d = 2$ as it happens for the examples in* [7], *where the invariant set is the whole manifold, and also for Anosov diffeomorphisms. In some sense this difficulty was already raised in section 4.5 in the Keller's paper* [15], *where an estimate like ours in terms of the Hölder exponent $\kappa$ was given and the subsequent question of the comparison with the SRB measure was addressed.*

## 4. The limiting law

4.1. **The Gumbel law.** We have now all the tools to compute the asymptotic distribution of $\mathcal{L}_n$. We need one more ingredient which will constitute our last assumption:

- **A5** Let us suppose that the following limit exist for any $k \geq 0$ :

$$q_k = \lim_{n\to\infty} q_{k,n} := \lim_{n\to\infty} \frac{Z\left([(\mathcal{L}-\mathcal{L}_n)\mathcal{L}_n^k(\mathcal{L}-\mathcal{L}_n)]\mu\right)}{\Delta_n} \tag{29}$$

Notice that

$$q_{k,n} = \frac{\mu(B_n \cap T^{-1}B_n^c \cap \cdots \cap T^{-k}B_n^c \cap T^{-(k+1)}B_n)}{\mu(B_n)}$$

and therefore by the Poincaré recurrence theorem

$$\sum_{k=0}^{\infty} q_{k,n} = 1.$$

Therefore if the limits (29) exist, the quantity

$$\theta = 1 - \sum_{k=0}^{\infty} q_k \leq 1. \tag{30}$$

is well defined and called the *extremal index* which modulates the exponent of the Gumbel's law as we will see in a moment. We have in fact by the perturbation theorem in [16]:

$$\lambda_n = 1 - \theta\Delta_n = \exp(-\theta\Delta_n + o(\Delta_n)),$$

or equivalently

$$\lambda_n^n = \exp(-\theta n\Delta_n + o(n\Delta_n)).$$

Therefore we have

$$\mu(M_n \leq u_n) = \mathcal{L}_n^n\mu(1) = \lambda_n^n\mu_n(1)Z_n(\mu) + \lambda_n^n Q_n(\mu)(1)$$

and consequently

$$\mu(M_n \leq u_n) = \exp(-\theta n\Delta_n + o(n\Delta_n))\mu_n(1) + \lambda_n^n Q_n(\mu)(1),$$

since $Z_n(\mu) = 1$. It has been proved in [16], Lemma 6.1, that $\mu_n(1) \to 1$ for $n \to \infty$. At this point we need an important assumption, which will allow us to get a non-degenerate limit for the distribution of $M_n$. We in fact ask that

$$n\ \Delta_n \to \tau,\ n \to \infty, \tag{31}$$

where $\tau$ is a positive real number. With this assumption $\lambda_n^n$ will be bounded and, by (4), we have

$$|Q_n^n(\mu)(1)| \leq \text{constant } sp(Q)^n||\mu|| \to 0.$$

In conclusion we got the Gumbel's law

$$\lim_{n\to\infty} \mu(M_n \le u_n) = e^{-\theta\tau},$$

as $o(n\Delta_n) = o(\tau) \to 0$, as $n \to \infty$.

4.2. **The extremal index.** We are now ready to compute the $q_{k,n}$ which will determine the extremal index. Let us first suppose that the center of the ball $B_n$ is not a periodic point; then the points $T^j(z), j = 1, \cdots, k$ will be disjoint from $z$. Let us take the ball so small that is does not cross the set $T^j\Gamma, j = 1, \cdots, k$. In this way the images of $B_n$ will be ellipses with the long axis along the unstable manifold and the short axis stretched by a factor $\gamma^k$. By continuity and taking $n$ large enough, we can manage that all the iterates of $B_n$ up to $T^k$ will be disjoint from $B_n$ and for such $n$ the numerator of $q_{k,n}$ will be zero. At this point we can state the following result:

**Proposition 4.1.** *Let $T$ be the baker transformation and consider the function $M_n(x) := \max\{\phi(x), \ldots, \phi(T^{n-1}x)\}$, where $\phi(x) = -\log d(x, z)$, and $z$ is chosen $\mu$-almost everywhere with respect to the SRB measure $\mu$. Then, if $z$ is not periodic, we have*

$$\lim_{n\to\infty} \mu(M_n \le u_n) = e^{-\tau},$$

*where the boundary level $u_n$ is chosen to satisfy $n\mu(B(z, e^{-u_n})) \to \tau$.*

Suppose now $z$ is a periodic point of minimal period $p$. By doing as above we could stay away from the discontinuity lines up to $p$ iterates and look simply to $T^{-p}(B_n) \cap B_n$. Since the map acts linearly, the $p$ preimage of $B_n$ would be an ellipse with center $z$ and symmetric w.r.t. the unstable manifold passing trough $z$. So we have to compute the SRB measure of the intersection of the ellipse with the ball shown in Fig. 2.

It turns out that this computation is not easy. The natural idea would be to disintegrate the SRB measure along the unstable manifolds belonging to the unstable partition $\mathcal{W}^u$. We index such fibers as $W_\nu$ and we put $\zeta(\nu)$ the associated counting measure. Let us recall that the conditional measures along leves $W_\nu$ are normalized Lebesgue measures: we denote them with $l_\nu$. If we call $\mathcal{E}_{in}$ the region of the ellipse inside the ball $B_n$, we have to compute

$$\frac{\int l_\nu(\mathcal{E}_{in} \cap W_\nu)\, d\zeta(\nu)}{\int l_\nu(B_n \cap W_\nu)\, d\zeta(\nu)} \tag{32}$$

Although simple geometry allows us to compute easily the length of $\mathcal{E}_{in} \cap W_\nu$ and $B_n \cap W_\nu$, and since they vary with $W_\nu$, it is not at the end clear how to perform the integral with respect to the counting measure, especially because we need asymptotic estimates, not bounds. We therefore proceed by introducing a different metric, a nice trick which was already used in [4]. We use the $l^\infty$ metric on $\mathbb{R}^2$, namely if $\overline{x} := (x, y) \in \mathbb{R}^2$ and $\overline{0} := (0, 0)$ then $d_\infty(\overline{x}, \overline{0}) = \max\{|x|, |y|\}$. In this way the ball $B_n$ will become a square with sides of length $r_n := e^{-u_n}$ and $T^{-p}(B_n)$ will be a rectangle with the long side of length $\gamma_a^{-p} r_n$ and the short side of length $\alpha^p r_n$. This rectangle will be placed symmetrically with respect to the square as indicated in Fig. 3. The ratio (32) can now be computed easily since the length in the integrals are constant and we get $\alpha^p$. In conclusion:

**Proposition 4.2.** *Let $T$ be the baker transformation and consider the function $M_n(x) := \max\{\phi(x), \ldots, \phi(T^{n-1}x)\}$, where $\phi(x) = -\log d_\infty(x, z)$, and $z$ is chosen $\mu$-almost everywhere with respect to the SRB measure $\mu$. Then, if $z$ is a periodic point of minimal period $p$, we have*

$$\lim_{n\to\infty} \mu(M_n \le u_n) = e^{-\theta\tau},$$

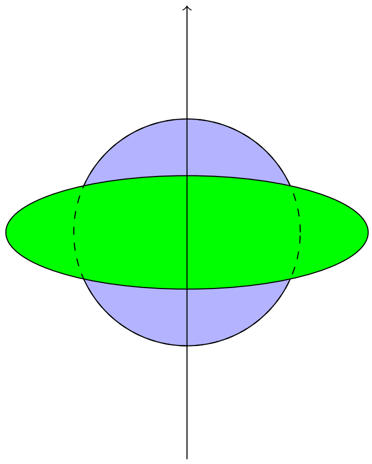

FIGURE 2. Computation of the extremal index around periodic point with the euclidean metric. The vertical line is an unstable manifold. We should compute the green area inside the circle.

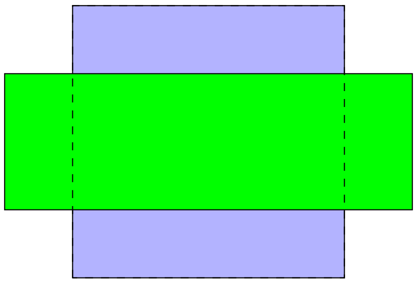

FIGURE 3. Computation of the extremal index around periodic point with the $l^\infty$ metric. We should compute the green area inside the square.

*where* $n\mu(B(z, e^{-u_n})) \to \tau$ *and*

$$\theta = 1 - \alpha^p.$$

**Remark 4.3.** *The baker map is probably the easiest example of a singular attractor. It is annoying that we could not compute analytically the extremal index with respect to the euclidean metric, which is that usually accessible in simulations and physical observations.*

## 5. Point process

As mentioned in the introduction, the spectral technique will not allow us to study the statistics of the number of visits in balls shrinking around a point. We instead use a recent approach developed in [14] and apply it to the baker's map. We will recover the usual dichotomy and get a pure Poisson distribution when the points are not periodic, and a Pólya-Aeppli distribution around periodic points and with the parameter giving the geometric distribution of the size the clusters which coincide with the extremal index computed in the preceding section. This last result is achieved in particular if we use the $l^\infty$ metric. This result is not surprising; what is interesting is the great flexibility of the technique of the proof which allow us to get easily the expected properties. In order to apply the theory in [14], we need to verify a certain number of assumptions, but otherwise defer to the aforementioned paper for precise definition. Here we recall the most important requirements and prove in detail one of them.

*Warning:* the next considerations are carried over with the euclidean metric which is more natural for applications. In order to cover visits to periodic points we will use the $l^\infty$ metric and the following computations are even easier.

*Decay of correlation.* There exists a decay function $\mathcal{C}(k)$ so that

$$\left|\int_M G(H\circ T^k)\,d\mu - \mu(G)\mu(H)\right| \le \mathcal{C}(k)\|G\|_{Lip}\|H\|_\infty \qquad \forall k\in\mathbb{N},$$

for functions $H$ which are constant on local stable leaves $W_s$ of $T$ and the functions $G: M\to\mathbb{R}$ being Lipschitz continuous. This is ensured by Theorem 2.5 in [6], where the role of $H$ is taken by the test functions in $C^\kappa(W,\mathbb{C})$ and $G\in\mathcal{B}$. The decay is exponential.

*Cylinder sets.* The proof requires the existence, for each $n\ge 1$ of a partition of each unstable leaf in subsets $\xi_k^{(n)}$, called $n$-cylinder (or cylinders of rank $n$,) and indexed with $k$, where $T^n$ is defined and the image $T^n\xi_k^{(n)}$ for each $k$, is an unstable leaf of full length. These cylinders are obtained by taking the $2^n$ preimages of $\alpha$ by the map $T_Y$ restricted to each leaf. In the following we will take $\alpha=1/2$ to simplify the exposition.

*Exact dimensionality of the SRB measure.* The following limit exists:

$$\lim_{r\to\infty}\frac{\log\mu(B(x,r))}{\log r} = d, \text{ for } x \text{ chosen } \mu\text{-a.e.}, \tag{33}$$

where $d$ was given above. We shall need the following result.

**Lemma 5.1.** *(*Annulus type condition*) Let $w>1$. If $x$ is a point for which the dimension limit* (33) *exists for a positive d, then there exists a $\delta>0$ so that*

$$\frac{\mu(B(x,r+r^w)\setminus B(x,r))}{\mu(B(x,r))} = O(r^\delta),$$

*for all $r>0$ small enough.*

Now we can apply the results of Section 7.4 in [14] to prove the following result which tracks the number of visits a trajectory of the point $x\in X$ makes to the set $U$ on a suitable normalized orbit segment:

**Proposition 5.2.** *Consider the counting function*

$$N_U^t(x) = \sum_{i=0}^{\lfloor t/\mu(U)\rfloor}\mathbf{1}_U\circ T^i(x),$$

*where $t$ is a positive parameter. Take $U := B_n = B(x,r_n)$ where $x$ is a point for which the limit (33) exists and $r_n$ is a sequence going to zero when $n\to\infty$, (in our EVT case $r_n = e^{-u_n}$.)*

- *If $x$ is not a periodic point and using the Euclidean metric, then we get a pure Poisson distribution:*

$$\mu(N_{B_n}^t = k)\to\frac{e^{-t}t^k}{k!},\ n\to\infty.$$

- *If $x$ is a periodic point of minimal period $p$ and using the $l^\infty$ metric, we get a compound Poisson distribution (Pólya-Aeppli):*

$$\mu(N_{B_n}^t = k)\to e^{-\theta t}\sum_{j=1}^{k}(1-\theta)^{k-j}\theta^{2j}\frac{s^j}{j!}\binom{k-1}{j-1},\ n\to\infty,$$

*where $\theta$ is given as above by $\theta = 1 - \lim_{n\to\infty} \frac{\mu(T^{-p}B_n \cap B_n)}{\mu(B_n)}$.*

*Proof of Lemma 5.1.* We have to prove the lemma in the two cases when (I) the norm is $\ell^2$ and (II) the norm is $\ell^\infty$ and the ball is geometrically a square.
(I) We now use the Eucliden metric and denote with $\mathcal{A}$ the annulus $\mathcal{A} = B(x, r + r^w) \setminus B(x, r)$ where $w > 1$. By disintegrating the SRB measure along the unstable manifolds we have:

$$\mu(\mathcal{A}) = \int l_\nu(\mathcal{A} \cap W_\nu)\, d\zeta(\nu).$$

We now split the subsets on each unstable manifold on the cylinders on rank $n$ and conditionate with respect to the Lebesgue measure on them:

$$l_\nu(\mathcal{A} \cap W_\nu) = \sum_{\xi_n;\xi_n\cap\mathcal{A}\neq\emptyset} \frac{l_\nu(\mathcal{A} \cap W_\nu \cap \xi_n)}{l_\nu(\xi_n)} l_\nu(\xi_n). \tag{34}$$

We then iterate forward each cylinder with $T^n$; they will become of full length equal to 1 and subsequently we get $l_\nu(T^n\xi_n) = 1$. Since the action of $T$ is locally linear and expanding by a factor $2^n$ (with the given choice of $\alpha = \frac{1}{2}$) on the unstable leaves and therefore has zero distortion, we have

$$\frac{l_\nu(\mathcal{A} \cap W_\nu \cap \xi_n)}{l_\nu(\xi_n)} = \frac{l_{\nu'}(T^n(\mathcal{A} \cap W_\nu \cap \xi_n))}{l_{\nu'}(T^n\xi_n)} = l_{\nu'}(T^n(\mathcal{A}) \cap W_{\nu'})$$

for some $W_{\nu'}$ so that $T^n(\mathcal{A} \cap W_\nu \cap \xi_n) \subset W_{\nu'}$. Therefore

$$l_\nu(\mathcal{A} \cap W_\nu) = \sum_{\xi_n;\xi_n\cap\mathcal{A}\neq\emptyset} l_{\nu'}(T^n(\mathcal{A} \cap W_\nu \cap \xi_n)) l_\nu(\xi_n).$$

By elementary geometry we see that the largest intersection of $\mathcal{A}$ with the unstable leaves will produce a piece of length $O(r^{\frac{w+1}{2}})$, where with the writing $a_r = O(b_r)$ we mean that there exists a constant $C$ independent of $r$ such that $a_r \leq Cb_r$ for $r$ small enough. Therefore $l_{\nu'}(T^n(\mathcal{A} \cap W_\nu \cap \xi_n)) = O(2^n r^{\frac{w+1}{2}})$, and:

$$\mu(\mathcal{A}) = O(2^n r^{\frac{w+1}{2}}) \int \sum_{\xi_n;\xi_n\cap\mathcal{A}\neq\emptyset} l_\nu(\xi_n)\, d\zeta(\nu)).$$

We now observe that in order to have our result, it will be enough to get it with a decreasing sequence $r_n$, $n \to \infty$, of exponential type, $r_n = b^{-t(n)}, b > 1$, and $t(n)$ increasing to infinity. We put $r = 2^{-n}$. With this choice and remembering that $2^{-n}$ is also the length of the $n$-cylinders, we have

$$\bigcup_{\xi_n;\xi_n\cap\mathcal{A}\neq\emptyset} \xi_n \subset B(x, r + r^w + 2^{-n}) \subset B(x, 2r + r^w) \subset B(x, 3r),$$

which, as the cylinders $\xi_n$ are disjoint, yields the estimate for the integral above:

$$\mu(\mathcal{A}) = O(2^n r^{\frac{w+1}{2}} r^{d-\epsilon}).$$

Now by the exact dimensionality of the SRB measure one has for any $\varepsilon > 0$

$$(2r + r^w)^{d+\varepsilon} \leq \mu(B(x, 2r + r^w)) \leq (2r + r^w)^{d-\varepsilon}$$

for all $r$ small enough i.e. $n$ large enough. With this we can divide $\mu(\mathcal{A})$ by the measure of the ball of radius $r$ and obtain the estimate

$$\frac{\mu(\mathcal{A})}{\mu(B(x,r))} = O(r^{\frac{w-1}{2}+d-\varepsilon-d-\varepsilon}) = O(r^{\frac{w-1}{2}-2\varepsilon}) = O(r^{\frac{w-1}{4}}),$$

since $w > 1$, and provided $\varepsilon$ is small enough.

(II) Now we shall use the $\ell^\infty$-distance and again denote by $\mathcal{A}$ the annulus $B(x, r + r^w) \setminus B(x, r)$. Since we are in two dimensions, we can cover the annulus by balls $B(y_j, 2r^w)$ of radii $2r^w$, with centres $y_j$ for $j = 1, \ldots, N$. The number $N$ of balls needed is bounded by $8\frac{r}{r^w}$. For any $\varepsilon > 0$ there exists a constant $c_1$ so that $\mu(B(y_j, 2r^w)) \le c_1 r^{w(d-\varepsilon)}$ for all $r$ small enough. Thus

$$\mu(\mathcal{A}) \le 8c_1 r^{1+w(d-1-\varepsilon)}$$

and since $\mu(B(x, r)) \ge c_3 r^{d+\varepsilon}$ for some $c_3 > 0$ we obtain

$$\frac{\mu(\mathcal{A})}{\mu(B(x, r))} \le c_4 r^{(d-1)(w-1)-\varepsilon(w+1)}.$$

The exponent $\delta = (d - 1)(w - 1) - \varepsilon(w + 1)$ is positive as $d, w > 1$ and $\varepsilon > 0$ can be chosen sufficiently small. □

The second statement of Proposition 5.2 about periodic points requires the neighbourhoods $B_n$ to be chosen in a dynamical relevant way. Here they turn out to be squares (or rectangles). If the measure has some mixing properties with respect to a partition then the sets $B_n$ can be taken to be cylinder sets as it was done in [13] for periodic points and in [12] Corollary 1 for non-periodic points. Here we show that for Euclidean balls one cannot in general expect the limiting distribution at periodic points to be Pólya-Aeppli and therefore cannot be described by the single value of the extremal value.

We assume that all parameters are equal, that is $\gamma_a = \gamma_b = \alpha = \beta = \frac{1}{2}$. This is the traditional baker's map for which the Lebesgue measure on $[0, 1]^2$ is the SRB measure $\mu$. Let $x$ be a periodic point with minimal period $p$. Then $\mu(B(x, r)) = r^2\pi$ and

$$\mu\left(\bigcap_{i=0}^{k} T^{-ip}B(x, r)\right) = 4r^2 2^{-kp}(1 + \mathcal{O}(2^{-2kp})).$$

This yields

$$\hat{\alpha}_{k+1} = \lim_{r\to 0} \frac{mu\left(\bigcap_{i=0}^{k} T^{-ip}B(x, r)\right)}{\mu(B(x, r))} = \frac{4}{\pi} 2^{-kp}(1 + \mathcal{O}(2^{-2kp}))$$

for $k = 1, 2, \ldots$. According to [14] Theorem 2 we then define the values $\alpha_k = \hat{\alpha}_k - \hat{\alpha}_{k+1}$ where the value $\alpha_1$ is the extremal index, i.e. $\theta = \alpha_1$. If the limiting distribution is Pólya-Aeppli then the probabilities $\lambda_k = \frac{\alpha_k - \alpha_{k+1}}{\alpha_1}$, $k = 1, 2, \ldots$, are geometrically distributed and must satisfy $\lambda_k = \theta(1 - \theta)^{k-1}$ which is equivalent to say that $\hat{\alpha}_{k+1} = (1 - \theta)^k$ for $k = 0, 1, 2, \ldots$ (see [14] Theorem 2). Evidently this condition is violated in the present case and we conclude that the limiting distribution given by the values $\hat{\alpha}_k$ is not Pólya-Aeppli and in fact obeys another compound Poisson distribution.

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