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Canonical product structure of equilibrium states

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Abstract: It will be shown that on the stable and unstable foliations of Axiom A flows there exist, associated to Hölder continuous potentials, transversal measures that satisfy smooth Margulis' type cocycle equations. Using these transversal measures, one then deduces that equilibrium states for Hölder continuous potentials are indeed of product form. It is then also shown that in the case of a suspended flow over a subshift of finite type, there are, corresponding to the poles of the weighted zeta function, transversal distributions which satisfy Margulis' type cocycle equations similar to the ones for the transversal measures.

1. Introduction

Transversal measures were first introduced to Dynamical Systems by Sinai [18] for Anosov diffeomorphisms and then by Margulis [12] for Anosov flows. These measures, which are now called Margulis' measures, are supported on the weak unstable leaves and are invariant under sliding transversally along the strong stable foliation. This was done by first finding a good class of measures and then to apply Tychonoff's fixed point theorem. Much of this approach has later been systematically developed for Ruelle's Perron-Frobenius type transfer operator which plays the central role in the subsequent generalisations of Sinai's and Margulis' original results. Ruelle and Sullivan proved in [16] the existence of transversal measures for Axiom A diffeomorphisms and Bowen and Markus [5] then showed how they are to be constructed for Axiom A flows.

In the case of an Axiom A diffeomorphism, the measure of maximal entropy, also called the canonical measure, is in fact the product of two Margulis' measures transversal to respectively the stable and unstable foliation. For Axiom A attractors one has a second canonical measure, the SBR (Sinai-Bowen-Ruelle) measure, which is the only invariant measure absolutely continuous on weak unstable leaves and is therefore considered the natural measure for attractors. As Parry [13] showed, these two measures can for C^2 flows be related by a

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velocity change of the flow. Changing the speed of the flow does not affect the SBR measure significantly (it is the equilibrium state for the Jacobian in the expanding direction), but changes Margulis' measure which in general is singular. However, the SBR measure can be made to coincide with the measure of maximal entropy if the speed of the flow is chosen appropriately (synchronisation).

In this paper we extend the context of transversal measures for Axiom A flows and shall introduce transversal measures that are associated to Hölder continuous functions that otherwise can be arbitrarily chosen. This will enable us to show that an equilibrium states for a Hölder continuous function is of a product form in similarity to the measure of maximal entropy which is a product of Margulis' measures on the stable and unstable foliations. In fact, these transversal measures which are associated to Hölder continuous potentials satisfy Margulis' type cocycle equations which, in the case of the zero potential, reduce to the well-known relations for the ordinary Margulis' measure.

We shall also show that for suspended flows over subshifts of finite type, there exists moreover a whole family of transversal functionals with similar transformation properties as the transversal measures.

As was pointed out to us, Series [17] constructed transversal measures on the stable and unstable foliations for Anosov flows much in the same way as it was done in [5]. However these measures don't satisfy Margulis' type transformation properties (cocycle equations) for the flow (see Lemma 8 and Proposition 9), although it is, of course, straightforward to write down such cocycle equations for the Poincaré return map as it is done in [17], section 2, for the special case in which the Markov cross sections are unions of local strong unstable leaves (or one-sided subshift).

2. Axiom A flows and main result

Let M be a compact riemannian manifold and Φ_t: $M \to M$ a smooth flow. A compact subset $\Omega \subset M$ which is invariant under the flow is called an Axiom A *basic set* if

(i) for every $x \in \Omega$ the tangent space $T_x\Omega$ is the Whitney sum $E^0 \oplus E^s \oplus E^u$, where E^0 is the one-dimensional direction of the flow, and such that there exists a constant $\lambda > 0$ satisfying

$$\|D\Phi_t v\| \leq Ce^{-\lambda t}\|v\|, \quad v \in E^s, \quad t \geq 0,$$

$$\|D\Phi_{-t} v\| \leq Ce^{-\lambda t}\|v\|, \quad v \in E^u, \quad t \geq 0,$$

where we can assume that the positive constant C is equal to 1. The number λ is called the *contraction parameter* of the flow.

(ii) $\Phi_t|\Omega$ is topologically mixing, that is for open $U, V \subset \Omega$ the intersection $\Phi_t(U) \cap V \neq \emptyset$ for all large enough t.

(iii) $\Omega = \bigcap_{-\infty<t<\infty} \Phi_t(U)$ for some open neighbourhood U of Ω.

(iv) Periodic orbits are dense in Ω.

If $\Omega = M$ then we speak of an Anosov flow.

Denote by d the metric on M, and let $\varepsilon > 0$ be some number, then

$$W_\varepsilon^{ss}(x) = \left\{y \in \Omega\colon\ d(\Phi_t(x), \Phi_t(y)) \le \varepsilon \ \forall\ t \ge 0 \text{ and } d(\Phi_t(x), \Phi_t(y)) \to 0 \text{ as } t \to \infty\right\},$$

$$W_\varepsilon^{uu}(x) = \left\{y \in \Omega\colon\ d(\Phi_{-t}(x), \Phi_{-t}(y)) \le \varepsilon \ \forall\ t \ge 0 \text{ and } d(\Phi_{-t}(x), \Phi_{-t}(y)) \to 0 \text{ as } t \to \infty\right\},$$

are the local strong stable respectively unstable manifold through the point $x \in \Omega$. The weak stable and unstable manifolds through x are given by

$$W^s(x) = \bigcup_{-\infty<t<\infty} \Phi_t(W^{ss}(x)),$$

$$W^u(x) = \bigcup_{-\infty<t<\infty} \Phi_t(W^{uu}(x)),$$

where $W^{ss}(x) = \bigcup_{\varepsilon>0} W_\varepsilon^{ss}(x)$ and $W^{uu}(x) = \bigcup_{\varepsilon>0} W_\varepsilon^{uu}(x)$. The strong stable manifolds form a foliation transversal to the weak unstable foliation and vice versa, W^{uu} (W^{ss}) is transversal to W^s (W^u). Put $W_\varepsilon^u(x) = \bigcup_{|t|\le\varepsilon} W_\varepsilon^u(\Phi_t(x))$, then if $d(x,x') \le \delta$, $0 < \delta < \varepsilon$ small enough, there is a bijection $\rho_{x,x'}$ from $W_{\varepsilon'}^u(x)$ to $W_\varepsilon^u(x')$ ($\varepsilon' < \varepsilon$ small enough) by shifting along the strong unstable foliation. The map $\rho_{x,x'}\colon W_{\varepsilon'}^u(x) \to W_\varepsilon^u(x')$ is given by

$$\rho_{x,x'}(z) = W_\varepsilon^{ss}(z) \cap W_\varepsilon^u(x'),$$

$z \in W_{\varepsilon'}^u(x)$.

We say a family of measures ν_x supported on the weak unstable foliation is transversal if for every $x' \in W_\delta^{ss}(x)$ the pull-back $\rho_{x,x'}{}^* \nu_x$ is a measure on $W_{\varepsilon'}^u(x')$ equivalent to $\nu_{x'}$, where ν_x lives on $W_\varepsilon^u(x)$.

The best known transversal measure for Axiom A flows is the Margulis measure, which is invariant under sliding along the strong unstable foliation. In fact, Margulis [12] originally introduced this measure for Anosov flows. (Recently U. Hamenstädt and B. Hasselblatt have given a geometrical interpretation of Margulis' measure as h-dimensional spherical measure (h is the topological entropy), in [8] for geodesic flows on negatively curved manifolds and in [9] for Anosov flows.) The generalisation to Axiom A diffeomorphism was subsequently done by

Ruelle and Sullivan [16], and the further extension to Axiom A flows is due to Bowen and Marcus [5]. The tools employed in the Axiom A case (diffeomorphism and flow) involves Markov partitions and the use of symbolic dynamics. We shall here follow this tradition and mainly work with flows suspended over subshifts of finite type. The general result then follows by standard arguments, however we have to show that the boundary set of the Markov partition has measure zero.

Let $F: \Omega \to \mathbb{R}$ be a Hölder continuous function, then one can associate to it a number $P(F)$, the *topological pressure*, which is given by the variational principle

$$P(F) = \sup_\rho (H(\rho) + \int F \, d\rho),$$

where the supremum is over all Φ_t–invariant probability measures ρ on Ω and $H(\rho)$ denotes the measure theoretic entropy with respect to ρ [4], [19]. There is exactly one such measure that attains the supremum and which is called the *equilibrium state* for F.

Theorem 1: Let $F: \Omega \to \mathbb{R}$ be Hölder continuous, then there exist transveral measures μ_x supported on the local weak unstable manifold $W_\varepsilon{}^u(x)$, $\varepsilon > 0$, such that

(i) $$\Phi_t{}^* \mu_x = e^{\tau_t} \mu_x,$$

where $\tau_t = {}_0\int^t (F \circ \Phi_s - P(F)) \, ds$

(ii) $$\mu_{x'} = e^{\omega_{x,x'}} \rho_{x,x'}{}^* \mu_x,$$

$x' \in W_\delta{}^{ss}(x)$, some $\delta > 0$, where $\omega_{x,x'} = {}_0\int^\infty (F \circ \Phi_s \circ \rho_{x,x'} - F \circ \Phi_s) \, ds$.

In a similar way there exist tranversal measures supported on the weak unstable foliation with the corresponding properties. We shall call the identities (i) and (ii) *Margulis' cocyle* equations for the potential F. Property (ii) implies in particular that μ_x and $\mu_{x'}$ have the same null sets under sliding along the strong stable foliation. This property is according to [11] called quasi-invariance.

Theorem 2: Let $\bar{\mu}$ be the equilibrium state for some Hölder continuous function $F: \Omega \to \mathbb{R}$. Then its equilibrium state $\bar{\mu}$ is (up to a normalising factor) locally given by the product $\mu^{ss} \times \mu^{uu} \times \ell$, where μ^{ss}, μ^{uu} are measures on the strong stable respectively unstable leaves and have the following properties:

(i) $\Psi_t{}^* \mu^{uu} = e^{\tau_t} \mu^{uu}$, $\Psi_t{}^* \mu^{ss} = e^{-\tau_t} \mu^{ss}$.

(ii) $d\mu^{uu}(y) = e^{\omega_{x,y}} d\rho_{x,y}{}^{*}\mu^{uu}(x)$, $d\mu^{ss}(y) = e^{\omega'_{x,y}} d\rho'_{x,y}{}^{*}\mu^{ss}(x)$.

Remark: Let G be a smooth function on M, and G' its derivative in the direction of the flow, then the functions F and $F + G'$ have the same equilibrium state $\tilde{\mu}$ (F and $F + G'$ are cohomologous functions as G' is a coboundary). The Margulis measures $\tilde{\mu}^{uu}$ and $\tilde{\mu}^{ss}$ for the modified potential $F + G'$ generally are related to the Margulis measures μ^{uu} and μ^{ss} for the potential F according to the identities: $\tilde{\mu}^{uu} = e^{G} \mu^{uu}$ and $\tilde{\mu}^{ss} = e^{-G} \mu^{ss}$. We moreover have that $\Psi_t{}^{*}\tilde{\mu}^{uu} = e^{\tilde{\tau}_t} \tilde{\mu}^{uu}$ and $d\mu^{uu}(y) = e^{\tilde{\omega}_{x,y}} d\rho_{x,y}{}^{*}\mu^{uu}(x)$, where $\tilde{\tau}_t = \tau_t + G - G\circ\Psi_t$ and $\tilde{\omega}_{x,y} = \omega_{x,y} + G\circ\rho_{x,y} - G$. Similar identities hold for the measures μ^{ss}.

In the following section we will discuss some special cases such as the SBR measure and the harmonic measure on compact manifolds with strictly negative curvature. The proof of the main results will be using symbolic dynamics which we shall introduce in section 4. In section 5 we shall prove Theorem 1 for suspended flows over subshifts of finite type. The essential part consists in finding the appropriate weight function in the flow direction. This is done in Proposition 9 and Lemma 10. In section 6 we prove Theorem 2, where we use the fact that in the case of Axiom A diffeomorphism equilibrium states are locally of product form. In section 8 we prove the theorems for the general Axiom A flows using standart results for constructing Markov partitions. In section 7, Theorem 11, we show that on the stable and unstable foliation there exist generalised functionals which satisfy Margulis cocyle equations similar to the ones for the measures.

3. Transversal measures for attractors and compact manifolds with negative curvature

(I) For $F = 0$ we recover Margulis measures which are invariant under $\rho_{x,x'}$ (since $\omega_{x,x'} = 0$) and have the scaling property $\Phi_t{}^{*} \mu_x = e^{-Ht} \mu_x$, where $H = P(0)$ is the topological entropy of the flow, that is, of its time one map Φ_1. In fact we see that the family of transversal measures is invariant under sliding along the stable foliation whenever F is constant along strong stable leaves.

(II) If condition (iii) is replaced by $\Omega = \bigcap_{0\le t} \Phi_t(U)$, (Ω,T) is an Axiom A attractor and carries the SBR measure which along the weak unstable manifolds is absolutely continuous with respect to the Riemann measure [6]. If we put $\lambda_t(x) = \det \|D\Phi_t(x)|E^u\|$, the function

$$F(x) = -\frac{d}{dt} \log \lambda_t(x)|_{t=0}$$

is Hölder continuous and its equilibrium state μ is called the SBR measure. It is the only Φ_t-invariant measure which is on the unstable weak foliation equivalent to Riemann measure.

Corollary 3: Denote by μ_x its restriction to the weak unstable leaf $W^u(x)$ through $x \in \Omega$. Then

(i) $$\Phi_t{}^*\mu_x = \frac{1}{\lambda_t}\mu_x,$$

(ii) $$\mu_{x'} = \tilde{\omega}_{x,x'}\, \rho_{x,x'}{}^*\, \mu_x,$$

where $\tilde{\omega}_{x,x'} = \lim_{t\to\infty} \lambda_t/\lambda_t \circ \rho_{x,x'}$ and $\rho_{x,x'}$: $W_{\varepsilon'}{}^u(x) \to W_\varepsilon{}^u(x')$ is as above the map sliding along the strong stable foliation.

Proof. To see (i) note that by [6], F has pressure zero and thus

$$\tau_t = \int_0^t -\frac{d}{ds}\log \lambda_s(x)\, ds = \log\frac{\lambda_0}{\lambda_t} = -\log \lambda_t,$$

as $\lambda_0 = 1$. The second statement (ii) follows from

$$\omega_{x,x'} = \lim_{t\to\infty} \int_0^t (F\circ\Phi_s\circ\rho_{x,x'} - F\circ\Phi_s)\, ds = \lim_{t\to\infty} \lambda_t/\lambda_t\circ\rho_{x,x'}.$$

The existence of the limit follows from the convergence of the integral. □

(III) Let M be compact connected n-dimensional ($n \geq 2$) manifold whose sectional curvatures lie in the interval $[-b,-a]$ for some $a, b > 0$. Denote by SM the spherical bundle of M and by π the projection $SM \to M$. Let N be the universal cover of M (π: $SN \to N$) and Φ_t be the geodesic flow (which is an Anosov flow). Two geodesics $\gamma(t)$, $\gamma'(t)$ are positively equivalent if $\sup_{t\geq 0} d(\gamma(t),\gamma'(t)) < \infty$. The ideal boundary ∂N is the set of equivalence classes of geodesics. For $x \in N$ let p_x: $S_xN \to \partial N$ be given by the equivalence class of the geodesic equivalent to $\Phi_t(X)$, where $X = (x,\xi)$, $\xi \in S_xN$. We can thus identify the point ξ with a point in ∂N and vice versa. Let f be a continuous function of ∂N, then by [2] the Dirichlet problem on N has a solution u_f such that $\Delta u_f = 0$ on N and $\lim_{z\to\xi} u_f(z) = f(\xi)$, $\xi \in \partial N$. For $x \in N$ the harmonic measure ν_x on ∂N is then defined by $\nu_x(\chi) = u_\chi(x)$, $\chi \in C(\partial N)$. Any two measures ν_x and ν_y are equivalent and their Radon–Nikodym derivative $\frac{d\nu_x}{d\nu_y}(\xi)$ is almost everywhere given by a unique function $k(x,y,\xi)$ which satisfies $k(x,x,\xi) = 1$, $k(x,z,\xi)k(z,y,\xi) = k(x,y,\xi)$, $\Delta_y k(x,y,\xi) = 0$ on N and $\lim_{y\to\xi'} k(x,y,\xi) = 0$ if $\xi \neq \xi'$, $\xi, \xi' \in \partial N$.

We have (strong) stable and unstable sub manifolds as introduced above. Put $F(X)$ for the Hölder continuous function $\frac{d}{dt} \log k(x,\pi\Phi_t(X),p_x\xi)|_{t=0}$. By [11] the pressure of F is zero. Then by Theorem 1 there exists a family of transveral measures μ_x for F. If $Y = (y,\eta) = \Phi_t(X)$, $\eta = p_y^{-1}p_x\xi$ we get

$$\frac{d\Phi_t{}^*\mu_x}{d\mu_x}(\xi) = \exp \tau_t(X) = \exp \int_0^t F\circ\Phi_s\, ds = k(x,y,p_x\xi) = \frac{d\nu_y}{d\nu_x}(p_x\xi).$$

Now if $Y = (y,\eta) \in W_\varepsilon^{ss}(X)$, then $d(\Phi_t(X),\Phi_t(Y)) \to 0$ as $t \to \infty$ and $p_x\xi = p_y\eta$, and thus

$$\omega_{X,Y} = \exp \int_0^\infty (F\circ\Phi_s\circ\rho_{X,Y} - F\circ\Phi_s)\, ds$$

$$= \lim_{t\to\infty} \frac{k(y,\pi\Phi_t(Y),p_y\zeta)}{k(x,\pi\Phi_t(X),p_x\xi)}$$

$$= k(y,x,p_x\xi),$$

since $\lim_{t\to\infty} k(\pi\Phi_t(Y),\pi\Phi_t(X),p_x\xi) = 1$ as the distance between $\pi\Phi_t(Y)$ and $\pi\Phi_t(X)$ goes to zero as $t \to \infty$ and k is continuous.

We obtain the following result which had been proven in [11] using an explicit construction borrowed from the proof of the SBR measure.

Corollary 4: The spherical harmonic measures $\mu_x = p_x{}^{-1*}\nu_x$ are the transversal measures for

$$F(X) = \frac{d}{dt} \log k(x,\pi\Phi_t(X),p_x\xi)|_{t=0},$$

defined for $X = (x,\xi) \in SN$.

4. Subshifts and suspended flows and their foliations

Let $T = \{1,2,\dots,n\}$, A be an $n\times n$ matrix of zeros and ones and define the *subshift* Σ as the set of points $x \in \prod_{-\infty<i<\infty} T$ which satisfy the transition condition $A_{x_i,x_{i+1}} = 1$ for all indices $i \in \mathbb{Z}$. On Σ we have the (two-sided) *shift transformation* σ defined by $(\sigma x)_i = x_{i+1}$ for all indices i. The topology on Σ is generated by the cylinder sets

$$U(x_{-n}\dots x_n) = \left\{y \in \Sigma:\ y_i = x_i,\ |i| \le n\right\},$$

where $x_{-n}\dots x_n$ runs over all allowed finite strings in Σ of lengths $2n+1$, $n \ge 1$.

The *variation* for a complex function f on Σ is given by

$$\mathrm{var}_n f(z) = \sup_{z\in\Sigma} \sup\{|f(z) - f(z')|:\ z'_i = z_i,\ |i| \le n\},$$

$n \ge 1$. Now let u: $\Sigma \to \mathbb{R}$ be a strictly positive continuous function on Σ, then if $\mathrm{var}\, f(x)$

decays fast enough, the quantity

$$\|f\|_u = \sup_{z\in\Sigma} \sup_{n\geq 1} e^{2\cdot\min(u^n(z),u^{-n}(z))} \operatorname{var}_n f(z),$$

is finite and called the *Hölder constant* of f. Here we have used the notation $u^n(x) = u(x) + u(\sigma x) + \cdots + u(\sigma^{n-1}x)$ and $u^{-n}(x) = u(\sigma^{-1}x) + \cdots + u(\sigma^{-n}x)$. We call $C_u(\Sigma)$ the Banach space of functions f which are finite with respect to the triple norm

$$|||f|||_u = \|f\|_u + \|f\|_\infty,$$

where $\|\cdot\|_\infty$ is the usual supremum norm. The function u is called *modulus of continuity* [10].

We can define a metric of Σ by putting $d(x,y) = e^{-\alpha(x,y)}$, $x, y \in \Sigma$, where

$$\alpha(x,y) = \inf\left\{\min\{u^n(z), u^{-n}(z)\}: \; z_i = x_i \;\; \forall \; |i| \leq n(x,y)\right\},$$

and $n(x,y)$ is the largest n such that $y_i = x_i$ for $|i| \leq n$. Obviously $d(x,y) = d(y,x)$. To verify the triangle inequality let $x, y, z \in \Sigma$ be such that $n(x,y) \leq m(y,z)$. Then, as $n(x,y) = n(x,z)$ and $\alpha(x,z) = \alpha(x,y) \leq \alpha(y,z)$, it follows that $d(x,z) = e^{-\alpha(x,z)} \leq e^{-\alpha(x,y)} + e^{-\alpha(y,z)} = d(x,y) + d(y,z)$. Note that in this metric f is in fact Lipshitz continuous with Lipshitz constant $\|f\|_u$.

Let us now define the one-sided shift spaces

$$\Sigma_1 = \left\{x \in \textstyle\prod_{j\geq 1} T: \; A_{x_i,x_{i+1}} = 1, \; i \geq 1\right\},$$

$$\Sigma_0 = \left\{x \in \textstyle\prod_{j\leq 0} T: \; A_{x_i,x_{i+1}} = 1, \; i < 0\right\}.$$

The topology is as in the two-sided case given by one-sided cylinders. There are shift maps on these one-sided sushifts, induced by σ and σ^{-1} which we again denote by the same symbol. However these maps are finite to one and only locally homeomorphism. As above we define the variation for functions on Σ_0 and Σ_1 and define triple norms $|||\cdot|||_{u_0}$, $|||\cdot|||_{u_1}$, with appropriate one-sided moduli of continuity u_0 on Σ_0 and u_1 on Σ_1, however with the slight difference that the Hölder constant $\|f\|_{u_1}$ is equal to $\sup_{y\in\Sigma_1} \sup_{n\geq 1} e^{u^n(y)} \operatorname{var}_n f(y)$ (similarly for $\|\cdot\|_{u_0}$). The spaces $C_{u_0}(\Sigma_0)$, $C_{u_1}(\Sigma_1)$ consist of the functions which are finite in the appropriate triple norm. Sometimes we shall use one-sided functions in a two-sided context, in which case they are understood to depend only on coordinates ≤ 0 respectively on positive coordinates.

We say two function f, g on Σ are *cohomologous* if there exists a function h on Σ such that $f - g = h - h\circ\sigma$. An expression of the form $f = h - h\circ\sigma$ which is cohomologous to zero is called a *coboundary*. We have the following result by Sinai which allows us to transform within a cohomology class two-sided functions into one-sided ones, however with some loss of regularity.

Lemma 5: If $f \in C_u(\Sigma)$, then there exist functions $f_1, v_1 \in C_{\frac{1}{2}u}(\Sigma)$ such that $f_1 = f + v_1 - v_1 \circ \sigma$ is independent of coordinates ≤ 0.

In the same way on can find $f_0, v_0 \in C_{\frac{1}{2}u}(\Sigma)$ such that $f_0 = f + v_0 - v_0 \circ \sigma$ only depends on coordinates ≤ 0. Thus f_0 and f_1 can be considered to lie in $C_{u_0}(\Sigma_0)$ and $C_{u_1}(\Sigma_1)$ respectively for appropriate and strictly positive moduli of continuity u_0, u_1 which we can assume to be cohomologous to u.

For a strictly postive real $r \in C_u(\Sigma)$ one defines on

$$\boldsymbol{\Sigma} = \left\{(x,t) \in \Sigma \times \mathbb{R}:\ 0 \leq t \leq r(x)\right\},$$

the suspended flow Ψ_t by $\Psi_t(x,s) = (x,t+s)$ if $0 \leq t+s \leq r(x)$ and extends it to all real t by identifying the point $(x,r(x))$ with $(\sigma x,0)$. According to Lemma 5 let $v_0, v_1 \in C_{\frac{1}{2}u}(\Sigma)$ be such that $r_0 = r + v_0 - v_0 \circ \sigma$ is a function on Σ_0 and $r_1 = r + v_1 - v_1 \circ \sigma$ is a function on Σ_1. Then we can define on

$$\boldsymbol{\Sigma}_0 = \left\{(x,t)\ \ 0 \leq t \leq r_0(x)\right\}$$

$$\boldsymbol{\Sigma}_1 = \left\{(x,t)\ \ 0 \leq t \leq r_1(x)\right\}$$

semiflows φ_{-t}, ψ_t for $t \geq 0$ by $\varphi_{-t}(x,s) = (x,s-t)$ if $0 \leq s-t \leq r(x)$ identifying $(x,0)$ with $(\sigma^{-1}x, r_0(x))$, and $\psi_t(y,s) = (x,s+t)$ if $0 \leq s+t \leq r_1(y)$ identifying $(y,r_1(y))$ with $(\sigma y,0)$ thus defining ψ_t for all $t \geq 0$.

Lemma 6: The strong stable direction $W^{ss}(xy,s)$ through a point $(xy,s) \in \boldsymbol{\Sigma}$ is locally given by the points $(x'y,\ s+v_1(xy)-v_1(x'y))$, where $x' \in \Sigma_0$ is close enough to x so that $0 \leq s + v_1(xy) - v_1(x'y) \leq r(x'y)$, and is extended by using the usual identification $(x'y,r(x'y)) = \sigma((x'y),0)$.

The strong unstable direction through a point $(xy,s) \in \Sigma$ is locally given by the points $(xy',\ s-v_0(xy)+v_0(xy'))$, where $y' \in \Sigma_1$ is close enough to y satisfying $0 \leq s + v_0(xy) - v_0(xy') \leq r(xy')$, and is extended as above.

Proof. It is sufficient to prove the first part of the lemma. We can assume that the metric d on Σ satisfies $d((z,s),(z,s')) \leq C \cdot |s-s'|$ for $z \in \Sigma$, $0 \leq s, s' \leq r(z)$ for some constant $C > 0$. (For a metric on $\boldsymbol{\Sigma}$ see for instance [7].) We have to show that $d(\Psi_t(xy,s), \Psi_t(x'y,\ s+v_1(xy)-v_1(x'y))) \rightarrow 0$ as $t \rightarrow \infty$. In fact it is sufficient to show this for an increasing sequence $\{t_j:\ j \in \mathbb{N}\}$ which goes to infinity as $j \to \infty$. For large t we have $\Psi_t(xy,s) = (\sigma^n(xy),s')$, where n is such that

$$0 \le s' = s + t - r^n(xy) \le r(\sigma^n(xy)),$$

or, since $r_1 = r + v_1 - v_1 \circ \sigma$,

$$s'(t) = s + t + v_1(xy) - r_1{}^n(y) - v_1(\sigma^n(xy)).$$

On the other hand $\Psi_t(x'y,\ s+v_1(xy)-v_1(x'y)) = (\sigma^m(x'y),s'')$, with m such that

$$0 \le s''(t) = s + t + v_1(xy) - v_1(x'y) - r^m(x'y)$$

$$\le s + t + v_1(xy) - r_1{}^m(y) - v_1(\sigma^m(x'y)) \le r(\sigma^m(x'y)).$$

If we choose t large enough we can find an increasing sequence $\{t_j\colon\ j \in \mathbb{N}\}$ so that $n(t_j) = m(t_j)$ for all j and moreover

$$0 \le s'(t_j),\ s''(t_j) \le \min(r(\sigma^{n(t_j)}(xy)),\ r(\sigma^{n(t_j)}(x'y))).$$

Thus

$$d(\Psi_t(xy,s),\ \Psi_t(x'y,\ s+v_1(xy)-v_1(x'y))) \le C\cdot|s'(t_j) - s''(t_j)|$$

$$\le C\cdot|v_1(\sigma^{n(t_j)}(xy)) - v_1(\sigma^{n(t_j)}(x'y))|$$

$$\le C\cdot \mathrm{var}_{n(t_j)}\, v_1(xy)$$

goes exponentially fast to zero if we choose the sequence so that $n(t_j) < n(t_{j+1})$ for all j. □

The lemma now allows us to identify Σ with $\Sigma_0{\times}\mathbf{\Sigma}_1$ or $\mathbf{\Sigma}_0{\times}\Sigma_1$ as follows. We can define maps $\pi_1\colon\ \Sigma_0{\times}\mathbf{\Sigma}_1 \to \mathbf{\Sigma}$ and $\pi_0\colon\ \mathbf{\Sigma}_0{\times}\Sigma_1 \to \mathbf{\Sigma}$ by

$$\pi_1(x,(y,t)) = \Psi_{t-v_1(xy)}(xy,0)$$

$$\pi_0((x,t),y) = \Psi_{t+v_0(xy)}(xy,0),$$

whenever $xy \in \Sigma$. From Lemma 6 it is clear that the one-sided shiftspace Σ_0 is by π_1 mapped to the local strong stable foliation in $\mathbf{\Sigma}$ and Σ_1 is by π_0 mapped to the local strong unstable foliation.

5. Proof of Theorem 1 for the suspended flow over a subshift of finite type

Let F be a real Hölder continuous function on $\mathbf{\Sigma}$ and let $\bar{\mu}$ be its equilibrium state. Thus $P = P(F) = H(\bar{\mu}) + \int F\, d\bar{\mu}$. Let μ be the equilibrium state on the subshift Σ for the function $f'(z) =$

$\int_0^{r(z)} F(z,s)\,ds$, $f' \in C_u(\Sigma)$ (that is μ assumes the supremum in the variational principle $p(f') = \sup_\rho (h(\rho) + \rho(f'))$, where ρ are shift invariant probability measures on Σ and $h(\rho)$ their measure theoretic entropies on the shiftspace Σ), then $\bar{\mu} = \frac{\mu \times \ell}{\mu(r)}$, ℓ being the Lebesgue measure on $\mathbb{R}$. By Abramov's formula [1] $H(\bar{\mu}) = h(\mu)/\mu(r)$ and therefore $P(F) = (h(\mu) + \mu(f'))/\mu(r) = p(f')/\mu(r)$. It follows that the pressure $p(f)$ of

$$f(z) = \int_0^{r(z)} (F(z,s) - P)\,ds = f'(z) - P \cdot r(z)$$

(ds is Lebesgue measure on $\mathbb{R}$) is zero since $F - P$ has pressure zero.

Define $F_1 = \pi_{1*}F$: $\Sigma_0 \times \Sigma_1 \to \mathbb{R}$ by $F_1(x,(y,t)) = F(xy, t - v_1(xy))$, and similarly $F_0 = \pi_{0*}F$: $\Sigma_0 \times \Sigma_1 \to \mathbb{R}$. We shall also need the following functions on the two-sided shift space

$$f_1(xy) = \int_0^{r_1(y)} (F_1(x,(y,s)) - P(F))\,ds,$$

$$f_0(xy) = \int_0^{r_0(y)} (F_0((x,t),y) - P(F))\,ds.$$

Lemma 7: The functions f_0, f_1, f have zero pressure and are cohomologous, that is $f_i = f + k_i - k_i \circ \sigma$, $i = 0,1$, where $k_1(xy) = \int_{-v_1(xy)}^{0} (F \circ \Psi_s(xy, 0) - P)\,ds$ and $k_0(xy) = \int_{v_0(xy)}^{0} (F \circ \Psi_s(xy, 0) - P)\,ds$

Proof. It is obvious that all three functions considered have pressure zero. We thus shall only show that $f_1 = f + k_1 - k_1 \circ \sigma$ for k_1. Since $r_1 = r + v_1 - v_1 \circ \sigma$ and $r_1 - v_1 = r - v_1 \circ \sigma$ we get

$$\begin{aligned} f_1(xy) &= \int_0^{r_1(y)} (F \circ \Psi_{s - v_1(xy)}(xy, 0) - P)\,ds \\ &= \int_{-v_1(xy)}^{r(xy) - v_1(\sigma(xy))} (F \circ \Psi_s(xy, 0) - P)\,ds \\ &= \int_0^{r(xy)} (F \circ \Psi_s(xy, 0) - P)\,ds + \int_{-v_1(xy)}^{0} (F \circ \Psi_s(xy, 0) - P)\,ds \\ &\qquad + \int_{r(xy)}^{r(xy) - v_1(\sigma(xy))} (F \circ \Psi_s(xy, 0) - P)\,ds \\ &= f(xy) + k_1(xy) - k_1(\sigma(xy)), \end{aligned}$$

where

$$k_1(xy) = \int_{-v_1(xy)}^{0} (F \circ \Psi_s(xy, 0) - P)\,ds$$

lies in $C_{\frac{1}{2}u}(\Sigma)$. In the same way one concludes that $f_0 = f + k_0 - k_0 \circ \sigma$ for $k_0 \in C_{\frac{1}{2}u}(\Sigma)$. □

By Lemma 5, there are $g_0, g_1, w_0, w_1 \in C_{\frac{1}{2}u}(\Sigma)$ such that $g_0 = f_0 + w_0 - w_0 \circ \sigma$ is independent of positive coordinates and $g_1 = f_1 + w_1 - w_1 \circ \sigma$ only depends on positive coordinates. We also have $g_0 \in C_{u_0}(\Sigma_0)$, $g_1 \in C_{u_1}(\Sigma_1)$. Note that $g_i = f + (w_i + k_i) - (w_i + k_i) \circ \sigma$, $i = 0,1$.

According to Ruelle [15] one defines a transfer operator L_1 on $C_{u_1}(\Sigma_1)$ by

$$(L_1\chi)(y) = \sum_{y' \in \sigma^{-1}y} e^{g_1(y')} \chi(y'),$$

$\chi \in C_{u_1}(\Sigma_1)$, where the summation is aver all $y' \in \Sigma_1$ such that $\sigma(y') = y$. By Ruelle's operator theorem [4], [15], L_1 has spectral radius $e^{p(g_1)} = 1$ (as $p(g_1) = 0$) where the largest eigenvalue 1 is simple and the remainder of the spectrum is contained in a disk of radius strictly smaller than 1. Let μ_1 be the probability measure that spans the eigenspace to the largest eigenvalue of L_1^*, that is $L_1^*\mu_1 = \mu_1$. For the following we shall agree upon giving the details for the right sided case only (which is identified by the subscript 1). With obvious modifications the same statements then also hold true in the left sided case.

For every $x \in \Sigma_0$ we define $L_{1,x}$: $C_{u_1}(\Sigma_1) \to C_{u_1}(\Sigma_1)$ by

$$(L_{1,x}\chi)(y) = \sum_{y' \in \sigma^{-1}y} e^{f_1(xy')} \chi(y'),$$

where we put $f_1(xy') = -\infty$ whenever xy' is not a point in Σ. Since f_1 and g_1 are cohomologous, $L_{1,x}$ and L_1 have the same spectrum and satisfy the identity

$$L_{1,x} = U_{w_1} \circ L_1 \circ U_{-w_1},$$

where U_w denotes the operator which multiplies by the function e^w.

Now we shall construct the transversal measures as follows. For $x \in \Sigma_0$ and $0 \le t+s \le r_1(y)$ put

$$\tau'_{x,t}(y,s) = \int_0^t (F_1(x,(y,s+p)) - P)\, dp = \int_0^t (F \circ \Psi_{s+p-v_1(xy)}(xy,0) - P)\, dp.$$

It is easy to see that the τ' satisfies a ψ 1-cocyle equation:

$$\tau'_{x,t+T}(y,s) = \tau'_{x,t}(y,s) + \tau'_{x,T} \circ \psi_t(y,s).$$

Define a measure $\nu_{1,x}$ on Σ_1 by putting

$$d\nu_{1,x}(y,t) = d\mu_{1,x}(y)\, dm(t) = e^{-w_1(xy)}\, d\mu_1(y)\, dm(t),$$

where $dm(t) = e^{-\tau'_{x,t}(y,0)}\, dt$, $0 \le t \le r_1(y)$, $y \in \Sigma_1$.

The Lebesque measure in the flow direction is thus given the weight $e^{-\tau'_{x,t}}$. This enables us to prove the Margulis type transformation formula in the next lemma. For the original Margulis measure, when the potential F and thus also the function f vanishes, the weight function in the t direction is the familiar factor e^{Ht}, with $H = P(0)$ being the topological entropy. (Series in [17] considered the weight $e^{P(F)t}$ which, except for constant functions, does however not yield the transformation properties of the following Poposition 9 and Lemma 10.) For the following lemma see also [5], Proposition 3.3, for the case where $F = 0$.

Lemma 8: $$\psi_T^* \nu_{1,x} = e^{\tau'_{x,T}} \nu_{1,x}.$$

Proof. Without loss of generality we can assume that T is some small positive number (less than $\inf r_1$) and χ a (test) function on Σ_1 so that ψ_T: $\mathrm{supp}\,\chi \to \psi_T(\mathrm{supp}\,\chi)$ is a homeomorphism. Then we get by the following manipulations

$$\tilde{\chi}(y) = \int_0^{r_1(y)} (\chi\circ\psi_{-T})(y,s)\, e^{-\tau'_{x,s}(y,0)}\, ds$$

$$= \int_T^{r_1(y)} \chi(y,s-T)\, e^{-\tau'_{x,s}(y,0)}\, ds + \int_0^T \chi(\eta y, r(\eta y)-T+s)\, e^{-\tau'_{x,s}(y,0)}\, ds$$

$$= \int_0^{r_1(y)-T} \chi(y,s)\, e^{-\tau'_{x,s}(y,0)-\tau'_{x,T}(y,s)}\, ds$$

$$+ \int_{r_1(\eta y)-T}^{r_1(xy)} \chi(\eta y,s)\, e^{-\tau'_{x,T}(\eta y,s)-\tau'_{x,s}(\eta y,0)}\, ds\, e^{\tau'_{x,r_1(\eta y)}(\eta y,0)},$$

where $\eta \in T$ is such that $\mathrm{supp}\,\chi \subset U_1(\eta) = \{y' \in \Sigma_1\colon y'_1 = \eta\}$ and $\eta y \in \Sigma_1$ is the concatenation of η with y. Since $\chi|U_1(\delta) = 0$ for $\delta \neq \eta$ we can in the second term sum over all $\eta \in T$ and identify Ruelle's operator since $\tau'_{x,r_1(y')}(y') = f_1(xy')$. Thus

$$\tilde{\chi}(y) = \int_0^{r_1(y)-T} \chi(y,s)\, e^{-\tau'_{x,T}(y,s)}\, dm(s) + L_{1,x}\Big(\int_{r_1-T}^{r_1} \chi\circ\psi_s\, e^{-\tau'_{x,T}\circ\psi_s}\, dm(s)\Big)(y,0),$$

and therefore

$$(\psi_{-T}^*\nu_{1,x})(\chi) = \mu_{1,x}(\tilde{\chi}) = (e^{-\tau'_{x,T}}\mu_{1,x})\Big(\int_0^{r_1} \chi\circ\psi_s\, dm(s)\Big) = (e^{-\tau'_{x,T}}\nu_{1,x})(\chi),$$

since $L_{1,x}^*\mu_{1,x} = \mu_{1,x}$. This identity is easily generalised to arbitrary Hölder continuous functions χ on Σ_1 and $T \geq 0$. □

We are now able to show that the transversal measures $\pi_1^*\, \nu_{1,x}$ satisfy Margulis' cocyle equations for the suspended flow Ψ_t. The following proposition concludes thus the proof of part (i) of Theorem 1 for the masures on the unstable foliation in the case where Φ_t is a suspended flow over a subshift of finite type.

Proposition 9: The measure $\mu_\zeta{}^u = \pi_1^*\, \nu_{1,x}$ which is supported on the weak unstable leaf through $\zeta \in \pi_1(x\times\Sigma_1)$ locally given by $\pi_1(x\times\Sigma_1)$, $x \in \Sigma_0$, has the scaling property

$$\Psi_T{}^* \mu_\zeta{}^u = e^{\tau_T} \mu_\zeta{}^u,$$

where $\tau_T(xy,t) = \int_0^T F\circ\Psi_s(xy,t)\, ds$. Moreover

$$d\mu_\zeta{}^u(xy,t) = e^{-w_1(xy)-k_1(xy)}\, d\mu_1(y)\, e^{-\tau_t(xy,0)}\, dt,$$

where as in Lemma 7 $k_1(xy) = -\tau_{-v_1(xy)}(xy,0)$.

Proof. Let us first note that

$$\begin{aligned} d\mu_\zeta{}^u(xy,t) &= d\pi_1{}^* \nu_{1,x}(y,t) \\ &= e^{-w_1(xy)}\, d\mu_1(y)\, e^{-\tau'_{x,t+v_1(xy)}(y,0)}\, dt \\ &= e^{-w_1(xy)}\, d\mu_1(y)\, e^{-\tau_t(y,0)-k_1(xy)}\, dt, \end{aligned}$$

since

$$\begin{aligned} \tau'_{x,t+v_1(xy)}(y,0) &= \int_0^{t+v_1(xy)} (F\circ\Psi_{s-v_1(xy)}(xy,0) - P)\, ds \\ &= \int_0^{t} (F\circ\Psi_s(xy,0) - P)\, ds + \int_{-v_1(xy)}^{0} (F\circ\Psi_s(xy,0) - P)\, ds \\ &= \tau_t(xy,0) + k_1(xy). \end{aligned}$$

This now yields the scaling property:

$$\begin{aligned} d\Psi_T{}^* \mu_\zeta{}^u(xy,t) &= d\pi_1{}^* \psi_T{}^* \nu_{1,x}(y,t) \\ &= d\pi_1{}^* e^{-\tau'_{x,T}(y,t)} \nu_{1,x}(y,t) \\ &= d\, e^{-\tau'_{x,T}(y,t+v_1(xy))} \nu_{1,x}(y,t) \\ &= e^{-\tau_T(xy,t)}\, d\mu_\zeta{}^u(xy,t), \end{aligned}$$

since

$$\tau'_{x,T}(y, t+v_1(xy)) = \int_{-v_1(xy)}^{T-v_1(\sigma^n(xy))} (F\circ\Psi_{s+t+v_1(xy)}(xy,0) - P)\, ds = \tau_T(xy,t). \ \square$$

The next lemma will prove part (ii) of Theorem 1 for suspensions over a subshift of finite type.

Lemma 10: Let $\xi = (x'y, t')$ be a point on the local strong stable manifold of $\zeta = (xy,t)$, $t' = t + v_1(xy) - v_1(x'y)$. Then

$$\mu_\xi{}^u = e^{\omega_{x,x'}} \mu_\zeta{}^u,$$

where $\omega_{x,x'}(xy,t) = \int_0^{\infty} (F\circ\Psi_s(xy,t) - F\circ\Psi_s(x'y,t'))\, ds$.

Proof. Since $g_1 = f_1 + w_1 - w_1\circ\sigma$ and by Lemma 7 $f_1 = f + k_1 - k_1\circ\sigma$, we have $g_1 = f + W_1 - W_1\circ\sigma$, where $W_1 = w_1 + k_1$ and $-W_1 = f - g_1 - W_1\circ\sigma = f^n - g_1{}^n - W_1\circ\sigma^n$. Therefore

$$W_1(x'y) - W_1(xy) = f^n(x'y) - f^n(xy) + W_1\circ\sigma^n(x'y) - W_1\circ\sigma^n(xy),$$

as g_1 only depends on positive coordinates. Moreover, as $W_1\circ\sigma^n(x'y) - W_1\circ\sigma^n(xy)$ dies exponentially fast as n goes to infinity, the following sum converges absolutely

$$W_1(x'y) - W_1(xy) = \sum_{j\geq 0} (f\circ\sigma^j(x'y) - f\circ\sigma^j(xy))$$

$$= \lim_{n\to\infty} \left(\int_0^{r^n(x'y)} (F\circ\Psi_s(x'y,0) - P)\, ds - \int_0^{r^n(xy)} (F\circ\Psi_s(xy,0) - P)\, ds \right)$$

since

$$f(\sigma^j xy) = \int_0^{r(\sigma^j xy)} (F\circ\Psi_s(\sigma^j xy, 0) - P)\, ds.$$

Thus we have shown that

$$\log \frac{d\mu_\xi{}^u(x'y,t')}{d\mu_\zeta{}^u(xy,t)} = W_1(x'y) - W_1(xy) - \tau_{t'}(x'y,0) + \tau_t(xy,0)$$

$$= \lim_{n\to\infty} \left(\int_{t'}^{r^n(x'y)} (F\circ\Psi_s(x'y,0) - P)\, ds - \int_t^{r^n(xy)} (F\circ\Psi_s(xy,0) - P)\, ds \right)$$

$$= \omega_{x,x'}(xy,t),$$

since

$$\tau_t(xy,0) - \tau_{t'}(x'y,0) = \int_0^{t} (F\circ\Psi_s(xy,0) - P)\, ds - \int_0^{t'} (F\circ\Psi_s(x'y,0) - P)\, ds. \qquad \square$$

Proposition 9 and Lemma 10 thus conclude the proof of Theorem 1 for suspension over subshifts and the measures on the unstable foliation. The Margulis measures on the stable foliations are constructed in a similar way.

6. Proof of Theorem 2 for suspensions

Bowen and Ruelle [5] showed that an equilibrium state $\bar{\mu}$ for some function $F: \mathbf{\Sigma} \to \mathbb{R}$ is in fact the (normalised) product of Lebesgue measure on the one-dimensional foliation generated by the flow with the equilibrium state μ for the function $f(z) = \int_0^{r(z)} (F(z,t) - P(F))\, dt$ on the subshift Σ. In this section we shall gc one step further and show that $\bar{\mu}$ is locally indeed of product form with factors that, apart from Lebesgue measure on the flow lines, are supported on the strong stable and unstable leaves respectively.

To prove Theorem 2 for the symbolic system let us first summarise in a more suitable form the results of the previous section for the two families of Margulis measures. We will now simply write μ^{uu} and μ^{ss} for measures supported on strong unstable respectively stable leaves. Similarly μ^{u} and μ^{s} will be transversal measures supported on the weak unstable respectively stable foliation. Let μ_0 and μ_1 be the equilibrium states for the one-sided functions $f_0 \circ \sigma^{-1}: \Sigma_0 \to \mathbb{R}$ and $f_1: \Sigma_1 \to \mathbb{R}$.

The transversal measure μ^{uu} on the strong unstable leaf $\pi_1((x,0),\Sigma_1)$ is then defined by $\mu^{uu} = \pi_1{}^*(e^{-w_1-k_1}\mu_1)$ and, for $t \in [0,r_1(x)]$, carried to the strong unstable leaf $\pi_1((x,t),\Sigma_1)$ using the identity $\Psi_t{}^*\mu^{uu} = e^{\tau_t}\mu^{uu}$, where $\tau_t = \int_0^t (F \circ \Psi_s - P)\, ds$ (and $\tau_{r(z)}(z,0) = f(z)$). The Margulis measure μ^{u} on the weak unstable foliation is then given by $d\mu^{u} = d\mu^{uu} \times d\ell$ (ℓ is Lebesgue measure).

Similarly, μ^{ss} on the strong stable leaf $\pi_0(\Sigma_0,(y,0))$ is defined by $\mu^{ss} = \pi_0{}^*(e^{w_0+k_0}\mu_0)$ and carried to the leaf $\pi_0(\Sigma_0,(y,t))$, $0 \le t \le r_0(y)$, by $\Psi_t{}^*\mu^{ss} = e^{-\tau_t}\mu^{ss}$. The Margulis measure μ^{s} on the weak stable foliation is then given by $d\mu^{s} = d\mu^{ss} \times d\ell$

Now let $F: \Sigma \to \mathbb{R}$ be as above Hölder continuous, $\bar{\mu}$ its equilibrium state and $f: \Sigma \to \mathbb{R}$ its associated Hölder continuous function on the shiftspace. Then we have $\bar{\mu} = \frac{\mu \times \ell}{\mu(r)}$, where μ is the equilibrium state for f on Σ. With $\rho = w_0 + k_0 - w_1 - k_1 \in C_{\theta/2}(\Sigma)$, we have by Corollary 5 of [10] that μ is up to a normalising factor given by the product $e^{\rho}\mu_0\mu_1$. We therefore obtain:

$$\begin{aligned} d\mu^{ss}(xy,t)\, d\mu^{uu}(xy,t)\, dt &= e^{-\tau_t(xy,0)}\, d\mu^{ss}(xy,0)\, e^{\tau_t(xy,0)}\, d\mu^{uu}(xy,0)\, dt \\ &= e^{w_0(xy)}\, d\mu_0(x)\, e^{-w_1(xy)}\, d\mu_1(y)\, dt \\ &= e^{\rho(xy)}\, d\mu_0(x) d\mu_1(y)\, dt = d\bar{\mu}(xy,t), \end{aligned}$$

for $(xy,t) \in \Sigma$. Thus, locally, $\bar{\mu} = \mu^{ss} \times \mu^{uu} \times \ell$ up to a normalising constant. □

7. Margulis functionals on suspensions

We shall now generalise Theorem 1 to a family of transversal functionals on the canonical foliations of a suspended flow. We shall use the notation established in the last two sections.

Let us denote by $L(\tau)$ the least period of a closed orbit τ of the suspended flow Ψ_t, and put $L_F(\tau) = \int_0^{L(\tau)} F\circ\Psi_s\,ds$ (evidently $L(\tau) = L_1(\tau)$) for $F: \Sigma \to \mathbb{R}$ a Hölder continuous function. The weighted dynamical zeta function $\zeta(s)$ which is defined by (see e.g. [13] or [10])

$$\zeta(s) = \prod_\tau \frac{1}{1 - e^{L_F(\tau) - sL(\tau)}},$$

where the Euler product is over all closed orbits τ of the suspended flow Ψ_t, converges for $\mathrm{Re}(s) > P(F)$ and has a meromorphic extension to a larger halfplane $\mathrm{Re}(s) > P - \gamma$ for some positive γ [10]. Its poles coincide with the values of s for which the transfer operator $L_{f_1-(s-P)r_1}$ has an eigenvalue 1. If we denote the values of s for which $\zeta(s)$ has poles by ω_j, $j = 0,1,2,\ldots$ (note that $P - \gamma < \mathrm{Re}(\omega_j) < \omega_0 = P$ for $j \geq 1$), then we can find Margulis' type functionals for every one of these values as follows. If we assume that the ω's are in fact simple poles, then $L_j = L_{f_1-(\omega_j-P)r_1}$ has a simple eigenvalue 1 and we can find a complex functional $\mu_{1,j} \in C_{u_1}(\Sigma_1,\mathbb{C})^*$ which spans the one-dimensional eigenspace of L_j^*; that is we have $L_j^*\mu_{1,j} = \mu_{1,j}$ (where $\mu_{1,0} = \mu_1$) for some $\mu_{1,j} \in C_{u_1}(\Sigma_1,\mathbb{C})^*$.

Thus, imitating our approach for the Margulis measures, $\mu_j^{uu} = \pi_1^*(e^{-w_1-k_1+(\omega_j-P)v_1}\mu_1)$ defines a functional on the strong unstable leaf $\pi_1((x,0),\Sigma_1)$ which can be extended to strong unstable leaves $\pi_1((x,t),\Sigma_1)$, $t \in [0,r_1(x)]$, using the identity $\Psi_t^*\mu^{uu} = e^{\tau_{j,t}}\mu^{uu}$, where here the weight function $\tau_{j,t}$ is given by

$$\tau_{j,t} = \int_0^t (F\circ\Psi_s - \omega_j)\,ds.$$

(Observe that we have the identity $\tau_{j,t(z)}(z,0) = f(z) - (\omega_j - P)r(z)) = \int_0^{r(z)} (F\circ\Psi_s - \omega_j)\,ds.$)

Thus, by putting $d\mu_j^u = d\mu_j^{uu} \times d\ell$ (ℓ is Lebesgue measure), we get a functional μ_j^u on the weak unstable foliation for which one easily checks that, with the appropriate modifications, Proposition 9 and Lemma 10 hold true. We therefore get the following generalisation of Theorem 1.

Theorem 11: Let $F: \Sigma \to \mathbb{R}$ be a Hölder continuous function on the suspension Σ, and let $\omega_0 = P(F)$, ω_1, ω_2, ... be the (simple) poles of its weighted zeta function $\zeta(s)$ in the strip of $P-\gamma < \mathrm{Re}(s) \leq P$, $\gamma > 0$, in which $\zeta(s)$ is meromorphic. Then for every j there exists a transversal functional μ_j^u supported on the weak unstable foliation such that

(i) $$\Psi_t^* \mu_j^u = e^{\tau_{j,t}} \mu_j^u,$$

where $\tau_{j,t} = \int_0^t (F \circ \Phi_s - \omega_j)\, ds$

(ii) $$\mu_j^u = e^{\omega_{x,x'}} \rho_{x,x'}{}^* \mu_j^u,$$

$x' \in W_\delta^{ss}(x)$, some $\delta > 0$, where $\omega_{x,x'} = \int_0^\infty (F \circ \Phi_s \circ \rho_{x,x'} - F \circ \Phi_s)\, ds.$

Similarly one can define transversal functionals μ_j^s on the weak stable foliation. Let us note that of particular importance are product functionals which are obtained by mating for instance μ_j^u, some j, with the transversal measure μ_0^s. Such product functionals determine the singular parts of the poles of the Fourier transformes of correlation functions [10]. However, because of their singular nature, this formulation of transversal functionals only applies to suspended flows over subshifts of finite type. As we shall see in the next section, only the generalised Margulis measures can be pushed down from the symbolic level to an arbitrary Axiom A flow by using Markov partitions.

8. Markov partitions and proof of the theorems

To prove Theorem 1 let us return to the notation of sections 4, 5 and 6. We will first show that the improper integral in (ii) converges. Let $\alpha > 0$ be a Hölder exponent of F. Thus, if y is close enough to x, we have, with some constant $C' > 0$,

$$|F \circ \Phi_s \circ \rho_{x,x'}(y) - F \circ \Phi_s(y)| \le C' \cdot d(\Phi_s \circ \rho_{x,x'}(y), \Phi_s(y))^\alpha$$

$$\le C' \cdot C^\alpha \cdot e^{-\lambda \alpha s},$$

and therefore the integral defining $\omega_{x,x'}$ converges.

The remainder of the proof consists of showing that the transversal measures constructed on the suspension implies the existence of such transversal measures for Axiom A flows in general. This is done by using standart arguments involving Markov partitions. By introducing finitely many pieces of hyperplanes H_i transversally to the flow Φ_t, one can construct a Markov partition such that the Poincare map $P: \bigcup_i H_i \to \bigcup_i H_i$ which maps a point $x \in \bigcup_i H_i$ to $P(x) = \Phi_{R(x)}(x) \in \bigcup_i H_i$, where $R(x) = \inf\{t > 0: \Phi_t(x) \in \bigcup_i H_i\}$, is semiconjugate to the shift on a subshift of finite type Σ via a map $\pi: \Sigma \to \bigcup_i H_i$; that is $\pi \circ P = \sigma \circ \pi$ [3], [14]. The flow Φ_t is thus via $\Pi: \Sigma \to \Omega$ semiconjugate to the suspended flow Ψ_t on Σ with ceiling function $r = \pi_* R$. In [10] it was shown that the ceiling function lies in the space $C_u(\Sigma)$ where the modulus of continuity is $u = \frac{1}{2}\lambda r$.

If the Markov partition is fine enough (finer than an expansive constant), then the projection Π is one to one almost everywhere with respect to ergodic measures that are positive on open sets, and therefore in particular with respect to equilibrium states. Hence by [5] section 4 we see that π is one to one almost everywhere with respect to the equilibrium state $\bar{\mu}$ for the function $\Pi_* F$ which is of the form $\bar{\mu} = \frac{\mu \times \ell}{\mu(r)}$, where ℓ is Lebesgue measure on $\mathbb{R}$ and μ is the equilibrium state on the shiftspace Σ for the function $f'(z) = \int_0^{r(z)} \Pi_* F(z,s)\, ds \in C_u(\Sigma)$. It follows that π is one to one with respect to the measure μ^u, since it is equivalent to the equilibrium state $\bar{\mu}$, and therefore $\Pi^* \mu^u$ is well defined on Ω. This concludes the proof of Theorem 1. Theorem 2 now follows immediately by the same considerations.

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