A NOTE ON BOREL–CANTELLI LEMMAS FOR NON-UNIFORMLY HYPERBOLIC DYNAMICAL SYSTEMS

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Abstract. Let (B_i) be a sequence of measurable sets in a probability space $(X, \mathcal{B}, \mu)$ such that $\sum_{n=1}^{\infty} \mu(B_i) = \infty$. The classical Borel–Cantelli lemma states that if the sets B_i are independent, then $\mu(\{ x \in X : x \in B_i \text{ infinitely often (i.o.)} \}) = 1$.

Suppose (T, X, μ) is a dynamical system and (B_i) is a sequence of sets in X. We consider whether $T^i x \in B_i$ i. o.for μ a.e. $x \in X$ and if so, is there an asymptotic estimate on the rate of entry. If $T^i x \in B_i$ i. o. for μ a.e. x we call the sequence (B_i) a Borel–Cantelli sequence. If the sets $B_i := B(p, r_i)$ are nested balls of radius r_i about a point p then the question of whether $T^i x \in B_i$ i. o. for μ a.e. x is often called the shrinking target problem.

We show, under certain assumptions on the measure μ, that for balls B_i if $\mu(B_i) \geq i^{-\gamma}$, $0 < \gamma < 1$, then a sufficiently high polynomial rate of decay of correlations for Lipschitz observables implies that the sequence is Borel–Cantelli. If $\frac{C_1}{i} \leq \mu(B_i) \leq \frac{C_2}{i}$ then exponential decay of correlations implies that the sequence is Borel–Cantelli. We give conditions in terms of return time statistics which quantitative Borel-Cantelli results for sequences of balls such that $\mu(B_i) \geq \frac{C}{i}$. Corollaries of our results are that for planar dispersing billiards and Lozi maps sequences of nested balls $B(p, 1/i)$ are Borel–Cantelli. We also give applications of these results to a variety of non-uniformly hyperbolic dynamical systems.

1. Introduction

Suppose $(X, \mathcal{B}, \mu)$ is a probability space. For a measurable set $A \subset X$, let 1_A denote the characteristic function of A. The classical Borel–Cantelli lemmas (see for example [11, Section 4]) state that

(1) if $(A_n)_{n=0}^{\infty}$ is a sequence of measurable sets in X and $\sum_{n=0}^{\infty} \mu(A_n) < \infty$ then $\mu(x \in A_n \text{ i.o.}) = 0$.
(2) if $(A_n)_{n=0}^{\infty}$ is a sequence of independent sets in X and $\sum_{n=0}^{\infty} \mu(A_n) = \infty$, then for μ a.e. $x \in X$

$$\frac{S_n(x)}{E_n} \to 1$$

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where $S_n(x) = \sum_{j=0}^{n-1} 1_{A_j}(x)$ and $E_n = \sum_{j=0}^{n-1} \mu(A_j)$.

Suppose now $T\colon X \to X$ is a measure-preserving transformation of a probability space (X, μ). If (A_n) is a sequence of sets such that $\sum_n \mu(A_n) = \infty$ it is natural in many applications to ask whether $T^n(x) \in A_n$ for infinitely many values of n for μ a.e. $x \in X$ and, if so, is there a quantitative estimate of the asymptotic number of entry times? For example, the sequence (A_n) may be a nested sequence of balls about a point, a setting which is often called the shrinking target problem.

In the setting of a dynamical system $T\colon X \to X$, we let $S_n = \sum_{j=0}^{n-1} 1_{A_j} \circ T^j$. The property $\lim_{n\to\infty} \frac{S_n(x)}{E_n} = 1$ for μ a.e. $x \in X$ is often called the Strong Borel–Cantelli (SBC) property in contrast to the Borel–Cantelli (BC) property that $S_n(x)$ is unbounded for μ a.e. $x \in X$.

W. Phillipp [25] established the SBC property for sequences of intervals in the setting of certain maps of the unit interval including the β transformation, the Gauss transformation and smooth uniformly expanding maps.

There have been some results on Borel–Cantelli lemmas for uniformly hyperbolic systems in higher dimensions. Chernov and Kleinbock [5] establish the SBC property for certain families of cylinders in the setting of topological Markov chains and for certain classes of dynamically-defined rectangles in the setting of Anosov diffeomorphisms preserving Gibbs measures. Dolgopyat [10] has related BC results for sequences of balls in uniformly partially hyperbolic systems preserving a measure equivalent to Lebesgue which have exponential decay of correlations with respect to Hölder observables.

Kim [21] has established the SBC property for sequences of intervals in the setting of 1-dimensional piecewise-expanding maps T with $\frac{1}{|T'|}$ of bounded variation.

Kim uses this result to prove some SBC results for non-uniformly expanding maps with an indifferent fixed point. In particular, he considers intermittent maps of the form

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \le x < \frac{1}{2}; \\ 2x - 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{1}$$

These maps are sometimes called Liverani–Saussol–Vaienti maps [22]. If $0 < \alpha < 1$ then T_α admits an invariant probability measure μ that is absolutely continuous with respect to Lebesgue measure m. We use the notation $x_n \sim y_n$ to denote $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$. The measure μ has an unbounded density $h(x) \sim Cx^{-\alpha}$ near 0. Kim shows that if (I_n) is a sequence of intervals in $(d, 1]$ for some $d > 0$ and $\sum_n \mu(I_n) = \infty$ then I_n is an SBC sequence if (a) $I_{n+1} \subset I_n$ for all n (nested intervals) or (b) $\alpha < (3 - \sqrt{2})/2$. Kim shows that the condition $I_n \subset (d, 1]$ for some $d > 0$ is in some sense optimal (with respect to the invariant measure μ) by showing that setting

$A_n = [0, n^{-1/(1-\alpha)})$ gives a sequence such that $\sum_n \mu(A_n) = \infty$ yet $T_\alpha^n(x) \in A_n$ for only finitely many values of n for μ a.e. $x \in [0,1]$.

For the same class of maps T_α, Gouëzel [13] considers Lebesgue measure m (rather than the invariant probability measure μ) and shows that if (I_n) is a sequence of intervals such that $\sum_n m(I_n) = \infty$ then (I_n) is a BC sequence. Assumptions (a) or (b) of Kim are not necessary for Gouëzel's result. Gouëzel uses renewal theory and obtains BC results but not SBC results.

At the end of this section we give an example of an intermittent type map which preserves Lebesgue measure m yet for which there exists a sequence of nested intervals I_n, $\sum_n m(I_n) = \infty$ yet (I_n) is not BC. Such maps may have arbitrarily high polynomial rate of decay of correlations for Hölder observables.

In the context of flows, Maucourant [23] has proved the BC property for nested balls in the setting of geodesic flows on hyperbolic manifolds of finite volume.

Recently, Gupta et al. [16] proved the SBC property for sequences of intervals in the setting of Gibbs–Markov maps and also sequences of nested balls in one-dimensional maps modeled by Young Towers. They also gave some applications to extreme value theory of deterministic systems, in particular the almost sure behavior of successive maxima of observables on such systems.

Fayad [12] has given an example of an analytic area preserving map of the three-dimensional torus which is mixing of all orders, yet for which there exists a sequence of nested balls (A_n) such that E_n diverges yet the sequence (A_n) is not BC.

In this short note we establish dynamical Borel–Cantelli lemmas using elementary arguments, we try to avoid dynamical assumptions beyond decay of correlations as much as we can. Our results are phrased in terms of the interplay between the measure $\mu(A_n)$ of the sets and the rate of decay of correlations of observables in various norms. In Section 6 make some dynamical assumptions in the guise of return time statistics to obtain more quantitative BC estimates, mainly for the shrinking target problem.

We define $E(\phi) := \int_X \phi \, d\mu$ for the expectation of an integrable observable on a dynamical system (T, X, μ) where X is a metric and probability space.

In applications it is common to have decay estimates for observables on a dynamical system in various Banach space norms, for example:

(a) Bounded variation (BV) versus L^1,

$$|E(\phi \ \psi \circ T^m) - E(\phi)E(\psi)| \le p(m) \|\phi\|_{BV} \ \|\psi\|_1,$$

(b) Lipschitz versus L^∞,

$$|E(\phi \ \psi \circ T^m) - E(\phi)E(\psi)| \le p(m) \|\phi\|_{\mathrm{Lip}} \ \|\psi\|_\infty,$$

(c) Lipschitz versus Lipschitz,

$$|E(\phi \ \psi \circ T^m) - E(\phi)E(\psi)| \le p(m) \|\phi\|_{\mathrm{Lip}} \ \|\psi\|_{\mathrm{Lip}},$$

where $p(m)$ is a rate function which tends to zero in m. Hölder norms are sometimes considered rather than Lipschitz but it is no essential loss to only consider Lipschitz.

In fact BV versus L^1 is not so common for non-uniformly hyperbolic systems i.e. those with a stable foliation. It is implicit in Kim [21] and explicitly stated in Gupta et al. [16] that summable rate of decay (i.e. $\sum_m p(m) < \infty$) for the norms BV versus L^1 implies the SBC property for any sequence of balls B_i such that $E_n := \sum_{i=0}^{n-1} \mu(B_i)$ diverges.

Proposition 1.1 ([21, 16]). *Suppose (T, X, μ) has summable decay of correlations with respect to BV versus L^1 in the sense that for $\phi \in BV$, $\psi \in L^1(\mu)$*

$$|E(\phi\ \psi \circ T^m) - E(\phi)\ E(\psi)| \leq p(m)\|\phi\|_{BV}\|\psi\|_1$$

where $\sum_m p(m) < \infty$. If (B_i) is a sequence of balls in X and $\sum_{i=0}^{\infty} \mu(B_i) = \infty$ then (B_i) is SBC.

The proof of this result is a straightforward application of a condition for SBC by W. Schmidt [28, 29] given in Sprindzuk [27] that we will soon state.

In this paper we will mainly consider decay of correlation of Lipschitz versus Lipschitz. The modifications of our results for Hölder versus Hölder, Lipschitz versus L^∞ or Hölder versus L^∞ are straightforward.

We will often use a proposition of W. Schmidt [28, 29] as stated by Sprindzuk [27]:

Proposition 1.2. *Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $f_k(\omega)$, $(k = 1, 2, \ldots)$ be a sequence of non-negative μ measurable functions and g_k, h_k be sequences of real numbers such that $0 \leq g_k \leq h_k \leq 1$, $(k = 1, 2, \ldots,)$. Suppose there exists $C > 0$ such that*

$$(*) \qquad \int \left(\sum_{m<k\leq n} (f_k(\omega) - g_k) \right)^2 d\mu \leq C \sum_{m<k\leq n} h_k$$

for arbitrary integers $m < n$. Then for any $\epsilon > 0$

$$\sum_{1\leq k\leq n} f_k(\omega) = \sum_{1\leq k\leq n} g_k(\omega) + O(\theta^{1/2}(n) \log^{3/2+\epsilon} \theta(n))$$

for μ a.e. $\omega \in \Omega$, where $\theta(n) = \sum_{1\leq k\leq n} h_k$.

Example 1.0.1. Cristadoro et. al. [9] consider an intermittent map T of the interval $[-1, 1]$ (actually on the unit circle as 1 and -1 are identified) with an unbounded derivative at the origin. The map is implicitly defined by the equation

$$x = \begin{cases} \frac{1}{2\gamma}(1 + T(x))^\gamma & \text{if } 0 \leq x \leq \frac{1}{2\gamma}; \\ T(x) + \frac{1}{2\gamma}(1 - T(x))^\gamma & \text{if } \frac{1}{2\gamma} \leq x \leq 1. \end{cases}$$

and extended as an odd function so that $T(-x) = -T(x)$. See Figure 1.

FIGURE 1. The graph of T.

We assume $\gamma > 1$. Let $\tau = \frac{1}{\gamma-1}$ and recall the notation $x_n \sim y_n$ denotes $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$.

The map T preserves Lebesgue measure m and is mixing with a polynomial rate of decay of correlations for Hölder observables. In fact

$$|\int \phi \circ T^n \psi dm - \int \phi dm \int \psi dm| \leq C(\phi, \psi) n^{-\tau}$$

for all Hölder ϕ, $\psi \in L^\infty(m)$.

We define $T_+ := T_{|(0,1)}$, $T_- := T_{|(-1,0)}$, $a_{0-} = \frac{-1}{2\gamma}$, $a_{-i} = T_-^{-i} a_{0-}$ and $b_i = T_+^{-1} a_{-(i-1)}$. Note that $T^{-1}(-1, a_{-n}) = (0, b_{n+1}) \cup (-1, a_{-n-1})$.

It is shown in [9, Lemma 2] that $m(-1, a_{-n}) \sim (2\gamma\tau)^\tau n^{-\tau}$, and $m(0, b_n) \sim \frac{1}{2\gamma}(2\gamma\tau)^{\gamma\tau}(n-1)^{-\gamma\tau}$.

If we choose $\gamma > 2$ then $0 < \tau < 1$ so $\sum_{n>0} m(-1, a_{-n})$ diverges. Since $m(0, b_n) \sim \frac{1}{2\gamma}(2\gamma\tau)^{\gamma\tau}(n-1)^{-\gamma\tau}$ and $\gamma\tau > 1$, $\sum_{n>0} m(0, b_n)$ converges. The only way for the orbit of a generic point under the map T to enter $(-1, a_{-n})$ infinitely often is to enter $(0, b_n)$ infinitely often. Since $\sum_{n>0} m(0, b_n)$ converges, m a.e. satisfies $T^n x \in (-1, a_{-n})$ for only finitely many n, although $\sum_{n>0} m(-1, a_{-n})$ diverges. This is an example of a sequence of nested sets $A_n := (-1, a_{-n})$ such that $\sum_{n>0} m(A_n)$ diverges yet $T^n x \in A_n$ at most finitely many times for m a.e. x. Contrary to the example of Kim, the map T preserves Lebesgue measure rather than a measure with an unbounded density. Note also that taking $\gamma - 1 := \delta > 0$ the rate of decay of correlation is $Cn^{-\frac{1}{\delta}}$ is at an arbitrarily high polynomial rate.

2. ASSUMPTIONS

Suppose (T, X, μ) is an ergodic measure preserving map of a probability space X which is also a metric space. We assume:

(A) For all Lipschitz functions ϕ, ψ on X we have summable decay of correlations in the sense that there exists a rate function $p(k) \to \infty$, $\sum_k p(k) < \infty$ (p is independent of ϕ, ψ) such that

$$|E(\phi\ \psi \circ T^k) - E(\phi)E(\psi)| < p(k)\|\phi\|_{\text{Lip}}\|\psi\|_{\text{Lip}}.$$

(B) There exists $r_0 > 0$, $0 < \delta < 1$ such that for all $p \in X$ and all $0 < \epsilon < r \leq r_0$

$$\mu\{\, x : r < d(x,p) < r + \epsilon \,\} < \epsilon^\delta.$$

Remark 2.0.1. If the balls $B_i = B(p, r_i)$, $(r_i \to 0)$, are nested balls centered at a point p then we would only require that there exist $\delta(p) > 0$, $r_0(p) > 0$ such that $\mu\{\, x : r < d(x, p) < r + \epsilon \,\} < \epsilon^{\delta(p)}$ for all $0 < \epsilon < r \leq r_0$.

Remark 2.0.2. We will call the sets B_i 'balls', but our results extend to any shapes such that the indicator function of the set may be approximated closely in the L^1 norm by a Lipschitz function of reasonable Lipschitz norm, for example our results extend immediately to rectangles of bounded side ratio.

If $p(k) \leq C\alpha^k$ for some constants $C > 0$, $0 < \alpha < 1$ then we say T has exponential decay of correlations. We will also consider polynomial decay, where $p(k) \leq Ck^{-q}$ for some constants $C > 0$, $q > 0$.

Our results will be formulated in terms of the measure of the balls B_i and the rate of decay of correlation. In Section 6 we make additional assumptions on return time distributions and short return times.

3. Sequences of sets (B_i) such that $\mu(B_i) \geq \frac{1}{i^\gamma}$ for $0 < \gamma < 1$.

Define

$$S_n(x) = \sum_{k=1}^{n} 1_{B_k} \circ T^k(x)$$

and

$$E_n = \sum_{1 \leq k \leq n} \mu(B_k).$$

Theorem 3.1. *Assume (T, X, μ) satisfies (A) and (B), with $p(k) \leq Ck^{-q}$. Let the (B_i) satisfy $\frac{C_1}{i^\gamma} \leq \mu(B_i)$ for some $\gamma \in (0, 1)$ and a constant $C_1 > 0$. If*

$$q > \frac{\frac{2}{\delta} + 1}{1 - \gamma} - 1$$

(where δ is from assumption (B)), then

$$\lim_{n \to \infty} \frac{S_n(x)}{E_n} = 1$$

for μ a.e. $x \in X$.

Remark 3.1.1. The proof of Theorem 3.1 gives the following asymptotic bounds.

$$\sum_{k=1}^{n} 1_{B_k} \circ T^k(x) = \sum_{1 \leq k \leq n} \mu(B_k) + O(\theta^{1/2}(n) \log^{3/2+\epsilon} \theta(n))$$

for μ a.e. $x \in X$, where $\theta(n) = \sum_{1 \leq k \leq n} \mu(B_k)$.

Remark 3.1.2. We note that if $\mu(B_i)=\frac{1}{i}$ then no degree of polynomial decay in the Lipschitz norm ensures that the sequence (B_i) is BC, even if the B_i are nested balls. To see this consider the intermittent map example of Kim [21], $T_\alpha\colon[0,1]\to[0,1]$ with the balls $B_i:=[0,i^{-1/(1-\alpha)})$ so that $\mu(B_i)\sim\frac{1}{i}$ (see Thaler [30]). The density $h(x)=\frac{d\mu}{dm}$ behaves like $x^{-\alpha}$ near 0 and the rate of decay of correlations with respect to Lipschitz versus L^∞ functions is $n^{1-\frac{1}{\alpha}}$. The constant δ in the statement of the theorem (from assumption (B)) may be taken as $1-\alpha$ by observing $\int_0^\epsilon x^{-\alpha}\,dx\le C\epsilon^{1-\alpha}$. As $\alpha\to0$, $\delta(\alpha)\to1$ and $q(\alpha)\to\infty$ yet for each $0<\alpha<1$, the sequence $B_i=[0,i^{-1/(1-\alpha)})$ is not BC.

Let $\{a_n:n\in\mathbb{N}\},\{b_n:n\in\mathbb{N}\}$ be two sequences of positive numbers. We write

$$a_n\prec b_n$$

if $\frac{a_n}{b_n}\to0$ as $n\to\infty$. In other words the sequence $\{a_n\}$ is negligible with respect to the sequence $\{b_n\}$. For the proof of the theorem we will need the following lemma:

Lemma 3.1. *(I) Let $\gamma\in(0,1)$ and $a_j\ge\frac{c}{j^\gamma}$, $j=1,2,\dots,$ be a sequence of numbers $(c>0)$. Then for any $\sigma>0$ and $0<\eta<\frac{1-\gamma}{1-\gamma+\sigma}$ we have $(n=1,2,\dots)$*

$$\left(\sum_{j=1}^n j^\sigma a_j\right)^\eta\prec\sum_{j=1}^n a_j.$$

(II) Let $a_j\ge c\frac{\log^\beta j}{j}$ $(c>0)$, be a sequence of numbers, where $\beta>0$. Then for any $\sigma>0$ and $0<\eta<\frac{1+\beta}{1+\beta+\sigma}$ we have $(n=1,2,\dots)$

$$\left(\sum_{j=1}^n a_j\log^\sigma j\right)^\eta\prec\sum_{j=1}^n a_j.$$

Proof. (I) One has that $\left(\sum_{j=1}^n j^{-\gamma}\right)^{\frac{1-\eta}{\eta}}\ge c_1 n^{(1-\gamma)\frac{1-\eta}{\eta}}\succ n^\sigma$ for some $c_1>0$ as $\sigma<(1-\gamma)\frac{1-\eta}{\eta}$. Hence

$$\sum_{j=1}^n j^\sigma a_j\le n^\sigma\sum_{j=1}^n a_j\prec\left(\sum_{j=1}^n\frac{1}{j^\gamma}\right)^{\frac{1-\eta}{\eta}}\sum_{j=1}^n a_j\le\left(\frac{1}{c}\right)^{\frac{1-\eta}{\eta}}\left(\sum_{j=1}^n a_j\right)^{\frac{1}{\eta}}.$$

(II) Similarly in this case

$$\sum_{j=1}^{n}(\log j)^\sigma a_j \leq (\log n)^\sigma \sum_{j=1}^{n} a_j \prec \log^{(1+\beta)\frac{1-\eta}{\eta}} n \sum_{j=1}^{n} a_j$$
$$\leq c_2 \left(\sum_{j=1}^{n} \frac{\log^\beta j}{j} \right)^{\frac{1-\eta}{\eta}} \sum_{j=1}^{n} a_j \leq c_3 \left(\sum_{j=1}^{n} a_j \right)^{\frac{1}{\eta}}$$

as $\sigma < (1+\beta)\frac{1-\eta}{\eta}$. □

Proof. Now we prove Theorem 3.1. For each k let $\tilde{f}_k$ be a Lipschitz function such that $\tilde{f}_k(x) = 1_{B_k}(x)$ if $x \in B_k$, $\tilde{f}_k(x) = 0$ if $d(B_k, x) > (k \log^2 k)^{-1/\delta}$, $0 \leq \tilde{f}_k \leq 1$ and $\|\tilde{f}_k\|_{\text{Lip}} \leq (k \log^2 k)^{1/\delta}$. Here, when we write $\log k$ then we intend it to mean $\max(1, \log k)$ (to avoid unessessary complications for low values of k). Clearly we may construct such functions by linear interpolation of 1_{B_k} and 0 on a region $r \leq d(p_k, x) \leq r + k(\log^2 k)^{-1/\delta}$.

In Proposition 1.2 we will take $f_k = \tilde{f}_k \circ T^k(x)$, $g_k = E(\tilde{f}_k)$, where $E(\phi) = \int \phi \, d\mu$ for any integrable function $\phi \in L^1(\mu)$. The constants h_k will be chosen later.

Note that for μ a.e. $x \in \Omega$, $f_i(x) = 1_{B_i}(T^i x)$ except for finitely many i by the Borel–Cantelli lemma as $\mu(x : f_i(x) \neq 1_{B_i}(T^i x)) = \mu(x : r_i < d(T^i x, p_i) < r_i + (i \log^2 i)^{-1/\delta}) < (i \log^2 i)^{-1}$ by assumption (B). Furthermore, since $g_k = E(f_k) \leq \mu(B_k) + \frac{1}{k \log^2 k}$ we obtain that $\sum_{k=1}^n g_k = \sum_{k=1}^n \mu(B_k) + O(1)$ for all n.

A rearrangement of terms in $(*)$ means that it suffices to show that

$$\sum_{i=m}^{n} \sum_{j=i+1}^{n} E(f_i f_j) - E(f_i)E(f_j) \leq \sum_{i=m}^{n} h_i$$

for arbitrary integers $n > m$ where h_i is given below.

We split each sum $\sum_{j=i+1}^n E(f_i f_j) - E(f_i)E(f_j)$ into the terms ($a \wedge b = \min(a, b)$)

$$I_i = \sum_{j=i+1}^{n \wedge (i + [i^\sigma])} E(f_i f_j) - E(f_i)E(f_j)$$

and (for $i + i^\sigma < n$)

$$II_i = \sum_{j=i+[i^\sigma]+1}^{n} E(f_i f_j) - E(f_i)E(f_j)$$

where $[i^\sigma]$ denotes the integer part of i^σ, where $0 < \sigma < 1 - \gamma$ is arbitrary.

The first term I_i is roughly estimated by (as $E(f_if_j) \leq g_j \leq \mu(B_j) + \frac{1}{j\log^2 j}$)

$$I_i = \sum_{j=i+1}^{n\wedge(i+[i^\sigma])} E(f_if_j) - E(f_i)E(f_j) \leq \sum_{j=i+1}^{n\wedge(i+[i^\sigma])} E(f_if_j) \leq \sum_{j=i+1}^{n\wedge(i+[i^\sigma])} \left(\mu(B_j) + \frac{1}{j\log^2 j}\right).$$

The second term II_i we bound using the decay of correlations. If $q > \frac{1}{\delta} + 1$, then

$$\begin{aligned}
II_i &\leq \sum_{j=i+[i^\sigma]}^{\infty} E(f_if_j) - E(f_i)E(f_j) \\
&= \sum_{j=i+[i^\sigma]}^{\infty} \int_X f_i(T^ix)f_j(T^jx)\,d\mu - E(f_i)E(f_j) \\
&= \sum_{j=i+[i^\sigma]}^{\infty} \int_X f_i(x)f_j(T^{j-i}x)\,d\mu - E(f_i)E(f_j) \\
&\leq \sum_{j=i+[i^\sigma]}^{\infty} \|\tilde{f}_i\|_{\mathrm{Lip}}\|\tilde{f}_j\|_{\mathrm{Lip}}p(j-i) \\
&\leq \sum_{k=0}^{\infty} c_1[(i+[i^\sigma]+k)\log^2(i+[i^\sigma]+k)]^{1/\delta}[\log^2 i]^{1/\delta}([i^\sigma]+k)^{-q} \\
&\leq c_2 i^{\frac{1}{\delta}+\varepsilon} \sum_{k=0}^{\infty} \frac{(i+k)^{\frac{1}{\delta}}}{(i^\sigma+k)^q} \\
&\leq c_2 i^{\frac{1}{\delta}+\varepsilon} \sum_{k=0}^{\infty} (i^\sigma+k)^{\frac{1}{\delta}-q}\left(\frac{i+k}{i^\sigma+k}\right)^{\frac{1}{\delta}} \\
&\leq c_3 i^{\frac{2}{\delta}-\sigma q+\sigma+\varepsilon},
\end{aligned}$$

for any arbitrarily small $\varepsilon > 0$ and some constants c_2, c_3 which depend on ε. The last estimate is because $\frac{i+k}{i^\sigma+k}$ is bounded by $i^{1-\sigma}$ for $\sigma \leq 1$.

In Proposition 1.2 we now take $h_i = I_i + II_i$. We now want to estimate $\theta(n) = \theta_1(n) + \theta_2(n)$, where $\theta_1 = \sum_{i=1}^n I_i$ and $\theta_2 = \sum_{i=1}^n II_i$. With the estimate on I_i from above we get

$$\theta_1(n) \leq \sum_{i=1}^{n} \sum_{j=i+1}^{n\wedge(i+[i^\sigma])} \left(\mu(B_j) + \frac{1}{j\log^2 j}\right) \leq \sum_{j=1}^{n} j^\sigma \mu(B_j) + \sum_{j=1}^{n} \frac{j^{\sigma-1}}{\log^2 j}.$$

Since $\sum_{j=1}^n \frac{j^{\sigma-1}}{\log^2 j} \le \sum_{j=1}^n j^{\sigma-1} \le c_4 n^\sigma \prec \sum_{j=1}^n \mu(B_j)$ and by Lemma 3.1(I) (with $a_j = \mu(B_j) \ge C_1 j^{-\gamma}$)

$$\left(\sum_{j=1}^n j^\sigma \mu(B_j)\right)^\eta \prec \sum_{j=1}^n \mu(B_j)$$

we get now

$$\theta_1(n)^\eta \le \left(\sum_{j=1}^n j^\sigma \mu(B_j)\right)^\eta + \left(\sum_{j=1}^n \frac{j^{\sigma-1}}{\log^2 j}\right)^\eta \prec \sum_{j=1}^n \mu(B_j),$$

where $\eta \in (\frac{1}{2}, \frac{1-\gamma}{1-\gamma+\sigma})$. Using the above estimate on II_i, the second term is

$$\theta_2(n) = \sum_{i=1}^n II_i \le c_4 n^{\frac{2}{\delta}-\sigma q+\sigma+1+\varepsilon}$$

which yields that

$$\theta_2(n)^\eta \prec n^{1-\gamma} \le c_5 \sum_{j=1}^n \mu(B_j)$$

holds for some $c_5 > 0$ and η sufficiently close to $\frac{1}{2}$, provided that $\frac{1}{2}(\frac{2}{\delta} - \sigma q + \sigma + 1 + \varepsilon) < 1 - \gamma$, where $\sigma < 1 - \gamma$.

Since ε can be arbitrarily small, $\theta(n)^\eta \le \theta_1(n)^\eta + \theta_2(n)^\eta$ is negligible with respect to $\sum_{j=1}^n \mu(B_j)$ whenever

$$q > \max\left(\frac{1}{\delta} + 1, \frac{\frac{2}{\delta}+1}{1-\gamma} - 1\right).$$

Thus we get the Strong Borel–Cantelli property as the error term $O(\sqrt{\theta(n)} \log^{3/2+\epsilon} \theta(n)) \prec \theta(n)^\eta$ is negligible with respect to $\sum_{j=1}^n g_j = \sum_{j=1}^n E(f_j) \ge \sum_{j=1}^n \mu(B_j)$.

In order to get the explicit lower bound on q let us note that the first term in the maximum expression above dominates, i.e. $\frac{1}{\delta} + 1 \ge \frac{\frac{2}{\delta}+1}{1-\gamma} - 1$, if $\gamma \le -\frac{1-\delta}{1+2\delta}$ which is a negative number. $\square$

Let us note that under the assumptions of the theorem one has $\liminf_n \frac{S_n(x)}{n^{1-\gamma'}} = \infty$ for μ a.e. $x \in X$ for any $\gamma' > \gamma$.

4. Sequences of sets (B_i) such that $\mu(B_i) \ge \frac{\log^\beta i}{i}$ $(\beta > 0)$

Now let us consider the more general case $\mu(B_i) \ge \frac{\log^\beta i}{i}$ for $\beta > 0$. In this case we assume exponential decay of correlations.

Theorem 4.1. *Assume (T,X,μ) satisfies assumptions (A) and (B) and has exponential decay of correlations. Suppose $\mu(B_i)\geq C\frac{\log^\beta i}{i}$ (>0) for some $\beta>0$, then*

$$\lim_{n\to\infty}\frac{S_n(x)}{E_n}=1$$

for μ a.e. $x\in X$.

Proof. Let $p(k)\leq C\alpha^k$ for some $0<\alpha<1$. We use the same approximations as in the proof of Theorem 3.1. In this setting we again split $\sum_{j=i+1}^n E(f_if_j)-E(f_i)E(f_j)$ as above into two sums I_i and II_i where

$$\begin{aligned} I_i &= \sum_{j=i+1}^{n\wedge(i+[\log^\sigma i])} E(f_if_j)-E(f_i)E(f_j)\\ II_i &= \sum_{j=i+[\log^\sigma i]+1}^{n} E(f_if_j)-E(f_i)E(f_j) \end{aligned}$$

where the parameter $\sigma\in(1,1+\beta)$ is arbitrary. As before we obtain

$$I_i\leq\sum_{j=i+1}^{n\wedge(i+[\log^\sigma i])}\left(\mu(B_j)+\frac{1}{\log^2 j}\right)$$

which gives (as in Theorem 3.1)

$$\theta_1(n)=\sum_{i=1}^n I_i\leq\sum_{j=1}^n(\log j)^\sigma\mu(B_j)+\sum_{j=1}^n\frac{1}{\log^{2-\sigma}j}.$$

For the second term we use the dominance $\sum_{j=1}^n\frac{1}{j\log^{2-\sigma}j}\leq c_1\log^{\sigma-1}n\prec\log^{1+\beta}n\leq c_2\sum_{j=1}^n\mu(B_j)$ and to the the first term we apply Lemma 3.1(II) (with $a_j=\mu(B_j)\geq C\frac{\log^\beta j}{j}$) to give us $\theta_1(n)^\eta\prec\sum_{j=1}^n\mu(B_j)$ where $\eta\in(\frac12,\frac{1+\beta}{1+\beta+\sigma})$ is arbitrary.

For the second term II_i above we get using decay of correlations,

$$\begin{aligned} II_i &= \sum_{j=i+1+[(\log i)^\sigma]}^{N}\int_X f_i(T^ix)f_j(T^jx)\,d\mu-E(f_i)E(f_j)\\ &\leq c_3\sum_{k=0}^\infty\alpha^{\log^\sigma i+k}(\log^\sigma i+i+k)^{\frac1\delta}\log[\log^\sigma i+i+k]^{\frac2\delta}(i(\log i)^2)^{\frac1\delta}\\ &\leq c_4 i^{\frac1\delta+\varepsilon}\sum_{k=0}^\infty\alpha^{\log^\sigma i+k}(i+k)^{\frac1\delta}\\ &\leq c_5 i^{\frac2\delta+\varepsilon}\alpha^{\log^\sigma i} \end{aligned}$$

for arbitrarily small $\varepsilon > 0$ and some constants $c_4, c_5 > 0$. Hence

$$\theta_2(n) = \sum_{j=1}^{n} II_i \leq c_5 \sum_{i=1}^{n} i^{\frac{2}{\delta}+\varepsilon}\alpha^{\log^\sigma i} = c_5 \sum_{i=1}^{n} e^{(\frac{2}{\delta}+\varepsilon)\log i + (\log\alpha)\log^\sigma i} \leq c_6$$

for some constant c_6 as $\sigma > 1$. Hence $\theta(n)^\eta \leq \theta_1(n)^\eta + \theta_2(n)^\eta \prec \sum_{j=1}^{n}\mu(B_j)$.

As above we put $h_i = I_i + II_i$ and obtain that the error $O(\sqrt{\theta(n)}\log^{3/2+\epsilon}\theta(n))$ is negligible with respect to $\sum_{j=1}^{n} g_j \geq \sum_{j=1}^{n}\mu(B_j)$ as $\theta(n)^\eta = \left(\sum_{j=1}^{n} h_j\right)^\eta \prec \sum_{j=1}^{n}\mu(B_j)$. □

As before we note that under the assumptions of the theorem $\liminf_n \frac{S_n(x)}{\log^{2-\epsilon} n} = \infty$ for μ a.e. $x \in X$ for any $\epsilon > 0$.

5. Exponential decay of correlations and sequences (B_i) such that $\frac{C_1}{i} \leq \mu(B_i) \leq \frac{C_2}{i}$

As in the previous section we assume exponential decay of correlations.

Theorem 5.1. *Assume (T, X, μ) satisfies assumptions (A) and (B) and has exponential decay of correlations. Suppose there exist $0 < C_1 \leq C_2$ such that for all $i > 0$ sufficiently large, $\frac{c_1}{i} \leq \mu(B_i) \leq \frac{C_2}{i}$, then*

$$\liminf \frac{S_n(x)}{\log E_n} > 0$$

for μ a.e. $x \in X$.

Proof. Since the decay rate is exponential $p(k) \leq C\alpha^k$ for some $0 < \alpha < 1$, $C > 0$. We choose $\sigma > 0$ such that $\alpha^{\sigma\log i} < i^{-6/\delta}$. We consider the subsequence $(B_{[\sigma i\log i]})$ and note that

$$\liminf_{n\to\infty} \frac{1}{\log\log n}\sum_{j=1}^{n}\mu(B_{[\sigma j\log j]}) > 0$$

We will consider the question of whether $T^{[\sigma i\log i]}(x) \in B_{[\sigma i\log i]}$ i. o. for μ a.e. x. By considering this subsequence we are introducing gaps of length $[\sigma(i+1)\log(i+1)] - [\sigma i\log i] \sim [\sigma\log(i)]$ between the iterates $T^{[\sigma(i+1)\log(i+1)]}$ and $T^{[\sigma(i)\log(i)]}$ we consider.

In Proposition 1.2 we put

$$h_i(x) = \sum_{j=i+1}^{N}\int_X f_{[\sigma i\log i]}(T^{[\sigma i\log i]}x) f_{[\sigma j\log j]}(T^{[\sigma j\log j]}x)d\mu - E(f_{[\sigma i\log i]})E(f_{[\sigma j\log j]})$$

Using the decay of correlations we estimate

$$\sum_{j=i+1}^{N}\int_X f_{[\sigma i\log i]}(T^{[\sigma i\log i]}x)f_{[\sigma j\log j]}(T^{[\sigma j\log j]}x)\,d\mu - E(f_{[\sigma i\log i]})E(f_{[\sigma j\log j]})$$

$$\leq c_1\sum_{k=0}^{\infty}\alpha^{\sigma\log(i+k)}[\sigma(i+k)\log(i+k)]^{\frac{1}{\delta}}\log[\sigma(i+k)\log(i+k)]^{\frac{2}{\delta}}(i\log i(\log(i\log i))^2)^{\frac{1}{\delta}}$$

Since $\alpha^{\sigma\log(i+k)}\leq (i+k)^{-6/\delta}$ the sum converges and is bounded by Ci^{-2}.

In Proposition 1.2 we have $h_i(x)\leq Ci^{-2}$ and hence the error $O(\sqrt{\theta(n)}\log^{3/2+\epsilon}\theta(n))$ is negligible with respect to $\sum_{j=1}^{n}\mu(B_{[\sigma j\log j]})$. The limit infinum estimate follows immediately from the fact that

$$\liminf_{n\to\infty}\frac{1}{\log\log n}\sum_{j=1}^{n}\mu(B_{[\sigma j\log j]})>0$$

□

5.1. **Applications of Theorem 3.1, Theorem 4.1 and Theorem 5.1.** We now list some dynamical systems which satisfy assumptions (A) and (B). Recall that although we have called the sets B_i balls, any sets B_i which are geometrically regular in the sense that their indicator function may be approximated in the L^1 norm by a Lipschitz function of reasonable Lipschitz norm (for example rectangles) also satisfy the conclusions of Theorem 3.1, Theorem 4.1 and Theorem 5.1.

Dispersing billiard systems: satisfy assumption (B) (as the invariant measure is equivalent to Lebesgue) and hence our results apply to these systems if they have sufficiently high rates of decay of correlations. The class of dispersing billiards considered by Young [31] and Chernov [4] have exponential decay of correlations.

Lozi maps: it is shown in Gupta et al. [15] that assumption (B) is satisfied by a broad class of Lozi mappings. Lozi mappings have exponential decay of correlations.

Compact group extensions of Anosov systems: Dolgopyat [10] has shown that compact group extensions of Anosov diffeomorphisms are typically rapid mixing i.e. have superpolynomial decay of correlations. These systems satisfy also assumption (B) as they are volume preserving.

Interval maps: one-dimensional non-uniformly expanding maps with an absolutely continuous invariant probabiility measure μ and density $h=\frac{d\mu}{dm}\in L^{1+\delta}(m)$ for some $\delta>0$ satisfy assumptions (A) and (B). For example the class of maps considered by Collet [7].

6. Return time distributions: quantitative BC results for sequences of balls (B_i) such that $\mu(B_i) \leq \frac{1}{i}$.

To obtain stronger quantitative estimates for BC lemmas we focus on the case of nested balls $B_{i+1} \subset B_i$ and make assumptions on return time statistics.

In this section we focus on the shrinking target problem in which $B_i(p) = B(p, r_i)$, a ball of radius r_i about a point $p \in X$, such that $\mu(B(p, r_i)) = i^{-1}$. Our proof and results generalize with no change to the setting where there positive constants C_1, C_2 such that $r_{i+1} < C_1 r_i$ and $\mu(B(p, r_i)) \geq \frac{C_2}{i}$.

We first observe, as remarked in Fayad [12], that if the balls $B_i(p)$ are nested then the set of points G such that $T^i x \in B_i(p)$ for infinitely many i has measure zero or one if (T, X, μ) is ergodic. This follows as $T^{i+1}x \in B_{i+1}(p)$ implies that $T^i(Tx) \in B_i(p)$ and hence G is T invariant. If (T, X, μ) is ergodic this implies that $\mu(G) = 0$ or $\mu(G) = 1$.

6.1. Return time distributions.

We now show that a return time distribution implies BC lemmas in certain settings. Fix $p \in X$ and set $B_i = B(p, r_i)$ where $r_i \to 0$ so that B_i is a sequence of nested balls with center p. Let τ_{B_i} be a random variable defined on the set $B_i = B(p, r_i)$ which gives the first return time i.e. $\tau_{B_i}(x) = \inf\{ n \geq 1 : T^n(x) \in B_i \}$. We define the conditional probability measure μ_{B_r} on B_r by $\mu_{B_r}(A) = \frac{\mu(A \cap B_r)}{\mu(B_r)}$.

For many dynamical systems distributional limit laws for return time statistics have been proven, of the form: for μ a.e. p, if $B_r = B(p, r)$ for $r > 0$ then

$$\lim_{r \to 0} \mu_{B_r}\{ y \in B_r : \tau_{B_r}(y)\mu(B_r) < t \} = F(t)$$

where $F(t)$ is a distribution function. Commonly $F(t) = 1 - e^{-t}$, an exponential law [18, 17].

Let $S_n = \sum_{j=0}^{n-1} 1_{B_j} \circ T^j$, $E_n = \sum_{j=0}^{n-1} \mu(B_j)$ with $\lim_{n\to\infty} E_n = \infty$. We now state a simple lemma.

Lemma 6.1. *Suppose that B_i is a sequence of balls (not necessarily nested). If $E(S_n - E_n)^2 \leq g(n)(E_n)^2$ for a sequence $g(n)$ such that $\lim_{n\to\infty} g(n) = 0$ then $\limsup \frac{S_n(x)}{E_n} \geq 1$ for μ a.e. $x \in X$.*

Proof. The assumptions imply that $E[(\frac{S_n}{E_n} - 1)^2] \leq g(n)$. Hence $E[|\frac{S_n}{E_n} - 1| > \epsilon] \leq \frac{E(S_n - E_n)^2}{\epsilon^2} \leq \frac{g(n)}{\epsilon^2}$ by Chebyshev's inequality. If we take a subsequence n_k such that $\sum_k g(n_k) < \infty$, then by Borel–Cantelli for μ a.e. $x \in X$, $|\frac{S_{n_k}(x)}{E_{n_k}} - 1| > \epsilon$ for only finitely many n_k. For any positive integer m, define $G_m := \{ x : \limsup \frac{S_n(x)}{E_n} \geq 1 - \frac{1}{m} \} = 1$. Taking $\epsilon = 1/m$ we see $\mu(G_m) = 1$ and hence $\mu(\cap_m G_m) = 1$, which implies the result.

□

In fact if the B_i are nested and (T,X,μ) is ergodic then the assumptions of the lemma above may be weakened to

Lemma 6.2. *Suppose that B_i is a nested sequence of balls and (T,X,μ) is ergodic. If $E(S_n-E_n)^2<\eta(E_n)^2$ for some $0<\eta<1$ then $T^n(x)\in B_n$ infinitely often for μ a.e. $x\in X$.*

Proof. Let G be the set of points x such that $T^i x\in B_i$ for infinitely many i. Then $\mu(G)=1$ or $\mu(G)=0$. We assume $\mu(G)=0$ and derive a contradiction.

$$\eta>\int_X (\frac{S_n(x)}{E_n}-1)^2\,d\mu=\int_{G^c}(\frac{S_n(x)}{E_n}-1)^2\,d\mu$$

Let $A_N:=\{x\in G^c: T^i(x)\notin B_i \text{ for all } i\geq N\}$. Then

$$\eta>\int_{G^c}(\frac{S_n(x)}{E_n}-1)^2\,d\mu\geq\int_{A_N}(\frac{S_n(x)}{E_n}-1)^2\,d\mu$$

But $\lim_{n\to\infty}\int_{A_N}(\frac{S_n(x)}{E_n}-1)^2\,d\mu=\mu(A_N)$. Hence $\eta>\mu(A_N)$ for all N, a contradiction as $\lim_{N\to\infty}\mu(A_N)=1$. □

We now state our main result of this section. Let $B_i(p)$ be a decreasing sequence of balls about a point p, $S_n=\sum_{j=1}^n 1_{B_j}$ and $E_n=\sum_{j=1}^n\mu(B_j)$.

Theorem 6.1. *Suppose that (T,X,μ) has exponential decay of correlations, that property (B) holds, that $B_i=B(p,r_i)$ for some point p and $\frac{C_1}{i}\leq\mu(B_i)\leq\frac{C_2}{i}$ for some positive constants C_1,C_2 and all i large enough. Also assume that*

$$\lim_{i\to\infty}\mu_{B_i}\{\,y\in B_i:\tau_{B_i}(y)\mu(B_i)<t\,\}=F(t)$$

for some distribution function $F(t)$ such that $\lim_{t\to 0^+}F(t)=0$. Then for μ a.e. $x\in X$

$$\limsup\frac{S_n(x)}{E_n}\geq 1.$$

Proof. We split $\sum_{j=i+1}^n E(f_if_j)-E(f_i)E(f_j)$ as above into two sums I_i and II_i where

$$I_i = \sum_{j=i+1}^{n\wedge(i+[c\log i])} E(f_if_j)-E(f_i)E(f_j)$$

$$II_i = \sum_{j=i+[c\log i]+1}^{n} E(f_if_j)-E(f_i)E(f_j).$$

Using decay of correlations we estimate the second term II_i for all i as follows ($c_1 > 0$):
(2)

$$\sum_{j>[i+c\log i]} (\mu(B_i \cap T^{-(j-i)}B_j) - \mu(B_i)\mu(B_j)) \leq \sum_{\beta=1}^{\infty} c_1(i+[c\log i]+\beta)^{\frac{2}{\delta}} i^{\frac{2}{\delta}} \alpha^{[c\log i]+\beta} \leq \frac{1}{i^2}$$

for a suitable choice of $c > 0$. Since $B_j \subset B_i$ for $j > i$ we obtain for the first term the estimate

$$I_i = \sum_{j=i}^{i+[c\log i]} \mu(B_i \cap T^{-(j-i)}B_j) \leq \sum_{j=i}^{i+[c\log i]} \mu(B_i \cap T^{-(j-i)}B_i).$$

Since $\lim_{t\to 0} F(t) = 0$, given $\eta > 0$ there exists t^* such that $c_2 F(t)c < \eta$ for all $0 < t \leq t^*$. As

$$\lim_{i\to\infty} \mu_{B_i}\{\, y \in B_i : \tau_{B_i}(y)\mu(B_i) < t \,\} = F(t)$$

we obtain

$$\begin{aligned} I_i &\leq \sum_{k=0}^{[c\log i]} \mu(B_i \cap T^{-k}B_i) \leq \mu(B_i) \sum_{k=0}^{[c\log i]} \mu_{B_i}(\tau_{B_i} \leq k) \\ &\leq cF(t^*)\mu(B_i)\log i \leq cF(t^*)c_2 \frac{\log i}{i} \end{aligned}$$

for i large enough so that $k \leq c\log i < t^*/\mu(B_i) < it^*$. In particular there exists a number n^* such that for $i \geq n^*$

$$\mu_{B_i}\{\, y \in B_i : \tau_{B_i}(y) < c\log i \,\} \leq F(t^*).$$

Hence

$$\sum_{i=n^*}^{n} I_i \leq \frac{\eta}{2}(\log n)^2.$$

Recalling that

$$E[(S_n - E_n)^2] = 2\sum_{i=1}^{n}\sum_{j>i}(\mu(B_i \cap T^{-(j-i)}B_j) - \mu(B_i)\mu(B_j)) + \sum_{i=1}^{n}(\mu(B_i) - \mu(B_i)^2)$$

we note first that $\sum_{i=1}^{n}(\mu(B_i)-\mu(B_i)^2)\le E_n$ and write

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j>i}(\mu(B_i\cap T^{-(j-i)}B_j)-\mu(B_i)\mu(B_j))&\\ =\sum_{n\ge i>n^*}\sum_{j>i}&(\mu(B_i\cap T^{-(j-i)}B_j)-\mu(B_i)\mu(B_j))\\ +\sum_{1\le i<n^*}\sum_{j>i}&(\mu(B_i\cap T^{-(j-i)}B_j)-\mu(B_i)\mu(B_j)).\end{aligned}$$

The term $\sum_{1\le i<n^*}\sum_{j>i}(\mu(B_i\cap T^{-(j-i)}B_j)-\mu(B_i)\mu(B_j))$ we bound by n^*E_n while

$$\begin{aligned}\sum_{n\ge i>n^*}\sum_{j>i}&(\mu(B_i\cap T^{-(j-i)}B_j)-\mu(B_i)\mu(B_j))\\ &\le\sum_{i=n^*}^{n}I_i+\sum_{i=n^*}^{n}\sum_{j=i+c\log i}^{n}(\mu(B_i\cap T^{-(j-i)}B_j)-\mu(B_i)\mu(B_j))\\ &\le\frac{\eta}{2}\log^2 n+c_3,\end{aligned}$$

where we have used the decay of correlations and the estimate (2) to bound the second term by a constant c_3. Thus for $n>n^*$, $E[(S_n-E_n)^2]<\eta E_n^2+2c_3+2n^*E_n$. Since η was arbitrary and E_n diverges this implies that given $\varepsilon>0$ there exists an N such that for all $n>N$, $E[(S_n-E_n)^2]<\varepsilon E_n^2$. Thus Lemma 6.1 implies the result. □

6.2. **Applications of Theorem 6.1.** We now give a brief list of systems, beside Axiom A [18], which have been shown to have the property that for μ a.e. p if $B_i=B(p,r_i)$ then

$$\lim_{r_i\to 0}(\mu_{B_i})\{\,y\in B_i:\tau_{B_i}(y)\mu(B_i)<t\,\}=F(t)$$

for a distribution function $F(t)$ such that $\lim_{t\to 0}F(t)=0$. In the examples below $F(t)=1-e^{-t}$, an exponential law. We abbreviate this property by saying that the system has an exponential law for first return times to balls (recall this holds only for μ a.e. point).

Dispersing billiard systems: These systems satisfy assumption (B) (as the invariant measure is equivalent to Lebesgue). The class of dispersing billiards considered by Young [31] and Chernov [4] have exponential decay of correlations. An exponential law for first return times to balls has been established by Gupta et al. [15] (and a Poisson distribution for further visits to balls by Chazottes and Collet [3]).

Lozi maps [8, 24]: it is shown in Gupta et al. [15] that assumption (B) and an exponential return time law is satisfied by a broad class of Lozi mappings (which have exponential decay of correlations).

Compact group extensions of Anosov systems: Dolgopyat [10] has shown that compact group extensions of Anosov diffeomorphisms are typically rapid mixing i.e. have superpolynomial decay of correlations. These systems satisfy also assumption (B) as they are volume preserving. Gupta [14] has shown the existence of an exponential law for the first return times to nested balls if the system is rapidly mixing.

Interval maps: one-dimensional non-uniformly expanding maps with an absolutely continuous invariant probability measure μ and density $h = \frac{d\mu}{dm} \in L^{1+\delta}(m)$ for some $\delta > 0$ satisfy assumptions (A) and (B). The class of maps considered by Collet [7] have been shown to have an exponential law for first return times to balls.

7. Short return times and sequences (B_i) such that $\mu(B_i) \geq (i \log i)^{-1}$

In the theory of return time statistics and extreme value theory a crucial role is played by short return times. Recent research has established that for certain chaotic dynamical systems short returns are rare in the sense that we call assumption (C):

Suppose $p \in X$ and $B_i(p)$ is a nested sequence of balls centered at a point p, with $\limsup_i \mu(B_i(p)) = 0$.

Assumption (C): We say $(B_i(p))$ satisfies assumption C if there exists $\zeta(p) > 0$ and $\kappa(p) > 1$ such that for all i sufficiently large

$$\mu(B_i(p) \cap T^{-r} B_i(p)) \leq \mu(B_i(p))^{1+\zeta}$$

for all $r = 1, \ldots, \log^\kappa(i)$.

We say a system (T, X, μ) satisfies assumption (C) if for μ a.e. $p \in X$ assumption C is satisfied for a nested sequence of balls $(B_i(p))$, with $\limsup_i \mu(B_i(p)) = 0$, centered at p.

Assumption (C) is a form of non-recurrence which implies certain limit laws in return time statistics and extreme value theory, to our knowledge first proved in the setting of non-uniformly expanding maps by Collet [7]. It has been verified for various systems, for example Sinai dispersing billiards and Lozi mappings [15]. We verify assumption (C) for Sinai dispersing billiards in the Appendix. Beside Axiom A systems, dynamical systems which satisfy assumption (C) include:

(1) *Billiard maps for dispersing billiards without cusps:* for example the systems described in [31, 4, 6]. For a proof of this property see the Appendix.

(2) *Lozi maps* [8, 24]: it is shown in Gupta et al. [15] that assumption (C) is satisfied by a broad class of Lozi mappings. Gupta [14] shows the same for toral extensions of certain non-uniformly and uniformly expanding intervals maps.

(3) *Certain one-dimensional maps:* Collet first verified assumption (C) in the setting of 1-d non-uniformly expanding maps with acip with exponential decay of

correlations [7]. These results have been generalized to a broad class of 1-d maps, including Lorenz like maps [15, 19].

Theorem 7.1. *Suppose (T,X,μ) has exponential decay of correlations and satisfies assumptions (B) and (C) with $\kappa > 1$ and exponent $\zeta \in (0,1)$. Then for μ a.e. $p \in X$ if $B_i = B(p, r_i)$ is a sequence of balls satisfying $\limsup_j \mu(B_j) \log^{\frac{1+\sigma}{\zeta}} < \infty$ for some $\sigma > 0$ and $E_n \to \infty$ then*

$$\lim_{n\to\infty} \frac{S_n(x)}{E_n} = 1$$

for μ a.e. $x \in X$, where as before $S_n = \sum_{j=1}^n 1_{B_i} \circ T^j$ and $E_n = E(S_n) = \sum_{j=1}^n \mu(B_j)$.

Corollary 7.1. *If (T,X,μ) is a Sinai dispersing billiard map (see Appendix for precise description) or a Lozi map (see [15]) then in the notation of Theorem 7.1 for μ a.e. $p \in X$ if B_i is a sequence of decreasing balls about p with $\sum_{i=1}^\infty \mu(B_i) = \infty$ and $\mu(B_i) \sim \frac{1}{i \log i}$ then*

$$\lim_{n\to\infty} \frac{S_n(x)}{E_n} = 1$$

for μ a.e. $x \in X$.

Proof of Corollary 7.1. The proof that assumption (C) holds for Sinai dispersing billiard maps with $k = 5$ is given in the Appendix. It is a very slight modification of the proof in Gupta et al. [15] where the sequence $\mu(B_i) = i^{-1}$ was considered. A similar straightforward modification of the proof of the corresponding result for sequences (B_i) such that $\mu(B_i) = i^{-1}$ given in Gupta et al. [15] in the setting of Lozi maps establishes the conclusions of the corollary for Lozi maps. □

Proof of Theorem 7.1. We define f_k and g_k as before and split up $\sum_{j=i+1}^n E(f_i f_j) - E(f_i)E(f_j)$ into the terms I_i and II_i. Take $\sigma \in (0, \kappa - 1)$ according to the Theorem. A rough estimate yields

$$\begin{aligned} I_i &= \sum_{j=i+1}^{n\wedge(i+[\log^{1+\sigma} i)])} E(f_i f_j) - E(f_i)E(f_j) \\ &\leq \sum_{j=i+1}^{n\wedge(i+[\log^{1+\sigma} i])} E(f_i f_j) \leq c_1 \sum_{j=i+1}^{n\wedge(i+[\log^{1+\sigma} i)])} \mu(B_j)^{1+\zeta} \end{aligned}$$

for large i where $c_1 > 1$. By the assumption $\mu(B_j) \leq c_2 \log^{-\frac{1+\sigma}{\zeta}} j$ we obtain

$$\theta_1(n) = \sum_{i=1}^n I_i \leq \sum_{j=1}^n \mu(B_j)^{1+\zeta} \log^{1+\sigma} j \leq c_2 \sum_{j=1}^n \mu(B_j)$$

which implies that $\theta_1(n)^\eta \prec E_n$ for any $\eta \in (\frac{1}{2}, 1)$.

For the second sum II_i we obtain as in Theorem 4.1 (for i so that $i + \log^{1+\sigma} i < n$)

$$\begin{aligned} II_i &= \sum_{j=i+1+[\log^{1+\sigma} i]}^{n} E(f_i f_j) - E(f_i)E(f_j) \\ &\leq \sum_{k=1}^{\infty} c_3 (i \log i)^{\frac{2}{\delta}} \log^{\frac{2}{\delta}} (i + \log^2 i + k) \alpha^{\log^2 i + k} \\ &\leq c_4 i^{\frac{2}{\delta}+\varepsilon} \alpha^{\log^{1+\sigma} i} \end{aligned}$$

for any $\varepsilon > 0$ and some constants c_3, c_4. As before we obtain $\theta_2(n) = \sum_{i=1}^n II_i$ is uniformly bounded in n and therefore $\theta(n)^\eta \leq \theta_1(n)^\eta + \theta_2(n)^\eta \prec E_n$ (as $E_n \to \infty$). As above in light of Proposition 1.2 this implies the conclusion of the theorem. □

8. Discussion

There are several questions that are prompted by this work. Here are some that we have considered but not resolved as yet.

(1) If (T, X, μ) is a smooth dynamical system with an absolutely continuous invariant probability measure and positive metric entropy is it true that for μ a.e. p if $B_i = B(p, r_i)$ is a nested sequence of balls about p then

$$\lim_{r_i \to 0} (\mu_{B_i})^{-1} \{\, y \in B_i : \tau_{B_i}(y) \mu(B_i) < t \,\} = F_p(t)$$

for a distribution function $F_p(t)$ such that $\lim_{t \to 0} F_p(t) = 0$?

(2) Chazottes and Collet [3] have shown that if (T, X, μ) is a dynamical system modeled by a Young tower, with exponential decay of correlations and a one-dimensional unstable foliation then for μ a.e. point p, if $B_i = B(p, r_i)$ is a nested sequence of balls about p then

$$\lim_{r_i \to 0} (\mu_{B_i})^{-1} \{\, y \in B_i : \tau_{B_i}(y) \mu(B_i) < t \,\} = 1 - e^{-t}$$

In fact they have shown much more, including a Poisson law for multiple returns. If such systems satisfied assumption (B) (respectively assumption (C)) then the conclusion of Theorem 6.1 (respectively Theorem 7.1) would hold. Is assumption (B) or (C) valid for such systems?

8.1. Appendix: Assumption C holds for Planar Dispersing Billiard Maps.

We first describe the class of billiards which we show satisfy assumption C. For a good general reference to billiards see [6].

Let $\Gamma = \{\Gamma_i, i = 1, \ldots, k\}$ be a family of pairwise disjoint, simply connected C^3 curves with strictly positive curvature on the two-dimensional torus $\mathbb{T}^2$. The billiard flow B_t is the dynamical system generated by the motion of a point particle in $Q = \mathbb{T}^2 / (\cup_{i=1}^k (\text{interior}\, \Gamma_i)$ with constant unit velocity inside Q and with elastic reflections at $\partial Q = \cup_{i=1}^{k} \Gamma_i$, where elastic means "angle of incidence equals angle of reflection". If each Γ_i is a circle then this system is called a periodic Lorentz gas. The billiard flow is Hamiltonian and preserves a probability measure (which is Liouville measure) $\tilde{\mu}$ given by $d\tilde{\mu} = C_Q dq\ dt$ where C_Q is a normalizing constant and $q \in Q$, $t \in \mathbb{R}$ are Euclidean coordinates.

We first consider the billiard map $T\colon \partial Q \to \partial Q$. Let r be a one-dimensional co-ordinatization of Γ corresponding to length and let $n(r)$ be the outward normal to Γ at the point r. For each $r \in \Gamma$ we consider the tangent space at r consisting of unit vectors v such that $(n(r), v) \geq 0$. We identify each such unit vector v with an angle $\theta \in [-\pi/2, \pi/2]$. The boundary M is then parametrized by $M := \partial Q = \Gamma \times [-\pi/2, \pi/2]$ so that M consists of the points (r, θ). $T\colon M \to M$ is the Poincaré map that gives the position and angle $T(r, \theta) = (r_1, \theta_1)$ after a point (r, θ) flows under B_t and collides again with M, according to the rule angle of incidence equals angle of reflection. Thus if (r, θ) is the time of flight before collision $T(r, \theta) = B_{h(r,\theta)}(r, \theta)$. The billiard map preserves a measure $d\mu = c_M \cos\theta dr d\theta$ equivalent to 2-dimensional Lebesgue measure $dm = dr\, d\theta$ with density $\rho(x)$ where $x = (r, \theta)$.

We say that the billiard map and flow satisfies the finite horizon condition if the time of flight $h(r, \theta)$ is bounded above. A good reference for background results for this section are the papers [1, 2, 31, 4, 20].

It is known (see [4, Lemma 7.1] for finite horizon and [4, Section 8] for infinite horizon) that dispersing billiard maps expand in the unstable direction in the Euclidean metric $|\cdot| = \sqrt{(dr)^2 + (d\phi)^2}$, in that $|DT_u^n v| \geq C\tilde{\lambda}^n |v|$ for some constants C, $\tilde{\lambda} > 1$ which is independent of v. In fact $|L_n| \geq C\tilde{\lambda}^n |L_0|$ where L_0 is a segment of unstable manifold (once again in the Euclidean metric) and L_n is $T^n L_0$.

We choose N_0 so that $\lambda := C\tilde{\lambda}^{N_0} > 1$ and then T^{N_0} (or DT^{N_0}) expands unstable manifolds (tangent vectors to unstable manifolds) uniformly in the Euclidean metric.

It is common to use the p-metric in proving ergodic properties of billiards. Young uses this semi-metric in [31]. Recall that for any curve γ, the p-norm of a tangent vector to γ is given as $|v|_p = \cos\phi(r)|dr|$ where γ is parametrized in the (r, ϕ) plane as $(r, \phi(r))$. The Euclidean metric in the (r, ϕ) plane is given by $ds^2 = dr^2 + d\phi^2$; this implies that $|v|_p \leq \cos\phi(r) ds \leq ds = |v|$. We will use $l_p(C)$ to denote the length of a curve in the p-metric and $l(C)$ to denote length in the Euclidean metric. If γ is a local unstable manifold or local stable manifold then $C_1 l(\gamma)_p \leq l(\gamma) \leq C_2\sqrt{l_p(\gamma)}$.

For planar dispersing billiards there exists an invariant measure μ (which is equivalent to 2-dimensional Lebesgue measure) and through μ a.e. point x there exists a local stable manifold $W^s_{loc}(x)$ and a local unstable manifold $W^u_{loc}(x)$. The SRB measure μ has absolutely continuous (with respect to Lebesgue measure) conditional measures μ_x on each $W^u_{loc}(x)$. The expansion by DT is unbounded however in the p-metric at $\cos\theta = 0$ and this may lead to quite different expansion rates at different points on $W^u_{loc}(x)$. To overcome this effect and obtain uniform estimates on the densities of conditional SRB measure it is common to define homogeneous local unstable and local stable manifolds. This is the approach adopted in [1, 2, 4, 31]. Fix a large k_0 and define for $k > k_0$

$$I_k = \{ (r,\theta) : \frac{\pi}{2} - k^{-2} < \theta < \frac{\pi}{2} - (k+1)^{-2} \},$$
$$I_{-k} = \{ (r,\theta) : -\frac{\pi}{2} + (k+1)^{-2} < \theta < -\frac{\pi}{2} + k^{-2} \}$$

and

$$I_{k_0} = \{ (r,\theta) : -\frac{\pi}{2} + k_0^{-2} < \theta < \frac{\pi}{2} - k_0^{-2} \}.$$

In our setting we call a local unstable (stable) manifold $W^u_{loc}(x)$, $(W^s_{loc}(x))$ homogeneous if for all $n \geq 0$ $T^n W^u_{loc}(x)$ $(T^{-n}W^s_{loc}(x))$ does not intersect any of the line segments in $\cup_{k>k_0}(I_k \cup I_{-k}) \cup I_{k_0}$. Homogeneous $W^u_{loc}(x)$ have almost constant conditional SRB densities $\frac{d\mu_x}{dm_x}$ in the sense that there exists $C > 0$ such that $\frac{1}{C} \leq \frac{d\mu_x(z_1)}{dm_x} / \frac{d\mu_x(z_2)}{dm_x} \leq C$ for all $z_1, \ z_2 \in W^u_{loc}(x)$ (see [4, Section 2] and the remarks following Theorem 3.1).

From this point on all the local unstable (stable) manifolds that we consider will be homogeneous. Bunimovich et al. [2, Appendix 2, Equation A2.1] give quantitative estimates on the length of homogeneous $W^u_{loc}(x)$. They show there exists $C, \ \tau > 0$ such that $\mu\{ x : l(W^s_{loc}(x)) < \epsilon \text{ or } l(W^u_{loc}(x)) < \epsilon \} \leq C\epsilon^{\tau}$ where $l(C)$ denotes 1-dimensional Lebesgue measure or length of a rectifiable curve C. In our setting τ could be taken to be $\frac{2}{9}$, its exact value will play no role but for simplicity in the forthcoming estimates we assume $0 < \tau < \frac{1}{2}$.

The natural measure μ has absolutely continuous conditional measures μ_x on local unstable manifolds $W^u_{loc}(x)$ which have almost uniform densities with respect to Lebesgue measure on $W^u_{loc}(x)$ by [4, Equation 2.4].

Let $A_{\sqrt{\epsilon}} = \{ x : |W^u_{loc}(x)| > \sqrt{\epsilon} \}$ then $\mu(A^c_{\sqrt{\epsilon}}) < C\epsilon^{\tau/2}$. Let $x \in A_{\sqrt{\epsilon}}$ and consider $W^u_{loc}(x)$. Since $|T^{-k}W^u_{loc}(x)| < \lambda^{-1}|W^u_{loc}(x)|$ for $k > N_0$ the optimal way for points $T^{-k}(y)$ in $T^{-k}W^u_{loc}(x)$ to be close to their preimages $y \in W^u_{loc}(x)$ is for $T^{-k}W^u_{loc}(x)$ to overlay $W^u_{loc}(x)$, in which case the map $T^{-k} : W^u_{loc}(x) \to W^u_{loc}(x)$ has a fixed point and it is easy to see $l\{, y \in W^u_{loc}(x) : d(y, T^{-k}y) < \epsilon \} \leq l\{y \in \mathbb{R} : d(y, \frac{y}{\lambda}) < \epsilon\} \leq (1-\lambda^{-1})\epsilon$. Accordingly $l\{ y \in W^u_{loc}(x) : d(y, T^{-k}y) < \epsilon \} \leq C\sqrt{\epsilon} l\{y \in W^u_{loc}(x)\}$. Recalling that the density of the conditional SRB-measure μ_x is bounded above and below with respect to one-dimensional Lebesgue measure we obtain $\mu_x(A^c_{\sqrt{\epsilon}}) < C\sqrt{\epsilon}$. Integrating over all unstable manifolds in $A_{\sqrt{\epsilon}}$ (throwing away the set $\mu(A^c_{\sqrt{\epsilon}})$) we have $\mu\{ x : d(T^{-k}x, x) < \epsilon \} < C\epsilon^{\tau/2}$. Since μ is T-invariant $\mu\{ x : d(T^k x, x) < \epsilon \} < C\epsilon^{\tau/2}$ for $k > N_0$. Hence for any iterate T^k, $k > N_0$

$$\mathcal{E}_k(\epsilon) := \mu\{ x : d(T^k x, x) < \epsilon \} < C\epsilon^{\tau/2}.$$

Define

$$E_k := \{\, x : d(T^j x, x) \le \frac{2}{\sqrt{k}} \ \text{ for some } \ 1 \le j \le (\log k)^5 \,\}.$$

We have shown that for any $\delta > 0$, for all sufficiently large k, $\mu(E_k) \le k^{-\tau/4+\delta}$. For simplicity we take $\mu(E_k) \le k^{-\sigma}$ where $\sigma < \tau/4 - \delta$.

Define the Hardy–Littlewood maximal function M_l for $\phi(x) = 1_{E_l}(x)\rho(x)$ where $\rho(x) = \frac{d\mu}{dm}(x)$, so that

$$M_l(x) := \sup_{a>0} \frac{1}{m(B_a(x))} \int_{B_a(x)} 1_{E_l}(y)\rho(y)\, dm(y).$$

A theorem of Hardy and Littlewood [26, Section 7.5] implies that

$$m(|M_l| > C) \le \frac{\|1_{E_l}\rho\|_1}{C}$$

where $\|\cdot\|_1$ is the L^1 norm with respect to m. Let

$$F_k := \left\{ x : \mu(B_{k^{-\gamma/2}}(x) \cap E_{k^{\gamma/2}}) \ge (k^{-\gamma\beta/2})k^{\gamma/2} \right\}.$$

Then $F_k \subset \{M_{k^{\gamma/2}} > k^{-\gamma\beta/2}\}$ and hence

$$m(F_k) \le \mu(E_{k^{\gamma/2}})k^{\gamma\beta/2} \le Ck^{-\gamma\sigma}k^{\gamma\beta/2}.$$

If we take $0 < \beta < \sigma$ and $\gamma > \sigma/2$ then for some $\delta > 0$, $k^{-\gamma\sigma}k^{\gamma\beta/2} < k^{-1-\delta}$ and hence

$$\sum_k m(F_k) < \infty.$$

Thus for m a.e. (hence μ a.e.) $x_0 \in X$ there exists $N(x_0)$ such that $x_0 \notin F_k$ for all $k > N(x_0)$. Thus along the subsequence $n_k = k^{-\gamma/2}$, $\mu(B_{n_k}(x_0) \cap T^{-j}B_{n_k}(x_0)) \le n_k^{-1-\delta}$ for $k > N(x_0)$. This is sufficient to obtain an estimate along the subsequence $(n \log n)^{-1}$. Since $\lim_{k\to\infty}(\frac{k+1}{k})^{\gamma/2} = 1$ if $k^{\gamma/2} \le n\log n \le (k+1)^{\gamma/2}$ then for sufficiently large n $\mu(B_{(n\log n)^{-1}}(x_0) \cap T^{-j}B_{(n\log n)^{-1}}(x_0)) \le \mu(B_{n_k}(x_0) \cap T^{-j}B_{n_k}(x_0)) \le n_k^{-1-\delta} \le 2(n\log n)^{-1-\delta}$. As $\frac{d\mu}{dm}(p)$ is finite for μ a.e. p this implies the result for $\mu(B_i) = i\log i$ by the Lebesgue density theorem. We now control the iterates $1 \le j \le N_0$. If x_0 is not periodic then $\min_{1\le i<j\le N_0} d(T^i x_0, T^j x_0) \ge s(x_0) > 0$ and hence for large enough n, for all $1 \le j \le N_0$, $\mu(B_{n^{-1}}(x_0) \cap T^{-j}B_{n^{-1}}(x_0)) = 0$.

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