Multiple returns for some regular and mixing maps

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Abstract

We consider the distributions of the number of visits in a given domain of the phase space and their limit behavior when the domain shrinks around a point. At first we perform a numerical analysis of the irrational rotations on the circle. Then we study a skew map defined on a cylinder which modelizes a shear flow, providing a theoretical explanation about the distributions found. Our analysis is finally extended to systems composed of invariant regions, and the results obtained allow us to investigate systems of higher physical relevance, like the standard map and the Hénon map.

This paper wishes to be the continuation of the previous one published on *Chaos* [2], where, among other things, we studied the statistics of the first return times for systems modeling the physical situation of an integrable motion. We pursue here our analysis by looking at the distribution of the number of visits, of which the preceding statistics represents the zeroth order. It is well-known that such a distribution exhibits a Poissonian behaviour for highly mixing systems. We consider, as a model of a shear flow on a cylinder, a skew map where almost all fibers are given by irrational rotations, so at first we investigate the distribution of the number of visits for irrational rotations. Then we propose for the shear

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flow on the cylinder a theoretical explanation about the power law decay shown by the distributions numerically computed. This same behaviour is also observed for the standard map in the integrable region, a new result to our knowledge. We consider also the situation in which two or more systems are coupled, showing that some transient effect could hide the real limit distribution. In particular, when a chaotic and a regular regions are glued together, in the limit of domains shrinking around a point the distributions follow a modified Poisson law, but before the limit is reached the polinomial behaviour of the regular component gives the main contribution for large times. It is our opinion that this analysis could be useful to understand and classify the complex dynamics in the regions where ergodic and integrable motions are intertwined, as can happen for the standard map. Finally, we study the distribution of the number of visits around generic and periodic points of the Hénon map. [DOI: 10.1063/1.1629191]

1 Introduction

The statistics of the first return times has been intensively studied in the last years, mainly to characterize the ergodic and statistical properties of dynamical systems. In two previous papers [1, 2], we considered such a statistics for a skew map defined on a cylinder and describing a shear flow. In this article we wish to extend our investigation to successive return times, with the aim to investigate the distributions of the number of visits for domains of the phase space which shrink around a point.

Let T be a transformation on the space Ω and μ be a probability invariant measure on Ω. Denoting by $\chi_{\mathcal{A}}$ the characteristic function of a measurable set $\mathcal{A} \subseteq \Omega$, we can define the "random variable" $\xi_{\mathcal{A}}$ in x (the symbol $\lfloor\,.\,\rfloor$ represents the integer part)

$$\xi_{\mathcal{A}}(t;x) = \sum_{j=1}^{\lfloor\langle\tau_{\mathcal{A}}\rangle\, t\rfloor} \big(\chi_{\mathcal{A}} \circ T^j\big)(x). \tag{1}$$

The value of $\xi_{\mathcal{A}}(t;x)$ measures the number of times the point $x \in \mathcal{A}$ returns into $\mathcal{A}$ within the iteration $\lfloor\langle\tau_{\mathcal{A}}\rangle\, t\rfloor$. Here $\langle\tau_{\mathcal{A}}\rangle$ is the conditional expectation of the first return time $\tau_{\mathcal{A}}(x)$ of the point x, namely

$$\tau_{\mathcal{A}}(x) = \min\Big(\big\{k \in \mathbb{N} : T^k(x) \in \mathcal{A}\big\} \cup \{+\infty\}\Big) \tag{2}$$

and

$$\langle\tau_{\mathcal{A}}\rangle = \int_{\mathcal{A}} \tau_{\mathcal{A}}(x)\, \mathrm{d}\mu_{\mathcal{A}}, \tag{3}$$

where $\mu_{\mathcal{A}}$ is the conditional measure with respect to $\mathcal{A}$ for any measurable set $\mathcal{B} \subseteq \Omega$

$$\mu_{\mathcal{A}}(\mathcal{B}) = \frac{\mu(\mathcal{B} \cap \mathcal{A})}{\mu(\mathcal{A})}. \tag{4}$$

In what follows, we will be interested in the distribution of the number of visits

$$F_{k,\mathcal{A}}(t) = \mu_{\mathcal{A}}\Big(x \in \mathcal{A} : \xi_{\mathcal{A}}(t;x) = k\Big) \tag{5}$$

in the limit $\mu(\mathcal{A}) \to 0$, denoting the limit distribution, whenever it exists, by

$$F_k(t) = \lim_{\mu(\mathcal{A})\to 0} F_{k,\mathcal{A}}(t). \tag{6}$$

Of particular interest is the distribution of order $k = 0$: in this case Eq. (5) gives the *statistics* (with respect to t) of the first return times which we quoted at the beginning

$$F_{0,\mathcal{A}}(t) = \mu_{\mathcal{A}}\Big(x \in \mathcal{A} : \tau_{\mathcal{A}}(x)/\langle\tau_{\mathcal{A}}\rangle > t\Big); \tag{7}$$

we will set $F(t) \equiv F_0(t)$ the respective limit distribution, if it exists. Correspondingly, we define the *distribution* of the first return times as

$$G_{r,\mathcal{A}}(t) = \mu_{\mathcal{A}}\Big(x \in \mathcal{A} : \tau_{\mathcal{A}}(x)/\langle\tau_{\mathcal{A}}\rangle \leq t\Big), \tag{8}$$

calling $G_r(t)$ the limit distribution, when it exists; of course $G_r(t) = 1 - F(t)$.

We recall that, for ergodic measures, $\langle\tau_{\mathcal{A}}\rangle$ is simply equal to $\mu(\mathcal{A})^{-1}$, as prescribed by Kac's theorem. The limit distributions $F_k(t)$ have been computed analytically for dynamical systems with strong mixing properties and whenever the set $\mathcal{A}$ is chosen as a ball or a cylinder (originating from a dynamical partition of Ω) shrinking around μ-almost all points of Ω [3, 4, 5, 6, 10, 12, 13, 9]; in these strong mixing situations the asymptotic laws are Poissonian

$$F_k(t) = \frac{e^{-t}\, t^k}{k!}. \tag{9}$$

The use of cylinders around arbitrary points requires different normalizations of the successive return times; see for instance [10] and [4] for a careful analysis of the statistics of the first return times around periodic points.

We will study the distributions of the number of visits for our skew map defined on the cylinder $\mathcal{C} = \mathbb{T} \times [0,1]$

$$R : \begin{cases} x' = x + y \bmod 1 \\ y' = y \end{cases} \tag{10}$$

which is not ergodic with respect to the invariant Lebesgue measure. As we argued in [2], by perturbing this simple map we obtain a tranformation that is integrable only for a subset of $\mathcal{C}$ whose Lebesgue measure approaches 1 as the amplitude of the perturbation vanishes, according to KAM theory.

It is interesting to note that our skew map represents a shear of the linearized dynamics at the fixed point of the following class of area preserving maps on $\mathbb{T}^2$ recently investigated in [23]

$$\begin{cases} x' = x + h(x) + y \\ y' = h(x) + y \end{cases} \mod 1, \tag{11}$$

where h is a smooth symmetric function vanishing at zero with its first derivative like, for example, $h(x) = x - \sin(x)$. These maps are not uniformely hyperbolic and the authors showed for them in [23] a polinomial decay of the correlations. Moreover, they argued that non-uniform hyperbolicity produces "regions in which the motion is rather regular and where the systems spend a substantial fraction of time (*sticky* regions)" [23]. This is also the physical framework which motivated our previous work [2] and this one too, as we will point out again in Sec. 4. Furthermore, the transformation (10) describes the flow on a square billiard [22].

Considering an arbitrary measurable domain $\mathcal{A} \subseteq \mathcal{C}$ of positive Lebesgue measure, the expectation (3) can be computed explicitly for the skew map

$$\langle \tau_{\mathcal{A}} \rangle = \frac{\mu^y(I_y)}{\mu(\mathcal{A})}, \tag{12}$$

where μ^x, μ^y represent the Lebesgue measure along the x-axis and y-axis, respectively, and

$$I_y = \{y : \mu^x(\mathcal{A}_y) > 0\} \tag{13}$$

with

$$\mathcal{A}_y = \{(x', y') \in \mathcal{A} : y' = y\} . \tag{14}$$

By choosing the set $\mathcal{A}$ as a square of size ϵ "around" the fixed point $(0, 0)$, $\mathcal{A} = [0, \epsilon] \times [0, \epsilon]$, we proved in [2] that $\langle \tau_{\mathcal{A}} \rangle = 1/\epsilon$ and that the limit law $F(t)$ is

$$F(t) = \begin{cases} 1 & \text{if } \ t = 0 \\ \dfrac{1}{2} & \text{if } \ 0 < t < 1 \\ \dfrac{1}{2t^2} & \text{if } \ t \geq 1 \end{cases} . \tag{15}$$

Of course, the same result holds for each of the fixed points $(x, 0)$ since the map enjoys a symmetry for translations. Furthermore, the algebraic decay for large value of t was numerically checked for square domains around arbitrary points of the cylinder.

Before going to compute the distributions of the number of visits, we observe that for almost all the ordinates y, the dynamics along the fiber placed at y is the *irrational* rotation $x' = x + y \bmod 1$. Therefore the first natural step should be to inquire whether limit distributions exist for irrational rotations. Since, as far as we know, surprisingly there is not an answer to this question yet, in the next section we will investigate numerically a special case, based on the analytical work of Coelho and De Faria [7] about the statistics of first entry times. Then we will return to our skew map, giving an heurisitic but quantitative argument to claim the existence of a limit distribution. Various applications will be presented in Sec. 4, where in particular we will study the behaviour of such a distribution when two invariant regions are coupled. The distribution of the number of visits will be also analyzed for the Hènon map.

2 Irrational rotations

Coelho and De Faria [7] were able to characterize the limit laws for the distribution of the first *entry* times for irrational rotations on the circle, provided the set $\mathcal{A}$ is chosen in a descending chain of renormalization intervals (see also [8] for further improvements and [14] for an independent related result). The entry time is defined by considering the first entrance $\tau_{\mathcal{A}}(x)$ of a point $x \in \Omega$ into the set $\mathcal{A}$; the corresponding distribution is given by

$$G_{e,\mathcal{A}}(t) = \mu\Big(x \in \Omega : \tau_{\mathcal{A}}(x)/\langle \tau_{\mathcal{A}} \rangle \leq t\Big) . \tag{16}$$

We call $G_e(t)$ the limit distribution for $\mu(\mathcal{A}) \to 0$, when it exists.

Let us now consider in detail one of the two asymptotic laws found in [7]. Let $R_\alpha(x) = x + \alpha \bmod 1$ be an irrational rotation with rotation number α. Without loss of generality, we can consider in what follows $0 < \alpha < 1$; thus the rotation number has the continued fraction expansion $\alpha = [0, a_1, a_2, a_3, \ldots]$. It is well known that $a_n = \lfloor 1/H^{n-1}(\alpha) \rfloor$, where $H : [0,1[\to [0,1[$ is the Gauss transformation defined by $H(0) = 0$, $H(x) = \{1/x\}$ for $x > 0$ ($\{\,.\,\}$ denotes the fractional part). The truncated expansion of order n of α is given by $p_n/q_n = [0, a_1, \ldots, a_n]$, where p_n and q_n verify the recursive relations

$$\begin{aligned} p_k &= a_k\, p_{k-1} + p_{k-2} \\ q_k &= a_k\, q_{k-1} + q_{k-2} \end{aligned} \tag{17}$$

with $p_{-2} = 0$, $p_{-1} = 1$ and $q_{-2} = 1$, $q_{-1} = 0$. Moreover, by setting $b_j = \lfloor 1/H^{j-1}(\beta) \rfloor$ for $j \geq 1$, with $\beta \in [0,1[$, we can construct the following quantities for $n \geq 1$

$$\Gamma^n(\alpha, \beta) = (H^n(\alpha),\ [0, a_n, a_{n-1}, \ldots, a_1, b_1, b_2, \ldots]); \tag{18}$$

of course the convergent subsequences of $\Gamma^n(\alpha, \beta)$ do not depend on β.

Now, let us consider how to construct the domains $\mathcal{A}$ used to get the distribution of the first entry times, that is the sets where the points of the circle will enter. We choose an arbitrary point z on the circle and define J_n as the closed interval of endpoints $R_\alpha^{q_{n-1}}(z)$ and $R_\alpha^{q_n}(z)$. The sequence of sets $\mathcal{A}$ shrinking to z is taken then as the descending chain of intervals J_n. Since the map R_α enjoys a symmetry for translations, it is clear that the choice of the point z does not matter; what is important is the sequence of the measures of the shrinking intervals.

Coelho and De Faria proved in particular that for each subsequence $\sigma = \{n_i\}$ of $\mathbb{N}$, the distribution functions $G_{e,J_{n_i}}(t)$ converge to a continuous piecewise linear function $G_e(t)$ if $\Gamma^{n_i}(\alpha, \beta) \stackrel{n_i \to \infty}{\longrightarrow} (\theta, \omega)$ with $\theta \in [0,1[$ and $\omega \in]0,1]$. Since we are mostly interested in return times, we would like to know whether it is possible to get the distribution of the first return times $G_r(t)$. This can be done by applying a very recent result by Lacroix et al. [11] which establishes the following relation between the two distributions

$$G_e(t) = \int_0^t (1 - G_r(s))\, \mathrm{d}s; \tag{19}$$

in the case considered we have

$$G_r(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < \dfrac{(1+\theta)\omega}{1+\theta\omega} \\[2ex] \dfrac{\omega}{1+\omega} & \text{if} \quad \dfrac{(1+\theta)\omega}{1+\theta\omega} \leq t < \dfrac{1+\theta}{1+\theta\omega} \\[2ex] 1 & \text{if} \quad t \geq \dfrac{1+\theta}{1+\theta\omega} \end{cases}. \tag{20}$$

Consequently, we immediately obtain the statistics of the first return times $F(t) = 1 - G_r(t)$, which represents the zero order ($k = 0$) distribution of the number of visits.

This result allows us to hope that the distribution for higher orders also exists. So we carried out a numerical investigation taking as rotation number the golden ratio $\gamma = \frac{\sqrt{5}-1}{2}$,

Figure 1: Distribution of the number of visits $F_{k,J_{20}}(t)$ of order $k = 1$, 2, 3.

Figure 2: Distribution of the number of visits $F_{k,J_{20}}(t)$ of order $k = 3$, 4, 5.

which exhibits the very simple continued fraction expansion $\gamma = [0, 1, 1, \ldots]$. It is easy to see that, in this case, $\Gamma^n(\gamma, \cdot)$ converges to $(\theta, \omega) = (\gamma, \gamma)$. The analysis has been performed for several orders k, computing for each of them the distribution $F_{k,J_n}(t)$, with n form 10 to 20. Of course, it is not possible to deal with limit distributions by means of numerical methods; nevertheless the results obtained strongly suggest the existence of the limit distribution $F_k(t)$. In fact, we found that the distributions with the same order k are very close to each other regardless of the value of n, and this despite the presence of statistical fluctuations and effects due to the finite size of the intervals J_n (the measure of J_n goes from about $2 \cdot 10^{-2}$ for $n = 10$ to $2 \cdot 10^{-4}$ for $n = 20$). Since we are interested in the limit for $\mu(\mathcal{A}) \to 0$, in Fig. 1, 2 are shown only the distributions referring to the smaller interval, namely J_{20}; however, the distributions of the same order computed for different values of n would appear practically indistinguishable in the graph.

It is worthwhile to note some of the features of the distributions $F_{k,J_n}(t)$ numerically obtained. First, their support is an interval and, for any k, it can be partitioned in three subintervals $I_k^{(l)}$ (the leftmost), $I_k^{(c)}$ (cental), $I_k^{(r)}$ (the rightmost) in such a way that $F_{k,J_n}(t)$ is constant on each of these subintervals; in particular $F_{k,J_n}(t) = 1$ if $t \in I_k^{(c)}$. Furthermore, the intervals $I_k^{(r)}$ and $I_{k+1}^{(l)}$ practically coincide (in this regard, the distribution with $k = 3$ is reported in both figures to clearly show that this is true even for $I_2^{(r)}$, $I_3^{(l)}$ and $I_3^{(r)}$, $I_4^{(l)}$). For every distribution studied we have that $\mu(I_k^{(l)}) \simeq \mu(I_k^{(r)}) \simeq 0.447$, and $\mu(I_k^{(c)}) \simeq 0.724$ except for $k = 2$ and $k = 5$ where $\mu(I_k^{(c)}) \simeq 0.276$. Interestingly enough, the measure of the support of $F_{k,J_n}(t)$ for $k = 1, 3, 4$ is about 1.618, that is near to $1 + \gamma$. As an example, we write the distributions $F_{k,J_{20}}(t)$ of order one and two (in the interval in which they differ from zero)

$$F_{1,J_{20}}(t) = \begin{cases} 0.38 & \text{if } \; 0.72 \leq t < 1.17 \\ 1 & \text{if } \; 1.17 \leq t < 1.89 \\ 0.24 & \text{if } \; 1.89 \leq t < 2.34 \end{cases} \tag{21}$$

and

$$F_{2,J_{20}}(t) = \begin{cases} 0.76 & \text{if } \; 1.89 \leq t < 2.34 \\ 1 & \text{if } \; 2.34 \leq t < 2.62 \\ 0.85 & \text{if } \; 2.62 \leq t < 3.06 \end{cases} , \tag{22}$$

It is our opinion that it would be interesting to get an analytic proof for the existence of the limit distributions $F_k(t)$ and an explanation of their properties.

3 Shear flow

The distribution $G_r(t)$ above found for the irrational rotations on the circle was, as we said, one of the two possible distributions that can be obtained, for each k, by looking at the renormalization intervals J_n. On the other side, nothing is known if one consider, for example, intervals of length ϵ going continuously to zero. What is surprising is that for our skew map, which is horizontally almost everywhere foliated by irrational rotations, a limit distribution exists for the first return time when we shrink the square domain $\mathcal{A}_\epsilon = [0, \epsilon] \times [0, \epsilon]$ toward the fixed point $(0, 0)$; thus one could wonder if, in such a situation, the limit distributions of the number of visits exist for a generic order $k > 0$. In this case however a geometric proof, as was performed for the first return, is exceedingly complicated; nevertheless the numerical computations suggest that a limit law still exists, as we will see in a moment. To understand this fact, we present here an heuristic, but quantitative, argument which produces predictions very close to the numerical observations.

In this respect, we have to consider another equivalent characterization of the distribution of the number of visits. Let us begin to introduce the k-th return time in $\mathcal{A}$ of a point $x \in \mathcal{A}$

$$\tau^k_{\mathcal{A}}(x) = \begin{cases} 0 & \text{if } \ k = 0 \\ \tau^{k-1}_{\mathcal{A}}(x) + \tau_{\mathcal{A}}\left(T^{\tau^{k-1}_{\mathcal{A}}(x)}(x)\right) & \text{if } \ k \geq 1 \end{cases} \tag{23}$$

(note that $\tau^1_{\mathcal{A}}(x) = \tau_{\mathcal{A}}(x)$). We then observe that Eq. (5) can be rewritten as [5]

$$\begin{aligned} F_{k,\mathcal{A}}(t) &= \mu_{\mathcal{A}}\left(x \in \mathcal{A} : \frac{\tau^k_{\mathcal{A}}(x)}{\langle \tau_{\mathcal{A}} \rangle} \leq t \ \wedge \ \frac{\tau^{k+1}_{\mathcal{A}}(x)}{\langle \tau_{\mathcal{A}} \rangle} > t \right) \\ &= \mu_{\mathcal{A}}\left(x \in \mathcal{A} : \frac{\tau^k_{\mathcal{A}}(x)}{\langle \tau_{\mathcal{A}} \rangle} \leq t \right) - \mu_{\mathcal{A}}\left(x \in \mathcal{A} : \frac{\tau^{k+1}_{\mathcal{A}}(x)}{\langle \tau_{\mathcal{A}} \rangle} \leq t \right). \end{aligned} \tag{24}$$

Since

$$\tau^k_{\mathcal{A}} = \tau_{\mathcal{A}} + (\tau^2_{\mathcal{A}} - \tau_{\mathcal{A}}) + \ldots + (\tau^k_{\mathcal{A}} - \tau^{k-1}_{\mathcal{A}}), \tag{25}$$

if we define $P_{k,\mathcal{A}}(t)$ as the distribution of the sum, normalised by $\langle \tau_{\mathcal{A}} \rangle^{-1}$, of the differences of consecutive return times until the k-th return, the preceding equation reads

$$F_{k,\mathcal{A}}(t) = P_{k,\mathcal{A}}(t) - P_{k+1,\mathcal{A}}(t). \tag{26}$$

We remark that the distribution of the difference between two consecutive return times (normalised by $\langle \tau_{\mathcal{A}} \rangle^{-1}$) follows the same law than the distribution of the first return [5], because the measure $\mu_{\mathcal{A}}$ is invariant with respect to the induced application on $\mathcal{A}$ and

$$\tau^k_{\mathcal{A}} - \tau^{k-1}_{\mathcal{A}} = \tau_{\mathcal{A}} \circ T^{\tau^{k-1}_{\mathcal{A}}}. \tag{27}$$

If the random variables $\tau_{\mathcal{A}}/\langle \tau_{\mathcal{A}} \rangle$, $(\tau^2_{\mathcal{A}} - \tau_{\mathcal{A}})/\langle \tau_{\mathcal{A}} \rangle$, $\ldots$, $(\tau^k_{\mathcal{A}} - \tau^{k-1}_{\mathcal{A}})/\langle \tau_{\mathcal{A}} \rangle$ were i.i.d. with the same distribution function $G_{r,\mathcal{A}}(t)$, it is well known that the distribution function of their sum would be the following convolution product

$$P_{k,\mathcal{A}}(t) = \underbrace{G_{r,\mathcal{A}}(t) * G_{r,\mathcal{A}}(t) * \ldots * G_{r,\mathcal{A}}(t)}_{k \text{ times}}. \tag{28}$$

In the case of highly mixing systems (for instance ϕ, α and (ϕ, f) mixing systems, see [5, 9, 10]) for which the limit distribution of the first return times $G_r(t)$ is given by $1 - e^{-t}$, the differences of the normalised successive return times become asymptotically independent when $\mu(\mathcal{A}) \to 0$; the strategy adopted in [5] to obtain the Poisson law

$$P_{k,\mathcal{A}}(t) - P_{k+1,\mathcal{A}}(t) \longrightarrow \frac{t^k \, e^{-t}}{k!}, \tag{29}$$

(for a suitable choice of the sets $\mathcal{A}$, as said in Sec. 1) was just based on this fact.

Now we proved in [2] that, although our skew map is not ergodic, it enjoys a "local" mixing-like property: for square domains $\mathcal{A}_\epsilon = [0,\epsilon] \times [0,\epsilon]$ we showed that

$$\left| \mu_\epsilon(\mathcal{A}_\epsilon \cap T^n(\mathcal{A}_\epsilon)) - \mu_\epsilon^2(\mathcal{A}_\epsilon) \right| = \mathrm{O}(n^{-1}) \,, \tag{30}$$

where $\mu_\epsilon = \mu/\epsilon$. This suggests to try to get the distributions of the number of visits by assuming that the differences of the normalised successive return times are *asymptotically independent*. We know that the limit statistics of the first return times $F(t)$, obtained for the sets $\mathcal{A}_\epsilon$ when $\epsilon \to 0$, is given by Eq. (15). Being $G_r(t) = 1 - F(t)$ the corresponding limit distribution, under the preceding assumption we can write

$$F_k(t) = P_k(t) - P_{k+1}(t) \tag{31}$$

with

$$P_k(t) = \underbrace{G_r(t) * G_r(t) * \ldots * G_r(t)}_{k \text{ times}} . \tag{32}$$

In particular, we have that

$$F_1(t) = G_r(t) - \int_{-\infty}^{+\infty} G_r(t-s) \, \mathrm{d}G_r(s), \tag{33}$$

and a rather straightforward computation of the Stieltjes integral gives

$$F_1(t) = \begin{cases} 0 & \text{if } \ t = 0 \\ 1/4 & \text{if } \ 0 < t < 2 \\ \dfrac{1}{4t^2} + \dfrac{1}{4(t-1)^2} + \dfrac{3}{2t^3} + \dfrac{3\log(t-1)}{t^4} + \dfrac{6-7t}{4t^3(t-1)^2} & \text{if } \ t \geq 2 \end{cases} . \tag{34}$$

Note that when t is large, $F_1(t)$ behaves like $1/2 \; t^{-2}$. With a similar but more cumbersome computation we can obtain $F_2(t)$

$$F_2(t) = \begin{cases} 0 & \text{if } \ t = 0 \\ 1/8 & \text{if } \ 0 < t < 1 \\ \dfrac{1}{4} - \dfrac{1}{8t^2} & \text{if } \ 1 \leq t < 2 \\ \mathrm{O}\big(t^{-2}\big) & \text{if } \ t \gg 2 \end{cases} \tag{35}$$

Using a recursive argument it is possible to show two interesting features of the distributions $F_k(t)$:

Figure 3: Distribution of the number of visits of order $k = 1$ for the domain $\mathcal{A}_\epsilon = [0, \epsilon] \times [0, \epsilon]$, with $\epsilon = 10^{-2}$. The dotted line represents the function given by Eq. (34).

Figure 4: Distribution of the number of visits of order $k = 2$ for the domain $\mathcal{A}_\epsilon = [0, \epsilon] \times [0, \epsilon]$, with $\epsilon = 10^{-2}$. The dashed line represents the function $1/2\, t^{-2}$.

Figure 5: Distribution of the number of visits of order $k = 1, 2, 5, 10, 20$ computed for the domain $\mathcal{A} = [0, \epsilon] \times [y_0, y_0 + \epsilon]$, with $y_0 = 0.35$ and $\epsilon = 10^{-2}$. The dashed line represents the function $1/2\, t^{-2}$.

Figure 6: Distributions of the best-fit parameter β, obtained through a least-squares method in the range $30 \leq t \leq 100$ from the distributions of the number of visits of order $k = 1, 2, 3$ for twenty domains of side $\epsilon = 10^{-2}$. The arrows represent the position of the mean value for each distribution.

- for $0 < t < 1$, the distributions present a *plateau* whose height is given by $1/2^{k+1}$. This is the only explicit dependence on k that we can easily detect.

- for $t \to \infty$ *all* the $F_k(t)$ exhibit the same behaviour, decaying like $1/2\, t^{-2}$.

We compared these distributions, computed under the assumption that the differences of the normalised successive return times are asymptotically independent, with the ones obtained through the numerical analyses. Although there is some discrepancy between them, nonetheless the qualitative features described above still seem to persist. In particular, both types of distribution show an initial plateaux, even if the actual one decreases to zero much faster for $k > 1$, and what is more interesting, all the numerical distributions appear to decay like $1/2\, t^{-2}$ after a transitory peak in the spectrum, see Fig. 3, 4. The discrepancy between the two kind of distribution could be reasonably due to the presence of some sort of weak correlation between the differences of successive returns. We met an analogous situation in the previous Section, where the same asymptotic independence assumption was made for the irrational rotations on the circle: again we did not recover the numerical distributions for $k = 1, 2$, although we were able to depict the qualitative behaviour of the limit laws. Note that this could be in agreement with a result of Coelho and De Faria [7], which shows that for $\theta > 0$ the limit joint distributions of the differences of successive *entry* times are not given by the product of the individual limit distributions.

We also investigated the behaviour of the distributions of the number of visits for sets $\mathcal{A} = [0, \epsilon] \times [y_0, y_0 + \epsilon]$ with $y_0 > 0$, using a least-squares method to fit the numerical distributions obtained for several values of y_0 and ϵ against the function $\alpha\, t^{-\beta}$. We found that an asymptotic power law decay preceded by a peak seems to hold even for such domains, as shown in Fig. 5. Note how the peaks narrow and shift toward larger t when k increases, while their height slightly decreases. However (Fig. 6) in this more general case the exponent β is usually greater than 2, and the mean value of its distribution appears to be correlated to the order k. Even for rectangular domains like $\mathcal{A} = [x_0, x_0 + \epsilon] \times [y_0, y_0 + \delta]$, the distributions computed numerically present features analogous to the ones described above. In particular the mean return time is $\langle \tau_\mathcal{A} \rangle = 1/\epsilon$, while their decay follows a power law with an exponent greater but near to 2.

In conclusion, we think that the inverse square decay in t, whatever the order k, is tipical for the fixed points (which lie along the x-axis) of our skew map, while as soon as we consider other points the exponent β changes weakly with k. We wish to remark that the differences between the distributions for periodic and generic points seems a general fact of recurreces, as we will also argue in the next Sections.

4 Mixed dynamical systems

We studied the skew map both in [2] and in this paper mainly for two reasons. First, we tried to understand, by means of a very simple model, why the statistics of the first return times exhibits a power law decay in systems which present regular components, as it was pointed out, for example, in [16, 17, 18, 19] (see also [20, 21] for an overview of recurrences in dynamical systems).

The second reason was to investigate whether this algebraic decay could be related to some finite size effect. In this respect, we considered in [2] a systems whose phase space was partitioned into two invariant regions, obtained by coupling the skew map, which models a shear flow, with some mixing map whose distribution of first returns decays exponentially. We were able to rigorously prove the following fact: for a particular domain $\mathcal{A}$ that intersect the boundary of the two invariant regions and whose measure goes to zero, the limit statistics of the first return times is simply ruled by the one of the mixing region, thus being exponential. Nevertheless, as long as the measure of $\mathcal{A}$ is finite (which is the only situation that can be analyzed by numerical investigations), the exponential and power laws are linearly superposed and weighted by coefficients proportional to the relative sizes of the chaotic and regular components of $\mathcal{A}$; in this case the polynomial part provides asymptotically the main contribution. The proof relies on the fact that, for the particular domain considered, we know analytically the statistics of the first return times, and this allow us to understand how the statistics converges to its limit value. Unfortunately we do not have such a knowledge for the distributions of order $k > 0$.

Here it will be presented a general result about the linear weighted superposition of the distribution of the number of visits when two invariant regions are coupled, which generalizes what we previously proved for the statistics of first return [2]. Such a result will be numerically checked for a one-dimensional system composed by two mixing maps. A similar investigation will also be performed for a two-dimensional system made by coupling our skew map with the Arnold's cat map and then for the so called 'standard map'.

The last part of our work is dedicated to show some results about the distributions of the number of visits for the Hénon map. Although this map is not area preserving, like the others considered above, it is a very famous example of a chaotic two-dimensional system which is of interest from a physical point of view too.

4.1 Distribution of the number of visits

Let T be a map acting on the measurable space Ω and μ a T-invariant measure. Suppose moreover that the dynamical system (Ω, T, μ) splits into two invariant components (Ω_1, T_1, μ_1) and (Ω_2, T_2, μ_2), where

$$\Omega = \Omega_1 \cup \Omega_2, \qquad \mu(\Omega_1 \cap \Omega_2) = 0; \tag{36}$$

T_1, T_2, defined over Ω_1, Ω_2 respectively, are such that they satisfy the following conditions

$$T_1 = T_{|\Omega_1 \setminus (\Omega_1 \cap \Omega_2)}, \qquad T_2 = T_{|\Omega_2 \setminus (\Omega_1 \cap \Omega_2)}, \tag{37}$$

that is they coincide with T except on the zero measure boundary $\Omega_1 \cap \Omega_2$.

We wish to consider the behaviour of the distribution of the number of visits for points belonging to a neighborhood of points on the common boundary of the invariant regions. To this end we take a neighborhood $\mathcal{A}$ of a point on the boundary, such that $\mu(\mathcal{A}) > 0$ and we denote with $\mathcal{A}_1$ and $\mathcal{A}_2$ the two different components of $\mathcal{A}$, that is $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ with $\mathcal{A}_1 = \mathcal{A} \cap \Omega_1$ and $\mathcal{A}_2 = \mathcal{A} \cap \Omega_2$. We choose this neighborhood such that the relative weights

$$w_1(\mathcal{A}) = \frac{\mu(\mathcal{A}_1)}{\mu(\mathcal{A})}, \qquad w_2(\mathcal{A}) = \frac{\mu(\mathcal{A}_2)}{\mu(\mathcal{A})} \tag{38}$$

have a finite limit when $\mu(\mathcal{A}) \to 0$, namely we will assume that the following limits exist and are different from zero

$$w_1 = \lim_{\mu(\mathcal{A})\to 0} w_1(\mathcal{A}), \qquad w_2 = \lim_{\mu(\mathcal{A})\to 0} w_2(\mathcal{A}). \tag{39}$$

By a procedure like the one we used in [2], it is possible to easily prove that the mean return time in $\mathcal{A}$ is related to the ones in $\mathcal{A}_1$ and $\mathcal{A}_2$ in the following way

$$\langle \tau_{\mathcal{A}} \rangle = w_1(\mathcal{A}) \, \langle \tau_{\mathcal{A}_1} \rangle + w_2(\mathcal{A}) \, \langle \tau_{\mathcal{A}_2} \rangle. \tag{40}$$

Let now $F_{k,\mathcal{A}_1}(t)$ and $F_{k,\mathcal{A}_2}(t)$ be the distribution of the number of visits in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively

$$F_{k,\mathcal{A}_i}(t) = \mu_{\mathcal{A}_i}\Big(x \in \mathcal{A}_i : \xi_{\mathcal{A}_i}(t;x) = k\Big), \quad i = 1,2. \tag{41}$$

It is straightforward to show that

$$F_{k,\mathcal{A}}(t) = w_1(\mathcal{A}) \, F_{k,\mathcal{A}_1}(w_1'(\mathcal{A})\, t) + w_2(\mathcal{A}) \, F_{k,\mathcal{A}_2}(w_2'(\mathcal{A})\, t) \tag{42}$$

where

$$w_1'(\mathcal{A}) = \frac{\langle \tau_{\mathcal{A}} \rangle}{\langle \tau_{\mathcal{A}_1} \rangle}, \qquad w_2'(\mathcal{A}) = \frac{\langle \tau_{\mathcal{A}} \rangle}{\langle \tau_{\mathcal{A}_2} \rangle}. \tag{43}$$

Assuming that the limits

$$F_{k,1}(t) = \lim_{\mu(\mathcal{A})\to 0} F_{k,\mathcal{A}_1}(t), \qquad F_{k,2}(t) = \lim_{\mu(\mathcal{A})\to 0} F_{k,\mathcal{A}_2}(t) \tag{44}$$

and

$$w_1' = \lim_{\mu(\mathcal{A})\to 0} \frac{\langle \tau_{\mathcal{A}} \rangle}{\langle \tau_{\mathcal{A}_1} \rangle}, \qquad w_2' = \lim_{\mu(\mathcal{A})\to 0} \frac{\langle \tau_{\mathcal{A}} \rangle}{\langle \tau_{\mathcal{A}_2} \rangle} \tag{45}$$

are well defined, it is possible to prove, in the same way as we did for the distribution of the first return times, that

Proposition 4.1 *Under the existence of (39), (44) and (45), the limit distribution of the number of visits exists and is given by*

$$F_k(t) = w_1 \, F_{k,1}(w_1' \, t) + w_2 \, F_{k,2}(w_2' \, t) \tag{46}$$

in the points of continuity of both $F_{k,1}$ *and* $F_{k,2}$.

4.2 Coupling of chaotic trasformations

In order to check Eq. (46), we consider a one-dimensional map T from the interval $\Omega = [-1, 1]$ into itself, defined in such a way that the two subintervals $\Omega_1 = [-1, 0]$ and $\Omega_2 = [0, 1]$ are invariant with respect to T:

$$T(x) = \begin{cases} T_1(x) & \text{if } -1 \le x < 0 \\ \\ T_2(x) & \text{if } 0 \le x \le 1 \end{cases} \tag{47}$$

where

$$T_1(x) = \begin{cases} -3x - 3 & \text{if } -1 \le x < -\frac{2}{3} \\ \\ 3x + 1 & \text{if } -\frac{2}{3} \le x < -\frac{1}{3} \\ \\ -3x - 1 & \text{if } -\frac{1}{3} \le x \le 0 \end{cases} \tag{48}$$

and

$$T_2(x) = \begin{cases} 3x & \text{if } 0 \le x < \frac{1}{3} \\ \\ 2 - 3x & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ \\ 3x - 2 & \text{if } \frac{2}{3} \le x \le 1 \end{cases} . \tag{49}$$

Note that T preserves the Lebesgue measure and is discontinuous in the point $x = 0$, which is a periodic point of period two for T_1 and a fixed point for T_2. Thus, in order to deal with Proposition 4.1, we need to know the distributions of the number of visits around periodic points. Since the two mixing maps T_1 and T_2 are conjugated with a Bernoulli shift on three symbols with equal weights, we can use a general formula recently proved in [15] which gives the limit distribution of order k for cylinders C_n around periodic points of period P

$$\begin{aligned} F_k(t) &= \lim_{n\to\infty} \mu_{C_n}\Big(x \in C_n : \xi_{C_n}(t; x) = k\Big) \\ &= (1 - p^P)\, e^{-(1-p^P)t} \sum_{j=0}^{k} \binom{k}{j} \frac{p^{P(k-j)}\,(1 - p^P)^{2j}}{j!}\, t^j , \end{aligned} \tag{50}$$

where p is the ratio of the measures of the cylinders C_{n+1} and C_n. Note that for $k = 0$ this formula coincides with the one found by Hirata [4]

$$F_0(t) = (1 - p^P)\, e^{-(1-p^P)\,t}. \tag{51}$$

For example, if we consider a sequence of shrinking cylinders C_n centered around the point $x = 0$ (therefore $\omega_1 = \omega_2 = 1/2$ and, by Kac's theorem, $w_1' = w_2' = 1$), the distribution of the number of visits for $k = 1$ should be well described by the following function

$$\begin{aligned} F_1(t)\Big|_{x=0} &= \frac{1}{2} e^{-(1-p^2)t}\,(1 - p^2)\left[p^2 + (1 - p^2)^2\, t\right] + \\ &+ \frac{1}{2} e^{-(1-p)t}\,(1 - p)\left[p + (1 - p)^2\, t\right] , \end{aligned} \tag{52}$$

Figure 7: Distributions of the number of visits of order $k = 1, 2$ computed for the map given by Eq. (47) in the interval $\mathcal{A} = [-\epsilon, \epsilon]$ with $\epsilon = 5 \cdot 10^{-4}$. The dashed lines represent the theoretical predictions; in particular the one for $k = 1$ is given by formula (52).

where in this case $p = 1/3$. As shown in Fig. 7, the agreement of the numerically computed distributions with the theoretical expectations is really good. As a final remark, we stress the fact that the distributions (50) can be obtained by convoluting $F_0(t)$ according to the procedure described in Sec. 3, under the assumption that the differences of the normalized successive return times are asymptotically independent.

4.3 Coupling of regular and mixing maps

We wish now to couple our skew map (10) with the hyperbolic automorphisms of the torus (Arnold's cat map) by constructing a map T on the two-dimensional space $\Omega = \Omega_1 \cap \Omega_2$, where $\Omega_1 = \mathbb{T} \times [0, 1]$ and $\Omega_2 = \mathbb{T} \times [-1, 0[$, so that Ω is obtained by gluing together the cylinder Ω_1 with the torus Ω_2. The transformation $T_1 = T_{|\Omega_1}$ is represented by the skew map, while $T_2 = T_{|\Omega_2}$ is the hyperbolic automorphism

$$T_2 : \begin{cases} x' = 2x + y \mod 1 \\ y' = (x + y \mod 1) - 1 \,. \end{cases} \tag{53}$$

Let us consider an arbitrary point $P = (x, 0)$ along the common boundary, taking the square domain $\mathcal{A} = [x - \epsilon/2, x + \epsilon/2] \times [(\lambda - 1)\epsilon, \lambda\epsilon]$ of side ϵ, with $0 < \lambda < 1$ fixed, as the neighborhood around P. Following the notations introduced in Sec. 4.1, we have $\mathcal{A}_1 = [x - \epsilon/2, x + \epsilon/2] \times [0, \lambda\epsilon]$ and $\mathcal{A}_2 = [x - \epsilon/2, x + \epsilon/2] \times [(\lambda - 1)\epsilon, 0]$; therefore using Eq. (38) and Eq. (43) it is straightforward to show that

$$w_1(\mathcal{A}) = \lambda, \qquad w_2(\mathcal{A}) = 1 - \lambda \tag{54}$$

and

$$w_1'(\mathcal{A}) = \lambda + \frac{1}{\epsilon}, \qquad w_2'(\mathcal{A}) = (1 - \lambda)(1 + \lambda\epsilon) \,. \tag{55}$$

Although we do not exactly know the distribution of the number of visits $F_{k,\mathcal{A}_1}(t)$ in $\mathcal{A}_1$, nevertheless the numerical computations of Sec. 3, supported by the heuristic explications presented there, strongly suggest that $F_{k,\mathcal{A}_1}(t)$ will dacay like $\alpha\, t^{-\beta}$, at least for $\mathcal{A}$ sufficiently small, with β slightly greater than 2. Similar considerations can be done for $F_{k,\mathcal{A}_2}(t)$: since the cat map, which is an Anosov diffeomorphism, enjoys the Poisson distribution (if we take balls or cylinders converging around almost all points) it is sensible to expect that the distribution $F_{k,\mathcal{A}_2}(t)$ will approach the function $e^{-t}\, t^k/k!$ when $\mu(\mathcal{A}) \ll 1$. Thus, using Eq. (42), we can write

$$F_{k,\mathcal{A}}(t) = w_1(\mathcal{A})\, \frac{\alpha}{(w_1'(\mathcal{A})\, t)^\beta} + w_2(\mathcal{A})\, \frac{(w_2'(\mathcal{A})\, t)^k}{k!}\, e^{-w_2'(\mathcal{A})\, t} \qquad (\beta \simeq 2). \tag{56}$$

This formula, despite the approximations employed in order to obtain it, describes quite well (for t sufficiently large) the distributions of the number of visits for domains that intersect the boundary, as shown in Fig. **??** and **??**. Note that a power law tail appears when the first term in Eq. (56) gives the main contribution. In particular, since the values of the normalized time t for which the polynomial decay prevails increase as the measure of $\mathcal{A}$ decreases (but is still different from zero), this means that if we numerically compute the distributions of the number of visits for a very small domain, we need to reach large values of t to see the power law tail.

Figure 8: Distribution of the number of visits of order $k = 1$ computed for a domain of side $\epsilon = 0.01$ and $\lambda = 0.5$. The dashed line represents the theoretical formula (56) with $\alpha = 0.5$ and $\beta = 2.05$.

We wish to stress that the appearence of this power law tail is a consequence of the fact that $\mu(\mathcal{A}) \neq 0$. Therefore, since it is not possible to deal with the limit of zero measure domains while performing a numerical analysis of the distributins, one should take into account the presence of these finite size effects before going to conclusions about the behaviour of the limit distributions from the numerical results, as we pointed out in our previous work [2].

Now it would seem that we could directly turn to Proposition 4.1 to get the limit distribution $F_k(t)$, but unfortunately it happens that one of the assumptions required for the existence of $F_k(t)$ fails, since $w_1'(\mathcal{A})$ has not a finite limit when $\mu(\mathcal{A}) = \epsilon^2 \to 0$. We were able to overcome this difficulty in the specific case presented in [2] because we knew how the statistics of the first return times for the skew map approached the limit law: this allowed us to directly prove that the behaviour of the mixing region prevails when $\mu(\mathcal{A}) \to 0$. Here we observe that, supposing the quantity α keeps a finite positive value, the limit of Eq. (56) for $\mu(\mathcal{A}) \to 0$ simply reads

$$F_k(t) = w_2 \, \frac{(w_2\, t)^k}{k!} \, e^{-w_2\, t} \,, \tag{57}$$

since $\lim_{\epsilon\to 0} w_2'(\mathcal{A}) = w_2$. As you can see, even if the power law contribution disappears from the distribution in the limit of zero measure domains, nevertheless the presence of the regular component of $\mathcal{A}$ is still revealed by the fact that the coefficient w_2 is different from 1.

XXX Aggiungere figura con coda spostata. XXX

4.4 Standard map

In this section we will investigate the distributions of the number of visits for the standard map

$$\begin{cases} y' = y - \dfrac{\eta}{2\pi}\sin(2\pi x) \\ \\ x' = x + y' \end{cases} \mod 1, \tag{58}$$

choosing the coupling parameter $\eta = 3$, such that the boundaries between the integrable orbits and the chaotic region are as sharp as possible, compared to the size of the sets used to compute the distributions for different orders k. We will consider three cases: domains wholly contained in the regular region, domains in the chaotic 'sea' and domains which overlap both.

4.4.1 Regular region

The numerical distributions obtained for domains wholly contained in the regular region seem to follow, for large values of the normalized time t, a power law, as shown in Fig. **??**. The least-squares fit estimate of the exponent gives a value near 2, usually between 1.95 and 2.05; there is moreover some indication that the exponent increases slightly with the order k.

4.4.2 Chaotic region

The behaviour of the distributions of the number of visits for domains included in the chaotic component of the phase space appears at first surprising. In fact, the computed mean return time is given, with a very good approximation, by the ratio of the measure of the chaotic region (which is about 0.88 with the choice $\eta = 3$ for the coupling parameter) and the measure of the considered domain, thus being in agreement with the value that would be provided by Kac's theorem (which holds for ergodic systems) if it could be applicable here. On the other hand, the distributions obtained reveal a departure from the expected Poisson law for sufficiently large values of t, showing an algebraic decay with an exponent near 2, see Fig. **??**.

The reasons for such a rather unexpected feature could sensibly be due to the fact that orbits originating even from points far away from the regular region usually approach it. In this way, the overall chaotic motion would be appreciably influenced by the regular component, leading to the appearence, in the distributions of the number of visits, of a power law decay which is typical of integrable systems. In this regard, it is interesting to note that Poincaré recurrences seem to provide a tool capable to capture some of the properties of the dynamics in a more "sensitive" way with respect to other quantities that could be used for the same purpose, like the mean return time into a given domain above seen.

4.4.3 Mixed regions

Considering the results of Sec. 4.1, it is reasonable to expect that the asymptotic behaviour of the distributions of the number of visits for domains that intersect the boundary between the regular and the chaotic regions should be given by a linear superposition of a power law (the contribution, above all, of the integrable orbits) and a Poisson distribution (from the chaotic sea). In order to test this conjecture we tried to fit the distributions computed for a given domain $\mathcal{A}$ against the following function, which should sensibly represent their behaviour (from now on, to semplify the notations, we drop the explicit dependence on $\mathcal{A}$)

$$F(t) = w_1\, S(\alpha, \beta; w_1' t) + w_2\, P(k; w_2' t), \tag{59}$$

where

$$P(k;t) = \frac{e^{-t}\, t^k}{k!}, \qquad S(\alpha, \beta; t) = \alpha\, t^{-\beta}. \tag{60}$$

¿From Eq. (38), (40) and (43) we have that the quantities w_1, w_2, w_1' and w_2' are related in the following way

$$w_1 + w_2 = 1, \qquad \frac{w_1}{w_1'} + \frac{w_2}{w_2'} = 1; \tag{61}$$

so, for instance, in Eq. (59) w_2 and w_2' can be expressed in terms of w_1 and w_1'

$$w_2 = 1 - w_1, \qquad w_2' = \frac{1 - w_1}{1 - w_1/w_1'}. \tag{62}$$

Using then w_1, w_1', α and β as fit parameters, we generally found a very good agreement between the numerical distributions and the formula (59), despite the fact that $P(k,t)$ and

Henon G1

Figure 9: Distributions of the number of visits $F_{k,\mathcal{A}}(t)$ of order $k = XXX$ computed for a circular domain of radius XXX around a generic point. The dotted lines represent the corresponding Poisson laws.

Henon G2

Figure 10: Distributions of the number of visits $F_{k,\mathcal{A}}(t)$ of order $k = XXX$ computed for a circular domain of radius XXX around a generic point. The dotted lines represent the corresponding Poisson laws.

$S(\alpha, \beta, t)$ do not take into account the domain's finite size effects: in this respect X Fig. XXX shows a distribution of order $k = 3$ with the corresponding fit function. X

It is worthwhile to mention that, for a given domain, the value w_1 obtained from the fit procedure generally is slightly greater than the geometrical estimate of the relative measure of the regular share of such a domain. This discrepancy seems reasonably caused by the fact that besides the points belonging to the regular share, which are responsible for the power law decay, there is a further contribution from the chaotic component, whose dynamics is influenced in some way by the integrable orbits, as seen in Sec. 4.4.2, so that the effective relative size of the regular component appears to be greater than expected.

5 Hénon dissipative map

In this section we will consider the following map, defined on $\mathbb{R}^2$, which was introduced by Hénon [24] as a simplification of the three-dimensional Lorenz's flow

$$\begin{cases} x' = 1 + y - ax^2 \\ y' = bx \end{cases} , \tag{63}$$

where $a = 1.4$ and $b = 0.3$. In order to compute the distributions of the number of visits $F_{k,\mathcal{A}}(t)$ we employed the so called physical measure (also known as SBR-measure), supposing it is well defined, despite the fact that no rigorous proof about its existence and its statistical properties has been given till now. The numerical analyses were performed by taking the domain $\mathcal{A}$ as a *little* ball centered both around generic and periodic points.

In the case of generic points, the distributions show a very good agreement with the Poisson law (9), see Fig. 9, 10. Since the Hénon map is not uniformly hyperbolic, we could not have taken for granted this result.

To investigate the distributions of the number of visits for periodic points we computed the periodic orbits of the Hénon map for every period from one to ten through a bisection method [25]. The decision to limit our study up to points of period ten was motivated by our believe that the lower periods are the ones with a stronger influence on return times. As expected, when $\mathcal{A}$ is a little ball centered around a periodic point, the Poisson law (9) does no more fit the numerical results, as is clear from Fig. 11 which shows some of the distributions computed for a point of period two. Instead, for every periodic point considered, the distributions of order $k = 0$ are very well described by the function [4]

$$F_0(t) = \alpha e^{-\alpha t} , \tag{64}$$

Henon P1

Figure 11: Distributions of the number of visits $F_{k,\mathcal{A}}(t)$ of order $k = XXX$ computed for a circular domain of radius XXX around a periodic point of period two.

Henon P2

Figure 12: Distributions of the number of visits $F_{k,\mathcal{A}}(t)$ of order $k = XXX$ computed for a circular domain of radius XXX around a periodic point of period two.

where α depends on the period P in a way which is still not completly understood: it grows with the period, being about 1/2 for $P = 1$ and becoming approximately 1 as soon as P is greater than $8 \div 10$. Supposing that the normalized differences of successive return times are indipendent random variables, we can use a procedure similar to the one employed in Sec. 3 to compute $F_k(t)$

$$F_k(t) = \alpha e^{-\alpha t} \sum_{j=0}^{k} \binom{k}{j} \frac{(1-\alpha)^{k-j}\, \alpha^{2j}}{j!}\, t^j. \tag{65}$$

As Fig. 12 shows, the numerical distributions obtained around periodic points agree in a very good way with this expression. It is easy to see that Eq. (65) becomes equal to Eq. (50) by the simple substitution $\alpha \to (1 - p^P)$ (here we refer to the notations introduced in Sec. 4.2). XXX Eventuali altre considerazioni qui!!! XXX. Moreover, note that $F_k(t)$ reduces to formula (9) when $\alpha = 1$. Since α approaches 1 as the period increases, this means that the distributions for high periods are practically undistinguishable from those concerning generic points, which are just described by Eq. (9).

Our work on Hénon map tells us the following remarkable facts. First, the distributions of the number of visits computed for generic points agree with the Poisson law (9) in an excellent way, as usually happens for systems with strong mixing properties (see [5]). Second, the effect that periodic points exercise on return times decreases with the period, thus confirming our hypothesis that lower periods have bigger effects, and is self-consistent, because an aperiodic orbit can be seen like an orbit with an infinite period. Furthermore, the distributions computed for periodic points follow very well the behaviour predicted by assuming that the differences of successive return times are indipendent. The importance of these features becomes more apparent if we consider that they hold for a system whose ergodic proprieties are not know analytically; in particular, they suggest that the Hénon map enjoys a rapid decay of the correlations.

The connection between these results and the work of Hirata [4] still remains an interesting open question.

6 Conclusions

Acknowledgments

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References

[1] N Buric, A Rampioni, G Turchetti, S Vaienti, Poincaré recurrences for area preserving maps, *Journ. Phys. A: Math. Gen.*, **36**, (2003), L209–L215, *Corrigendum*: *Journ. Phys. A: Math. Gen.*, **36**, (2003), 7223

[2] H. Hu, A. Rampioni, L. Rossi, G. Turchetti, S. Vaienti: Statistics of Poicaré recurrences for area-preserving maps with integrable and ergodic components, *Chaos*, **14**, (2004), 160–171

[3] B Pitskel: Poisson law for Markov chains; Ergod. Th. Dynam. Syst. 11 (1991), 501–513

[4] M Hirata: Poisson law for Axiom A diffeomorphisms; Ergod. Th. & Dynam. Sys. 13 (1993), 533–556

[5] M Hirata, B Saussol and S Vaienti: Statistics of return times: a general framework and new applications. Commun. Math. Phys. 206 (1999), 33–55

[6] N T A Haydn: Statistical properties of equilibrium states for rational maps; Ergo. Th. & Dynam. Sys. 20 (2000), 1371–1390

[7] Z. Coelho, E. De Faria, Limit laws of entrance times for homeomorphisms of the circle. *Israel J. Math.*, **93**, 93-112, (1996)

[8] Z. Coelho, The loss of tightness of time distribution for homeomorphisms of the circle, to appear in Trans. Math. Soc.

[9] N. Haydn, S. Vaienti, The limiting distribution and error terms for return time of hyperbolic maps, *Discrete and Continuous Dynamical Systems*, **10**, (2004), 584-616

[10] M Abadi: Exponential Approximation for Hitting Times in Mixing Processes; Math. Phys. Elec. J. 7 No.2 (2001)

[11] N. Haydn, Y. Lacroix, S. Vaienti, Hitting and return times in ergodic dynamical systems, to appear in Ann. of Prob.

[12] M Abadi and A Galves: Inequalities for the occurence times of rare events in mixing processes. The state of the art; Markov Proc. Relat. Fields 7 (2001) 97–112

[13] A Galves and B Schmitt: Inequalities for hitting times in mixing dynamical systems; Random and Computational Dynamics 1997

[14] N. Buric, A. Rampioni, G. Turchetti, Statistics of Poincaré recurrences for a class of smooth circle maps, to appear in *Chaos, solitons and fractals*

[15] N Haydn, S. Vaienti, in preparation

[16] G. M. Zaslavsky, M. K. Tippet, Connection between recurrence-time statistics and anomalous transport, *Phys. Rev. Lett.*, **67**, 3251, (1991)

[17] G. M. Zaslavsky, M. Eldeman, Weak mixing and anomalous kinetics along filamented surfaces, *Chaos*, **11**, 295-305, (2001)

[18] Phys. Rev. E**55**, 5418 (1997)

[19] Lect. Notes Phys. **511** (1998), 59

[20] J.D. Meiss, Rev. Mod. Phys, **64**, (1992), 795

[21] J.D. Meiss, Average exit times for volume preserving maps, Boulder, CO, 80304, (1996)

[22] S. Tabachnikov, Billiards, *Panoramas et Synthèse*, **1**, Soc. Math. France, Paris, (1995)

[23] C. Liverani, M. Martens *Convergence to equilibrium for intermittent symplectic maps*, preprint

[24] M. Hénon, A two-dimensional mapping with a strange attractor, *Communications in Mathematical Physics*, **50**, 69–77, (1976)

[25] Polimilis, Servizi, Skokos, Turchetti, Vrahatis, Topological degree theory and local analysis of area preserving maps, *Chaos*, **13**, 94 (2003)