Exam 1 will cover §5.6–8, §6.1–6.3, §6.6 and §8.1. Equivalently, Homework 2a–5b.

Exercise 1. Evaluate the following limits. Clearly indicate whether or not they converge and why.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin^{-1}(x^2)}{\sinh^2(x)}$.

(b) $\displaystyle\lim_{x\to\infty} (xe^{1/x} - x^2)$.

(c) $\displaystyle\lim_{x\to\infty} (1+ax)^{1/x}$ where $a > 0$.

(d) $\displaystyle\lim_{x\to 0^+} x^p(\ln x)^2$ where $p > 0$ is a constant.

(e) $\displaystyle\lim_{x\to 0} \frac{\lambda - \lambda\cos\lambda x}{\mu - \mu e^{\mu x}}$ where λ and μ are non-zero constants.

Exercise 2. Evaluate the following integrals:

(a) $\displaystyle\int x\cosh x\, dx$.

(b) $\displaystyle\int x^k \ln x\, dx$ where $k \geq 1$ is an integer.

(c) $\displaystyle\int e^{3x+e^x}\, dx$.

(d) $\displaystyle\int x\tan^{-1}(3x)\, dx$.

(e) $\displaystyle\int \frac{1}{(x-1)^2(x^2+4)}\, dx$.

(f) $\displaystyle\int \frac{K}{P^2(K-P)}\, dP$ where $K > 0$ is a constant.

(g) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\, dx$.

(h) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}$.

Exercise 3. Determine if the following improper integrals converge or diverge. Evaluate any that converge.

(a) $\displaystyle\int_1^3 \frac{1}{\sqrt{3-x}}\,dx.$

(b) $\displaystyle\int_0^\infty \frac{mg}{(x+c)^2}\,dx$ where m, g and c are positive constants.

(c) $\displaystyle\int_0^1 \frac{(\ln x)^k}{x}\,dx$ where $k \geq 1$ is an integer.

(d) $\displaystyle\int_0^\infty x^3 e^{-x^2}\,dx.$

(e) $\displaystyle\int_{-2}^2 \frac{1}{x^2-1}\,dx.$

Exercise 4. Determine if the following improper integrals converge or diverge. Clearly state any test(s) you use.

(a) $\displaystyle\int_1^\infty \frac{2+\sin x}{x^2+\frac{1}{2}x}\,dx.$

(b) $\displaystyle\int_2^\infty \frac{x-\cos x}{\sqrt{x^\pi - x - 1}}\,dx.$

(c) $\displaystyle\int_2^\infty \frac{\ln x}{\sqrt{x-1}}\,dx.$

(d) $\displaystyle\int_e^\infty \frac{1}{x\ln x}\,dx.$

Exercise 5. The velocity of a particle in the Large Hadron Collider is given by

$$v(t) = \frac{100}{m}\tan^{-1}\left(\frac{t}{mg}\right)$$

where t is time in seconds, m is the mass of the particle, and g is the gravitational constant.

(a) Show that the acceleration of the particle is never 0.

(b) Find the terminal velocity of the particle, that is find $\lim_{t\to\infty} v(t)$.

(c) What happens to the velocity as the mass of the particle tends to zero? (Compute $\lim_{m\to 0+} v(t)$. Note your answer will be a function of t).

(d) How far does a particle move in the first 10 seconds?

Exercise 6. Consider a function f defined on an interval $[a, b]$. We call f a *probability density* on $[a, b]$ if $f(x) \geq 0$ for every x with $a \leq x \leq b$, and

$$\int_a^b f(x)\,dx = 1.$$

We define E, the *expected value* of f on $[a, b]$, by

$$E = \int_a^b x f(x)\,dx.$$

Let $f(x) = k \cos x$ where k is a constant.

(a) Find k so that $f(x)$ is a probability density on $[0, \frac{\pi}{2}]$.

(b) Compute the expected value of $f(x)$ on $[0, \frac{\pi}{2}]$.

Exercise 7. Consider the sequence $a_n = \dfrac{3^n}{n!\sqrt{n+1}}$.

(a) Find an expression for the general term of na_n.

(b) Compute $\lim\limits_{n\to\infty} a_n$.

(c) Compute $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$.