# Gian-Carlo Rota (1932–1999)

*Edwin F. Beschler, David A. Buchsbaum, Jacob T. Schwartz, Richard P. Stanley, Brian D. Taylor, and Michael Waterman*

***Editor's Note.** Gian-Carlo Rota—combinatorialist, probabilist, phenomenologist, philosopher, editor, premier lecturer, thesis advisor to dozens—died in his sleep about April 18, 1999. Born in Vigevano, Italy, on April 27, 1932, he came to the United States in 1950 and obtained a Ph.D. degree in mathematics under Jacob T. Schwartz from Yale University in 1956. He was a postdoctoral research fellow at the Courant Institute at New York University in 1956-57 and a Benjamin Peirce Instructor at Harvard University in 1957-59. In 1959 he took a faculty position at the Massachusetts Institute of Technology (MIT), where he remained—except for a stay in 1965-67 at the Rockefeller University—until his death.*

*He had a number of visiting faculty positions—among them at the University of Colorado, the University of Florida, the University of Southern California, the University of Paris VII, the University of Buenos Aires, the University of Strasbourg, and the Scuola Normale Superiore in Pisa. He was a long-time consultant for the Los Alamos National Laboratory and was a Director's Office Fellow there starting in 1971.*

*He was the founding editor of the* Journal of Combinatorial Theory *(1966),* Advances in Mathematics *(1967), and* Advances in Applied Mathematics *(1979), and he remained as editor of all these journals until his death. He was, in addition, editor of several book series and served on the editorial boards of a dozen other journals at various times.*

*He had more than forty doctoral students (see sidebar) and was a consummate lecturer, eagerly sought as a guest lecturer around the world. In fact, his death was discovered on a Monday when he failed to arrive for a series of three guest lectures in Philadelphia. The AMS honored his extraordinary talents by choosing him as its Colloquium Lecturer for 1998.*

*He was a fellow of the American Academy of Arts and Sciences, a member of the National Academy of Sciences of the USA, the 1988 winner of the AMS Steele Prize for a Seminal Contribution to Research, and an invited lecturer at the International Congress of Mathematicians in Helsinki in 1978.*

## *Jacob T. Schwartz*

In the recollections of Yale included in his *Indiscrete Thoughts*, Gian-Carlo mentions the 1954 functional analysis seminar at which we met. This seminar, in memory yet green, was organized by Nelson Dunford and addressed by an outstanding group of young researchers, including John Wermer, William Bade, Robert Bartle, and Henry Helson. It was a high point of early functional analysis at Yale: interesting new results were presented by their discoverers almost every week. Even though he was then in his first graduate year, Gian-Carlo's talents became apparent at once, and he was immediately recruited, along with Bartle, Bade, and me, as a junior member of the group then working on the "Dunford project" that subsequently became *Linear Operators*. In 1957 Dunford decided to take a year's sabbatical at NYU (New York University). Support from the ONR (Office of Naval Research) being available, Rota and I were able to tag along for this one-year visit, which turned into a three-year stay for Gianco and forty-two years for me. Though only a few years older than Gian-Carlo, I assumed for the first time the ponderous dignity of thesis advisor. This was a period of youthful friendship, punctuated by frequent risotto Milanese garnished by Asti Spumante at Gian-Carlo's bottom-price, sixth-floor walkup apartment in what was then something of a Mafia-dominated slum just south of NYU but which has since been gentrified.

The Dunford connection, the general prestige of functional analysis at Yale, the ONR contract, and the pattern of my own interests led Gian-Carlo to an initial specialization in functional analysis. His dissertation, "Extension theory of differential operators I", appeared in *Communications in Pure and Applied Mathematics* in 1958. (Not untypically for papers whose titles bear the fatal Roman digit "I", there never was a "II".) A series of other papers on operator theory followed: "Note on the invariant subspaces of linear operators", "On the spectra of singular boundary value problems", "On models of linear operators", "On the eigenvalues of positive operators"—all in the period 1958-61. But already

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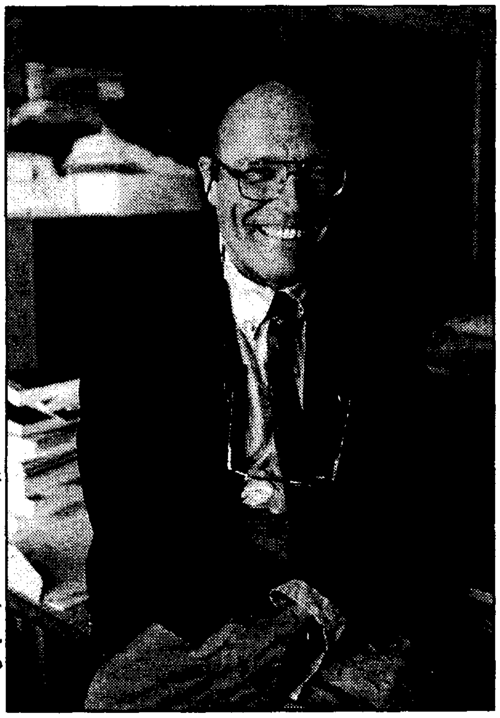
in his two 1959-60 papers on Reynolds operators, Gianco had stepped onto the bridge of ergodic theory, often a highly combinatorial corner of analysis, that carried him into purer combinatorics. The first of his landmark papers in his new area, "On the foundations of combinatorial theory I", dates from 1964. (Here the "I" had many following integers.) To its own good fortune, combinatorics (and later philosophy) had captured him completely. His distinguished book with Garrett Birkhoff on *Ordinary Differential Equations* was a farewell to analysis. Unlike many, he had the courage and curiosity to move on, with consequences which others will speak of in this memorial.



Photograph by Donna Coveney/MIT.

**Gian-Carlo Rota**

Our lifelong friendship made me not only an admirer of the depth, scholarship, and sheer energy of his mathematical work (and of his ceaseless activities as an editorial entrepreneur on behalf of mathematics) but one in awe of his status as the ultimate relaxed sophisticate. Gian-Carlo could always state with easy authority not only the current standing of all the top restaurants in Paris, Rome, Boston, and Milan but where to get the hottest and best chili in New Mexico and even what local hash house had the most unexpected culinary surprises. I shall miss him greatly.

## *Michael Waterman*

"I have never known Stan Ulam to last longer than ten minutes of anyone else's lecture," Gian-Carlo Rota wrote, mimicking the famous and irreverent first sentence of Chapter 1 of James Watson's *The Double Helix*. Nevertheless, Rota tells of meeting Ulam in New York City in 1964 when Mark Kac prevailed on Ulam to attend a lecture of Rota's; Ulam made it through twenty minutes before bolting, and one need not be an expert on extreme value distributions to know that was a rare event. Kac and Ulam were great mathematicians born in Poland who each came to the U.S. at the beginning of World War II. They both had broad European educations and did not observe boundaries between mathematics and other sciences, let alone between mathematical subfields. It is natural that they each took up with Rota with his multiple languages and wide-ranging intellect.

Soon after New York, Rota was invited to Los Alamos National Laboratory, known as the Lab, the Hill, the Mesa, and most famously as Santa Fe Box 1663 during the war, when brilliant men of science, physics especially, worked feverishly to create the atomic bomb. By 1964 Stan Ulam was one of those who retained a regular association with Los Alamos. The Lab, at 7,400 feet, is on a mesa top in ponderosa pines just above the pinion-juniper zone. The crisp clear air has a distinctive incense of cedars, pine, ozone, and sun-baked tuff, and one can see for tens of miles. In Santa Fe, thirty-five miles distant, is Native American and Caucasian culture, with good restaurants and art galleries. This exotic high-altitude, sun-drenched locale captivated Rota, but surely it was Ulam who kept him coming back. One can find each of them writing about the other in several places, such as Ulam's *Adventures of a Mathematician* and Rota's *Indiscrete Thoughts*.

Rota soon became part of Los Alamos. He gave lectures that were deeply informative, polished works of art that made him known throughout the Lab. The topics were wide-ranging: differential equations, ergodic theory, nonstandard analysis, probability, and of course, combinatorics. I attended the series on nonstandard analysis, and it was the equivalent of a course with an approach that had not yet appeared in print. These notes exist as a Los Alamos report. Over the years Rota helped organize several conferences: History of Computing in the Twentieth Century (1979), Science and the Information Onslaught (1981), and Frontiers of Combinatorics (1998). He was made a consultant of the Lab in 1966 and Director's Office Fellow in 1971. When asked what he did, he said, "I wish I knew. I manage to snoop around, and every once in a while I pop into the director's office and have a chat with him." (Rota loved and absorbed gossip about mathematicians and scientists!) Director's Fellows could come whenever they chose and could stay as long as they wished. For Rota this meant at least a week in January (Rota hated Boston winters even more than New Mexico winters!) and a month in the summer. As a Fellow he quickly became involved with high-level Lab politics. In the

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late 1970s he was at a dinner party in my home when a new director was being chosen. He received so many lengthy telephone calls that I feared he would not get enough to eat.

Other than Ulam his closest collaboration at Los Alamos was with Nick Metropolis, an elegant man who had a long association with the Lab. Metropolis was educated as a physicist at the University of Chicago, where he took many mathematics courses. He had a distinguished career as a physicist and pioneer in the development of modern computers; he passed away on October 17, 1999. In wartime Los Alamos he and Feynman repaired Marchant manual calculators to the disapproval of Hans Bethe. In 1945, at von Neuman's invitation, Metropolis began to work with the ENIAC, and in 1947 he started a computer research group at Los Alamos that produced the remarkable series of MANIAC computers. At Los Alamos I used the MANIAC II, which was a joy. The MANIAC III, based on significance arithmetic, was built at the University of Chicago. For the last twenty years of his career, Metropolis worked in mathematics, much of it with Rota. One of their major contributions was in using concepts created for computers such as binary representation of numbers and "carry" operations and applying them to the foundations of real numbers. They brought forward a new idea, distinct from the usual Peano and Dedekind construction. There are four papers on those topics. They also studied the lattice of the faces of the $n$-cube, and they gave an explicit decomposition of the lattice into a minimal number of chains of lattice faces. And they had the good fortune to discover a fact missed by all the early workers in symmetric functions: that every function in three variables is uniquely expressible as a sum of a symmetric function, a skew symmetric function, and a cycle symmetric function [14]. The underlying idea was extended to $n$ variables in several papers, including an introduction of two new classes of symmetric functions.

Innumerable people gave Rota private lectures, which he carefully inscribed in one of his heavy notebooks. "It's my job," he would say with pride. It was much more than a passive activity; here is an example of one of those exchanges. Jim Louck, a Los Alamos physicist, listened to Rota lecture in the late 1960s on the set $M_{m,n}(\alpha,\beta)$ of $m \times n$ matrices with nonnegative integer entries having vector row sum $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and vector column sum $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ with $\sum_i \alpha_i = \sum_j \beta_j = N$. During the lecture Rota remarked that he knew of no physical applications of the set $M_{m,n}(\alpha,\beta)$. During this same period physicists were very active in developing explicit unitary irreducible representations of the general unitary group for physical applications, and one of the popular physical models for this theory was a collection of independent harmonic oscillators as realized through the Heisenberg algebra of creation and annihilation operators. Many physical problems can be modeled in this way because of the generality of the property that quantum states can be created from the ground state by the action of the creation operators, the ground state itself being defined by its annihilation by the action of the annihilation operators. The simplest of such

**Ph.D. Students of Gian-Carlo Rota**

Peter L. Duren (1960)
Richard M. Moroney (1961)
John Markham Freeman (1963)
Henry Crapo (1964)
Norton Starr (1964)
Martin Billik (1965)
Nick Metas (1966)
Stephen Grossberg (1967)
Patrick Eugene O'Neil (1969)
Thomas H. Brylawski (1970)
Thorkell Helgason (1971)
Kenneth Lamar Lange (1971)
Richard Stanley (1971)
Walter John Whiteley (1971)
Steve Fisk (1972)
Neil White (1972)
Sergio L. de Braganca (1973)
Peter Doubilet (1973)
Stephen Tanny (1973)
Kenneth Holladay (1975)
Hien Quang Nguyen (1975)
Joseph PeeSin Kung (1978)
Joel A. Stein (1980)
Mark Haiman (1984)
Zambrano Oscar Nava (1986)
William R. Schmitt (1986)
Domenico Senato (1988)
Daniel Loeb (1989)
Rosa Q. Huang (1990)
William Y. C. Chen (1991)
Julia S. Yang (1991)
Shunhuang Peter Zhuang (1991)
Richard Ehrenborg (1993)
Susana Mondschein (1993)
Joe Oliviera (1993)
Mike Hawrylycz (1994)
Dan Klain (1994)
Wendy Chan (1995)
Jozsef Losonczy (1995)
Brian Taylor (1997)
Catherine Huafei Yan (1997)
Matteo Mainetti (1998)

The *Notices* is grateful to Linda Okun for helping to assemble the above list. In addition, Rota's records indicate that Robert McCabe and O. Murru were his doctoral students; the *Notices* has no further information about doctoral degrees for these people.

models is a system of $mn$ identical 1-dimensional harmonic oscillators, which may also be viewed as $n$ oscillators, each of which is an $m$-dimensional isotropic oscillator in Euclidean $m$-space. If the total energy of such a system is $N$ energy quanta, and the number of these quanta associated with the motion of all $n$ oscillators in the $i$-th direction is the nonnegative integer $\alpha_i$, while the number of quanta associated with the $j$-th component oscillator of each of the $m$-dimensional oscillators is the nonnegative integer $\beta_j$, then $\alpha$ and $\beta$ are respectively the row and column sums of the $m \times n$ matrix $(a_{i,j})_{1 \le i \le m, 1 \le j \le n}$. Here $a_{i,j}$ is the number of energy quanta associated with oscillator $(i, j)$ in the set of $mn$ 1-dimensional oscillators. In this way the set $M_{m,n}(\alpha, \beta)$ enters almost universally into the physical theory of quantum systems. It was this observation, which emerged after Rota's lecture, that led to thirty years of interactions between Louck, L. C. Biedenharn, and Rota. Louck and Biedenharn gave many informal presentations on the tensor operator theory they had created. "Rota never really bought it," Louck told me, and he and Biedenharn wrote no joint papers with Rota. But when Rota's student W. Y. C. Chen came to the Lab, Rota said, "Go to Los Alamos and look up Jim Louck. He's a gold mine for mathematicians." Chen was delighted to find this to be true, and he and Louck went on to mine that rich ore in an ongoing series of papers.

Biology is another area that Rota helped along, although he did not entirely buy into biology either. (I refer especially to his doubts about Darwin's theory of evolution.) When my first paper on sequence matching was rejected, Rota placed it in *Advances in Mathematics*. It is still being quoted, and I (along with Bill Beyer and Temple Smith) have Rota to thank for the timely appearance of that paper. "There are so few people working on those problems," he said many years later. A few years ago David Torney began to give Rota lectures about his work that arose in classification of DNA sequences. The results were an elegant joint paper on probability set functions and help in organizing a conference.

It is of course impossible to list all of Rota's interactions. Some of the most unexpected (to me at least) are those relating to Rota's interest in philosophy. David Sharp is a multitalented mathematical scientist who shared Rota's passion for philosophy. Their dialogue "Mathematics, Philosophy, and Artificial Intelligence" in *Los Alamos Science*, No. 12, is fascinating. Rota had a tremendous impact on students who took his philosophy classes. Mark Ettinger and David McComas are two of those MIT physics students who went to the Lab because of Rota. McComas went on to become director of the Center for Space Science and Exploration.

Rota served on the Advisory Board for Non-Proliferation and International Security (there is a Lab division of that name), but it is next to impossible to learn any details. While he did write short classified reports on national security issues, they are not available to "unclassified eyes". At Los Alamos this activity, just as with almost everything else there, has gone under various names, but it is often called "the Spook Shop". It will be many years before much more is known. For example, I am curious about whether Rota's relationship with the Spook Shop or the National Security Agency came first.

Let me return to Rota's vital connection with Ulam. The fascinating essay "The Lost Café", the final version of which appeared as Chapter VI of *Indiscrete Thoughts*, is a sketch of Ulam's life, with details of his health, work habits, mathematical abilities, and state of mind; and some of it was far from complimentary. "The Lost Café" was controversial at the Lab, with the Ulam family, and elsewhere. "It's a scandal," Rota told me with evident satisfaction. The editor Palombi writes, "...one does not say this kind of thing about great men." I can almost hear Rota use those exact words! I believe "The Lost Café" is filled with respect and love, but it is radical. Among other things Rota writes that Ulam was lazy. I like a remark Carson Mark made at a reception at Los Alamos, "Ulam was thinking all the time," and I doubt that Rota would have disagreed. I believe Ulam's widow remains bitter about the article and has not forgiven Rota, not even after his death. And at Los Alamos and elsewhere there are resentments, grudges, and judgments; although Rota would say, "We should tell it like it is," I have not space to list them here.

In winter, snowstorms come to northern New Mexico, and the following day dawns clear with deep-blue sky and subzero temperatures. Every snow crystal reflects light, and the vast landscape is dazzling. Rota planned a Los Alamos article entitled "The Desert Is Covered with Snow." It too would have dazzled and, just as likely, shocked and upset some. We can never know all that we have lost, what Gian-Carlo Rota would have revealed to us about mathematics and about ourselves.

## *Edwin F. Beschler*

Gian-Carlo's involvement with publishing was complex and intense. His motivations, as I perceived them, were multiple and intertwined: mathematical, scientific, intellectual, sociological, political. He was at various times author, editor, consultant, or advisor to many publishers, some of the best known to this audience being

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Academic Press, Addison-Wesley, Birkhäuser, Harper and Row, and Springer-Verlag. Some of the individuals who benefited from Gian-Carlo's insights and initiative, who epitomize the close ties between him and the publishing industry, and who are well known to many mathematicians attended the memorial meeting at MIT on April 30, 1999, and allowed me to speak on their behalf: Klaus Peters, who supported Gian-Carlo as founding editor of the archival series Contemporary Mathematicians, the collected works of leading mathematicians of our times; Ann Kostant, who carries on the administration of that series and was my coeditor in publication of Gian-Carlo's award-winning book *Indiscrete Thoughts*; Peter Renz, a mathematician and publisher who produced a revision of the influential book *Discrete Thoughts* (coauthored by Mark Kac, Gian-Carlo Rota, and Jacob Schwartz) and who worked with him in various capacities both editorial and mathematical.

We few are only a small percentage of the publishing professionals who were proud to be part of Gian-Carlo's editorial network, a group that included people in the American Mathematical Society and such institutional publishers as MIT Press and Cambridge University Press, with, in particular, the highly acclaimed *Encyclopedia of Mathematics* that Gian-Carlo edited over many years. I know they all join me in honoring him as a brilliant writer, sagacious editor, incisive critic, and—in addition to all this—a colleague and friend.

My relationship with Gian-Carlo began in the 1960s, when I was learning my trade as mathematics editor at Academic Press, and extended to Birkhäuser in the late 1980s. Gian-Carlo was one of a very small number of close advisors during those years of unprecedented growth in scientific publications, in particular in mathematical books and journals, and even more particularly in the establishment of "specialized mathematical journals", a term we can use with some amusement in the 1990s when considering their titles: the *Journal of Algebra*, the *Journal of Differential Equations*, the *Journal of Number Theory*, and so on.

Our first enterprise together, which should not be surprising, was the *Journal of Combinatorial Theory*, a publication whose time had come but whose birth pangs reflected the divided nature of the field itself. The editorial structure of JCT was a delicate coalition, given the chaos and lack of direction of the discipline. An even more delicate task arose in the eventual division into Parts A and B, a bit of intellectual surgery that saved the journal from imminent collapse and that was an unapologetically political move, made possible by Gian-Carlo's commanding position in the field, sense of ongoing mathematical history, and steadfast belief in combinatorics.

Gian-Carlo was also the creative force behind the *Journal of Functional Analysis*. He not only suggested it but guided and advised me through the intricate process of identifying, convincing, and bringing together the team of Irving Segal, Ralph Phillips, and Paul Malliavin. The continued success of these journals and the numerous others in which he played an advisory role is ample testimony to his vision.

In those days Academic Press had a faltering publication called *Advances in Mathematics*, which was to have been a yearly volume of expository papers in mathematics modeled on a successful formula of such publications in the physical and biological sciences. The model was not working, due mainly to the long-standing difficulty of writing expository articles in mathematics. I asked Gian-Carlo's help, and he offered to take responsibility for it, contingent on our transforming it into a journal and giving him complete editorial license to publish papers on any topic and of any length he chose, assisted by an editorial board and any necessary refereeing but dependent almost 100 percent on his personal judgment. The publication as it now exists is a successful journal with one of the highest prestige factors in the mathematics literature—and a wicked reputation for pithy book reviews—backed up by the later *Advances in Applied Mathematics*. It was Gian-Carlo's particular genius that he could transform an intractable set of dynamics sheerly by force of his ability to recognize superior work and his willingness to "break the rules" in the interests of publishing it expeditiously, thus furthering mathematics. He was a communicator of the highest degree, and he believed in the power of the written word and the necessity—even to proliferation—of publishing thoughts, ideas, and information.

In reflecting on my relationship and friendship with Gian-Carlo—not always easy in the 1960s, but rich and comfortable in the 1990s—I belatedly recognized a previously unarticulated erroneous assumption I once lived with about the nature of his inner forces. His role as "kingmaker" in constructing editorial boards seemed to me Machiavellian, his concept of priority in the publishing queue often looked to me quixotic, his directions and demands sometimes came across tinged with a dictatorial flavor. And this I imagined grew from an ego that needed constant nourishment and that was a leading motivation for his intense and personal involvement in so many editorial and publishing initiatives. In the leisure of retirement, from the perspective of reexamined years, I have come to realize fully how wrong I had been to attribute so much to that undeniably present component of Gian-Carlo's persona. I now appreciate more richly how much he was motivated by a desire for something he simply believed was crucial for mathematics—expansion of the literature in the hands of competent and dedicated people. I profoundly wish that I could have the opportunity to tell him

### The Foundations Papers

These are the ten papers published by Rota, all with the title "On the foundations of combinatorial theory". All but the first have coauthors. Below are the coauthors, subtitle, and year for each of the ten.


#### References

[I] Theory of Möbius functions (1964).

[II] Henry Crapo, Combinatorial geometries (1970).

[III] Ronald Mullin, Theory of binomial enumeration (1970).

[IV] Jay Goldman, Finite vector spaces and Eulerian generating functions (1970).

[V] George Andrews, Eulerian differential operators (1971).

[VI] Peter Doubilet and Richard Stanley, The idea of generating function (1972).

[VII] Peter Doubilet, Symmetric functions through the theory of distribution and occupancy (1972).

[VIII] David Kahaner and Andrew Odlyzko, Finite operator calculus (1973).

[IX] Peter Doubilet and Joel Stein, Combinatorial methods in invariant theory (1974).

[X] Flavio Bonetti, Domenico Senato, and Antonnieta M. Venezia, A categorical setting for symmetric functions (1992).


of this insight into my youthful misjudgment, to revisit the days of our stimulating, and sometimes stormy, dialogues, and to acknowledge to him my mature understanding of the complementary roles we played, as well as to tell him how much he was appreciated and how much he will be missed.

Mathematicians and philosophers share with poets a critical dependence on the written word. Structure of language, style of discourse, nuance of expression are the tools with which their ideas are made manifest, given form, and communicated. Gian-Carlo Rota was a mathematician and a philosopher, and the richness of his writing in these fields was known to both communities. I like also to think of him as a poet—not in a formal sense, since to the best of my knowledge he never wrote a poem—but in the larger sense of a person who expresses himself with imaginative power and beauty of thought, even when many of these thoughts were sardonic reflections on people, ideas, institutions, and the general condition of humanity. His sense of humor was biting and deep—and full of truth. And his modes of expression poetic in a fundamental sense of the word.

## *Richard P. Stanley*

Hermann Weyl has described Cayley's development of invariant theory as "[coming] into existence somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley's jovian head." A similar statement could be made about the work of Gian-Carlo Rota on the foundations of combinatorics. Though led into combinatorics by his work on functional analysis (as briefly explained by Jacob Schwartz in his segment above), Rota's work on combinatorics was from the beginning a completely fresh combination of innovation and synthesis. His first paper in this area had the audacious title "On the foundations of combinatorial theory, I. Theory of Möbius functions". The title was by no means pretense; it was the first in a series of seminal *Foundations* papers that lifted the subject of combinatorics from disrepute to eminent respectability.


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*Foundations I* established partially ordered sets (posets) as a fundamental concept in combinatorics. Its tremendous influence remains unabated to this day. The primary object of study of *Foundations I* is the Möbius function of a poset (with suitable finiteness properties). It is the function $\mu : \mathcal{I}(P) \to \mathbb{Z}$, where

$$\mathcal{I}(P) = \{(x, y) \in P \times P : x \le y\},$$

defined recursively by

$$\mu(x,x) = 1, \quad \text{for all } x \in P$$
$$\sum_{t : x \le t \le y} \mu(x,t) = 0, \quad \text{if } x < y \text{ in } P.$$

Rota was the first to realize that the Möbius function was a fundamental invariant of posets and not just an enumerative tool. Of special concern is the Möbius inversion formula for posets, a vast generalization of the classical Inclusion-Exclusion Formula and the classical Möbius inversion formula of number theory. It asserts that if $f$ and $g$ are functions from $P$ to some abelian group related by

$$f(y) = \sum_{x \le y} g(x)$$

(where it is assumed that this sum has finitely many terms for all $y \in P$), then

$$g(y) = \sum_{x \le y} f(x)\mu(x,y).$$

As Rota points out, the first coherent version of the Möbius inversion formula for posets is due to Louis Weisner and later, independently, to Philip Hall. Rota remarks that "strangely enough, however, these authors did not pursue the combinatorial implications of their work; nor was an attempt made to systematically investigate the properties of Möbius functions." It took an exceptional imagination to carry out exactly such a unification and systemization, as well as great courage to proceed in such an unfashionable direction.

*Foundations I* planted many seeds that have produced bounteous fruit. If $c_i(x, y)$ is the number of chains $x < x_1 < \cdots < x_{i+1} < y$ in $P$ between $x$ and $y$, then a formula of Philip Hall asserts that

$$\mu(x,y) = \sum_{i \geq -1} (-1)^i c_i(x,y).$$

This formula shows that $\mu(x,y)$ is the (reduced) Euler characteristic of a certain abstract simplicial complex, the complex of chains between $x$ and $y$. Moreover, if the closed interval $[x,y]$ is a lattice (a poset for which any pair of elements have a least upper bound and greatest lower bound) and $A$ is the set of atoms (minimal elements of the open interval $(x,y)$) of $[x,y]$, then the subsets of $A$ whose least upper bound is not $y$ form another simplicial complex $\Delta(A)$. A formula of Louis Weisner can be interpreted as saying that $\mu(x,y)$ is the reduced Euler characteristic of $\Delta(A)$. The realization of Rota that the Möbius function of a lattice could be interpreted as an Euler characteristic in two different ways immediately raises a host of topological questions and gave rise to the subject of *topological combinatorics*, which has now achieved a high level of sophistication. See, for example, the recent survey [2].

The discussion in *Foundations I* concerning geometric lattices played a significant role in the revitalization of matroid theory, with many further contributions appearing in *Foundations II* and its subsequent elaboration [7], both written jointly with Rota's student Henry Crapo. The concept of matroid, originally due to Hassler Whitney, is an abstraction of linear algebra: one specifies that certain subsets of a set $S$ are "independent" (an abstraction of linear independence). The only condition on the independent sets is that for any subset $T$ of $S$ all maximal independent subsets of $T$ have the same cardinality. Again, Rota was exactly on target in realizing intuitively the immense contributions that matroid theory could make to combinatorics and other branches of mathematics. For instance, deep connections between matroid theory, topology, and algebraic geometry pervade the two books [3, 15].

*Foundations III-VIII* are concerned primarily with enumerative combinatorics and played an important role in the subsequent development of this area. *Foundations III* (with R. Mullin) and *VIII* (with D. Kahaner and A. Odlyzko) are concerned with "finite operator calculus", an exceptionally elegant recasting and generalization, based on linear algebra, of the nineteenth-century subject of "umbral calculus". In particular, the formal similarities between the differentiation and difference operators $d/dx$ and $\Delta$ are demystified and vastly extended. As with the other *Foundations* papers, the finite operator papers have stimulated much further research.

*Foundations VI*, entitled "The idea of generating function", is a direct sequel to *Foundations I* and shows how a coherent theory of generating functions can be based on the incidence algebra of a poset. Why, for instance, does one encounter in enumerative combinatorics generating functions of the type

$$\sum_{n\geq 0} f(n)x^n, \quad \sum_{n\geq 0} f(n)\frac{x^n}{n!}, \quad \text{and} \quad \sum_{n\geq 0} f(n)\frac{x^n}{2^{\binom{n}{2}}n!},$$

but never

$$\sum_{n\geq 0} f(n)\frac{x^n}{n^2+1} \quad \text{or} \quad \sum_{n\geq 0} f(n)\frac{x^n}{(n+1)^n}?$$

*Foundations VI* is the only paper I ever wrote jointly with Gian-Carlo (and also Peter Doubilet), a priceless experience that I regret can never be repeated. *Foundations VII* (written jointly with Peter Doubilet) was devoted to enumerative aspects of symmetric functions and anticipated the prodigious role that symmetric functions would later play in combinatorics. (See, for instance, Chapter 7 of [20].) Rota returned to symmetric functions in *Foundations X*, the last of the *Foundations* papers.

*Foundations IV* and *V*, written jointly with Jay Goldman and George Andrews respectively, foresaw what is now a thriving cottage industry within mathematics and mathematical physics—the theory of $q$-analogues (or, in more stylish terminology, "quantum" mathematics). In general, if $A_q$ is the $q$-analogue of some object $A$, then in some sense it should be true that $A = A_1$ or $A = \lim_{q\to 1} A_q$. The theory of $q$-analogues began in the work of Euler and Gauss with the lowly factorials and binomial coefficients and now extends to such objects as the Gamma function, the Lagrange inversion formula, and a host of algebraic structures typified by semisimple Lie algebras (via the theory of quantum groups).

The remaining *Foundations* paper to be discussed is *IX* (with P. Doubilet and J. Stein), entitled "Combinatorial methods in invariant theory". It was the first of over twenty papers by Rota and his collaborators as part of a monumental effort to bring the moribund subject of classical invariant theory into mainstream mathematics. Further discussion of this aspect of Rota's work appears in the segment by David Buchsbaum and Brian Taylor.

All but the first *Foundations* paper were jointly written. In fact, twelve different persons served as collaborators for these nine papers. For Rota mathematics was a social endeavor, and he generously shared both his time and his creativity with anyone who partook in his enthusiasm for beautiful mathematics. Combinatorics, and indeed all of mathematics, has become a poorer subject with the passing of such a singular leader.

## *David A. Buchsbaum*

Although I met Gian-Carlo Rota in the late 1950s, it was not until the summer of 1990, when we met by chance in Rome, that we decided to get together fairly regularly once we were back in the Boston area, for it was in Rome that we discovered that in our own very different ways we were interested in and working on very closely related problems. Perhaps this should not have been too surprising, given that in the late 1970s his paper with Doubilet and Stein [10] had given tremendous impetus to the work that Akin, Weyman, and I were engaged in. And Gian-Carlo had always had a soft spot for homological algebra (hence, in part, the name of his long-running seminar, "Syzygy Street"). In addition to these affinities, we both shared a love for what we liked to call multilinear algebra, although many might say that considering Hopf algebras, superalgebras, homotopy, and cohomology theory as "multilinear algebra" is stretching the meaning of the term a bit.

Working together fairly regularly from the fall of 1990 until Rota's death last April, we got to know each other pretty well. It was during this period that I experienced firsthand his gentleness, kindness, intellect, and passion for mathematics of all kinds. I also learned to appreciate his work on the straightening formula and invariant theory. It was in connection with the straightening formula that our trajectories first significantly intersected.

## *David A. Buchsbaum and Brian D. Taylor*

This segment of the article contains a description of some of Gian-Carlo Rota's work on the straightening formula and invariant theory.

### The Straightening Formula and the First and Second Fundamental Theorems

In *The Classical Groups* Hermann Weyl considered vector invariants of the special linear, orthogonal, and symplectic groups. He described explicitly the generators of the various rings of invariants along with the relations between them. These descriptions constitute the first and second fundamental theorems of invariant theory. Since Weyl considered only fields of characteristic 0, it was natural to ask how much of this work remains valid for fields of arbitrary characteristic. For the special linear group, J.-I. Igusa proved the appropriate theorems in 1954. Then, in the early 1970s, Rota and various collaborators came up with an extraordinarily simple and powerful way to achieve the same results [8, 10]. Their method introduced several fundamental tools for working with polynomials in the entries of a matrix.

First, they found a symbolism for products of minors, which they called *bitableaux*. In their notation the determinant of the minor indexed by rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$ of a matrix $X$ is written as a *biproduct*, $(i_1 \dots i_k \mid j_1 \dots j_k)$, multiplied by a sign factor of $(-1)^{\binom{k}{2}}$. They described a product of minors by stacking biproducts vertically. The product $(3\ 1 \mid 3\ 4) \cdot (2 \mid 1)$ is written as the bitableau $\left(\begin{array}{l|l} 3\ 1 & 3\ 4 \\ 2 & 1 \end{array}\right)$. This is, up to sign, the product $\begin{vmatrix} X_{3,3} & X_{3,4} \\ X_{1,3} & X_{1,4} \end{vmatrix} \cdot X_{2,1}$ of determinants. The sign appearing in front of each minor is part of a system of sign rules that Rota and his collaborators established to simplify calculations with biproducts. For ease of definition, however, we have stripped the remaining rules from the bitableaux appearing in this presentation.

Second, using this symbolism, they introduced the idea of *standard bitableaux*, namely, those bitableaux $(D \mid E)$ whose component Young diagrams, $D$ and $E$, strictly increase across rows and weakly increase down columns. They then proved that, assuming the entries of $X$ to be algebraically independent, these bitableaux form a basis of the polynomial algebra over the integers generated by the entries of $X$.

In the above example the bitableau was built out of nonstandard Young diagrams, but $\left(\begin{array}{l|l} 3\ 1 & 3\ 4 \\ 2 & 1 \end{array}\right) = -\left(\begin{array}{l|l} 1\ 3 & 3\ 4 \\ 2 & 1 \end{array}\right)$ since we are effectively just switching two rows in a determinant. Nevertheless, the right-hand Young diagram remains nonstandard. Applying the identity

$$\left(\begin{array}{l|l} 1\ 3 & 3\ 4 \\ 2 & 1 \end{array}\right) = -\left(\begin{array}{l|l} 1\ 3 & 1\ 4 \\ 2 & 3 \end{array}\right) + \left(\begin{array}{l|l} 1\ 3 & 1\ 3 \\ 2 & 4 \end{array}\right) + (1\ 2\ 3 \mid 1\ 3\ 4)$$

expresses the original bitableau as a linear combination of standard ones.

The expression of a polynomial as an integer linear combination of standard bitableaux was given by repeated application of identities similar to those in the preceding example, and this algorithm was referred to as *straightening*. In the introduction to [8] Rota said that the straightening algorithm was the result of a train of thought "developed most notably by Alfred Young, and the Scottish invariant theorists."

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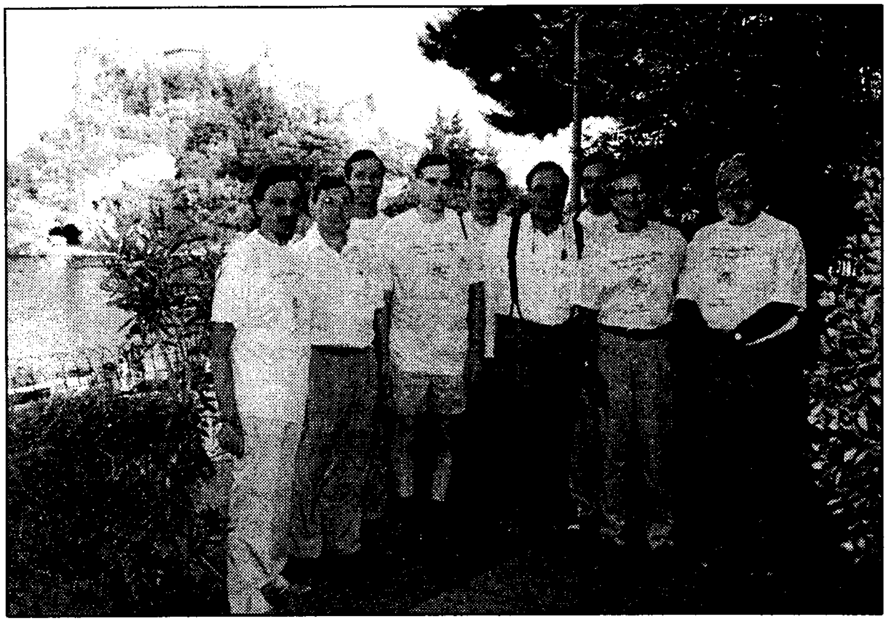
In [10] the authors use straightening to provide the first characteristic-free proof of the first fundamental theorem for an $m \times d$ matrix under the action of invertible $d \times d$ matrices. They prove that the (quasi) invariant polynomials in the matrix entries consist of all homogeneous linear combinations of determinants of $d \times d$ matrix minors. In the language of bitableaux, this is all linear combinations of bitableaux

$$\left(\begin{array}{c|c} i_{1,1} \dots i_{1,1} & 1 \dots d \\ i_{1,2} \dots i_{1,2} & 1 \dots d \\ \dots & \dots \\ i_{1,k} \dots i_{1,k} & 1 \dots d \end{array}\right) \tag{1}$$

of some fixed number of rows $k$. Rota's interest in this problem stemmed from the view of these invariants as describing the incidence relations of a set of $m$ vectors in $d$-space, that is, the incidence relations of a "representable matroid". With other collaborators in [8] Rota generalized the preceding result to describe the incidence relations between $m$ vectors in $d$-dimensional space $V$, no longer in terms of a fixed basis for $V$, but in terms of $m$ "covectors" in $V^*$. The resulting version of the first fundamental theorem describes invariants as linear combinations of bitableaux, each with the same number of rows and with each row of length $d$.



Attending a meeting in honor of A. Garsia, Taormina, Sicily, July 1994. Left to right: C. Procesi, I. Macdonald, X. Viennot, R. Stanley, G. Andrews, G.-C. Rota, C. Greene, A. Björner, A. Garsia.

Photograph courtesy of Richard Stanley, MIT.

In the preceding situations, the second fundamental theorem is given constructively by the straightening law. The relations between invariants are generated by the straightening relations and by the vanishing of biproducts longer than $d$.

The straightening formula is one of the most significant contributions of multilinear algebra to combinatorial and constructive methods. We sample below a few of the many analogues to and applications of the straightening formula.

In the middle to late 1970s straightening laws for the algebra of Gramians and Pfaffians (the case of invariants for the orthogonal and symplectic groups respectively) were given by De Concini and Procesi, and applications to the geometry of Pfaffian varieties were developed by Abeasis and Del Fra.

Formanek and Procesi applied the techniques introduced in Doubilet-Rota-Stein [10] for their proof that the general linear group is geometrically reductive. This is a special case of a conjecture of Mumford solved independently and contemporaneously by Haboush in 1975.

Pommerening in the early to mid-1980s described a class of subgroups of the general linear group whose algebra of invariants is spanned by standard bitableaux. This allowed him to show that these algebras are finitely generated and thus to prove that various rings of invariants are generated by a finite number of elements. He thus provided a positive answer for Hilbert's Fourteenth Problem for various nonreductive subgroups of the general linear group.

In addition to the applications of straightening to invariant theory, the representation theory of the general linear group can be studied entirely in terms of modules spanned by bitableaux. In this formulation the straightening law has been used by Brini and Barnabei, Brini and Teolis, Boffi, Clausen, and others to provide characteristic-free versions of such standard tools in representation theory as the Littlewood-Richardson formula and the branching rule. The application of the straightening law over arbitrary ground rings played a crucial part in the program—begun in the early 1980s by Akin, Buchsbaum, and Weyman—of understanding the representation theory of the general linear group as it relates to resolutions of determinantal ideals outside of characteristic 0. It is in this context that the work in [4, 5] developed. Applications of these techniques to finding intertwining numbers can be found in works of Buchsbaum with Akin and with Flores.

The standard basis theorem for bitableaux was reformulated in the late 1980s and early 1990s by Brini and Teolis, who applied their generalization to the study of $\mathbb{Z}$-forms for the universal enveloping algebra of the general linear group.

A long-standing desire of Rota's, to obtain a description of the Robinson-Schensted-Knuth (RSK) bijection between monomials and pairs of Young diagrams in terms of straightening, was achieved by Leclerc and Thibon. They formulated the problem in a quantized algebra of functions on matrices to which Huang and Zhang had already extended the straightening law. Applying the theory of crystal bases, they derived a description of the RSK bijection from the $q = 0$ term in the expansion of a monomial under straightening.

Because of the variety of situations in which the straightening formula presents itself and turns out to be extremely useful, the notion of an "algebra with straightening law" (ASL) was developed. This was a concept due essentially, and initially, to De Concini (in collaboration with Eisenbud and Procesi) and, independently, to K. Baclawski. It has proved to be a powerful tool in establishing the Cohen-Macaulay property for many classes of algebras of general interest.

Two key properties of straightening that are generalized in the formal notion of an ASL are:

- Straightening is always applied to a pair of adjacent rows.
- The top (or longer) row weakly increases in length after straightening.

For determinantal ideals, this implies that a polynomial is in the ideal generated by all $k \times k$ minors of $X$ if and only if the bitableaux in their expansion all have top row of length at least $k$. Not surprisingly then, the straightening law is a key tool in the study of determinantal ideals; indeed, this is precisely what was studied in [8], albeit in slightly different language. Bruns and Vetter's *Determinantal Rings* is an excellent source for this field.

## Invariant Theory and the Symbolic Method

### Binary Forms and Symmetric Tensors

One of the popular approaches to invariant theory today links invariants to algebraic transformation groups and then uses the machinery of modern algebraic geometry and algebraic group theory. In this regard, too much cannot be said concerning the influence of W. V. D. Hodge. But Gian-Carlo's approach was inspired by his study of the works of algebraists of the last century and the first part of this one (e.g., P. Gordan, A. Capelli, and A. Young). In [16] he says:

> [T]he program of invariant theory, from Boole to our day, is precisely the translation of geometric facts into invariant algebraic equations expressed in terms of tensors. This program of translation of geometry into algebra was to be carried out in two steps. The first step consisted of decomposing a tensor algebra into irreducible components under changes of coordinates. The second step consisted in devising an efficient notation for the expression of invariants for each irreducible component.

In his work on the second step, Rota was led to the study of a technique developed by the invariant theorists of the nineteenth century: the symbolic method. In [16] he interpreted their use of this device in the following way:

> The hidden purpose of the symbolic method in invariant theory was not simply that of finding easy expressions for invariants. A deeper faith was guiding this method. It was the expectation that the expression of invariants by the symbolic method would eventually guide us to single out the "relevant" or "important" invariants among an infinite variety.

Whether or not this was indeed the deeper purpose for developing the symbolic method, the fact is that it soon becomes clear to anyone working with invariants that their polynomial expressions are extremely complicated. To deal with this problem, the symbolic method was devised and used to both describe invariants explicitly as well as to handle important theoretical problems, such as finite generation. But over the past hundred years, standards of rigor and exposition have changed, and new ideas were called for. In [13], "On the invariant theory of binary forms", the authors reconstruct and remodel, in elementary terms, P. Gordan's work on this topic. In their development two ideas are central: first, the symbols are elements of a commutative algebra where generalizations of bitableaux and straightening are valid; second, a linear transformation, called the "umbral operator" (after Sylvester), from this algebra to the usual polynomial algebra, translates facts about the symbols into explicit formulae for invariants. The umbral operator is the natural generalization to invariant theory of methods Rota first applied in [17] to make rigorous the "representative notation" developed by Blissard and popularized by Bell and Riordan for calculating with sequences of numbers.

Consider the simplest nontrivial example. Take a quadratic polynomial in two variables (a quadratic binary form), $b(x_1, x_2) = a_2x_1^2 + a_1x_1x_2 + a_0x_2^2$. We want to consider properties of this polynomial that do not depend on the choice of coordinates, and in particular we want to describe such properties by the vanishing of polynomials in the coefficients of $b$. Suppose we impose a linear change of coordinates and write $b(x_1 + cx_2, x_2)$. This is equivalent to replacing the coefficients $a_0$, $a_1$, and $a_2$ with $a_1c + a_0 + a_2c^2$, $a_1 + 2a_2c$, and $a_2$ respectively. In present-day notation, if $g$ acts by change of variables on a quadratic, $b(x_1, x_2)$, we define the action of $g$ on a polynomial $p$ in the coefficients of $b(x_1, x_2)$ to be $p$ evaluated on the coefficients of $g^{-1}(x_1, x_2)$. So when $p$ is invariant (up to scalar multiples) under this action, the quadratic forms on which $p$ vanishes must share some properties that are invariant under linear change of coordinates.

The symbolic method (or umbral operator) is introduced to make the above situation more amenable to direct manipulation and computation. Consider the polynomial ring in variables $x_1, x_2$ with coefficients in $\Bbbk[\alpha_1, \alpha_2, \beta_1, \beta_2]$. Rota defined an umbral operator, U, to be a $\Bbbk[x_1, x_2]$-linear map on this ring such that

$$\begin{aligned} \mathrm{U}\Big((\alpha_1 x_1 + \alpha_2 x_2)^2\Big) &= b(x_1, x_2) \\ &= \mathrm{U}\Big((\beta_1 x_1 + \beta_2 x_2)^2\Big) \end{aligned}$$

and $\mathrm{U}(MN) = \mathrm{U}(M)\mathrm{U}(N)$ whenever $M, N$ are monomials in $x_1, x_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ and the pair $M$ and $N$ share none of the variables $\alpha_1, \alpha_2, \beta_1, \beta_2$. If we write $a_{i,j}$ for $U(\alpha_1^i \alpha_2^j)$, we can compute

$$\begin{aligned} b(x_1, x_2) &= \mathrm{U}(\alpha_1 x_1 + \alpha_2 x_2)^2 \\ &= a_{2,0}x_1^2 + 2a_{1,1}x_1x_2 + a_{0,2}x_2^2 \end{aligned} \tag{2}$$

to see precisely how Rota encoded the ill-defined notion of a nineteenth-century "lowering operator" into U; the new coefficients are related to the old by $\binom{2}{i} a_{i,j} = a_i$. Further employing this notation, one calculates

$$\begin{aligned} &\mathrm{U}\Big((\alpha_1(x_1 + cx_2) + \alpha_2 x_2)^2\Big) \\ &\qquad = \mathrm{U}\Big((\alpha_1 x_1 + (\alpha_2 + c\alpha_1)x_2)^2\Big), \end{aligned}$$

so that the change of variables $x_1 \mapsto x_1 + cx_2$ becomes a change of variables in the *umbrae*, $\alpha_2 \mapsto \alpha_2 + c\alpha_1$.

Following through with this reasoning, Rota showed that if a polynomial $p(\alpha_1, \alpha_2)$ is invariant under linear changes of variable in the $\alpha_i$'s, then $\mathrm{U}\big(p(\alpha_1, \alpha_2)\big)$ is invariant under the action that change of variables induces on the coefficients of a quadratic binary form. Indeed, this result holds for polynomials $p(\alpha_1, \alpha_2, \beta_1, \beta_2)$ invariant under application of the same change of variables to the pair $\alpha_1, \alpha_2$ and the pair $\beta_1, \beta_2$. Think of the polynomials $p$ as being on variables arrayed in a matrix whose rows are indexed $\{\alpha, \beta\}$ and whose columns are indexed by $\{1, 2\}$. We are now looking for polynomials in the matrix entries invariant under linear action from the right.

But these are precisely the invariants written down in [10] and discussed in connection with (1) above! So, for instance, $\mathrm{U}(\alpha\ \beta \mid 1\ 2)$ is an invariant. Of course, $\mathrm{U}(\alpha\ \beta \mid 1\ 2) = \mathrm{U}(\beta\ \alpha \mid 1\ 2)$ since $\alpha, \beta$ behave identically under U, and, further, $\mathrm{U}(\alpha\ \beta \mid 1\ 2) = \mathrm{U}(-(\alpha\ \beta \mid 1\ 2))$. Thus, by linearity of U this invariant is 0. The next simplest invariant we can construct is

$$\begin{aligned} \mathrm{U}\left(\begin{matrix} \alpha\ \beta \\ \alpha\ \beta \end{matrix} \middle| \begin{matrix} 1\ 2 \\ 1\ 2 \end{matrix}\right) &= \mathrm{U}(-\alpha_1\beta_2 + \alpha_2\beta_1)^2 \\ &= \mathrm{U}\left(\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2 - 2\alpha_1\alpha_2\beta_1\beta_2\right). \end{aligned} \tag{3}$$

Applying the rules defining U above gives us

$$a_{2,0}a_{0,2} + a_{0,2}a_{2,0} - 2a_{1,1}a_{1,1}, \tag{4}$$

which, on comparison with (2), we find to be twice the discriminant. So the invariant in (3) and (4) vanishes on the coefficients of $b(x, y)$ in (2) precisely when $b(x, y)$ is a perfect square.

The above constructions are independent of characteristic and generalize to polynomials in more than two variables. In characteristic 0 one obtains all invariants in this fashion. In the preceding example this lets us verify that the discriminant is the only invariant of quadratic polynomials in two variables. More precisely, the (graded) ring of (quasi) invariants in the coefficients under change of variables is generated by the discriminant. The application of the umbral operator and another variant of the straightening law in [13] provides an explicit construction for a finite set of such generators in the case of binary forms.

One could reasonably ask what happens if we compute $\mathrm{U}(\alpha\ \beta \mid 1\ 2)^2$ when $\mathrm{U}(\alpha_1 x_1 + \alpha_2 x_2)^2 = b(x_1, x_2)$ and $\mathrm{U}(\beta_1 x_1 + \beta_2 x_2)^2 = c(x_1, x_2)$ for some quadratic binary forms $b, c$. By the same reasoning as above, this is a joint invariant of $b$ and $c$, and it turns out to be the simplest example of the "a polar covariant". This covariant was applied by Sylvester to finding canonical forms of homogeneous polynomials. This technique is itself covered and refined in [13]; extensions to forms in more than two variables are given in [11] (although invariant theory is not explicitly used here). Applications to finding the ranks of symmetric and skew-symmetric tensors can be found in recent work of R. Ehrenborg.

**Gian-Carlo and the Letterplace (Super)algebra**

As Gian-Carlo loved to relate, the idea of the "letterplace (super)algebra" was suggested to him in a conversation with R. Feynman. The idea is very simple: in order to handle complicated multilinear algebra, a multiplicative algebra is defined by double variables (letters *and* places), subjected to certain suitable commutation properties.

This simple trick makes it possible to treat many seemingly disparate situations in a unified way. As an example, relying on the straightening law and interpreting the results about the letterplace superalgebra on appropriate homogeneous subspaces, one obtains as special cases the principal results:

- of ordinary representation theory of the symmetric group;

- of the representation theory of $GL(n)$ and $S_n$ on the space of homogeneous tensors of order $n$;
- of the theory of Berele-Regev on the actions of the general linear Lie superalgebra $pl(r, s)$ on a space of $\mathbb{Z}_2$-graded tensors on the algebra of polynomial functions on the space, which is a direct sum of a space with itself.

The superalgebra version of Rota's work began with Doubilet and Rota's extension of the straightening law to exterior algebras of letterplaces [9]. The extension to more general superalgebras was performed by Grosshans, Rota, and Stein in their monograph [12], where the letterplace superalgebra plays a central role in the invariant theory of mixed skew-symmetric and symmetric tensors.

### The Symbolic Method for Skew-Symmetric Tensors

With classical constructive techniques it is in principle possible to find invariants for any representation of $GL_n(\mathbb{C})$. However, Gian-Carlo felt that this technique did not provide an *effective* expression of invariants. The classical symbolic method extends to the representation of $GL_n(\mathbb{C})$ on symmetric tensors without much difficulty [12]. Consequently, Grosshans, Rota, and Stein turned their attention to the representation of $GL_n(\mathbb{C})$ on skew-symmetric tensors. Here the appropriate definitions of the symbols and umbral operator were less evident, but they found that the general steps in formulating the symbolic method for binary forms remain valid for skew-symmetric tensors. They encoded the symbols as elements in a (noncommutative) letterplace superalgebra with bitableaux and straightening and then found the appropriate umbral operator. The result was a truly effective method for expressing the invariants of skew-symmetric tensors [12].

For example, let $\omega$ be a skew-symmetric tensor of degree 2 in the exterior algebra on four generators. It is easy to verify that $\omega$ can be written as a product of two degree 1 elements if and only if $\omega^2 = 0$. Rota and his collaborators observed that if one starts by the suggestive notation of $a^{(2)}b^{(2)}$ for the product, where $a$ and $b$ are "letters" associated with the tensor $\omega$, then one can apply an umbral operator to the superalgebra bitableau $(a^{(2)}b^{(2)} \mid 1\ 2\ 3\ 4)$ and recover the Grassmann condition. More explicitly, we expand $(a^{(2)}b^{(2)} \mid 1\ 2\ 3\ 4)$ inside the exterior algebra generated by the letterplaces to get

$$(a^{(2)} \mid 1\ 2)(b^{(2)} \mid 3\ 4) + (a^{(2)} \mid 1\ 4)(b^{(2)} \mid 2\ 3) - (a^{(2)} \mid 1\ 3)(b^{(2)} \mid 2\ 4).$$

Then we apply the umbral operator, U, to "lower indices" and get

$$a_{12}b_{34} + a_{14}b_{23} - a_{13}b_{24},$$

where $a_{i,j} = b_{i,j}$ is the coefficient of the basis element $\mathbf{i} \cdot \mathbf{j}$ in $\omega$. In characteristic 0 one can write any invariant or even any joint covariant of symmetric and skew-symmetric tensors in this fashion. For example, an element $\omega$ of the exterior algebra on $n$ generators can be written as a product of linear terms precisely when the letters $a$ and $b$ both represent $\omega$ and when the covariant represented by

$$\mathrm{U}\begin{pmatrix} a^{(k)}b^{(2)}c^{(n-k-2)} & 1\ 2 \ldots n \\ b^{(k-2)}d^{(n-k+2)} & 1\ 2 \ldots n \end{pmatrix}$$

vanishes irrespective of the tensors that $c$ and $d$ represent.

For further exposition and more complicated examples of these techniques, the reader can consult [6, 12, 19] and the work of Howe and Huang in the mid-1990s describing the invariants of an arrangement of four subspaces.

One of the keys to understanding invariants produced by the supersymmetric symbolic method is the Grassmann-Cayley algebra. This was developed by Rota and his colleagues in various works, notably [10, 1], as a system of computation with subspaces of a vector space. The application to invariants of arbitrary skew-symmetric tensors can be found in [12]. Indeed, Grosshans pointed out to the authors that Rota's interest in skew-symmetric tensors arose from his interest and research into the Cayley-Grassmann algebra. He offered the following quote from the work of Csurka and Faugeras in computer vision:

> The Grassmann-Cayley algebra introduced in the 1970's by Rota and his collaborators [[1], as well as [12]] is a modern version of the Grassmann algebra. During the last few years it has regained interest because of its wide applicability to "effective projective geometry"...and computer vision. The reason is that it can be seen as an algebra of geometric incidence relations...

The interested reader is well advised to consult the third chapter of Sturmfels's *Algorithms in Invariant Theory* for a cogent account of the theory and computational applications of the Grassmann-Cayley algebra, but we include the following brief paragraph to illustrate the point.

Consider a triple $(\omega_1, \omega_2, \omega_3)$ of degree 2 elements of the exterior algebra generated by three variables, and let $a, b, c$ be letters associated with these tensors. The previous discussion implies that

$$\mathrm{U}\begin{pmatrix} a^{(2)}b & 1\ 2\ 3 \\ bc^{(2)} & 1\ 2\ 3 \end{pmatrix}$$

is an invariant of these three tensors. Since any homogeneous tensor in an exterior algebra generated by three variables can be written as a product of degree 1 tensors, this invariant can equally well be considered an invariant of three lines in projective space. The Grassmann-Cayley algebra directly rewrites the above invariant as $\omega_1 \wedge \omega_2 \wedge \omega_3$, where, subject to technical conditions on nondegeneracy, $\wedge$ can be read as intersection. Thus this invariant vanishes precisely when the three lines meet in a common point.

**Capelli Operators and Superalgebras**

Another substantial reformulation done by Rota involved the work of A. Capelli. The action of the general linear Lie (super) algebra on the letterplace algebra by Capelli operators was introduced as a combinatorial tool in [10], was developed in [8], and has been the point of departure for the reformulation of Capelli's method of auxiliary variables by Brini and Teolis. In intuitive terms, the idea of Capelli consisted of adding to a polynomial algebra "supplementary variables" by application of certain derivations, called polarizations, and then removing these variables by further polarization. This procedure allowed for the simplification of the combinatorial complexity of many proofs in the theory of invariants and in representation theory, e.g., the famous "Capelli Identities" found in Weyl's book. This was due to the metafact that a polarization operator constructed as above via the auxiliary variables is shown to have the same action on the original algebra as some operator constructed via polarizations that does not contain the auxiliary variables. This operator then naturally belongs to the action of the universal enveloping algebra of the general linear Lie algebra and with care can be constructed to belong to the algebra generated by the action of the general linear group (or of some subgroup). A typical example is the description by Weyl of the classical Capelli operator as a true determinantal operator in the "pseudo polarizations" (Weyl's terminology).

In the final analysis, Capelli's method suggests the idea of treating, via polarizations with respect to auxiliary variables, questions of symmetry in a kind of virtual mode. Since Capelli did not have the notion of superalgebra at hand, his method proved effective for treating the problems of symmetry but was less effective for those involving skew-symmetry, which is equally important in studying the representations of the classical groups.

The strength of Rota's idea of passing to the superalgebra shows up clearly in this setting. Here the auxiliary variables have a $\mathbb{Z}_2$-grading, possibly different from the original variables. We can now consider the action, as a Lie superalgebra, of (super)polarizations on both the new and the auxiliary variables. This permits the Capelli method to work in the same way for symmetry and skew-symmetry.

For example, both the permanent and determinant of a generic matrix can be regarded as

Rota with Rotafest organizers, April 1996. Back, left to right: R. Ehrenborg, D. Loeb, A. diBucchianico, N. White. Front: Rota, R. Stanley.

Photograph courtesy of Richard Stanley, MIT.

polarizations of *virtual monomials*, as can the Schur and Weyl modules. More generally, the symmetrized determinantal Young bitableaux, fairly complex combinatorial objects that are basic in representation theory, can also be treated as the image under polarizations of *virtual bracket monomials*.

The same method can be applied to the umbral map for skew-symmetric tensors and to various symmetrization operators of some importance: Capelli operators, generators of the Schur algebra, Young symmetrizers. All of these operators can be represented, by means of the virtual superalgebra method originated by the ideas of Rota, as monomials in the polarizations, thereby simplifying enormously the combinatorial study of their actions.

We close with a combined application of supersymmetry and Capelli operators. Consider the superalgebra bitableau $\left(\begin{matrix} a\,b \\ a\,b \end{matrix}\middle|\begin{matrix} 1\,2 \\ 1\,2 \end{matrix}\right)$, which is an element of an exterior letterplace algebra. The superalgebra version of the standard basis theorem says that this must be a constant multiple of the bitableau $\left(\begin{matrix} a\,a \\ b\,b \end{matrix}\middle|\begin{matrix} 1\,2 \\ 1\,2 \end{matrix}\right)$. Polarizing the positive letters $a$ and $b$ to negative letters $x_1, x_2$ and $y_1, y_2$ in the first, unstraightened, expression yields

$$\left(\begin{matrix} x_2\, y_1 \\ x_1\, y_2 \end{matrix}\middle|\begin{matrix} 1\,2 \\ 1\,2 \end{matrix}\right) - \left(\begin{matrix} x_2\, y_2 \\ x_1\, y_1 \end{matrix}\middle|\begin{matrix} 1\,2 \\ 1\,2 \end{matrix}\right) - \left(\begin{matrix} x_1\, y_1 \\ x_2\, y_2 \end{matrix}\middle|\begin{matrix} 1\,2 \\ 1\,2 \end{matrix}\right) + \left(\begin{matrix} x_1\, y_2 \\ x_2\, y_1 \end{matrix}\middle|\begin{matrix} 1\,2 \\ 1\,2 \end{matrix}\right).$$

Applying the same polarizations to the second expression above (the straightened bitableau) yields the identity

$$2\begin{pmatrix} x_2\, y_1 & 1\,2 \\ x_1\, y_2 & 1\,2 \end{pmatrix} - 2\begin{pmatrix} x_2\, y_2 & 1\,2 \\ x_1\, y_1 & 1\,2 \end{pmatrix}$$
$$= c \cdot \begin{pmatrix} x_1\, x_2 & 1\,2 \\ y_1\, y_2 & 1\,2 \end{pmatrix},$$

for some constant $c$. Now consider a $2 \times 2$ array of vectors in $\mathbb{Q}^2$, each of whose rows lists a basis for $\mathbb{Q}^2$. From the preceding identity, together with the fact that $c$ turns out to be nonzero, it is easy to see that the entries in each row can be permuted so that each column also indexes a basis. This kind of technique may also be applied for larger arrays; the only substantial difficulty is that $c$ becomes extremely difficult to compute. This computation led to Rota's famous Basis Conjecture:

> Take any $n^2$ vectors in $\mathbb{Q}^n$ and arrange them in an $n \times n$ array. If each column forms a basis, then the entries can be permuted inside the columns so that each row also forms a basis.

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