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Matching Rectangles in d-Dimensions: Algorithms and Laws of Large Numbers

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For each point of the integer lattice Z^d, let X and Y be independent identically distributed random variables with $P(X = Y) = p \in (0, 1)$. Let $S(n)$ be the volume of the largest d-dimensional cube in $\{1,..., n\}^d$ with the property that $X = Y$ at every point of the cube; $R(n)$ is similarly defined to be the maximum volume of perfectly matching rectangles. It is proved that, if all possible shifts of the X lattice relative to the Y lattice are allowed, $P(\lim_{n\to\infty} S(n)/\log n = \lim_{n\to\infty} R(n)/\log n = 2d) = 1$, where log is to base $(1/p)$. The corresponding limit without shifts is d. Algorithms to find largest squares and rectangles, with and without shifts, are also given.

1. INTRODUCTION

The length of the longest head run in a sequence of coin tosses is a random variable of much interest in probability theory. In addition to the obvious appeal of problems like runs of red in roulette, this random variable has been repeatedly suggested by other important, if somewhat less colorful, applications. Mosteller [9] recommends a test for randomness which is based on the length of the longest head run and is included in Gibbons [5]. Naus [10, 11], in his studies of the birthday problem, mentions and gives approximations for several related problems. His papers have a number of interesting references with topics ranging from vision (Glatz [6]) to uranium prospecting (Conover *et al.* [2]). In a recent paper, Schwager [13] mentions applications to DNA sequencing, psychology, sociology, ecology, and radar astronomy.

A new approach was made to this problem by Erdös and Rényi [3]. Let

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$R(n)$ be the length of the longest head run in n independent tosses of a coin with P (Heads) $= p \in (0, 1)$. Among other results they show that

$$P(\lim_{n\to\infty} R(n)/\log_{1/p}(n) = 1) = 1. \tag{1.1}$$

Later Erdös and Révész [4] give more precise results, and Guibas and Odlyzko [7] obtain even more careful estimates.

Motivated by DNA sequence comparison, Arratia and Waterman [1] consider a related problem: Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots,$ be independent and identically distributed with $P(X_i = Y_j) = p \in (0, 1)$. If all possible shifts of one sequence relative to the other are allowed, the relevant random variable is

$$M(n) = \max_{0 \leqslant i,j \leqslant n} \{m: X_{i+k} = Y_{j+k} \text{ for } k = 1, 2, \ldots, m;\ m \leqslant n - i;\ m \leqslant n - j\}.$$

Then the result corresponding to (1.1) is

$$P(\lim_{n\to\infty} M(n)/\log_{1/p}(n) = 2) = 1. \tag{1.2}$$

Two random sequences of length n, not allowing shifts, have a longest match length which behaves like $R(n)$ above. Allowing shifts involves the random variables $M(n)$ and effectively doubles the length of the longest match.

There are also results for two dimensions. At (i, j), where i and j are positive integers, let $P(X_{ij} = 1) = p \in (0, 1)$ and $P(X_{ij} = 0) = 1 - p$. Révész proved that the area $S(n)$ of the largest square of all 1's satisfies

$$P(\lim_{n\to\infty} S(n)/\log_{1/p}(n) = 2) = 1. \tag{1.3}$$

Allowing rectangles greatly increases the number of elements to be maximized, but Nemetz and Kusolitsch [12] show that $R(n)$, the maximum area of the rectangles of all 1's, also satisfies

$$P(\lim_{n\to\infty} R(n)/\log_{1/p}(n) = 2) = 1. \tag{1.4}$$

In this paper, we give algorithms to find the largest square and rectangle of 1's in an $n \times n$ lattice, each in time $O(n^2)$. In addition, we find $O(n^4)$ algorithms for the largest matching square and rectangle between two lattices. These algorithms are of interest in their own right since they provide practical means of computer calculation of the random variables $S(n)$ and $R(n)$. The algorithms are generalized to the case of finding squares and rectangles of "mostly" 1's.

The last part of the paper generalizes the results of Erdös, Rényi, Révész, Nemetz, and Kusolitsch to d-dimensions, and allows shifts. The counterpart of (1.2) is that

$$P(\lim_{n\to\infty} S(n)/\log_{1/p}(n) = \lim_{n\to\infty} R(n)/\log_{1/p}(n) = 2d) = 1,$$

where $S(n)$ and $R(n)$ denote d-dimensional volume. If shifts are not allowed, the Nemetz and Kusolitsch result generalizes to show that the above limit becomes d, so that shifting, just as for $d = 1$, simply doubles the limiting volume.

2. Algorithms

For $x_{ij} \in \{0, 1\}$, $1 \leqslant i,j \leqslant n$, algorithms are presented to find maximum area squares and rectangles composed entirely or mostly of 1's. The case of squares is simpler than that of rectangles and will be treated first. We will consider the position $(i,j) = (1, 1)$ to be the upper, left-hand corner of the $n \times n$ array.

2.1. *Largest Square of* 1's

The length of the (horizontal) run of 1's in the row beginning with x_{ij} and extending left is h_{ij}; v_{ij} is the length of the (vertical) run of 1's beginning at x_{ij} and extending up the column; s_{ij} is the side length of the largest square of 1's with lower right corner at x_{ij}. More formally,

$$h_{ij} = \max\{m: x_{il} = 1 \text{ for } 1 \leqslant j - m + 1 \leqslant l \leqslant j\},$$

$$v_{ij} = \max\{m: x_{kj} = 1 \text{ for } 1 \leqslant i - m + 1 \leqslant k \leqslant i\},$$

and

$$s_{ij} = \max\{m: x_{kl} = 1 \text{ for all } i - m + 1 \leqslant k \leqslant i \text{ and } j - m + 1 \leqslant l \leqslant j\}.$$

The row run of 1's ending at $x_{i,j-1}$ is extended by one at x_{ij} if and only if $x_{ij} = 1$. This observation justifies the equations

$$h_{ij} = x_{ij}(h_{i,j-1} + 1),$$

and

$$v_{ij} = x_{ij}(v_{i-1,j} + 1).$$

The side of the maximum square with lower right corner at (i, j) can be no larger than $s_{i-1,j-1}+1$. The side is that large if and only if the row and column run of 1's are at least $s_{i-1,j-1}+1$. Therefore

$$s_{ij}=\min\{s_{i-1,j-1}+1,\, h_{ij},\, v_{ij}\}.$$

The number of steps indicated by this algorithm is $O(n^2)$. The area of the largest square is of course

$$\mathrm{AS}=(\max_{1\leqslant i,j\leqslant n} s_{ij})^2.$$

2.2. *Largest Rectangle of* 1's

Let h_{ij} and v_{ij} be as above. In addition define

$$r_{ijk}=\max\{m\colon x_{qr}=1 \text{ for } j-m+1\leqslant r\leqslant j \text{ and } i-k+1\leqslant q\leqslant i\}.$$

which is the "width" of the largest rectangle of 1's with lower right corner at (i,j) and "depth" k. Reasoning as above yields

$$r_{ijk}=\min\{r_{i-1,j,k-1},\, h_{ij}\}.$$

The largest area of all rectangles with lower right corner at (i,j) is given by

$$l_{ij}=\max_k\,\{kr_{ijk}\}$$

and the largest area rectangle is

$$\mathrm{AR}=\max_{1\leqslant i,j\leqslant n} l_{ij}.$$

The running time calculation for the indicated algorithm appears to be

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i)=O(n^3)$$

but that is not correct. Observe that if $r_{ijk}=0$, then $r_{ijk^*}=0$ for all $k\leqslant k^*$. Therefore "i" in the above equation can be replaced by a random variable Y, equal to the length of 1's run in the jth column (ending at (i,j)). The Y is a truncated geometric random variable with expectation bounded by a constant; therefore expected running time is seen to be $O(n^2)$.

2.3. *Largest Figures with z Zeros*

Now consider the problem of finding the maximum area square with no more than z zeros. It is necessary to introduce more notation: Let

$$h_{ijz} = \max\{m\colon x_{il} \text{ is zero exactly } z \text{ times in } j - m + 1 \leqslant l \leqslant j\}$$
$$v_{ijz} = \max\{m\colon x_{kj} \text{ is zero exactly } z \text{ times in } i - m + 1 \leqslant k \leqslant i\}.$$

If there are less than z zeros in the sets above which define h_{ijz} or v_{ijz}, then set $h_{ijz} = j$ and $v_{ijz} = i$, respectively. If $z = -1$, set $h_{i,j,-1} = v_{i,j,-1} = -1$. Recursion relations for h_{ijz} and v_{ijz} are easily derived:

$$h_{ijz} = x_{ij}(h_{i,j-1,z} + 1) + (1 - x_{ij})(h_{i,j-1,z-1} + 1),$$
$$v_{ijz} = x_{ij}(v_{i-1,j,z} + 1) + (1 - x_{ij})(v_{i-1,j,z-1} + 1).$$

To proceed, define S_{ijz} to be the maximum side width of a square with lower right corner at (i,j) containing no more than z zeros. By reasoning similar to that of Subsection 2.1,

$$S_{ijz} = \max\{\min(v_{ijz_2}, h_{ijz_1}, S_{i-1,j-1,z_3} + 1)\colon$$
$$0 \leqslant z_1 + z_2 + z_3 + 1 - x_{ij} \leqslant z\}.$$

In fact z_1, z_2, z_3 can be further restricted by

$$S_{i,j,z-1} \leqslant h_{i,j,z_1},$$
$$S_{i,j,z-1} \leqslant v_{i,j,z_2},$$

and

$$S_{i,j,z-1} \leqslant S_{i-1,j-1,z_3} + 1.$$

To handle the situation for rectangles requires a similar extension of Subsection 2.2.

2.4. *Matching Two Lattices*

For the case of matching two planar $n \times n$ lattices, a matching function or criterion must be given. We take this to be of the form $z(i,j;k,l) \in \{0, 1\}$, where the (i,j)th position of lattice x is considered to match the (k,l)th position of lattice y if $z(i,j;k,l) = 1$. To perform the matching takes time $O(n^4)$. We assume that when $z(i,j;\, k, l)$ is computed, all $z(i',j';\, k',\, l')$ are known where $i' + j' < i + j$ and $k' + l' < k + l$.

It is a routine matter to convert the algorithms given above for largest square, rectangle,.... To illustrate this, we define $h(i,j;k,l)$ to be the length of the longest (horizontal) run of matches beginning at $z(i,j;\, k,\, l)$ and extending to the left. Formally

$$h(i,j;k,l) = \max\{m\colon z(i,j-a;k,l-a) = 1 \text{ for } a = 0, 1,\ldots, m-1\}.$$

To recursively compute h,

$$h(i,j;k,l)=z(i,j;k,l)(h(i,j-1;k,l-1)+1).$$

Similar equations convert the recursions for v and other quantities above.

3. Laws of Large Numbers

For ease of exposition, we present our results for $d=2$, in the case of matching with shifts. The proofs for d-dimensions present notational difficulties only. Denote logarithms to the base $1/p$ simply by log.

THEOREM A (Largest matching rectangle). *Let* X_{ij}, $1 \leqslant i, j < \infty$, *and* Y_{ij}, $1 \leqslant i,j < \infty$, *be independent, identically distributed random variables such that* $0 < p \equiv P(X_{11} = Y_{11}) < 1$. *Let* $M \equiv M(n)$ *be the area of the largest matching rectangle between the finite lattices* $(X_{ij}\colon 1 \leqslant i,j \leqslant n)$ *and* $(Y_{ij}\colon 1 \leqslant i,j \leqslant n)$, *allowing for shifts, i.e.,*

$$M(n)=\max\{A=ab\colon X_{i+r,j+s}=Y_{k+r,l+s}, r=0 \text{ to } a-1, s=0 \text{ to } b-1, \text{ for some } 1 \leqslant i,k \leqslant n-a+1, 1 \leqslant j,l \leqslant n-b-1\}.$$

Then as $n \to \infty$

$$M(n)/\log n \xrightarrow{\text{a.s.}} 4. \tag{3.1}$$

Remark. For simplicity of exposition, we state Theorem A for the case of random variables on the lattice Z^2. The corresponding result for the lattice Z^d, $d \geqslant 1$, is that if M is the volume of the largest matching rectangular box, allowing for shifts, then $M/\log n \longrightarrow^{\text{a.s.}} (2d)$. If shifts are *not* allowed, the result is that $M/\log n \longrightarrow^{\text{a.s.}} d$ (see [12] for $d=2$). Our methods are closely related to [1] and [12].

THEOREM B. (Largest matching square). *With the notation of Theorem* A, *let* $L \equiv L(n)$ *be the area of the largest matching square, allowing for shifts, i.e.,*

$$L(n)=\max\{a^2\colon X_{i+r,j+s}=Y_{k+r,l+s}, r=0 \text{ to } a-1, s=0 \text{ to } a-1, \text{ for some } 1 \leqslant i,j,k,l \leqslant n-a+1\}.$$

Then as $n \to \infty$,

$$L(n)/\log(n) \xrightarrow{\text{a.s.}} 4. \tag{3.2}$$

Proof of Theorem A (*assuming Theorem B*).

Evidently $M(n) \geqslant L(n)$, so (3.2) implies

$$P(\lim_n (M(n)/\log n) \geqslant 4) = 1.$$

Hence it suffices to prove

$$P(\lim_n (M(n)/\log n) \leqslant 4) = 1. \tag{3.3}$$

Define

$$Q(a, n, i, k) = \max \{b: X_{i+r,j+s} = Y_{k+r,l+s}, r = 0 \text{ to } a-1, s = 0 \text{ to } b-1, \\ \text{some } 1 \leqslant j, l \leqslant n-b+1\}.$$

In other words, $Q(a, n, i, k)$ is the maximum height of a matching rectangle of base a, starting at x coordinate i among the X array and x coordinate k among the Y array. Fix a constant $0 < \varepsilon < 1$, and define

$$G_n = \{(M(n)/\log n) > 4(1+\varepsilon)\},$$

$$B_{a,n,i,k} = \{(aQ(a, n, i, k)/\log n) > 4(1+\varepsilon)\}.$$

Evidently

$$G_n = \bigcup_{a=1}^{n} \bigcup_{i,k=1}^{n-a+1} B_{a,n,i,k}. \tag{3.4}$$

We pause for

LEMMA. *For all integers* a,

$$P(B_{a,n,i,k}) \leqslant n^{-2-4\varepsilon}$$

and

$$P(B_{a,n,i,k}) \leqslant n^{-6} \qquad \textit{if} \quad a > 8 \log n.$$

Proof. For any integer m,

$$\begin{aligned} &P(Q(a, n, i, k) \geqslant m) \\ &\quad = P\left(\bigcup_{j,l=1}^{n-m+1} \{X_{i+r,j+s} = Y_{k+r,l+s}, r = 0 \text{ to } a-1, s = 0 \text{ to } m-1\}\right) \\ &\quad \leqslant (n-m+1)^2 p^{am}. \end{aligned} \tag{3.5}$$

Let $m = [(4(1+\varepsilon)\log n)/a] + 1$, where $[x]$ denotes the greatest integer $\leqslant x$. Then $m \geqslant (4(1+\varepsilon)\log n)/a$ and

$$\left(\frac{1}{p}\right)^{am} \geqslant n^{4+4\varepsilon}; \qquad p^{am} \leqslant n^{-4-4\varepsilon}$$

so

$$P(Q(a,n,i,k) > 4(1+\varepsilon)\log n/a) \leqslant (n-m-1)^2\, n^{-4-4\varepsilon};$$

which implies

$$P(B_{a,n,i,k}) \leqslant n^{-2-4\varepsilon}.$$

On the other hand, if $a > 8\log n$, then

$$m = [(4(1+\varepsilon)\log n)/a] + 1 = 1,$$
$$p^{am} = p^a = (1/p)^{-a} < (1/p)^{-8\log n} = n^{-8};$$

so using (3.5)

$$P(B_{a,n,i,k}) = P(Q(a,n,i,k) > 4(1+\varepsilon)\log n/a) < n^{-6},$$

and the lemma is proved.

Continuing from (3.4),

$$G_n = \left(\bigcup_{a=1}^{8\log n}\ \bigcup_{i,k=1}^{n-a+1} B_{a,n,i,k}\right) \cup \left(\bigcup_{a=8\log n+1}^{n}\ \bigcup_{i,k=1}^{n-a+1} B_{a,n,i,k}\right).$$

The lemma now implies

$$P(G_n) \leqslant \frac{8\log n}{n^{4\varepsilon}} + n^{-3},$$

which implies

$$\sum_{m=0}^{\infty} P(G_{(2m)}) < \infty.$$

From this inequality, we can deduce, by means of the Borel–Cantelli lemma,

$$P\left(\limsup_m \left(\frac{M(2^m)}{\log(2^m)}\right) > 4(1+\varepsilon)\right) = 0.$$

Since this holds for all $\varepsilon > 0$,

$$P\left(\limsup_m \left(\frac{M(2^m)}{\log(2^m)}\right) \leqslant 4\right) = 1.$$

If n is an integer with $2^{m-1} < n \leqslant 2^m$, then

$$\frac{M(n)}{\log(n)} \leqslant \frac{M(2^m)}{\log(2^{m-1})} = \frac{M(2^m)}{\log(2^m) - \log 2}$$

and so assertion (3.3) follows.

Proof of Theorem B.

Step I. We prove that $P(\limsup_n (L(n)/\log n) \leqslant 4) = 1$. Fix a small ε, positive or negative, and define

$$l \equiv l(n, \varepsilon) \equiv [((4 + \varepsilon) \log n)^{1/2}] + 1.$$

Consequently

$$p^{(l^2)} \leqslant n^{-4-\varepsilon}.$$

Let A_{ijrs} be the event that there is a matching square of side at least l, starting at the (i, j) position among the X's and the (r, s) position among the Y's:

$$A_{ijrs} = \{X_{i+t, j+u} = Y_{r+t, s+u} \text{ for } t = 0 \text{ to } l - 1, \text{ and } u = 0 \text{ to } l - 1\}.$$

Now

$$P(A_{ijrs}) = p^{(l^2)} \leqslant n^{-4-\varepsilon}.$$

For ε positive,

$$P(L(n) \geqslant l^2) = P\left(\bigcup_{i,j,r,s=1}^{n-l+1} A_{ijrs}\right) \leqslant \sum_{i,j,r,s} P(A_{ijrs})$$
$$\leqslant (n - l + 1)^4 \, n^{-4-\varepsilon} < n^{-\varepsilon}.$$

Consequently

$$\sum_{m=0}^{\infty} P(L(2^m)/\log(2^m) > 4 + \varepsilon) < \infty,$$

and the Borel–Cantelli lemma now implies that

$$P\left(\limsup_m \left(\frac{L(2^m)}{\log(2^m)}\right) \leqslant 4\right) = 1.$$

If n is an integer with $2^{m-1} < n \leqslant 2^m$, then

$$\frac{L(n)}{\log(n)} \leqslant \frac{L(2^m)}{\log(2^{m-1})} = \frac{L(2^m)}{\log(2^m) - \log 2}$$

and we have proved that

$$P(\limsup_n (L(n)/\log n) \leqslant 4) = 1.$$

Step II. We now prove that $P(\liminf_n (L(n)/\log n) \geqslant 4) = 1$. This time, define

$$q \equiv q(n, b) \equiv [(4 \log n - 6 \log \log n)^{1/2}].$$

Consider matches between squares of side q, whose top left corners have coordinates which are multiples of q: more precisely, let

$$C_{ijrs} \equiv \{X_{qi+t,qj+u} = Y_{qr+t,qs+u} \text{ for } t = 0 \text{ to } q-1, \text{ and } u = 0 \text{ to } q-1\}.$$

Let D be the sum of the indicators of these events:

$$D \equiv D(n, \varepsilon) \equiv \sum_{i,j,r,s=1}^{[n/q]} 1_{C_{ijrs}}.$$

Note that

$$\{L(n) < q^2\} \subset \{D = 0\}.$$

Also note that

$$p^{(q^2)} \geqslant n^{-4}(\log n)^6, \qquad q^{-4} \geqslant (1/16)(\log n)^{-2}$$

and

$$ED \sim (n/q)^4 p^{(q^2)} \geqslant (n/q)^4 n^{-4}(\log n)^6.$$

This implies

$$ED \geqslant \tfrac{1}{16}(\log n)^4 \tag{3.6}$$

so that $ED \to \infty$ as $n \to \infty$.

In the expansion of $\text{Var}(D)$ as a sum of covariances of indicator functions, there are $\sim(n/q)^4$ diagonal terms, whose combined contribution is less than ED, and there are $\sim(n/q)^8$ terms which are all zero, corresponding to the independent pairs of events C_{ijrs} and C_{kltu} with $(i,j) \neq (k,l)$ and $(r,s) \neq (t,u)$. Finally there are $\sim 2(n/q)^6$ nontrivial terms from pairs of events such as C_{ijrs} and C_{ijtu} with $(r,s) \neq (t,u)$, or C_{ijrs} and C_{klrs} with $(i,j) \neq (k,l)$. Let $\{p_1, p_2, \ldots,\}$ be the weights of the atoms in the common distribution of X and Y, so that $p = \sum_a (p_a)^2$. By a version of Holder's inequality (see Hardy, Littlewood, and Polya [8])

$$P(C_{ijrs} \cap C_{ijtu}) = \left[\sum_a (p_a)^3\right]^{(q^2)}$$

$$\leqslant \left[\sum_a (p_a)^2\right]^{(3q^2)/2} = p^{(3q^2)/2}$$

so that the combined contribution to $\text{Var}(D)$ from these $\sim 2(n/q)^6$ nontrivial pairs is less than or equal to $2(n/q)^6\, p^{(3q^2)/2}$, which $\sim 2(ED)^{3/2}$. Combining these estimates, we have

$$\text{Var}(D) < ED + 2(ED)^{3/2}.$$

Using Chebychev's inequality,

$$P(L(n) < q^2) \leqslant P(D = 0) \leqslant \text{Var}(D)/(ED)^2 < (1/ED) + 2(ED)^{-1/2}$$

$$< 16(\log n)^{-4} + 4(\log n)^{-2} < 20\,(\log n)^{-2}.$$

Therefore

$$\sum_{m=0}^{\infty} P(L(2^m) < q(2^m)^2) < 20 \sum_{m=0}^{\infty} (m \log 2)^{-2} < \infty.$$

The Borel–Cantelli lemma implies that

$$P(\limsup_m \{L(2^m) < q(2^m)^2\}) = 0,$$

so that

$$P(\liminf_m (L(2^m)/q(2^m)^2) \geqslant 1) = 1.$$

If n is an integer with $2^m \leqslant n < 2^{m+1}$, then

$$L(n)/q(n)^2 \geqslant L(2^m)/q(2^{m+1})^2 \geqslant L(2^m)/((q(2^m) + 1)^2 + 4 \log 2).$$

This implies

$$P(\liminf_n (L(n)/q(n)^2) \geqslant 1)$$

$$\geqslant P(\liminf_n (L(2^m)/(q(2^m) + 1)^2) \geqslant 1) = 1,$$

and

$$P(\liminf_{n} \left(\frac{L(n)}{\log n}\right) \geq 4) = 1$$

follows as desired.

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