Mathematical Geology, Vol. 11, No. 1, 1979

Modeling and Optimizing a Gas–Water Reservoir: Enhanced Recovery with Waterflooding[1]

Mark E. Johnson,[2] Ellis A. Monash,[3] and Michael S. Waterman[2]

Accepted practice dictates that waterflooding of gas reservoirs should commence, if ever, only when the reservoir pressure has declined to the minimum production pressure. Analytical proof of this hypothesis has yet to appear in the literature however. This paper considers a model for a gas–water reservoir with a variable production rate and enhanced recovery with waterflooding and, using an initial dynamic programming approach, confirms the above hypothesis. KEY WORDS: gas reservoirs, waterflooding, dynamic programming.

Nomenclature

g	gas entrapment factor (dimensionless)
N_0	initial amount of gas in the reservoir in moles
$N(t)$	moles of gas in the reservoir at time t
$N_p(t)$	moles of gas produced up to time t
$N_\tau(t)$	moles of gas entrapped up to time t
P_c	cutoff pressure in psia
P_0	initial pressure of gas in psia
$P(t)$	pressure of gas in psia at time t
r	constant production rate of gas in moles per year
$r(t)$	production rate at time t in moles per year
R	ideal gas constant
s	constant rate of water injection in cubic feet per year
$s(t)$	rate of water injection at time t in cubic feet per year
T_0	temperature in °K
V_c	volume of the reservoir in cubic feet, below which gas production ceases
V_0	initial reservoir volume in cubic feet
$V(t)$	reservoir volume at time t in cubic feet
$V_\tau(t)$	reservoir volume of trapped gas at time t in cubic feet
$V_w(t)$	reservoir volume of invaded water at time t in cubic feet

[1] Manuscript received 23 January 1978; revised 23 August 1978.
[2] Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545.
[3] Conservation Division, U.S. Geological Survey, Lakewood, Colorado 80226.

INTRODUCTION

Enhanced recovery of oil and gas from existing and depleted reservoirs is now recognized as an important energy strategy. This is due, in part, to increased demand for petroleum products, an evolving international situation, and the fact that production from newly discovered reserves may not yield as much petroleum as was anticipated. In spite of the importance of maximizing recovery, many models of oil and gas reservoirs remain largely descriptive and are not used to analytically obtain optimal production strategies.

The reservoir model uses volume and mass balances with the ideal gas law to obtain a simple, nonstochastic set of differential equations; the problem of enhanced recovery of natural gas from such a reservoir is then addressed. The objective is to maximize total recovery with respect to two control functions: the rate of gas withdrawal and the rate of water injection. A discrete, dynamic programming formulation of the problem is presented which explores the optimal production strategy. Essentially, this strategy is to refrain from waterflooding until the minimum production pressure is reached. Then waterflooding should begin with production keyed to maintain the reservoir at this minimum pressure. Finally, an analytical proof is given which shows this production strategy to be optimal.

THE GAS–WATERFLOOD RESERVOIR MODEL

The mathematical details of the gas–waterflood reservoir per se model are discussed, and the closed form solutions to the differential equations of the state variables of the system are obtained. In a gas–waterflood reservoir system, water is injected into the gas reservoir at the rate of s ft^3/yr so that additional amounts of gas can be recovered from the "exhausted" reservoir. Gas is withdrawn from the reservoir at the rate of r moles/yr. As the water is injected into the reservoir, the gas is "pushed" ahead of it. Some of the water will entrap pockets of gas that become unavailable for production. The gas entrapment factor g represents the "strength" of the trapping process. Specifically, if V_w ft^3 of water is injected into the reservoir, then $g \cdot V_w$ ft^3 of gas is entrapped by the water. Initially the gas reservoir has a volume V_0, a pressure P_0, and a quantity of material N_0 at a temperature T_0. Production of gas ceases when the volume or pressure of gas falls below some low value.

The physical model describes only the reservoir per se. The assumptions used in this basic model are listed below.

1. The gas obeys the ideal gas law.
2. The reservoir is homogeneous with respect to pressure.
3. The reservoir is isothermal and nonadiabatic.

4. Water can be injected at a variable rate.
5. The gas withdrawal rate can be constant or variable.
6. The ratio g of the volume of gas entrapped behind the injected water to the volume of the injected water is constant in time.
7. Gas is entrapped behind the injected water at the current pressure in the reservoir and, after entrapment, has no effect on the reservoir.
8. Production ceases when the pressure becomes some low pressure or the volume becomes some low volume, whichever occurs first.

While these assumptions could preclude application of this model to any *real* reservoirs, the model should be useful in making technical assessments as to the nature and magnitude of ultimate recovery of gas from a gas reservoir–waterflood model.

The gas–waterflood reservoir per se model is described by a set of first-order linear differential equations for the state variables. In this section, the differential equations will be obtained. Consider the volume balance equation for the gas–waterflood system:

$$V_0 = V(t) + V_w(t) + V_\tau(t) \tag{1}$$

where V_0 is the initial volume of gas in the reservoir at time 0, $V(t)$ is the volume of gas in the reservoir at time t, $V_w(t)$ is the volume of water injected into the reservoir up to time t, $V_\tau(t)$ is the volume of gas that has been trapped behind the encroaching water.

The rate at which water is injected into the reservoir is given by

$$dV_w(t)/dt = s(t) \tag{2}$$

and the rate at which the volume of gas is entrapped is

$$dV_\tau(t)/dt = g\,dV_w(t)/dt = gs(t) \tag{3}$$

Differentiating Equation (1) with respect to t yields

$$0 = (dV(t)/dt) + (dV_w(t)/dt) + (dV_\tau(t)/dt) \tag{4}$$

Substituting equations (2) and (3) into equation (4) yields

$$dV(t)/dt = -(1+g)s(t) \tag{5}$$

Now consider the mass balance relation

$$N_0 = N(t) + N_\tau(t) + N_p(t) \tag{6}$$

where N_0 is the initial amount of gas in the reservoir, $N(t)$ is the amount of gas in the reservoir at time t, $N_\tau(t)$ is the amount of gas that has been entrapped behind the encroaching water at time t, $N_p(t)$ is the amount of gas production up to time t.

The rate at which gas is withdrawn from the reservoir is

$$dN_p(t)/dt = r(t) \tag{7}$$

and the rate at which the gas is being entrapped is

$$dN_\tau(t)/dt = (P(t)/RT_0)(dV_\tau(t)/dt) + (V_\tau(t)/RT_0)(dP(t)/dt) \tag{8}$$
$$\doteqdot (P(t)/RT_0)\,(dV_\tau(t)/dt)$$

where $P(t)$ is the pressure at time t, T_0 is the reservoir temperature, and R is the universal gas constant. The second equation is due to assumption 7. Substituting equation (3) into equation (8) yields

$$dN_\tau(t)/dt = (gs(t)/RT_0)\;P(t) \tag{9}$$

Differentiating equation (6) with respect to time t,

$$0 = (dN(t)/dt) + (dN_\tau(t)/dt) + (dN_p(t)/dt) \tag{10}$$

By substituting equations (7) and (9) into equation (10),

$$dN(t)/dt = -r(t) - (gs(t)/RT_0)\;P(t) \tag{11}$$

The ideal equation of state, relating pressure to quantity and volume, is

$$P(t) = RT_0 N(t)/V(t) \tag{12}$$

The reservoir model is thus given by equations (5), (11), and (12).

In subsequent sections, the reservoir model is equivalently given as follows:

$$dV(t)/dt = -s(t)\cdot(1+g) \tag{13}$$

$$dP(t)/dt = (-r(t)RT_0 + P(t)s(t))/V(t) \tag{14}$$

$$N(t) = P(t)V(t)/RT_0 \tag{15}$$

By differentiating equation (12) with respect to time, it can be shown that this model is equivalent to the earlier representation as given in equations (5), (11), and (12).

The solutions to the differential equations (13)–(15) are now obtained. Consider the volume equation:

$$dV(t)/dt = -(1+g)s(t)$$

with the initial value condition

$$V(0) = V_0$$

Separation of variables leads to the solution

$$V(t) = V_0 - (1+g)\int_0^t s(t)dt \tag{16}$$

Now consider equation (11) and substitutions from equation (12) and equation (16).

$$dN(t)/dt = -r(t) - gs(t)(RT_0N(t)/(RT_0V(t))$$
$$= -r(t) - gs(t)N(t)/\left(V_0 - (1+g)\int_0^t s(t)dt\right) \quad (17)$$

Thus,

$$dN(t)/dt + \alpha(t)N(t) = -r(t) \quad (18)$$

where

$$\alpha(t) = gs(t)/\left(V_0 - (1+g)\int_0^t s(t)dt\right)$$

Equation (18) is a linear differential equation of the first order. Equation (18) is solved using the "integrating factor"

$$I = e^{\int \alpha(t)dt} \quad (19)$$

Multiplying equation (18) by equation (19) and integrating yields:

$$N(t) = e^{-\int \alpha(t)dt}\int -r(t)e^{\int \alpha(t)dt}dt + Ke^{-\int \alpha(t)dt} \quad (20)$$

In this paper the special case $s(t) = s =$ constant and $r(t) = r =$ constant is particularly important. Making these substitutions and performing the algebraic simplications yields the following closed-form solution to the model:

$$V(t) = V_0 - s(1+g)t \quad (21)$$

$$N(t) = (r/s)[V_0 - s(1+g)t] + [N_0 - (r/s)\ V_0]\,[1-(s(1+g)/V_0)\ t]^{(g/1+g)} \quad (22)$$

$$P(t) = rP_0V_0/(sN_0) + [P_0 - (rP_0V_0/(sN_0))]\,[1-(s(1+g)/V_0)t]^{-(1/1+g)} \quad (23)$$

DYNAMIC PROGRAMMING APPROACH

In this section we formally state the optimization problem—maximize ultimate recovery subject to the production rate and water injection rate controls. This problem is formulated in a general framework, as follows:

$$\max_{r,s} \int r(t)dt$$

subject to the constraints

$$dV(t)/dt = -s(t)\cdot(1+g)$$
$$dP(t)/dt = [-r(t)RT_0 + P(t)s(t)]/V(t)$$

$$N(t)=P(t)V(t)/RT_0$$

$$\left.\begin{array}{l}P(t)\geqslant P_c>0\\ V(t)\geqslant V_c>0\end{array}\right\}\text{ for all } t$$

where $g\geqslant 0$, $P(0)=P_0$, $V(0)=V_0$, $N(0)=N_0$, $r(t)$; $s(t)\geqslant 0$ are continuous for all except a finite number of $t\geqslant 0$, and P_c and V_c are pressure and volume cutoffs, respectively.

The controls in this formulation are continuous functions of time so that the optimization problem as stated is infinite dimensional. Our initial approach in addressing this dimensionality difficulty is to *discretize* the optimization problem. This means that we restrict the controls in the sense that we assume that they are piecewise constant functions. Discretizing allows us to apply powerful dynamic programming solution methods, as described below.

Traditional approaches would discretize the range of values of the controls. However, for this problem it is more convenient to discretize on the values of pressure and volume, since these variables are nonnegative and bounded above (i.e., the set of admissible values is contained in a finite interval). Working in the volume–pressure space (henceforth, V–P space) is convenient since trajectories between two points in V–P space that require a fixed time uniquely determine corresponding constant production and water injection rates. From the previous section, the solutions for volume and pressure with r and s constant are

$$V(t)=V_0-s\cdot t\cdot(1+g) \tag{24}$$

$$P(t)=rP_0V_0/(sN_0)+[P_0-(rP_0V_0/(sN_0))]\,[1-(s(1+g)/V_0)t]^{-(1/1+g)} \tag{25}$$

Consider the trajectory in V–P space from the point (V_0,P_0) to the point (V_1, P_1). From equation (24), we must have the constant water injection rate

$$s=(V_0-V_1)/t\cdot(1+g)$$

Equation (25), which is linear in r, implies that we also must have the constant production rate

$$r=\frac{P_1-P_0\left[1-\dfrac{s(1+g)}{V_0}t\right]^{-(1/1+g)}}{\dfrac{P_0V_0}{sN_0}\cdot\left[1-\left(1-\dfrac{s(1+g)}{V_0}t\right)^{-(1/1+g)}\right]}$$

The recovery corresponding to this trajectory is

$$u=rt$$

Given a sequence of points in V–P space, say (V_0,P_0), (V_1,P_1), . . ., (V_n,P_n), the corresponding ultimate recovery is

$$u = \sum_{i=1}^{n} u_i$$

where u_i is the recovery from point (V_{i-1}, P_{i-1}) to point (V_i, P_i). Maximizing ultimate recovery is therefore a matter of selecting the optimal set of points on the path in V–P space.

Figure 1 illustrates a sample discretizing or gridding of the V–P space. Our approach is to consider the set of paths through this grid of points and to select that path (i.e., a collection of points in Fig. 1) which yields the maximum ultimate recovery. It is unnecessary to consider all possible paths for the grid as in Figure 1. We use dynamic programming to examine these paths implicitly and to determine the one that is optimal.

For the dynamic programming formulation, we consider an $(n+1)\times(n+1)$ grid of points. Suppose we have $V_0 \geqslant V_1 \geqslant \cdots \geqslant V_n$ and $P_0 \geqslant P_1 \geqslant \cdots \geqslant P_n$. The point (i, j) in the grid corresponds to (V_i, P_i) in V–P space. The dynamic programming approach is to solve the following problem:

$$\max_{i,j}\left\{\max_{\{(i^*,j^*):\, i^*\leqslant i, j^*\leqslant j\}}\left\{R\Big((i^*,j^*),(i,j)\Big)+R^*\Big((1,1),(i^*,j^*)\Big)\right\}\right\}$$

where $R\,\big((i^*,j^*),\,(i,j)\big)$is the recovery from point (i^*,j^*) to point (i,j), and

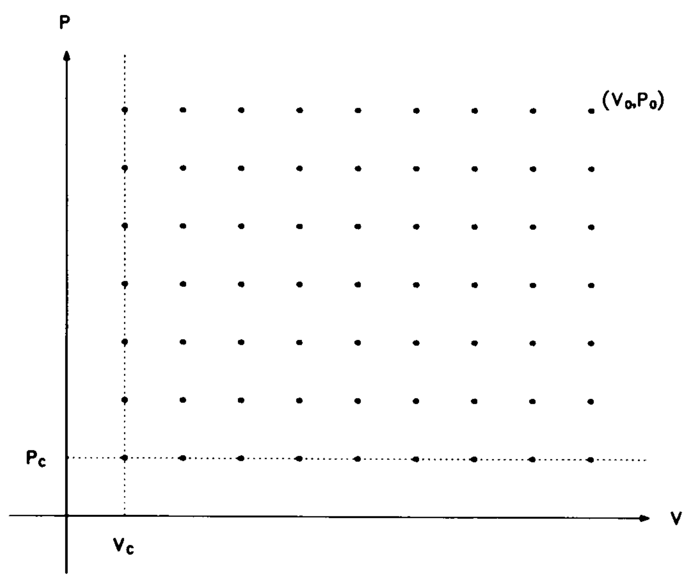

Figure 1. Discretizing V–P space where (V_0,P_0) is the initial state of the reservoir and V_c and P_c are volume and pressure cutoffs (terminal), respectively.

$R^*((1,1),(i^*,j^*))$ is the optimal recovery from the starting point to the point (i^*,j^*).

This dynamic programming formulation induces an algorithm for computing the maximum recovery over this grid of points. This algorithm is, as follows:

0. Initialize an $(n+1)\times(n+1)$ array A to zeroes.
1. Proceed through the grid by considering the points in the order: (1,1), (1,2), . . . , (1, $n+1$), (2,1), (2,2), . . . , (2, $n+1$), . . . , ($n+1$,1), ($n+1$,2), . . . , ($n+1$,$n+1$).
2. For each successive point (i,j), compute
$$A(i,j)=\max_{\substack{i^*=1,2,\ldots,i\\ j^*=1,2,\ldots,j}}\{R((i^*,j^*),(i,j))+A(i^*,j^*)\}$$
3. The maximum recovery is the largest value in the matrix A.

A simple FORTRAN program was written on a LASL 6600 CDC machine to implement this algorithm. Figure 2 illustrates the optimal path we obtained for all trial combinations of initial values, time increments, entrapment factors, and so forth. We also noted that if the starting point were (V_i,P_i),

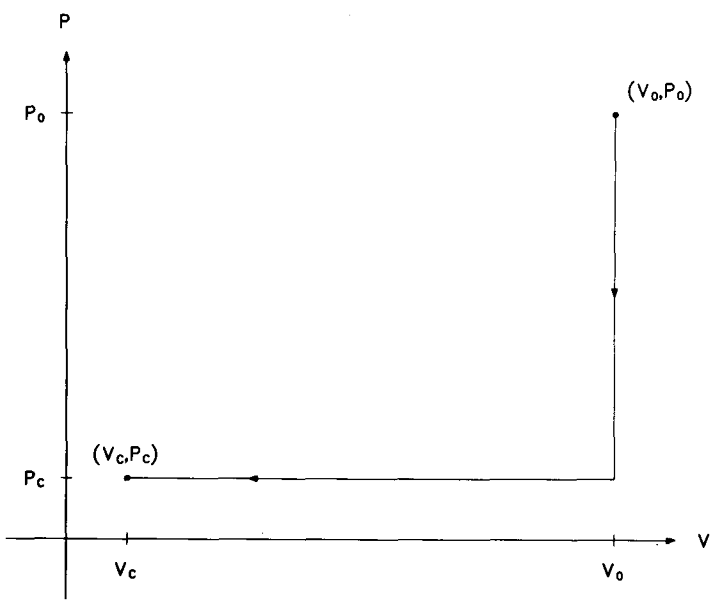

Figure 2. Optimal path where (V_0,P_0) is the initial state of the reservoir and V_c and P_c are volume and pressure cutoffs (terminal), respectively.

then the resulting optimal path would follow the path from (V_i,P_i) to (V_i,P_c) to (V_c,P_c). In terms of the gas production from the reservoir, this leads to the following optimal production strategy:

> Withdraw gas from the reservoir without waterflooding until the pressure declines to the cutoff pressure. Then withdraw gas with waterflooding so as to maintain the pressure (at the cutoff pressure) until the volume declines to the cutoff volume.

In the next section this result is established by means of a rigorous analytical proof. The dynamic programming approach, however, was valuable for two reasons: it led to an interesting result that we could then prove analytically, and it helped us to formulate the problem in V–P space, an essential aspect of the proof.

THE OPTIMAL CONTROLS

The goal of this section is to prove a theorem describing the controls that give the maximum recovery of gas from a gas–water reservoir. The physical intuition associated with the optimal path is that water should never be injected into the reservoir if gas can be removed without water injection. This intuition is shown to be correct at several stages of the proof.

Recall that our system is determined for $t \geqslant 0$ by

$$\begin{aligned} &dV(t)/dt = [-s(t)\,(1+g) \\ &dP(t)/dt = [-r(t)RT_0 + P(t)s(t)]/V(t) \\ &P(t)V(t) = N(t)RT_0 \end{aligned} \tag{26}$$

where $P(0)=P_0$, $V(0)=V_0$, $N(0)=N_0$, $g \geqslant 0$, and

$$\left.\begin{aligned} P(t) \geqslant P_c > 0 \\ V(t) \geqslant V_c > 0 \end{aligned}\right\} \text{ for all } t$$

We restrict (r,s) to be members of

$\mathcal{S} = \{(r,s)$: $r(t)$, $s(t) \geqslant 0$ are continuous for all except a finite number of $t \geqslant 0$, $P(t) \geqslant P_c$ and $V(t) \geqslant V_c$ for all $t\}$

Let t_c = maximum $\{t$: $P(t) \geqslant P_c$ and $V(t) \geqslant V_c\}$ ($t_c = \infty$ is possible) for a given $(r,s) = S \in \mathcal{S}$. Then, define the *recovery associated with S, R(S)*, by

$$R(S) = \int_0^{t_c} r(t)\,dt \tag{27}$$

The problem is to find $S_M \in \mathcal{S}$ such that

$$R(S_M) = \max_{S \in \mathcal{S}} R(S)$$

Our solution of this problem is given in the following theorem which explicitly gives all such S_M.

Theorem: For all $S \in \mathscr{S}$,

$$R(S) \leqslant R(S_M)$$

where $S_M = (r,s)$ satisfies

$$s(t) = 0, \qquad 0 \leqslant t \leqslant t_1$$

$$\int_0^{t_1} r(t)dt = V_0(P_0 - P_c)/RT_0$$

$$\int_{t_1}^{t_2} s(t)dt = (V_0 - V_c)/(1+g)$$

$$r(t) = P_c s(t)/(RT_0), \qquad t_1 < t \leqslant t_2$$

and

$$r(t) = s(t) = 0, \qquad t_2 < t < \infty$$

where

$$0 \leqslant t_1 \leqslant t_2 \leqslant \infty$$

The path that S_M induces in V–P space is sketched in Figure 2. Before outlining a proof of the theorem, we recall some results of the previous section. Let $s = s(t)$ and $r = r(t)$ be constant for $t \in [t_{i-1}, t_i]$ where $V(t_{i-1}) = V_{i-1}$, $P(t_{i-1}) = P_{i-1}$, $V(t_i) = V_i$, and $P(t_i) = P_i$. Then, if $s > 0$,

$$V_i = V_{i-1} - s(1+g)\Delta t \tag{28}$$

and

$$P_i = (rRT_0/s) + [P_{i-1} - (rRT_0/s)]\,[1 - (s(1+g)\Delta t/V_{i-1})]^{-(1/1+g)} \tag{29}$$

where $\Delta t = t_i - t_{i-1}$. If $s = 0$,

$$V_i = V_{i-1} \tag{30}$$

and

$$P_i = (-rRT_0\Delta t/V_{i-1}) + P_{i-1} \tag{31}$$

Next we outline the proof of the theorem:

A. Approximate an arbitrary $(r^*,s^*) \in \mathscr{S}$ by $(r,s) \in \mathscr{S}$ where $r(t)$ and $s(t)$ are piecewise constant.
B. Establish inequalities indicated by Figure 3. This will allow replacing the (V,P) path of A by horizontal and vertical segments.

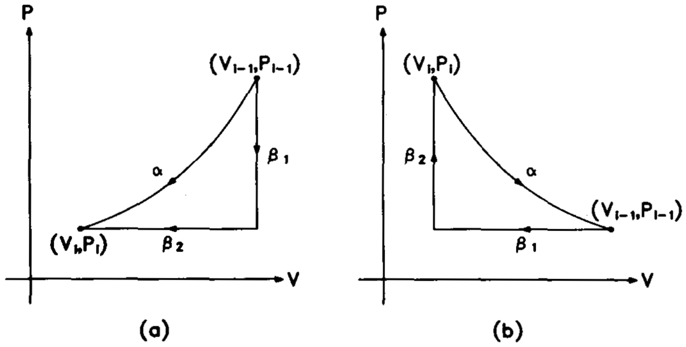

Figure 3. Paths (α) in V–P space for constant (r,s) which have recoveries less than or equal to the paths $(\beta_1+\beta_2)$, $(R(\alpha)\leqslant R(\beta_1)+R(\beta_2))$.

C. Complete the proof by applying B to the general case.
D. Derive the form of S_M from the shape of the optimal (V,P) path.

The details in proving A through D are lengthy and are available from the authors in a technical report (Johnson, Waterman, and Monash, 1978).

We instead present an elegant proof of this theorem which was obtained by W. Fleming (personal communication, 1977). His argument is now stated. Consider any curve $V=V(t)$ and $P=P(t)$ which lies in the admissible region $V\geqslant V_c$ and $P\geqslant P_c$ with $dV/dt=-(1+g)s$. The total differential of PV is

$$\begin{aligned} d(PV)/dt &= d(NRT_0)/dt \\ &= -rRT_0 - gsP \\ &= -rRT_0 - (g/1+g)\,(1+g)sP \\ &= -rRT_0 + cP(dV/dt) \end{aligned}$$

where $c=g(1+g)^{-1}$. Since $\int[d(PV)/dt]\,dt$ depends only on the endpoints of a curve, maximizing $\int r dt$ is equivalent to maximizing $\int P(dV/dt)\,dt$. However, $\int(PdV/dt)\,dt$ is equal to $\int PdV(t)$, which is the negative of the area under the curve given in Figure 2. This curve obviously has the minimum area of all curves satisfying the constraints. Hence, the proof is complete.

CONCLUSIONS

This paper has given the details of modeling and optimizing a gas–water reservoir with a variable production rate and enhanced recovery with waterflooding. The dynamic programming empirical solutions were verified analytically. The optimal two control (production rate and water injection rate) strategy for maximizing recovery is to produce without water injection until the attainment of a pressure cutoff, and then to waterflood so as to maintain the pressure until attainment of the volume cutoff. This strategy concurs with the industry practice of not waterflooding gas reservoirs due to the entrapment problem. A study by Johnson, McFarland, Monash, and Lohrenz (1977) has also suggested that even in the absence of entrapment (i.e., $g=0$), waterflooding appears to be *economically* infeasible.

In an extension of our results, Mantini and Beyer (1977) have shown that the same set of controls are optimal for four other situations. For example, they consider maximizing the present value of net revenues and the internal rate of return. Their techniques are similar to those of this paper.

An obvious modification of this work is to consider optimizing gas reservoirs with "natural" water influx. For this situation, water from the surrounding aquifer will invade the reservoir as gas is withdrawn. Water invasion helps to maintain reservoir pressure which can assist in production, but the water also traps gas. Hence, an interesting tradeoff in these properties occurs. The primary advantage of modeling and optimizing these natural water-drive reservoirs is that real data is available. Thus, the efficacy of this approach can be examined.

Also, we have an interest in adding a stochastic component to the above model and in extending our results to the much more difficult problem of waterflooding gas-oil reservoirs.

ACKNOWLEDGMENTS

The authors would like to express their gratitude to Dr. John Lohrenz for his continuing support and encouragement during the course of this research. Thanks are due also to S. Juan and the referees for some insightful comments.

REFERENCES

Johnson, M. E., McFarland, J. W., Monash, E. A., and Lohrenz, J., 1977, MER (maximum efficient rate) using a gas reservoir model with waterflooding: Los Alamos Scientific Laboratory report LA-6989-MS.

Johnson, M. E., Waterman, M. S., Monash, E. A., 1978, Modeling and optimizing a gas–water reservoir: Los Alamos Scientific Laboratory report LA-7088.

Mantini, L. A. and Beyer, W. A., 1977, Optimization of natural gas production by waterflooding: Los Alamos Scientific Laboratory report, LA-UR-77-2273. Accepted for publication in Applied Mathematics and Optimization.