f factorial moments and using

$$- \sum_{x_2=r_2}^{s} x_2^{(r_2)} p_2(x_2) \Big)$$

$$x_2^{(r_2)} p_2(x_2) k_r(x_2). \qquad (A17)$$

$p_2(x_2)k_r(x_2)$

$$p_2(x_2) j_r(x_2). \qquad (A18)$$

stituting it into (A18) reduces to

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COMMENTS, CONJECTURES, AND CONCLUSIONS

Section Editor: *I. J. Good*

Please be succinct but lucid and interesting.

C1. LEAST SQUARES WITH NONNEGATIVE REGRESSION COEFFICIENTS

In the literature, various types of inequality restrictions have been considered for regression problems. In Waterman (1974), a solution utilizing existing regression programs is given for the problem with the restriction that all regression coefficients be nonnegative. Armstrong and Frome (1976) give a branch-and-bound solution to this problem, while Khuri (1976) shows that more general inequality constraints can be reduced to the nonnegative restrictions mentioned above.

In this note, the procedure of Waterman is modified to reduce the number of unrestricted problems that must be solved. First we review the original procedure.

Let X be a given m x n matrix of rank $m \leq n$ and Y an n-dimensional column vector of responses. The problem is to minimize

$$L(\lambda) = ||Y - X^T\lambda|| = (Y - X^T\lambda)^T(Y - X^T\lambda) \qquad (1)$$

subject to the constraints

$$\lambda_i \geq 0 \text{ for } i = 1, 2, \ldots, m . \qquad (2)$$

If the usual, unconstrained, solution $\hat{\lambda} = (XX^T)^{-1}XY$ satisfies (2), then $\hat{\lambda}$ solves the constrained problem. Otherwise, the solution to the constrained problem must be a boundary point

F*

of $([0, \infty))^m$ and therefore at least one $\lambda_i = 0$. To proceed, perform all possible regressions where one or more λ_i in (1) are set equal to zero. The nonnegative problem is solved by picking the λ satisfying (2) such that $L(\lambda)$ is smallest.

The modification proposed here is based on the following:

THEOREM Let λ^* denote the solution to the nonnegative problem and $\hat{\lambda}$ denote the solution to the unconstrained problem. If j $(1 \leq j < m)$ of the components of λ are negative, then at least one of the corresponding j components in λ^* is zero.

Proof Assume j of the components of $\hat{\lambda}$ are negative, and, without loss of generality, let us assume they are the first j components. That is, $\hat{\lambda}_i \geq 0$ for all $i > j$, and $\hat{\lambda}_i < 0$ for all $1 \leq i \leq j$.

Then $\lambda_i \geq 0$ and $\lambda_i^* \geq 0$ if $i > j$, and thus $\alpha\hat{\lambda}_i + (1-\alpha)\lambda_i^* \geq 0$ if $i > j$ and $0 \leq \alpha \leq 1$.

Suppose $\lambda_i^* > 0$ for all $i \leq j$. Then, for $i \leq j$, there exists α_i, $0 < \alpha_i < 1$, so that $\alpha_i\hat{\lambda}_i + (1-\alpha_i)\lambda_i^* > 0$. Now, choose $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_j\}$ and consider the vector $\alpha\hat{\lambda} + (1-\alpha)\lambda^* = \xi$. Then for $1 \leq i \leq m$, $\xi_i \geq 0$ and

$$
\begin{aligned}
L(\xi) &= ||Y - \xi X|| = ||(\alpha Y - \alpha\hat{\lambda}X) + ((1-\alpha)Y - (1-\alpha)\lambda^* X)|| \\
&\leq ||\alpha(Y - \hat{\lambda}X)|| + ||(1-\alpha)\ (Y - \lambda^* X)|| \\
&= \alpha||Y - \hat{\lambda}X|| + (1-\alpha)||Y - \lambda^* X|| \\
&\leq \alpha||Y - \lambda^* X|| + (1-\alpha)||Y - \lambda^* X|| \\
&= ||Y - \lambda^* X|| = L(\lambda^*) ,
\end{aligned}
$$

where the first inequality comes from the triangle inequality for norms and the second inequality is due to $\hat{\lambda}$ being the best unconstrained solution.

Thus, we have $L(\xi) \leq L(\lambda^*)$. But, if $L(\xi) = L(\lambda^*)$, we have a contradiction, since λ^* is the unique best constrained solution, and, if $L(\xi) < L(\lambda^*)$, we contradict the fact the λ^* is the best constrained solution.

Hence, we cannot have $\lambda_i^* > 0$ for all $i \leq j$, and thus at least one of the λ_i^*, $i \leq j$, is zero.

As an immediate consequence of this result, if $\hat{\lambda}$ has exactly one negative component, then the corresponding component in λ^* is zero. Although $\hat{\lambda}$ will frequently have more than one negative component, the result can still be utilized. Suppose $\hat{\lambda}$ has negative values in the $\ell_1, \ell_2, \ldots, \ell_k$ components. Then, solve the k (nonnegative) problems where exactly one of the $\ell_1, \ldots, \ell_k$ components is set equal to zero. Of course, the result is to be reapplied to each of these problems and care is to be taken not to duplicate computation.

The proof of our result is similar to that of Armstrong and Frome (1976) but our result is much stronger. The four-variable example they give is reduced to 13 of $2^4=16$ regressions. Using our technique, the problem is reduced to 4 of 16 regressions. (Their "Nodes" 1, 2, 4, 5.)

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