SYMMETRIES FOR CONDITIONED RUIN PROBLEMS

BY
W. A. BEYER AND M. S. WATERMAN

Reprinted from the MATHEMATICS MAGAZINE
Vol. 50, No. 1, January 1977
pp. 42–45

Symmetries for Conditioned Ruin Problems

W. A. Beyer
M. S. Waterman
Los Alamos Scientific Laboratory

In this paper we discuss some interesting symmetries that arise in conditioning random walks on the integers with absorbing boundaries. These results are ancillary to a study done on conditional expected duration of walks used in a mathematical model of cancer tumors [1].

A tumor is an abnormal mass of tissue which is not inflammatory. A cancer tumor is thought of as arising from perhaps a single wayward cell which has lost the ability to control itself. At least this is a model for certain simple cases. The wayward cell starts reproducing without regard to the presence of other cells. Of course, the wayward cell may die before catastrophe overtakes the host organism. On the other hand, the single wayward cell may produce a family tree of descendants (called a clone) large enough to be noticeable to the host organism. In this case the descendents become a tumor. It is the tumor which is noticeable, and not the early cells that died without progeny. Our interest in the tumor growth is to estimate the expected time for the cancerous clone to reach tumor size from a single wayward cell, given that tumor size was reached.

Suppose each cell has probability $\lambda\,(0<\lambda<1)$ of dividing to produce two identical new cells and probability $\mu = 1-\lambda$ of dying. We are interested in the process by which a single cell becomes, by this chance mechanism, a macroscopic clone of N cells. Although the natural model for this problem is a birth and death process with linear growth, we will employ the simpler model of classical gambler's ruin. The classical ruin problem concerns a random walk on $\{0,1,\dots,N\}$ where, if $0<r<N$, the probability of moving from r to $r-1$ is $\mu = 1-\lambda$. The walk terminates when either absorbing state 0 or N is reached. If the initial state is z, where $0<z<N$, the duration of the walk will be the number of steps from z to absorption. The quantity in which we are interested is the conditional duration F_z of the walk, given that the walk terminates at N. For the cancer tumor problem, $z=1$.

Recently F. Stern [**5**] derived formulae for F_z and for E_z, the conditional duration of those walks terminating at 0. He also showed that $E_z = F_z$ when $2z = N$ and $0<\lambda<1$. S. M. Samuels [**4**] showed that, under these conditions, even more is true: the duration of the game and the absorption point are independent random variables. For a numerical example relevant to the cancer tumor model, let $z=1$, $N=10^4$, and $\lambda = .99$. Then, using Stern's formula with $r = .01/.99$,

$$F_z = \frac{(.99-.01)^{-1}}{1-r}\left[(10^4-1)(r-1)+2\cdot 10^4\left(\frac{r-r^{10^4}}{r^{10^4}-1}\right)\right] = 10203.04\dots.$$

If the formula for F_z is used with $\lambda = .01$ and $\mu = .99$, it turns out that, again, $F_z = 10203.04\dots$. We observed this symmetry in μ and λ from computer results for conditional expected duration before Stern's results appeared. That the conditional duration was equal for $\lambda = .99$ and $\lambda = .01$ seemed to us a surprising and paradoxical result. The purpose of this paper is to explain and generalize this phenomenon.

We begin with a random walk $\mathcal{W}$ on $\{0,1,\dots,N\}$ with absorbing barriers at 0 and N and for $0<r<N$, transition probabilities λ_r for $r\to r+1$ and $\mu_r = 1-\lambda_r$ for $r\to r-1$. Our major result can be summarized as follows:

THEOREM. *Conditioning the random walk $\mathcal{W}$ on absorption at one of the barriers yields a new random walk of the same type. If the transition probabilities λ and μ of $\mathcal{W}$ are independent of position, transition probabilities of the conditioned walk depend on position and are symmetric in μ and λ.*

It follows from this theorem that when the transition probabilities of $\mathcal{W}$ are independent of position, any function of the transition probabilities of the conditioned walk is symmetric in μ and λ. For example, the expected duration of the conditioned walk is — as we noted earlier — symmetric in μ and λ. This result underlies the results of Stern and Samuels and will be discussed more fully later in this note. But before doing that we want to prove our theorem.

Proof. If q_z is the probability of absorption at 0 for a walk starting at z, then $q_z = \lambda_z q_{z+1} + u_z q_{z-1}$ with $q_0 = 1$ and $q_N = 0$. Parzen ([**3**], p. 233) gives the solution to this difference equation with these boundary conditions as

$$q_z = \left(\sum_{i=z}^{N-1}\prod_{j=1}^{i} x_j\right)\Big/\left(\sum_{i=0}^{N-1}\prod_{j=1}^{i} x_j\right) \tag{1}$$

where $x_j = \mu_j/\lambda_j$ and, by convention, $\Pi_{j=1}^{0} x_j \equiv 1$. If p_z is the probability of absorption at N for a walk starting at z, the value of p_z may be obtained from (1) by interchanging λ_j and μ_j (thereby replacing x_j by x_j^{-1}) and replacing z by $N - z$:

$$p_z = \left(\sum_{i=0}^{z-1} \prod_{j=1}^{i} x_j\right) \Big/ \left(\sum_{i=0}^{N-1} \prod_{j=1}^{i} x_j\right).$$

Clearly $p_z + q_z = 1$. If $\lambda_z = \lambda$ and $\mu_z = \mu$ for $0 < z < N$, then we obtain the well-known formula (Feller [2], p. 345) that

$$q_z = \begin{cases} \dfrac{(\mu/\lambda)^N - (\mu/\lambda)^z}{(\mu/\lambda)^N - 1}, & \lambda \neq \mu, \\ 1 - z/N, & \lambda = \mu. \end{cases}$$

Let us now compute the probability of a left step conditioned on absorption at a given boundary, say at $z = n$. Let P denote the probability measure associated with $\mathscr{W}$, L the event of the first step left, B_N the event of absorption at N, and S_z the event of being at z for $0 < z < N$. Then

$$P(L \mid S_z \cap B_N) = P(L \cap S_z \cap B_N)/P(S_z \cap B_N)$$

(2)
$$= \mu_z \frac{P(S_{z-1} \cap B_N)}{p_z} = \mu_z \frac{p_{z-1}}{p_z} = \mu_z \left(\sum_{i=0}^{z-2} \prod_{j=1}^{i} x_j\right) \Big/ \left(\sum_{i=0}^{z-1} \prod_{j=0}^{i} x_j\right)$$

$$= \mu_z \frac{1 + x_1 + x_1x_2 + \cdots + x_1x_2\cdots x_{z-2}}{1 + x_1 + x_1x_2 + \cdots + x_1x_2\cdots x_{z-1}}.$$

To investigate whether this formula for $P(L \mid S_z \cap B_N)$ is symmetric in λ and μ, we replace x_j by x_j^{-1} and μ_z by $\lambda_z = \mu_z x_z^{-1}$ to obtain

(3)
$$\mu_z \left(\sum_{i=0}^{z-2} \prod_{j=i+1}^{z-1} x_j\right) \Big/ \left(\sum_{i=0}^{z-1} \prod_{j=i+1}^{z} x_j\right) = \mu_z \frac{x_1x_2\cdots x_{z-1} + x_2x_3\cdots x_{z-1} + x_3x_4\cdots x_{z-1} + \cdots + x_{z-1}}{x_1x_2\cdots x_z + x_2x_3\cdots x_z + x_3\cdots x_z + \cdots + x_z}.$$

In general, (2) and (3) will not have the same value unless x_j is independent of j. But if the x_j are independent of j, they will agree and have as a common value:

$$P(L \mid S_z \cap B_N) = \mu \frac{P_{z-1}}{P_z} = \begin{cases} \dfrac{\mu^z\lambda - \mu\lambda^z}{\mu^z - \lambda^z} & (\mu \neq \lambda) \\ \mu \dfrac{N - z + 1}{N - z} & (\mu = \lambda) \end{cases}$$

A similar analysis for the first step right, R, leads to the formula

$$P(R \mid S_z \cap B_N) = \lambda \frac{p_{z-1}}{p_z} = \begin{cases} \dfrac{\mu^{z+1} - \lambda^{z+1}}{\mu^z - \lambda^z} & (\mu \neq \lambda) \\ \lambda \dfrac{N - z - 1}{N - z} & (\mu = \lambda) \end{cases}$$

Therefore, in either case, the conditional transition probabilities for uniform x_j are symmetric in μ and λ. This completes the proof of the theorem.

In the remainder of the note, the transition probabilities λ, μ are independent of position. The formulae in the case of absorption at zero are

$$P(R \mid S_z \cap B_0) = \frac{\lambda^{N-z}\mu - \lambda\mu^{N-z}}{\lambda^{N-z} - \mu^{N-z}} \quad (\mu \neq \lambda)$$

$$P(L \mid S_z \cap B_0) = \frac{\lambda^{N-z+1} - \mu^{N-z+1}}{\lambda^{N-z} - \mu^{N-z}} \quad (\mu \neq \lambda).$$

(These formulae are derived by interchanging λ and μ, interchanging L and R, and replacing z by $N - z$. But λ is *still* the probability of a right step.) If we assume $\mu < \lambda$ and let $N \to \infty$, we obtain the probabilities conditioned on absorption at zero of a random walk on the nonnegative integers:

$$P(R \mid S_z \cap B_0) \to \mu \quad \text{and} \quad P(L \mid S_z \cap B_0) \to \lambda.$$

This yields a new random walk in which left and right probabilities are interchanged, a result noted by O'N. Waugh [6] who attributes it to D. G. Kendall.

Our theorem applies directly to the classical gambler's ruin problem, and it is in that context that we can relate our work to that of Samuels and Stern. In the ruin problem players A and B start with a and b dollars, respectively. They repeatedly toss a coin which has probability λ of heads, $0 < \lambda < 1$. A wins one dollar from B whenever heads occur, while B wins one dollar from A whenever tails occur. The game continues until one of the players has no money left. Samuels [4] showed that if $a = b$, the duration of the game is independent of who wins it. That is, if D is the duration of the game (i.e., the number of tosses in the game) and $W = 1$ or 0 according to whether A or B wins, then $P(D = d \mid W = 1) = P(D = d \mid W = 0)$ for all d. (Stern derives explicit expressions for the conditional expected duration, and notes symmetry if $a = b$).

Suppose now that our gamblers begin with equal fortunes $a = b$. We denote by $\bar{g}$ the reflection of a game history g obtained by interchanging heads and tails. This permits us to distinguish between a **symmetric** event, a set of game histories S for which $g \in S$ implies $\bar{g} \in S$, and an **antisymmetric** event in which $g \in S$ implies $\bar{g} \notin S$. (For example, the set of game histories for which $D = 10$ is a symmetric event, whereas the set of game histories in which A wins is an antisymmetric event.) With this terminology we can generalize Samuel's theorem in the following way: *In the classic gambler's ruin with equal initial fortunes, if E is a symmetric event and F is either of the antisymmetric events* "A wins" *or* "B wins", *then E and F are independent.* If E is the event "$D = d$" and F is the event "A wins", then one obtains Samuels' theorem that the duration of the game is independent of who wins.

To prove this denote by P_λ the probability measure for the games with probability λ of heads. Then since $P_\lambda(g) = P_\mu(\bar{g})$, one has $P_\lambda\{E \mid F\} = P_\mu\{\bar{E} \mid \bar{F}\} = P_\mu\{E \mid \sim F\}$. By the theorem, $P_\mu\{E \mid \sim F\} = P_\lambda\{E \mid \sim F\}$ and thus $P_\lambda\{E \mid F\} = P_\lambda\{E \mid \sim F\}$. Hence

$$\frac{P_\lambda\{E \cap F\}}{P_\lambda\{F\}} = \frac{P_\lambda\{E \cap \sim F\}}{P_\lambda\{\sim F\}} = \frac{P_\lambda\{E\} - P_\lambda\{E \cap F\}}{1 - P_\lambda\{F\}}.$$

This relation simplifies to the desired statement of independence: $P_\lambda\{E \cap F\} = P_\lambda\{E\}P_\lambda\{F\}$. (The case of $P_\lambda\{F\} = 0$ or $P_\lambda\{\sim F\} = 0$ can be handled separately.)

We are indebted to Dr. G. I. Bell for suggesting this problem and for discussions and to Professor S. M. Samuels for pointing out connections betweeen our results and other work. Work on this paper was performed under the auspices of the United States Energy Research and Development Administration.

References

[1] G. I. Bell, Models of carcinogenesis as an escape from mitotic inhibitors, Science, 192 (1976) 569–572.

[2] W. Feller, An Introduction to Probability Theory and its Applications, vol. I, 3rd ed., Wiley, New York, 1968.

[3] E. Parzen, Stochastic Processes, Holden-Day, San Francisco, 1962.

[4] S. M. Samuels, The classical ruin problem with equal initial fortunes, this MAGAZINE, 48 (1975) 286–288.

[5] F. Stern, Conditional expectation of the duration in the classical ruin problem, this MAGAZINE, 48 (1975) 200–203.

[6] W. O'N. Waugh, Conditioned Markov processes, Biometrika, 45 (1958) 241–249.