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F-Expansions of rationals

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The ergodic theorem has been used to deduce results about the F-expansions of almost all x in $(0, 1)^n$. A simple lemma from measure theory yields some corresponding statements about the expansions of the rationals, a set of measure zero.

§1. Introduction

There have been several papers dealing with the ergodic properties of F-expansiosn of n-dimensional reals ([3, 8, 10, 11]). For $x \in (0, 1)^n$, we have

$$x = \lim_{k \to \infty} F(a_1(x) + F(a_2(x) + \cdots + F(a_k(x)) \ldots)).$$

Under certain conditions (see [11]) we have the shift T ergodic and there exists a measure $\mu \sim \lambda$, n-dimensional Lebesgue measure, such that T is ergodic and measure preserving with respect to μ. The individual ergodic theorem yields, for $f \in L_1(\mu)$,

$$\frac{1}{m} \sum_{i=0}^{m-1} f(T^i(x)) \to \int f \, d\mu \quad \text{for} \quad \text{a.a. } x.$$

In particular, if $f = I_{B(a)}$, where $B(a) = \{x : [F^{-1}(x)] = a\}$, then

$$\frac{1}{m} \sum_{i=0}^{m-1} I_{B(a)}(T^i(x)) \to \mu(B(a)) \quad \text{for} \quad \text{a.a. } x.$$

Thus ergodic theory yields a statement concerning the distribution of digits in the expansion of almost all x. Much information of a similar character can be derived.

This paper is an attempt to utilize these ergodic theory results to deduce related results concerning the rationals, a set of Lebesgue measure zero. Our results are known for one-dimensional continued fractions [5, p. 328] but do not appear to have been deduced for the general case. In this connection, see [2] for a computational treatment of one- and two-dimensional continued fractions.

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§2. One-dimensional F-expansions

Our proof is based on a measure theoretic lemma and follows Knuth [5]. We define $\nu(E)$ to be the cardinality of the set E. The lemma is proved for n-dimensions for use in §3.

LEMMA 1. *Let* $A_m = \{0/m, 1/m, \ldots, (m-1)/m\}^n$. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be countable unions of disjoint rectangles in* $[0, 1]^n$, $\mathbf{R} \cap \mathbf{S} = \phi$, *and* $\lambda(\mathbf{R} \cup \mathbf{S}) = 1$. *Then*

$$\lim_{m \to \infty} \frac{\nu(A_m \cap \mathbf{R})}{m^n} = \lambda(\mathbf{R}). \tag{1}$$

Proof. Now $\mathbf{R} = \bigcup_{i=1}^{\infty} R_i$ and $\mathbf{S} = \bigcup_{i=1}^{\infty} S_i$ where each R_i and S_i is of the form $E = \times_{l=1}^{n} I_l$, I_l an interval. Now

$$\prod_{l=1}^{n} (m\lambda(I_l) - 1) \leqslant \nu(E \cap A_m) \leqslant \prod_{l=1}^{n} (m\lambda(I_l) + 1). \tag{2}$$

Choose N such that $\lambda(\mathbf{R}) < \lambda(\mathbf{R}_N) + \varepsilon$ and $\lambda(\mathbf{S}) < \lambda(S_N) + \varepsilon$, where $\mathbf{R}_N = \bigcup_{i=1}^{N} R_i$ and $\mathbf{S}_N = \bigcup_{i=1}^{N} S_i$. Let $\mathbf{U}_N = (\mathbf{R} \cup \mathbf{S})' \cup \bigcup_{i>N} (R_i \cup S_i)$, where the prime $'$ denotes complement. Define $r_m = \nu(\mathbf{R}_N \cap A_m)$, $s_m = \nu(\mathbf{S}_N \cap A_m)$, $u_m = \nu(\mathbf{U}_N \cap A_m)$. Note that $r_m + s_m + u_m = m^n$. We easily have

$$\frac{r_m}{m^n} \leqslant \frac{r_m + u_m}{m^n} = 1 - \frac{s_m}{m^n}. \tag{3}$$

From (2) we can show

$$\lambda(\mathbf{R}_N) - \frac{N2^n}{m} \leqslant \frac{r_m}{m^n}$$

and obtain

$$\lambda(\mathbf{R}) - \varepsilon - \frac{N2^n}{m} \leqslant \frac{r_m}{m} \leqslant \lambda(\mathbf{R}) + \frac{N2^n}{m}.$$

Thus, from (3) we obtain $\lim_{m \to \infty} r_m/m^n = \lim_{m \to \infty} (r_m + u_m)/m^n = \lambda(\mathbf{R})$ and (1) therefore holds.

For our discussion of F-expansions we refer to [10] and [11]. In this section we deal one-dimensional F-expansions. In n-dimensions some measure theoretic difficulties arise. Let F be a function defining such an F-expansion. Then we define $T(x) = F^{-1}(x) - [F^{-1}(x)]$ and $a_\nu(x) = [F^{-1}(T^{\nu-1}(x))]$ where $x \in (0, 1)$. Define

$$B_\nu = B(k_1, k_2, \ldots, k_\nu) = \{x \in (0, 1) \colon a_i(x) = k_i,\ i = 1, \ldots, \nu\}.$$

Condition (A) or (B) of Rényi [7] assumes F either decreasing or increasing and in these cases each B_v is an interval. The addition of another condition allows Rényi to show there exists a measure μ such that μ is invariant with respect to T and $C^{-1}\lambda(E)\leqslant\mu(E)\leqslant C\lambda(E)$ for all measurable sets E and some constant C depending on F only. This, of course, implies λ and μ are equivalent. Finally let $\mathbf{C}_k(a_1,\ldots,a_l)=$ $=\bigcup_{b_1,\ldots,b_k} B(b_1,\ldots,b_k,a_1,\ldots,a_l)$ which is the set of all points $x\in(0,1)$ such that $a_{k+1}(x)=a_1,\ldots,a_{k+l}(x)=a_l$.

THEOREM 1. *Suppose F satisfies condition* (A) *or* (B). *Then*

$$\lim_{m\to\infty}\frac{\nu(\mathbf{C}_k(a_1,\ldots,a_l)\cap A_m)}{m}=\lambda(\mathbf{C}_k(a_1,\ldots,a_l))=\lambda(T^{-k}B(a_1,\ldots,a_l)). \tag{4}$$

Proof. Let $\mathbf{R}=\mathbf{C}_k(a_1,\ldots,a_l)$ and $\mathbf{S}$ equal the union of all B_{k+l} not included in $\mathbf{R}$. Both $\mathbf{R}$ and $\mathbf{S}$ are unions of disjoint intervals, $\mathbf{R}\cap\mathbf{S}=\phi$. Also $U=\{x: a_i(x)$ is undefined for some $1\leqslant i\leqslant k+l\}$ is countable and therefore $\lambda(\mathbf{U})=0$. The requirements of Lemma 1 are satisfied and (4) follows.

COROLLARY 1. *If F satisfies condition* (A) *or* (B) *and the assumptions of Kuzmin's Theorem* [10], *then*

$$\lim_{k\to\infty}\lim_{m\to\infty}\frac{\nu(\mathbf{C}_k(a_1,\ldots,a_l)\cap A_m)}{m}=\mu(B(a_1,\ldots,a_l)). \tag{5}$$

where μ is the invariant measure associated with the transformation $T(x)=F^{-1}(x)-[F^{-1}(x)]$.

Proof. Theorem 2 of [10] states $\lambda(T^{-k}B(a_1,\ldots,a_l))\to\mu(B(a_1,\ldots,a_l))$.

The previous results relate the number of $x\in A_m$ with digits $a_{i+k}(x)=a_i$, $i=1,\ldots,l$, to the measure of $B(a_1,\ldots,a_l)$. The next Theorem considers the average value of a function of the digits. It is possible to prove a more general theorem here.

THEOREM 2. *Let F be as in Corollary* 1. *Suppose g is real valued and $\sum_a|g(a)|\times|\lambda(B(a))|<\infty$ where the summation is over all values a such that $a=a_i(x)$ for some $x\in(0,1)$: Then*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{x\in A_m}g(a_{k+1}(x))=\sum_a g(a)\,\lambda(T^{-k}B(a)), \tag{6}$$

where the first summation in (6) *is over $x\in A_m$ such that $a_{k+1}(x)$ is defined.*

Proof. Let μ be the invariant measure associated with T. From $C^{-1}\lambda(E)\leqslant\mu(E)\leqslant C\lambda(E)$ and $\mu(T^{-k}E)=\mu(E)$, we have the following equivalent inequalities:

$\sum_a |g(a)| \, \mu(B(a)) < \infty$, $\sum_a |g(a)| \, \lambda(B(a)) < \infty$, and $\sum_a |g(a)| \, \lambda(T^{-k}(B(a)) < \infty$. Since

$$\frac{1}{m} \sum_{x \in A_m} g(a_{k+1}(x)) = \sum_a g(a) \frac{\nu(C_k(a) \cap A_m)}{m},$$

the result follows from Theorem 1.

COROLLARY 2. *If F satisfies the assumptions of Corollary* 1 *and Theorem* 2, *then*

$$\lim_{k \to \infty} \lim_{m \to \infty} \frac{1}{m} \sum_{x \in A_m} g(a_{k+1}(x)) = \sum_a g(a) \, \mu(B(a)).$$

Proof. Since $\lambda(T^{-k}(B(a))) \leqslant C\lambda(B(a))$ by an argument of Rényi [7] and $\sum g(a) \, C\lambda(B(a)) < \infty$, the Bounded Convergence Theorem gives us

$$\lim_{k \to \infty} \sum_a g(a) \, \lambda(T^{-k}(B(a))) = \sum_a g(a) \lim_{k \to \infty} \lambda(T^{-k}(B(a))) = \sum_a g(a) \, \mu(B(a)).$$

The last equality is by Theorem 2 of [10].

As an application take F such that all $a_i \geqslant 1$ and $g(m) = \log(m)$ satisfies the hypothesis of Theorem 2. Then

$$\left(\prod_{i=0}^{m-1} a_{k+1} \left(\frac{i}{m} \right) \right)^{1/m} \to \exp \left\{ \sum_{l=1}^{\infty} \log(l) \, \lambda(T^{-k} B(l)) \right\}$$

and

$$\lim_{k \to \infty} \lim_{m \to \infty} \left(\prod_{i=0}^{m-1} a_{k+1} \left(\frac{i}{m} \right) \right)^{1/m} = \lim_{k \to \infty} \exp \left\{ \int \log(a_{k+1}(x)) \, d\mu(x) \right\}$$
$$= \exp \left\{ \int \log a_1(x) \, d\mu(x) \right\}.$$

Since $T(x) = 1/x - [1/x]$ on (0, 1) is the shift for the digits of continued fractions, our results hold for continued fraction expansions of rationals. Theorem 1 and Corollary 1 are known in the case of continued fractions [5, p. 328]. Another example is q-adic expansions where q is an integer greater than 1 and $T(x) = qx - [qx]$.

§3. The Jacobi-Perron algorithm

The Jacobi-Perron algorithm can be used to expand almost all $x \in (0, 1)^n$. The F is defined by

$$F(x) = F(x_1, x_2, \ldots, x_n) = \left(\frac{1}{x_n}, \frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n} \right).$$

Associated with this F we have

$$T(x)=T(x_1,\dots,x_n)=\left(\frac{x_2}{x_1}-\left[\frac{x_2}{x_1}\right],\frac{x_3}{x_1}-\left[\frac{x_3}{x_1}\right],\dots,\frac{1}{x_1}-\left[\frac{1}{x_1}\right]\right).$$

$$a_1(x)=\left(\left[\frac{x_2}{x_1}\right],\left[\frac{x_3}{x_1}\right],\dots,\left[\frac{1}{x_1}\right]\right),$$

and

$$a_\nu(x)=a_1(T^{\nu-1}(x)).$$

The convergence result is

$$x=\lim_{\nu\to\infty} F(a_1(x)+F(\cdots+F(a_\nu(x))\dots)).$$

F. Schweiger has examined measure theoretic properties of the Jacobi-Perron algorithm. Among other results, he has shown there exists $\mu\sim\lambda$ such that $\mu(T^{-1}E)=$ $=\mu(E)$ for all measurable $E\subset(0,1)^n$. In [8, 10, 11] there are references and a discussion of these and related topics.

The problem here is to establish results for the Jacobi-Perron algorithm corresponding to those of §2. Certain measure theoretic difficulties exist in the proof of those results for general F-expansions in n-dimensions but they can be overcome in certain cases. The notation of the next theorem is the n-dimensional analogue of that in §2.

THEOREM 3. *Let F and T be the transformations associated with the Jacobi-Perron algorithm. Then*

$$\lim_{m\to\infty}\frac{\nu(\mathbf{C}_k(a_1,\dots,a_l)\cap A_m)}{m^n}=\lambda(T^{-k}B(a_1,\dots,a_l)). \tag{7}$$

Proof. $\mathbf{C}_k(a_1,\dots,a_l)=\bigcup_{b_1,\dots,b_k}B(b_1,b_2,\dots,b_k,a_1,\dots,a_l)$ where the B_{k+l} are convex polytopes in $(0,1)^n$. It is this property of the Jacobi-Perron algorithm which allows us to write **R** and **S** defined below as countable unions of rectangles and apply Lemma 1. We delete $\mathfrak{U}=\{x:a_i(x)$ is undefined for some $i=1,\dots,k+l\}$. $\mathfrak{U}$ is a countable set of hyperplanes and hence $\lambda(\mathfrak{U})=0$. Thus $\mathbf{R}=\mathbf{C}_k(a_1,\dots,a_l)$ can be obtained as a countable union of rectangles as can **S** equal the union of all B_{k+l} not included in **R**. Equation (2) follows from Lemma 1.

The remaining Corollaries and Theorem of §2 hold for the Jacobi-Perron algorithm. [10] contains the results necessary in these proofs. For example, we have

$$\lambda(T^{-1}B(a))=\mu(B(a))+0(\varrho^k),$$

where $\varrho=(1-1/(n+1)^n)^{1/n}$. See [4].). Thus Corollary 1 follows.

§4. Remarks

These methods can easily be extended to include β-expansions [7] and Cantor's series [9]. Most ergodic properties of expansions of reals will yield a corresponding statement about the expansion of k/m, $0 \leqslant k < m$.

The invariant measure μ for the Jacobi-Perron algorithm is unknown if $n > 1$. Corollary 1 would provide a foundation for a numerical approximation of this invariant measure. See [2] for a related treatment.

Consider any sequence $b_m \to +\infty$. Then if we define $A_m = \{0/b_m, 1/b_m, \ldots, [b_m]/b_m\}^n$, Lemma 1 holds. Therefore, Theorems 1 and 3 are valid for expansions of numbers like $k/\sqrt[3]{m}$, and, in this sense, algebraic numbers have expansions such that the digit a occurs with frequency $\mu(B(a))$. Thus, Corollary 1 holds for any such sequence. We are indebted to Professor T. S. Pitcher for this observation. See [1, p. 9] for some remarks of Professor Leon Bernstein regarding these matters.

Questions related to our work have been treated in a thesis by David B. Preston [6]. He notes that Theorem 2 holds in the case of 1-dimensional continued fractions with $g(a_1(x), \ldots, a_k(x)) = Q_{k-1}(x)/Q_k(x)$. Finally we wish to gratefully acknowledge several useful comments and suggestions by the referees.

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