# CANTOR'S SERIES FOR VECTORS

BY

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Decimal expansions have been generalized in several directions. First, there are the well-known expansions with arbitrary integer bases. For a sequence of integers $q_1, q_2, \cdots$ ($q_i \geqq 2$) Cantor [1] obtained the expansion

$$x = \sum_{m=1}^{\infty} \frac{a_m}{q_1 q_2 \cdots q_m}, \tag{1}$$

where $x \in [0,1)$ and $a_m \in \{0, 1, \cdots, q_m - 1\}$. See [5] for a survey on expansions with references to Cantor's series. There are also expansions with non-integer bases [4, 5, 8, 9], negative bases [7], and similar expansions exist for complex numbers [2]. In $n$-dimensional Euclidean space, matrix expansions and their associated transformations have been studied extensively [3, 8]. The purpose of this note is to generalize Cantor series (1) to matrix expansions and to give a list of examples of such expansions.

Let $n$ be a fixed positive integer and $Q_1, Q_2, \cdots$ be a sequence of nonsingular $n \times n$ matrices. We take $|\ \ |$ to be a norm on $n$-dimensional Euclidean space and take $\|\ \ \|$ to be a compatible matrix norm (i.e., $|Qx| \leqq \|Q\| \cdot |x|$). (See Lancaster [6] for material on matrix norms.) Let $I = \times_{i=1}^{n} [0,1)$ and assume $|x| \leqq B$ for $x \in I$. Next we make some fundamental definitions.

$$\begin{aligned} T^{(i)}(x) &= Q_i x - [Q_i x] & (i \geqq 1), \\ T_m &= T^{(m)} \circ T^{(m-1)} \circ \cdots \circ T^{(1)} & (m \geqq 1), \\ a_1(x) &= [Q_1 x], & \\ a_m(x) &= [Q_m(T_{m-1}(x))] & (m \geqq 2), \end{aligned}$$

where $[(y_1, y_2, \cdots, y_n)^T] = ([y_1], [y_2], \cdots, [y_n])^T$ and $[\ \ ]$ is the usual greatest integer function in this last expression.

THEOREM. *Assume* $\sup_i \|Q_i^{-1}\| < 1$. *Then*

$$x = \sum_{i=1}^{\infty} (Q_i Q_{i-1} \cdots Q_1)^{-1} a_i(x). \tag{2}$$

*Proof.* $x = Q_1^{-1} Q_1 x = Q_1^{-1}(a_1(x) + T_1(x)) = Q_1^{-1} a_1(x) - Q_1^{-1} T_1(x)$. For our induction, assume for a positive integer $m$ that

$$x = \sum_{i=1}^{m} Q_1^{-1} Q_2^{-1} \cdots Q_i^{-1} a_i(x) + Q_1^{-1} Q_2^{-1} \cdots Q_m^{-1} T_m(x). \tag{3}$$

Then $Q_1^{-1}Q_2^{-1}\cdots Q_m^{-1}T_m(x) = Q_1^{-1}Q_2^{-1}\cdots Q_m^{-1}Q_{m+1}^{-1}(a_{m+1}(x) + T_{m+1}(x))$ and (3) holds for all $m$. Therefore, since $T_m(x) \leqq B$,

$$\lim_{m\to\infty} \left| x - \sum_{i=1}^{m} Q_1^{-1}Q_2^{-1}\cdots Q_i^{1}a_i(x) \right| = \lim_{m\to\infty} \left| Q_1^{-1}Q_2^{-1}\cdots Q_m^{-1}T_m(x) \right|$$

$$\leqq \lim_{m\to\infty} B \| Q_1^{-1} \| \| Q_2^{-1} \| \cdots \| Q_m^{-1} \| \leqq B \lim_{m\to\infty} (\sup_i \| Q_i^{-1} \|^m) = 0.$$

Thus (2) holds and the theorem is proved.

Clearly the theorem holds when $\|(Q_mQ_{m-1}\cdots Q_1)^{-1}\| \to 0$ as $m \to \infty$. Under the assumptions of the theorem, the convergence is geometric with rate $B(\sup_i \| Q_i^{-1} \|)^m$.

Next we give some examples, both general and numerical, of the theorem. As noted below, examples 1, 2, and 3 are known. In the case $n > 1$ and the $Q_i$ are not all identical, the result was not known. Example 4 gives a numerical example of this situation.

*Example* 1. If $n = 1$ and $|\ \ |$, $\|\ \ \|$ are both the usual absolute value on the real line, we can obtain the results mentioned above for any $x \in [0, 1)$. If $Q_i \equiv q$ is a positive integer then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{q^m}, \quad a_m(x) \in \{0, 1, \cdots, q-1\}. \tag{4}$$

This is the usual $q$-adic expansion. If $Q_i \equiv \beta$ where $\beta > 1$ and $\beta$ is not an integer, then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{\beta^m}, \quad a_m(x) \in \{0, 1, \cdots, [\beta]\}. \tag{5}$$

These $\beta$-expansions have been extensively studied [3, 4, 5, 9, 10]. If we let $Q_i = \gamma_i$, $\gamma_i \in R$, $\inf_i |\gamma_i| > 1$, then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{\gamma_1\gamma_2\cdots\gamma_m}. \tag{6}$$

This last formulation allows expansions with negative bases (e.g., $-10$) as well as mixtures of integral and non-integral positive and negative numbers. For some material on expansions with negative radices, see [7].

*Example* 2. Let $n = 2$. If $Q_i \equiv \begin{bmatrix} r & -q \\ q & r \end{bmatrix} = Q$, then $Qx = (rx_1 - qx_2, qx_1 + rx_2)^T$ which is equivalent to $(x_1 + ix_2)(r + iq) = (rx_1 - qx_2) + i(qx_1 + rx_2)$. Therefore if we take

$$|x| = \sqrt{x_1^2 + x_2^2} \text{ and } \|Q\| = \sqrt{r^2 + q^2},$$

we have $|Qx| \leqq \|Q\| |x|$ by the theory of complex numbers. Of course, $\|Q^{-1}\| = (r^2 + q^2)^{-1/2}$. Thus

$$x = \sum_{m=1}^{\infty} Q^{-m} a_m(x), \tag{7}$$

and the expansion is valid for complex numbers $x = x_1 + x_2 i$ ($|x_j| < 1$) with complex base satisfying $r^2 + q^2 > 1$. This transformation and the expansion have been studied by Fischer [2, 3].

*Example* 3. If $|x|^2 = \sum_{i=1}^{n} x_i^2$, then $\|Q\|^2 = \sum_{i,j} q_{ij}^2$ is a compatible matrix norm. As a specific numerical example with $n = 2$, we take $x^T = (1/2, 3/4)$ and $Q = \begin{bmatrix} 2 & 0 \\ 2/3 & 4/3 \end{bmatrix}$. Then $Q^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/4 & 3/4 \end{bmatrix}$ and $\|Q^{-1}\|^2 = 7/8 < 1$. We compute

$$\begin{aligned} a_1(x) &= (1,1)^T \\ a_2(x) &= a_3(x) = a_4(x) = (0,0)^T \\ a_5(x) &= (0,1)^T. \end{aligned}$$

The fifth order approximation to $x$ is $Q^{-1}a_1(x) - Q^{-5}a_5(x) = (1/2,\ 755/1024)^T$. Finally we check the rate of convergence.

$$|x - Q^{-1}a_1(x) - Q^{-5}a_5(x)| = |(0, 13/1024)^T| = (13/1024) < 2(7/8)^{5/2}.$$

The last inequality is by the guaranteed rate of convergence given in the proof of the theorem where $B = \sup_{x \in I} |x| = 2$.

*Example* 4. If $|x| = \max\{|x_i| : 1 \leqq i \leqq n\}$, then $\|Q\| = n \max\{|q_{ij}| : 1 \leqq i, j \leqq n\}$ is a compatible matrix norm. For the matrices given in Table 1 below, we have

TABLE 1. Expansion of $(1/2, 1/2)^T$

| $m$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $Q_m$ | $\begin{bmatrix} 3/2 & 3/2 \\ -3/2 & 3/2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 0 \\ 4/3 & 8/3 \end{bmatrix}$ | $\begin{bmatrix} 8 & -4 \\ -8 & 8 \end{bmatrix}$ | $\begin{bmatrix} 3/2 & -3/2 \\ 3/2 & 3/2 \end{bmatrix}$ |
| $Q_m^{-1}$ | $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$ | $\begin{bmatrix} 1/4 & 0 \\ -1/8 & 3/8 \end{bmatrix}$ | $\begin{bmatrix} 1/4 & 1/8 \\ 1/4 & 1/4 \end{bmatrix}$ | $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$ |
| $\|Q_m^{-1}\|$ | 2/3 | 3/4 | 1/2 | 2/3 |
| $a_m^T(x)$ | (0, 1) | (2,0) | (−3, 5) | (0, 1) |
| $A_m^T(x)$ | (1/3, 1/3) | (7/12, 5/12) | (97/192, 91/192) | (1/2, 1/2) |

$$(1/2,1/2)^T = Q_1^{-1}(1,0)^T + (Q_2Q_1)^{-1}(2,0)^T + (Q_3Q_2Q_1)^{-1}(-3,5)^T + (Q_4Q_3Q_2Q_1)^{-1}(0,1)^T.$$

Let $A_m(x) = \Sigma_{m=1}(Q_i \cdots Q_1)^{-1} a_i(x)$ in Table 1.

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