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A NOTE ON THE REPARAMETRIZATION OF AN EXPONENTIAL FAMILY

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1. Introduction. The exponential family is one of the more important classes of distributions considered in statistics (see [1] or [5]). In this note we restrict attention to one-dimensional exponential families. For μ, a σ-finite measure on R, we define $a = \inf\{x:\mu([x, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}$ and $b = \sup\{x:\mu((x-\varepsilon, x]) > 0$ for all $\varepsilon > 0\}$. If a and b are finite, we define $\Lambda(\mu)$ by $\Lambda(\mu) = [a, b]$. If a or b are infinite, we use open or half-open intervals. The parameter space of $\mu, \Omega(\mu)$, is defined by

$$\Omega(\mu) = \{\omega: 0 < \int e^{\omega x}\, d\mu(x) = \mathscr{B}(\omega) < +\infty\}.$$

For each $\omega \in \Omega(\mu)$ we define a probability measure by

$$P_\omega(A) = (\mathscr{B}(\omega))^{-1} \int_A e^{\omega t}\, d\mu(t).$$

The collection $\{P_\omega(\cdot):\omega \in \Omega(\mu)\}$ is known as the exponential family generated by μ. (For a more general definition see Lehmann [5].) Also let $m(\omega) = \int x dP_\omega(x)$. It is well known [5] that $\Omega(\mu)$ is a convex set, all moments of P_ω exist, and $dm(\omega)/d\omega = \sigma^2(\omega)$, the variance of P_ω. Suppose $\Omega(\mu) \neq \phi$. Then the μ-measure of any bounded measurable set is finite. It is also easy to see that $b < +\infty$ implies $\Omega(\mu)$ unbounded on the right and $-\infty < a$ implies $\Omega(\mu)$ unbounded on the left.

It is clear that $\Lambda(\mu)$ contains the range of the mean $m(\cdot)$ and that $m(\cdot)$ is strictly increasing whenever $a \neq b$. Guthrie and Johns [4] assume there exists a function $\omega(\lambda)$ such that $m(\omega(\lambda)) = \lambda$ for each $\lambda \in \Lambda(\mu)$. Using methods distinct from ours, Girshick and Savage [3] show this property holds whenever $\Lambda(\mu) = [0, b]$, $b < +\infty$. This note extends their result and characterizes measures such that the associated exponential families permit reparametrization in terms of the mean.

2. Reparametrization. The technique employed in the proof of our theorem is motivated by Laplace's method ([2] page 36). The statement of our theorem concerns the interiors of $\Lambda(\mu)$ and $\Omega(\mu)$ since measures with the same $\Lambda(\mu)$ can have open, half-open, or closed parameter spaces (see example E in Section 3). We will use Int (A) to denote the interior of A.

THEOREM. *Let μ be a σ-finite measure such that* Int $\Lambda(\mu) \neq \phi$ *and* Int $\Omega(\mu) = (\omega_0, \omega_1) \neq \phi$. *Then $\omega(\cdot) = m^{-1}(\cdot)$ exists on* Int $\Lambda(\mu)$ *and therefore*

$$\{P_\omega(\cdot):\omega \in \text{Int}\,\Omega(\mu)\} = \{P_{\omega(\lambda)}(\cdot):\lambda \in \text{Int}\,\Lambda(\mu)\}$$

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unless

(i) $\omega_1 < +\infty$ *and* $x\, e^{\omega_1 x} \in L_1(\mu)$, *or*

(ii) $-\infty < \omega_0$ *and* $x\, e^{\omega_0 x} \in L_1(\mu)$.

PROOF. Since $m(\cdot)$ is strictly increasing, we need only show that the range of the mean contains Int $\Lambda(\mu)$. It is sufficient to consider only the right-hand end points, b and ω_1. We must consider several cases. First, suppose $\omega_1 = +\infty$. Then, for any $A < b$,

$$\lim_{\omega \to +\infty} \frac{\int_{[a,\infty)} x^n e^{\omega x}\, d\mu(x)}{\int_{[A,\infty)} x^n e^{\omega x}\, d\mu(x)} = 1,$$

for $n = 0, 1$. Thus

$$\lim_{\omega \to \infty} m(\omega) = \lim_{\omega \to \infty} \frac{\int_{[A,\infty)} x\, e^{\omega x}\, d\mu(x)}{\int_{[A,\infty)} e^{\omega x}\, d\mu(x)} \geqq A,$$

and we have $\lim_{\omega \to \infty} m(\omega) = b$.

Next assume $\omega_1 < +\infty$. We then have $b = +\infty$. If $x\, e^{\omega_1 x} \notin L_1(\mu)$ and $\omega_1 \in \Omega(\mu)$, the result is obvious. If $x\, e^{\omega_1 x} \notin L_1(\mu)$ and $\omega_1 \notin \Omega(\mu)$, the above technique yields $\lim_{\omega \to \omega_1 -} m(\omega) = +\infty = b$. The only case left is (i) in which $\lim_{\omega \to \omega_1 -} m(\omega) < +\infty = b$.

3. Examples. The following examples exhibit $\omega = m^{-1}$ for some simple cases. The form of $\omega(\cdot)$ in (B), (C), and (D) can be obtained as a direct application of Guthrie and Johns [4].

(A) Let μ be Lebesgue measure on R^+. Then $\omega(\lambda) = -1/\lambda$ for $\lambda \in \text{Int}\,\Lambda(\mu) = (0, \infty)$. Thus $P_{\omega(\lambda)}(\cdot)$ is the distribution of an exponential random variable with parameter $1/\lambda$.

(B) Let μ be counting measure on the nonnegative integers. Then $\omega(\lambda) = \ln(\lambda/(1+\lambda))$ for $\lambda \in (0, \infty)$ and $P_{\omega(\lambda)}(\cdot)$ is the distribution of a geometric random variable with parameter $\lambda/(1+\lambda)$.

(C) Let $\mu(\{k\}) = \binom{n}{k}$, $k = 0, 1, \cdots, n$. Then $\omega(\lambda) = \ln(\lambda/(n-\lambda))$ for $\lambda \in (0, n)$ and $P_{\omega(\lambda)}$ is the distribution of a binomial random variable with parameter λ/n.

(D) Let $\mu(\{k\}) = 1/k!$, $k = 0, 1, \cdots$. Then $\omega(\lambda) = \ln \lambda$ for $\lambda \in (0, \infty)$ and $P_{\omega(\lambda)}$ is the Poisson distribution with parameter λ.

(E) Finally, to illustrate (i) of the theorem, we consider $\mu(\{k\}) = ((k)^{1+\delta} e^{\omega_1 k})^{-1}$, $k = 1, 2, \cdots$. If $\delta > 0$, $\Omega(\mu) = (-\infty, \omega_1]$. If $\delta = 0$, $\Omega(\mu) = (-\infty, \omega_1)$. Thus it is clear that we cannot always expect to map $\Lambda(\mu)$ $1-1$ onto $\Omega(\mu)$. P_{ω_1} has exactly n moments if $n < \delta \leqq n+1$ so (i) holds if $1 < \delta$.

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