ACTA ARITHMETICA
XIX (1971)

A Kuzmin theorem for a class of number theoretic endomorphisms

by

MICHAEL S. WATERMAN (Pocatello, Idaho)

Recently several papers ([3], [4], [5], [6], [7]) have been concerned with generalizations of a 1928 theorem of Kuzmin. His result gives a rate of $e^{-\lambda\sqrt{n}}$ for the convergence of the iteration of an arbitrary function to the invariant measure for the continued fraction. The present paper gives a generalized Kuzmin theorem for a class of multi-dimensional F-expansions which includes the n-dimensional continued fraction. An earlier paper ([6]) presented such a theorem with a rate of $\left(e^{-\lambda\sqrt{\nu}}+\sigma(\sqrt{\nu})\right)$. Our present theorem improves the rate to $\sigma(\nu)$.

Our F-expansions were first considered in [6], and we include a short summary of notation and assumptions here. Let A be a fixed convex subset of R^n. Suppose F is a one-to-one continuous map of A onto $(0,1)^n$. We assume $J_F(\cdot)$, the Jacobian of F, exists, the components of F have continuous first order partial derivatives, and $J_F(x)\neq 0$ for almost all $x\in A$. Let $D=F^{-1}$, $T(x)=D(x)-[D(x)]$, and $a_\nu(x)=[D(T^{\nu-1}x)]$ (where $[z]=([z_1],[z_2],\ldots,[z_n])$). We call $a_\nu(x)$ the *ν-th coordinate of the F-expansion of x*. Letting

$$(0,1)^n_F=\{x\in(0,1)^n\colon T^\nu(x)\in(0,1)^n \text{ for all } \nu\geqslant 1\},$$

we impose the assumption $m(0,1)^n_F=1$, where m denotes n-dimensional Lebesgue measure. We will write $F\in\mathcal{F}$ to indicate the satisfaction of these assumptions.

We define the cylinder of order ν generated by a realizable set of coordinates $k_1,k_2,\ldots,k_\nu$ as

$$B_\nu=B_\nu(k_1,k_2,\ldots,k_\nu)=\{x\in(0,1)^n_F\colon a_i(x)=k_i,\ i=1,\ldots,\nu\},$$

and the cylinder of order ν generated by $x\in(0,1)^n_F$ as

$$B_\nu=B_\nu(x)=\{y\in(0,1)^n_F\colon a_i(y)=a_i(x),\ i=1,\ldots,\nu\}.$$

Of course $T(B_\nu(k_1, k_2, \ldots, k_\nu)) \subseteq B_{\nu-1}(k_2, \ldots, k_\nu)$ so that T is the shift on the coordinates of the expansion. If B_ν is generated by $k_1, k_2, \ldots, k_\nu$ and we let $f_{a_i}(t) = F(a_i + t)$, then we define

$$f_\nu(t) = f_{k_1} \circ f_{k_2} \circ \ldots \circ f_{k_\nu}(t) = \prod_{i=1}^{\nu} \circ f_{k_i}(t), \quad t \in T^\nu B_\nu.$$

Below are three additional assumptions on F. The first generalizes condition (C) of Renyi [2].

$$\text{(C)} \qquad \frac{\sup\limits_{t \in T^\nu B_\nu} |J_{f_\nu}(t)|}{\inf\limits_{t \in T^\nu B_\nu} |J_{f_\nu}(t)|} \leqslant C < +\infty$$

uniformly where f_ν runs over all $\nu \geqslant 1$ and all realizable cylinders $B_\nu(k_1, k_2, \ldots, k_\nu)$.

If $m(T^\nu B_\nu) = 1$ we say that B_ν is *proper*; otherwise B_ν is said to be *improper*. Difficulties with improper cylinders necessitate the next two conditions.

$$\text{(L)} \qquad 0 < L \leqslant m(T^\nu B_\nu(x)) \quad \text{for all } x \in (0,1)^n_F, \quad \nu \geqslant 1.$$

For each $B_\nu(x)$, there exists $\hat{B}_{\nu+1}$, a collection of proper cylinders of order $\nu+1$ contained in $B_\nu(x)$, such that

$$\text{(q)} \qquad 0 < q \leqslant \frac{m(\hat{B}_{\nu+1})}{m(B_\nu(x))} \quad \text{for all } x \in (0,1)^n_F, \quad \nu \geqslant 1.$$

The following theorem appears in [6] and is basic to the problem considered here.

THEOREM 1. *Suppose $F \in \mathscr{F}$ satisfies condition* (C), *condition* (L), *and condition* (q). *Then there exists a unique probability measure μ on $(0,1)^n$ such that $\mu \ll m$ and T is a measure preserving transformation for μ. If we let $\varrho(x) = \frac{d\mu}{dm}(x)$, we have*

$$\frac{q}{C} \leqslant \varrho(x) \leqslant \frac{C}{L}.$$

Of course we could conclude $\mu \sim m$ but we will only need $\mu \ll m$ in our proof. Also, adding the assumption $m\{x\colon \operatorname{diam} B_\nu(x) \to 0\} = 1$ allows us to conclude T ergodic. This assumption is included in Theorem 2 below.

To formulate a Kuzmin theorem for $\mathscr{F}$ we need to partition $(0,1)^n_F$. For each cylinder of order 1, $B(k)$, we have $TB(k) \subset (0,1)^n_F$. We use the collection $TB(k)$ to partition $(0,1)^n_F$ and assume the partition is essentially countable. Denote this partition by $\{A_i\}_{i\geq 1}$. With each A_i we associate

$$\mathscr{E}_i = \{k\colon\ TB(k) \supset A_i\}.$$

This allows us to calculate

$$\varrho(x) = \sum_{k\in\mathscr{E}_i} \varrho\big(f_k(x)\big)\,|J_{f_k}(x)|, \qquad x\in A_i. \tag{1}$$

The two lemmas below are taken from [6] and depend only on the properties of f_k. Both are related to the form of equation (1).

LEMMA 1. *Suppose $F\in\mathscr{F}$ and assume $TB_{\nu+1} = B_\nu$, $\nu\geq 1$. Let Ψ_0 be given and Ψ_ν be defined by*

$$\Psi_\nu(x) = \sum_{k\in\mathscr{E}_i} \Psi_{\nu-1}\big(f_k(x)\big)\,|J_{f_k}(x)|, \qquad x\in A_i \ \ (i = 1, 2, \ldots).$$

Then

$$\Psi_\nu(x) = \sum^{(i)} \Psi_0\big(f_\nu(x)\big)\,|J_{f_\nu}(x)|, \qquad x\in A_i \ \ (i = 1, 2, \ldots),$$

where the last summation is over all realizable cylinders $(k_1, \ldots, k_\nu)$ where $k_\nu\in\mathscr{E}_i$.

LEMMA 2. *Let F, $\{\Psi_\nu\}_{\nu\geq 0}$ be as in Lemma 1. Then*

$$\int_{(0,1)^n} \Psi_\nu(x)\,dx = \int_{(0,1)^n} \Psi_0(x)\,dx \quad \textit{for} \quad \nu\geq 1.$$

The theorem below was motivated by a paper of Schweiger ([5]) in which he proves a Kuzmin theorem for a class of F-expansions which has the restriction that all cylinders be proper. Since the n-dimensional Jacobi algorithm has improper cylinders, it was not included. Difficulties are encountered in our proof which do not exist if all cylinders are proper. The assumption $\lim_{\nu\to\infty}\operatorname{diam} B_\nu(x) = 0$ almost everywhere is to insure our F-expansions converge and $\sigma(\nu)\to 0$ as $\nu\to\infty$. To circumvent notational difficulty, we will tacitly assume $x\in(0,1)_F^n$ implies $\lim_{\nu\to\infty}\operatorname{diam} B_\nu(x) = 0$, which involves the deletion of a set of measure zero from the conclusion of our theorem.

THEOREM 2. *Let $F\in\mathscr{F}$ satisfy conditions* (C), (q), (*L*) *and $m\{x\colon \lim_{\nu\to\infty}\operatorname{diam} B_\nu(x) = 0\} = 1$. In addition, suppose $TB_{\nu+1}(x) = B_\nu(x)$, $\nu\geq 1$, $x\in(0,1)_F^n$. Assume there is a constant A such that*

$$\left|\frac{\partial(f_\nu)_k}{\partial x_j}\right| \leq A \quad \textit{uniformly in } \nu,\ k, \textit{ and } j.$$

Also suppose there exists a constant D such that

$$\big||J_\nu(x)|-|J_\nu(y)|\big| \leqslant Dm(B_\nu)\|x-y\| \quad (x, y \in T^\nu B_\nu)$$

uniformly in ν. Let $\{\Psi_\nu\}_{\nu\geqslant 0}$ be a sequence of functions recursively defined by

$$\Psi_\nu(x) = \sum_{k\in\mathscr{E}_i} \Psi_{\nu-1}\big(f_k(x)\big)|J_{f_k}(x)|, \quad x\in A_i,\ i\geqslant 1,$$

where Ψ_0 is an arbitrary measurable function satisfying

$$0 < m \leqslant \Psi_0(x) \leqslant M$$

and

$$|\Psi_0(x)-\Psi_0(y)| \leqslant N\|x-y\|.$$

Then

$$|\Psi_\nu(x) - a\varrho(x)| < b\sigma(\nu)$$

where ϱ is the density of the invariant measure for F,

$$a = \int_{(0,1)^n} \Psi_0(x)\,dx \text{ and } b \text{ are constants,}$$

and

$$\sigma(\nu) = \sup\{\operatorname{diam} B_\nu(y):\ y\in(0,1)^n_F\}.$$

Proof. By Lemma 1, we have

$$\Psi_\nu(x) = \sum^{(i)} \Psi_0\big(f_\nu(x)\big)|J_\nu(x)|, \quad x\in A_i.$$

Using this formula and the bounds assumed above, we can show, for $x, y \in A_i$,

$$|\Psi_\nu(x)-\Psi_\nu(y)| \leqslant N\sum^{(i)}\|f_\nu(x)-f_\nu(y)\|\cdot|J_\nu(x)| + MD\|x-y\|\sum^{(i)} m(B_\nu).$$

Now, applying the mean value theorem to the components of f_ν, we obtain

$$\|f_\nu(x)-f_\nu(y)\| \leqslant nA\|x-y\|,$$

and use of condition (C) and condition (L) yields

$$\sum^{(i)} |J_\nu(x)| \leqslant \frac{C}{L}.$$

Therefore

$$|\Psi_\nu(x)-\Psi_\nu(y)| \leqslant (NnACL^{-1}+MD)\|x-y\| = C_1\|x-y\| \quad \text{for } x, y\in A_i.$$

Application of Lemma 1, equation (1), and condition (C) yields

(2) $$0 < m_1 = mLqC^{-2} < \Psi_\nu(x) < M_1 = C^2(MLq)^{-1}, \quad \text{uniformly in } \nu, x.$$

This allows us to obtain $0 < g_0 < G_0$ such that

(3) $$g_0\Psi_\nu(x) < \Psi_{\mu+\nu}(x) < G_0\Psi_\nu(x) \quad \text{uniformly in } x, \mu, \text{ and } \nu.$$

For $\mu \geqslant 0$, we define

(4) $$\Phi_\nu(x) = \Psi_{\mu+\nu}(x) - g_0\Psi_\nu(x)$$

and

(5) $$\zeta_\nu(x) = G_0\Psi_\nu(x) - \Psi_{\mu+\nu}(x).$$

By application of Lemma 1, we have

$$\Phi_\nu(x) = \sum^{(i)} \Phi_0\big(f_\nu(x)\big)\,|J_\nu(x)|$$

and

$$\zeta_\nu(x) = \sum^{(i)} \zeta_0\big(f_\nu(x)\big)\,|J_\nu(x)|, \qquad x \in A_i.$$

We obtain

$$\Phi_\nu(x) \geqslant C^{-1}\sum^{(i)} \Phi_0\big(f_\nu(x)\big)\, m(B_\nu)$$

from condition (C). We let

$$\mathscr{C}_i^\nu = \bigcup_{k_\nu \in \mathscr{E}_i} B_\nu(k_1, \ldots, k_\nu).$$

By the mean value theorem for integrals

$$\int_{\mathscr{C}_i^\nu} \Phi_0(y)\,dy = \sum^{(i)} \Phi_0(y'_\nu)\, m(B_\nu).$$

Therefore

$$\Phi_\nu(x) - C^{-1}\int_{\mathscr{C}_i^\nu} \Phi_0(y)\,dy \geqslant C^{-1}\sum^{(i)}\big\{\Phi_0\big(f_\nu(x)\big) - \Phi_0(y'_\nu)\big\}\, m(B_\nu)$$

$$\geqslant -C^{-1}C_1(1+g_0)\,\sigma(\nu)\sum^{(i)} m(B_\nu) \geqslant -C_2\,\sigma(\nu).$$

That is,

$$\Psi_{\nu+\mu}(x) - g_0\Psi_\nu(x) \geqslant C^{-1}\int_{\mathscr{C}_i^\nu}\big(\Psi_\mu(x) - g_0\Psi_0(x)\big)\,dx - C_2\,\sigma(\nu).$$

In the same manner we obtain ($x \in A_i$)

$$\zeta_\nu \geqslant C^{-1} \sum^{(i)} \zeta_0 (f_\nu(x)) m(B_\nu),$$

$$\int_{\mathscr{C}_i^\nu} \zeta_0(x)\, dx = \sum^{(i)} \zeta_0(y_\nu') m(B_\nu),$$

and

$$G_0 \Psi_\nu(x) - \Psi_{\mu+\nu}(x) \geqslant C^{-1} \int_{\mathscr{C}_i^\nu} (G_0 \Psi_0(x) - \Psi_\mu(x))\, dx - C_3 \sigma(\nu).$$

Letting

$$l_i = C^{-1} \int_{\mathscr{C}_i^\nu} (\Psi_\mu(x) - g_0 \Psi_0(x))\, dx,$$

and

$$l_i' = C^{-1} \int_{\mathscr{C}_i^\nu} (G_0 \Psi_0(x) - \Psi_\mu(x))\, dx,$$

we can show

(6) $$\Psi_{\nu+\mu}(x) \geqslant \Psi_\nu(x) \left(g_0 + \frac{l_i}{M_1} - (m_1)^{-1} C_2 \sigma(\nu) \right) = g_1 \Psi_\nu(x)$$

and

(7) $$\Psi_{\nu+\mu}(x) \leqslant \Psi_\nu(x) \left(G_0 - \frac{l_i'}{M_1} + (m_1)^{-1} C_3 \sigma(\nu) \right) = G_1 \Psi_\nu(x).$$

There exists ν_0 such that for $\nu \geqslant \nu_0$,

$$g_0 < g_1 < G_1 < G_0.$$

Now

$$l_i + l_i' = C^{-1} \int_{\mathscr{C}_i^\nu} (G_0 - g_0) \Psi_0(x)\, dx \geqslant C^{-1} (G_0 - g_0) m_1 \sum^{(i)} m(B_\nu)$$

$$\geqslant C^{-1} (G_0 - g_0) m_1 \sum' m(B_\nu) = C^{-1} mq (G_0 - g_0),$$

where $\sum'$ denotes summation over proper cylinders of order ν and the last inequality is by condition (q). The importance of this bound is its independence of both μ and ν.

From these results we obtain

(8) $$G_1 - g_1 = G_0 - g_0 - \frac{1}{M_1} (l_i' + l_i) + (m_1)^{-1} (C_2 + C_3) \sigma(\nu)$$

$$\leqslant (G_0 - g_0) (1 - m_1 q (C M_1)^{-1}) + C_4 \sigma(\nu).$$

We note that

$$0 < \lambda = 1 - m_1 q (CM_1)^{-1} < 1,$$

since without loss of generality $C > 1$.

Now we summarize the result just obtained. From

$$g_0 \Psi_\nu(x) < \Psi_{\nu+\mu}(x) < G_0 \Psi_\nu(x)$$

we have proved (for $\nu \geqslant \nu_0$)

$$g_1 \Psi_\nu(x) < \Psi_{\nu+\mu}(x) < G_1 \Psi_\nu(x)$$

where

$$g_0 < g_1 < G_1 < G_0$$

and

$$G_1 - g_1 \leqslant (G_0 - g_0)\lambda + C_4 \sigma(\nu).$$

The argument for g_1 and G_1 can be repeated to obtain

$$g_r \Psi_\nu(x) < \Psi_{\nu+\mu}(x) < G_r \Psi_\nu(x)$$

where

$$g_0 < g_1 < \ldots < g_r < G_r < \ldots < G_1 < G_0$$

and

$$\begin{aligned} G_r - g_r &\leqslant (G_{r-1} - g_{r-1})\lambda + C_4 \sigma(\nu) \\ &\leqslant \lambda^r (G_0 - g_0) + C_4 \sigma(\nu)\,(1 + \lambda + \ldots + \lambda^r) \\ &\leqslant \lambda^r (G_0 - g_0) + \frac{C_4}{1-\lambda}\,\sigma(\nu). \end{aligned}$$

It should be emphasized that G_r and g_r are functionally dependent on r, i, ν, μ, and Ψ_0.

Now

$$\lim_{\nu, r \to \infty} (G_r - g_r) = 0$$

implies

$$\lim_{\nu, r \to \infty} G_r = \lim_{\nu, r \to \infty} g_r = Q(\mu).$$

Thus we can write

$$|\Psi_{\nu+\mu}(x) - Q(\mu)\Psi_\nu(x)| < (G_r - g_r)\Psi_\nu(x) \leqslant M_1 C \left(\lambda^r (G_0 - g_0) + \frac{C_4}{1-\lambda}\,\sigma(\nu) \right)$$

which implies (letting $r \to \infty$)

$$|\Psi_{\nu+\mu}(x) - Q(\mu)\Psi_\nu(x)| < b\sigma(\nu). \tag{9}$$

At this point we employ (9) to conclude $Q(\mu) \equiv 1$. Take $\nu \geqslant \nu_0$, we have the following inequalities

$$\begin{gathered} |\Psi_{\nu+l\mu}(x) - Q(\mu)\Psi_{\nu+(l-1)\mu}(x)| < b\sigma\big(\nu+(l-1)\mu\big), \\ |\Psi_{\nu+(l-1)\mu}(x) - Q(\mu)\Psi_{\nu+(l-2)\mu}(x)| < b\sigma\big(\nu+(l-2)\mu\big), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ |\Psi_{\nu+\mu}(x) - Q(\mu)\Psi_\nu(x)| < b\sigma(\nu). \end{gathered}$$

By multiplying row 1 by $Q^0(\mu)$, row 2 by $Q^1(\mu), \ldots,$ row l by $Q^{(l-1)}(\mu)$, noting $\sigma(\cdot)$ is a decreasing function, and applying the triangle inequality, we have $(Q(\mu) \neq 1)$

$$|\Psi_{\nu+l\mu}(x) - Q^l(\mu)\Psi_\nu(x)| < b\sigma(\nu)\left(\frac{1-Q^l(\mu)}{1-Q(\mu)}\right). \tag{10}$$

Suppose $Q(\mu) < 1$. Then from (10)

$$\Psi_{\nu+l\mu}(x) < Q^l(\mu)\Psi_\nu(x) + \frac{b\sigma(\nu)}{1-Q(\mu)}.$$

Since $\Psi_\nu(\cdot)$ is bounded above, $Q^l(\mu) \to 0$ as $l \to \infty$, and $\sigma(\nu) \to 0$ as $\nu \to \infty$, we have ν_1, l_1 such that

$$\Psi_{\nu_1+l_1\mu}(x) < mLqC^{-2}.$$

This contradicts (2) so that $Q(\mu) \geqslant 1$.

Next suppose $Q(\mu) > 1$. Then from (10)

$$\frac{b\sigma(\nu)}{Q(\mu)-1} + Q^l(\mu)\left(\Psi_\nu(x) - \frac{b}{Q(\mu)-1}\,\sigma(\nu)\right) < \Psi_{\nu+l\mu}(x).$$

Applying (2) we have

$$Q^l(\mu)\left(mLqC^{-2} - \frac{b}{Q(\mu)-1}\,\sigma(\nu)\right) < \Psi_{\nu+l\mu}(x).$$

By choosing $\nu \geqslant \nu_2$ we have the expression in parentheses positive so that there exists l_2 such that

$$C^2(MLq)^{-1} < \Psi_{\nu_2+l_2\mu}(x)$$

which is a contradiction of (2). Therefore $Q(\mu) \leqslant 1$.

Finally, since $Q(\mu) \equiv 1$, we have by (9)

$$|\Psi_{\nu+\mu}(x) - \Psi_\nu(x)| < b\sigma(\nu), \qquad \nu \geqslant \nu_0. \tag{11}$$

Therefore $\{\Psi_\nu(x)\}_{\nu\geqslant 1}$ is a Cauchy sequence. Letting $\Psi(x) = \lim_{\nu\to\infty}\Psi_\nu(x)$, $a = \int \Psi(x) = \int \Psi_0(x)$, and $\varrho(x) = a^{-1}\Psi(x)$, we have

$$|\Psi_\nu(x) - a\varrho(x)| < b\varrho(\nu).$$

Since $\varrho(x)$ satisfies (1), $\int \varrho(x)dx = 1$, ϱ is the unique invariant measure $\ll m$. This completes the proof.

The following corollary corresponds to F. Schweiger's result ([5]) of $\varrho \in \mathrm{Lip}^1(0,1)^n$.

COROLLARY 1. *The density function, $\varrho(\cdot)$, of Theorem 2 satisfies a Lipschitz condition of order* 1 *on each of the sets A_i. That is,*

$$|\varrho(x) - \varrho(y)| \leqslant K\|x-y\|, \qquad x, y \in A_i.$$

Proof. The result follows directly from $\Psi_\nu(\cdot) \in \mathrm{Lip}^1(A_i)$, and the conclusion of Theorem 2. Note that K has the same value for each of the A_i.

COROLLARY 2. *Let $F \in \mathscr{F}$ and Ψ_0 be as in Theorem 2. Then for all $\mu \geqslant 0$ and $i \geqslant 1$,*

$$\lim_{\nu\to\infty} \frac{\int_{\mathscr{C}_i^\nu} \Psi_\mu(x)dx}{\int_{\mathscr{C}_i^\nu} \Psi_0(x)dx} = 1.$$

Proof. We remark that if $\mathscr{C}_i^\nu \equiv (0,1)_F^n$ (for fixed i), then the result is obvious from Lemma 2. In general, however, it seems necessary to return to an explicit determination of g_r.

$$g_0 = \frac{m_1}{2M_1},$$

$$g_r = g_{r-1}\big(1-(CM_1)^{-1}\big)\int_{\mathscr{C}_i^\nu} \Psi_0(x)dx + (CM_1)^{-1}\int_{\mathscr{C}_i^\nu}\Psi_\mu(x)dx - C_5\sigma(\nu)$$

$$= ag_{r-1} + b = a^r g_0 + b(1+a+\ldots+a^{r-1}).$$

By choosing $\nu \geqslant \nu_3$ and making M_1 sufficiently large, we have $0 < a, b < 1$. Therefore,

$$\lim_{r\to\infty} g_r = \frac{b}{1-a} = \frac{(CM_1)^{-1}\int_{\mathscr{C}_i^\nu}\Psi_\mu(x)dx - C_5\sigma(\nu)}{(CM_1)^{-1}\int_{\mathscr{C}_i^\nu}\Psi_0(x)dx} \tag{12}$$

and

$$1 = Q(\mu) = \lim_{\nu\to\infty}(\lim_{r\to\infty} g_r) = \lim_{\nu\to\infty} \frac{\int_{\mathscr{C}_i^\nu} \Psi_\mu(x)\,dx}{\int_{\mathscr{C}_i^\nu} \Psi_0(x)\,dx}.$$

A formula very similar to (12) exists for $\lim_{r\to\infty} G_r$, so that an alternate method of proving our theorem would be to conclude Corollary 2 without benefit of $Q(\mu)\equiv 1$. However, this is essentially asking for an explicit calculation of the nature of $\lim_{\nu\to\infty}\mathscr{C}_i^\nu$ which does not seem to be easy. If, for example, $m(\lim_{\nu\to\infty}\mathscr{C}_i^\nu) = 1$, the result would follow.

To apply our theorem to the Jacobi algorithm we refer to [6]. There we showed that

$$F(x) = \left(\frac{1}{x_n}, \frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right)$$

belongs to $\mathscr{F}$ and the assumptions are satisfied with

$$C = (1+2n)^{n+1},$$

$$L = \frac{1}{n!},$$

and

$$q = \frac{1}{n!(1+n)^{n+1}(1+2n)^{n+1}}.$$

Also, following Schweiger [3], we can verify the assumptions on f_ν and J_ν. Thus our Kuzmin theorem holds for the Jacobi algorithm.

The author would like to express his appreciation to F. Schweiger for making available a manuscript containing a corrected version of his Kuzmin theorem ([5]). The work referred to in [6] will appear elsewhere.

References

[1] R. O. Kuzmin, *Sur un probleme de Gauss*, Atti del Congresso Internazionale del Matematici Bologna 6 (1928), pp. 83–89.

[2] A. Renyi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungaricae 8 (1957), pp. 477–493.

[3] F. Schweiger, *Ein Kuzminscher Satz über den Jacobischen Algorithmus*, J. Reine Angew. Math. 232 (1968), pp. 35–40.

[4] — *Metrische Theorie einer Klasse zahlentheoretischer Transformationen*, Acta Arith. 15 (1968), pp. 1–18.

[5] F. Schweiger, *Metrische Theorie einer Klasse zahlentheoretischer Transformationen (Corrigendum)*, Acta Arith. 16 (1969), pp. 217–219.
[6] M. Waterman, *Some ergodic properties of multi-dimensional F-expansions*, Z. Wahrscheinlichkeitstheorie verw. Geb. 16 (1970), pp. 77–103.
[7] T. Vinh-Hien, *The central limit theorem for stationary processes generated by number theoretic endomorphisms* (in Russian), Vestnik Moskov Univ. Ser. I. Mat. Meh. 5, 1 (1963), pp. 28–34.

IDAHO STATE UNIVERSITY

Received on 2. 2. 1970 (30)