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TITLE — Allocating Defense Missiles to Maximize the Expected Distance Between the Target and the Closest Offensive Missile Penetrating the Defense

MM 66 - 4263 - 12

DATE — September 22, 1966

CASE CHARGED — 27540-5

FILING CASES — 27540-5

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FILING SUBJECTS — Defense Tactics
Probability

ABSTRACT

A point target is to be defended by allocating one or more defensive missiles to each of a set of offensive missiles. We assume that we can measure (or predict) the distance between the impact point of each offensive missile and the point target. The target is assumed to be destroyed if an offensive missile lands within a distance D of it, but we assume D is not known with enough precision to be of use in designing the defense.

In the general case, we allocate defensive missiles to maximize the expected distance between the target and the nearest offensive missile penetrating the defense. Let p be the kill probability of the defensive missiles, and A_i denote the probability that the ith closest missile penetrates the defense. We recursively derive the A_i (and hence the number of defensive missiles to be allocated to the ith offensive

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missile) as a function of the offensive missile distances from the target. This derivation is carried out for certain defense resources: the maximum resources associated with the engagement of exactly (n-1) offensive missiles.

In the second part of this memorandum, we specialize the general results outlined above: specifically, we maximize the expected rank of the nearest offensive missile penetrating the defense. In the limiting case of small p, we tabulate the A_i and the expected rank for defense resources associated with 2, 3, ..., 40 offensive missiles. For comparison, an even allocation of defense missiles is evaluated, and found to be nearly as effective. When values of p near unity are considered, however, we are no longer able to allocate our resources in an arbitrary manner. We first find (as a function of p) how many offensive missiles must be engaged before we allocate two defensive missiles (instead of one) to the nearest offensive missile. To obtain similar results concerning the allocation of three defensive missiles to the nearest offensive missile, we resort to numerical studies. Tables are presented (as a function of p) indicating how these allocations should be carried out.

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SUBJECT: Allocating Defense Missiles to Maximize the Expected Distance Between the Target and the Closest Offensive Missile Penetrating the Defense - Case 27540-5

DATE: September 22, 1966

FROM: M. S. Waterman

MM 66-4263-12

MEMORANDUM FOR FILE

A. Introduction

A point target is to be defended by allocating one or more defensive missiles to each one of a set of offensive missiles directed against the target. We assume the target is destroyed if an offensive missile lands within a distance D of it. If D is known, our defense is easy to design; we simply allocate our defense missiles as evenly as possible among the offensive missiles that land nearer than a distance D to the target. However, we assume here that D is not known with enough precision to be of use in designing the defense. Intuitively, we would like to allocate more of the defense resources to nearby missiles than to more distant ones.

We also assume that the offensive missiles can be detected and their impact points determined by some tracking device. Thus ℓ, the number of offensive missiles, is known, and we define $X_1, X_2, \ldots, X_\ell$ by letting X_i be the distance of the ith nearest missile from the target. Since this study utilizes only these distances, it is clear that our problem can be set in any number of dimensions. The two-dimensional target is the usual assumption in the literature and is of the greatest practical importance (for example, bombing problems are usually considered in two dimensions). However, one-dimensional problems may also occur (for example, the aiming error may be much larger in one of the dimensions). The first part of this memorandum treats the general case; the second part treats a special case occurring when the tracking device can only determine which offensive missile will be the ith closest to the target. The special case fits into the above discussion by letting $X_i = i$ and will be referred to below as the rank case.

B. The General Case

The purpose of this discussion is to discover allocations of the available defense missiles that maximize the expected distance between the point target and the nearest offensive missile penetrating the defense. It should be noted that we will be concerned with large scale attacks. (That is, ℓ is assumed to be large in comparison with the defense resources.) It is clear, then, that the parameters we are to be involved with are those of the defense. We define k to be the number of defensive missiles and p to be the kill probability of these missiles.

Suppose the X_i are defined as above and let A_i = Pr (ith offensive missile penetrates our defense). If k_i defensive missiles are allocated to the ith nearest offensive missile, we have $A_i = (1-p)^{k_i}$. (Note that we assume independence of the launching of the defensive missiles.) Suppose we engage the n nearest offensive missiles $\left(\sum_{i=1}^{n} k_i = k\right)$. The expected distance to the closest offensive missile penetrating is

$$E(\text{Closest missile}) = X_1A_1 + X_2A_2(1-A_1) +\dots+ X_nA_n \prod_{i=1}^{n-1} (1-A_i)$$

$$+ X_{n+1} \prod_{i=1}^{n} (1-A_i). \qquad (1)$$

Equation (1) is of interest throughout this memorandum. By a remark above,

$$\prod_i A_i = \prod_i (1-p)^{k_i} = (1-p)^k$$

Thus we subject E(closest missile) to the constraint $\prod_i A_i = A$ for $0 \leq A \leq 1$. Now

$$E(\text{Closest missile}) = X_1A_1 + X_2A_2(1-A_1) +\dots+$$

$$+ X_nA(A_1\cdots A_{n-1})^{-1} \prod_{i=1}^{n-1} (1-A_i)$$

$$+ X_{n+1}\left(1-A(A_1\cdots A_{n-1})\right)^{-1} \prod_{i=1}^{n-1} (1-A_i). \qquad (2)$$

It is clear that, while we may vary the k_i, they must be integral. However, we formally differentiate equation (2) with respect to the k_i and obtain the following system of equations

$$\frac{\partial E}{\partial k_i} = 0 \qquad i = 1,2,\ldots,n-1. \tag{3}$$

The solution of this system will give the optimal allocation of resources for shooting at the n closest offensive missiles. However, this is not quite all that we could hope for. As an illustration consider $p = 0.9$ and $k = 10$. An optimal allocation of these resources for $n = 2$ yields an E(closest missile) of nearly 3 (let $k_1 = k_2 = 5$). Yet, for $n = 3$, we can obtain an E(closest missile) quite near 4 (let $k_1 = 5$, $k_2 = 3$, $k_3 = 2$, for example). Thus it is very clear that it would be desirable to "optimize on n" as well as optimize the defensive missile allocation for a specified value of n.

However, let us first return to system (3) for $n = 2$. Differentiating with respect to n_1 yields

$$X_1A_1 - X_2AA_1^{-1}(1-A_1) - X_2A - X_3A\left(1-AA_1^{-1}\right)$$
$$+ X_3(1-A_1)AA_1^{-1} = 0.$$

This can be solved to yield

$$A_1 = A^{1/2}\left(\frac{X_3-X_2}{X_3-X_1}\right)^{1/2}$$

or

$$k_1 = \frac{k}{2} + \frac{1}{2}\Big(\ln(1-p)\Big)^{-1} \ln\left(\frac{X_3-X_2}{X_3-X_1}\right) . \qquad (4)$$

These results can be expressed in another form by letting $A_1 = (X_3-X_2)\beta^{-1}$. It then follows that $A_2 = (X_3-X_1)\beta^{-1}$ (where $\beta \geq X_3-X_1$).

However, solving (3) when n = 3 leads to solving the equation below for y.

$$(X_4-X_3)y^4 + (X_3-X_4)y^3 + A\left(\frac{X_4-X_3}{X_4-X_2}\right)^2 (X_2-X_4)y$$
$$+ A\left(\frac{X_4-X_3}{X_4-X_2}\right)^2 (X_4-X_1) = 0.$$

This equation does not seem to be easy to solve (even letting $X_i = i$) so that this approach to optimal allocations is not pursued further here. It is of course possible to study the problem numerically.

Due to the difficulties noted above, we now let $A = \prod_{i=1}^{n-1} A_i$ in (3). This is equivalent to letting $A_n = 1$ and should yield the probabilities at which the optimal allocation is changing from engaging n-1 missiles to engaging n missiles. Since we can form n-1 equations in which A does not appear, we can, after solving for the A_i, determine the A (or, equivalently, the number of defensive missiles) at which we should begin engaging n offensive missiles. The practical value of these boundary points is clear. The first few coefficients are given by

$$A_{n-1} = \frac{X_{n+1}-X_n}{X_{n+1}-X_{n-1}} \; ; \tag{5}$$

$$A_{n-2} = \frac{(X_{n+1}-X_n)(X_n-X_{n-1})}{(X_{n+1}-X_{n-1})(X_n-X_{n-2})} \; ; \tag{6}$$

$$A_{n-3} = \frac{(X_{n+1}-X_n)(X_n-X_{n-1}) \; [(X_n-X_{n-2})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)]}{(X_{n+1}-X_{n-1})(X_n-X_{n-2}) \; [(X_n-X_{n-3})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)]} \; ; \tag{7}$$

and

$$\begin{aligned} A_{n-4} = A_{n-3} \Big\{ & (X_{n+1}-X_{n-1})(X_n-X_{n-2})[(X_n-X_{n-3})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)] \\ & - (X_{n+1}-X_n)(X_n-X_{n-1})[(X_n-X_{n-2})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)] \Big\} / \\ \Big\{ & (X_{n+1}-X_{n-1})(X_n-X_{n-2})[(X_n-X_{n-4})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)] \\ & - (X_{n+1}-X_n)(X_n-X_{n-1})[(X_n-X_{n-2})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)] \Big\} \; . \end{aligned} \tag{8}$$

It should be noted that equation (4) and $k = k_i$ imply $A = A_1 = (X_3-X_2)/(X_3-X_1)$ which is equation (5) with $n = 2$.

When we examine equations (5) through (8), we notice a general pattern in getting from A_{i+1} to A_i. A_i is the product of A_{i+1} and another fraction. To obtain this fraction subtract the numerator of A_{i+1} from the denominator of A_{i+1} and divide this difference by the same quantity with a substitution of X_i for X_{i+1}. Although this procedure is easy to follow, we must not reduce our fractions in a numerical example until we have calculated the desired coefficients. This general procedure has not been proved, but the derivation of A_{n-1} through A_{n-4} is indicated in Appendix I.

To illustrate the use of these coefficients, suppose $p = 0.4$, $k = 7$, $X_1 = 1$, $X_2 = 4$, $X_3 = 5$, $X_4 = 7$, and $X_5 = 8$. For $n = 2$, $A_1 = 0.25$ or $k_1 = 2.71$. This does not use up our 7 defensive missiles so we consider $n = 3$. Then $A_1 = 1/6$ and $A_2 = 2/3$ or $k_1 = 3.52$ and $k_2 = 0.795$. This still does not use all our resources. For $n = 4$, we find $k_1 = 4.56$, $k_2 = 2.94$, and $k_3 = 2.15$. This allocation required 9.65 missiles so that the optimal allocation we desire requires allocation to the 3 nearest offensive missiles. By linear interpolation between the k's we find $k_1 = 4.047$, $k_2 = 1.873$, and $k_3 = 1.08$. It is clear that an allocation of 0.047 of a defensive missile is nonsense and that a practical allocation would be $k_1 = 4$,

$k_2 = 2$, and $k_3 = 1$. A comment on the interpolation should be made here. By the remark following equation (4), we see that linear interpolation between the A_i for allocation to the nearest 2 offensive missiles is exact.

The formulas for E(closest missile) on the boundary points are now easily found from the coefficients given above, and give an indication of E(closest missile) as a function of the X_i. For example, if E_i = E(closest missile) as the allocation shifts from i to i+1 missiles, we have

$$E_1 = \Big(X_1(X_3-X_2) + X_2(X_2-X_1)\Big)/(X_3-X_1). \tag{9}$$

E_i for $i > 1$ becomes more complicated.

It would be of interest to find the expected value of the E_i with respect to a distribution of the order statistics $X_1, X_2, \ldots, X_\ell$. A reasonable assumption would be that the X_i are random variables from the Rayleigh distribution. However, the calculation of this expected value is not an easy task, and difficulties arise even for E_1. Due to these problems, this aspect of the problem is not pursued further.

C. <u>The Rank Case</u>

It was noted above that the rank case fits into the general framework when $X_i = i$. We also mentioned how this case could arise in practice. The fact that we can study optimal allocation of resources without the problem of generating sets of X_i is an important feature of this case. Perhaps the strongest motivation for using the rank case is its computational ease. Later in this section we see a situation that in general involves the calculation of $\binom{40}{2}$ coefficients reduced to the calculation of 40 coefficients. Also, series that are not generally expressible in closed form have relatively easily calculated sums in the rank case.

In this section, our object will be to maximize the expected rank of the nearest penetrating offensive missile. If we allocate defensive missiles to the nearest n offensive missiles,

$$E(\text{rank}) = \sum_{i=1}^{n} iA_i \prod_{j=1}^{i-1} (1-A_j) + (n+1) \prod_{i=1}^{n} (1-A_i). \qquad (10)$$

Of course, the solution for the A_i on boundary points holds here. For example, the allocation changes from one to two missiles when $A_1 = \frac{X_3-X_2}{X_3-X_1} = 1/2$. Here, however, emphasis is placed on two limiting cases: (a) many unreliable missiles and (b) a few reliable missiles. As we noticed above, it is usually impossible to allocate defense resources exactly, and the effect of the practical constraint of allocations of integral numbers of missiles is studied in each case.

a. <u>Many Unreliable Missiles</u>

The case of many defensive missiles with small kill probability is of interest because the available resources can be allocated in a very nearly optimal manner. As an example, one can consider artillery with large numbers of shells defending a target against incoming aircraft.

If p is near 0, the decrease in E(rank) that results from allocating integral numbers of defensive missiles is not of practical interest. To illustrate this we give a rough upper bound (for $n = 2$) between E, the optimal E(rank) for the k missiles available, and $\hat{E}$, the "practical" allocation suggested by the optimal one. Let δ_1 be the number of missiles assigned to the nearest offensive missile by the optimal allocation. Then, if $k_2 = k - [\delta_1]$,

$$0 \leq E - \hat{E} \leq (X_3 - X_1) \cdot (1-p)^{k_2 - 1} p = 2(1-p)^{k_2 - 1} p, \qquad (11)$$

and it is clear that $\lim_{p \to 0^+} (E - \hat{E}) = 0$. Appendix II gives the derivation of (11). For a numerical example, let $p = .1$ and $k = 20$. Then $\delta_1 = 10.329$ and $k_2 = 10$. Thus $0 \leq E - \hat{E} \leq 2(.9)^9(.1) = .0768$.

If p is small (0.1 or less) and m_i is large in comparison with p (so that $k_i p > 3$), we may approximate $(1-p)^{k_i}$, the probability that the ith offensive missile penetrates the defense, by $e^{-k_i p}$. We write $\alpha_i = k_i p$ and use this quantity as a measure of defense resources allocated to the ith offensive missile. It is then convenient to use $\ln A = \ln(1-p)^k = kp = \alpha$

as a measure of total defense resources. These approximations easily apply to the results of the previous section. For example, since $\ln(1/2) = -0.69315$, we should have $n \geq 2$ whenever $\alpha > 0.69315$.

Since it is difficult to see the behavior of the A_i and the corresponding E(rank), some computation was undertaken. As the allocation changes from engaging n-1 to engaging n offensive missiles, the A_i have been calculated for $n = 2,3,\ldots,40$ according to the procedure described in Section A. It can be pointed out that $A_{n-1} = 1/2$, $A_{n-2} = 1/4$, $A_{n-3} = 3/20,\ldots$ regardless of the value of n. Therefore, these coefficients are invariant in n so that only 40 coefficients need to be calculated instead of $\binom{40}{2}$. In Table 1, we read A_1 in the second column directly opposite the rank of the first offensive missile not engaged, A_2 directly above A_1, and proceed in this fashion until we reach $A_{n-1} = 0.5000$ at the head of the second column. The third column represents the total defense resources necessary to allocate defensive missiles to n offensive missiles. In the final column, the expected rank of the closest penetrator is given.

Table 1

Optimum Allocation of Defense Resources
to (n-1) Closest Offensive Missiles
(Expected Rank of Nearest Penetrator)

Rank of First Offensive Missile Unengaged	Probability of Penetration for ith Missile $A_i = (1-p)^{k_i}$	Maximum Resources to Engage (n-1) Missiles $\log_e \prod A_i$	Expected Rank of Nearest Penetrator
2	0.5000	0.69	1.50
3	0.2500	2.08	2.13
4	0.1500	3.98	2.81
5	0.1020	6.26	3.52
6	0.0752	8.85	4.26
7	0.0586	11.68	5.01
8	0.0474	14.73	5.77
9	0.0395	17.96	6.54
10	0.0337	21.35	7.32
11	0.0292	24.89	8.11
12	0.0257	28.55	8.90
13	0.0229	32.33	9.69
14	0.0205	36.21	10.50
15	0.0186	40.19	11.30
16	0.0170	44.26	12.11
17	0.0156	48.43	12.92
18	0.0144	52.66	13.73
19	0.0134	56.97	14.55
20	0.0125	61.35	15.36
21	0.0117	65.80	16.19
22	0.0110	70.31	17.01
23	0.0103	74.89	17.83
24	0.0098	79.51	18.66
25	0.0092	84.20	19.48
26	0.0088	88.93	20.31
27	0.0084	93.72	21.14
28	0.0080	98.55	21.98
29	0.0076	103.43	22.81
30	0.0073	108.36	23.64

Rank of First Offensive Missile Unengaged	Probability of Penetration for ith Missile $A_i = (1-p)^{k_i}$	Maximum Resources to Engage (n-1) Missiles $\log_e \prod A_i$	Expected Rank of Nearest Penetrator
31	0.0070	113.32	24.48
32	0.0067	118.33	25.31
33	0.0064	123.38	26.15
34	0.0062	128.47	26.99
35	0.0059	133.59	27.83
36	0.0057	138.75	28.67
37	0.0055	143.95	29.51
38	0.0053	149.18	30.35
39	0.0052	154.45	31.20
40	0.0049	165.07	32.89

As a comparison with the optimal allocation described above for the rank case, we consider allocating α/h of our resources to each of the nearest h offensive missiles. We let $E(\alpha,h)$ be the E(rank) for this allocation and write $B = e^{-\alpha/h}$.

$$E(\alpha,h) = B + 2B(1-B) + \ldots + hB(1-B)^{h-1} + (h+1)(1-B)^h$$

$$= B\left(1+2(1-B) + \ldots + h(1-B)^{h-1}\right) + (h+1)(1-B)^h .$$

If we apply the formula for the sum of a differentiated geometric series, we can write $E(\alpha,h)$ in closed form.

$$E(\alpha,h) = B^{-1} + (1-B)^h(1-B^{-1}) . \qquad (12)$$

It is desirable to maximize $E(\alpha,h)$ with respect to h for a fixed α, but $\partial E(\alpha,h)/\partial h = 0$ is difficult to solve analytically. However, the simplicity of equation (12) and a restriction of h to integer values allows a maximum to be found (for fixed α) numerically. In Table 2, we allocate the defensive missile resources used in Table 1 evenly among the optimum number of offensive missiles (given in the second column). Thus, the final columns of the two tables can be directly compared. It is interesting to note that the expected rank for the even allocation is approximately 0.94 of the expected rank for the optimum allocation - very little efficiency is lost. Note also that slightly fewer offensive missiles are engaged in the even allocation - typically, about 0.9 of those engaged in the optimum allocation.

Table 2

Even Allocation of Defense Resources
to (n-1) Closest Offensive Missiles

Rank of First Offensive Missile Unengaged: Optimum Allocation	Even Allocation	Probability of Penetration for any Missile	Expected Rank of Nearest Penetrator
2	2	0.5000	1.50
3	3	0.3535	2.06
4	4	0.2658	2.67
5	4	0.1342	3.31
6	5	0.1095	4.02
7	6	0.0966	4.72
8	7	0.0858	5.43
9	8	0.0769	6.15
10	9	0.0695	6.87
12	11	0.0574	8.32
14	12	0.0365	9.83
16	14	0.0332	11.35
18	16	0.0299	12.87
20	18	0.0271	14.40
25	22	0.0181	18.27
30	27	0.0155	22.20
40	36	0.0104	30.14

b. Few Reliable Missiles

In this section, we assume the defense has a small number of missiles with kill probability near one. If we consider the limiting case of $p = 1$, the allocation is obvious and E(closest missile) = X_{k+1}. However, difficulties arise when $p < 1$. We have noted above that very nearly optimal allocations are made when we have many unreliable missiles;

of course, the practical constraint of integral allocations is not to be dismissed so easily here. Since our resources cannot be divided arbitrarily, the coefficients $A_1, A_2, \ldots, A_{n-1}$ are not as useful as in the case of small p.

It is thus clear that some other approach must be found. If we calculate E(rank) with $.9 \leq p < 1$ for $k \leq 5$, we notice that we optimize the expectation simply by assigning one defensive missile to each of the nearest k offensive missiles. For what value of k does the optimal allocation assign two missiles to the nearest offensive missile? More generally, how large must k be before we assign three (or four) defensive missiles to the nearest offensive missile? The mathematical analysis of this section consists simply of algebra and summations of geometric and differentiated geometric series, and therefore is not closely detailed.

We wish to decide the p at which an allocation of one defensive missile to each of the n+1 nearest offensive missiles is equal to an allocation of two defensive missiles on the nearest offensive missile and one defensive missile on each of the next n-1 nearest offensive missiles. We calculate E(rank) for each allocation below. Let (1^{n+1}) denote the first allocation and $(2^1, 1^{n-1})$ denote the second. This type of notation will also be used below.

$$E(1^{n+1}) = (1-p) + 2p(1-p) + \ldots + (n+1)p^n(1-p) + (n+2)p^{n+1}$$

$$= 1 + p + \ldots + p^n + p^{n+1}$$

$$= (1-p^{n+2})/(1-p).$$

$$E(2^1,1^{n-1}) = (1-p)^2 + \left(1-(1-p)^2\right)\left[2(1-p) + 3(1-p)p + \ldots + n(1-p)p^{n-2} + (n+1)p^{n-1}\right]$$

$$= (1-p)^2 + \left(1-(1-p)^2\right)\left[2 + p + p^2 + \ldots + p^{n-1}\right]$$

$$= 1 + (2p-p^2)(1-p^n)/(1-p)$$

From $E(1^{n+1}) = E(2^1,1^{n-1})$ we obtain

$$(1-p) + (2p-p^2)(1-p^n) = 1 - p^{n+2}$$

or

$$p - p^2 = 2p^{n+1} - p^{n+2}$$

which yields

$$p = \left(\tfrac{1}{2}\right)^{1/n}. \qquad (13)$$

It should be clear that $E(1^{n+1}) > E(2^1,1^{n-1})$ whenever $p > (1/2)^n$. Thus we assign 2 defensive missiles to the nearest offensive missile whenever our stockpile has more than $\left(\ln(1/2)(\ln p)\right)^{-1}$ missiles (rounded to the nearest integer).

For example, suppose $p = 0.9$. Then $\ln(1/2)/\ln(.9) \approx 6.65$ so that we assign two missiles to the nearest offensive missile when we have more than 7 defensive missiles.

How do we allocate 11 defensive missiles when $p = 0.9$? To give a general answer to this question, compare the allocation $(n_1, n_2, \ldots, n_k, 1^{n+1})$ with the allocation $(n_1, n_2, \ldots, n_k, 2^1, 1^{n-1})$. We set the expected ranks equal, subtract the first k terms, and cancel out the common factor. The equation immediately reduces to $E(1^{n+1}) = E(2^1, 1^{n-1})$ which was solved above. Thus if $m = 11$ and $p = .9$, we should assign 2 defensive missiles to each of the nearest 2 offensive missiles, and one defensive missile each to the next 7 nearest offensive missiles. This sort of invariance will not be discussed again, but the same principle obviously applies in each of the problems discussed below.

Now, having solved the problem of allocations of one versus two defensive missiles, we ask when we should begin to allocate 3 defensive missiles (instead of 2) to the nearest offensive missile. This suggests a comparison of $(2^{m+1}, 1^{n-1})$ with $(3^1, 2^{m-1}, 1^n)$.

$$E(2^{m+1},1^{n-1}) = (1-p)^2 + 2(1-p)^2\left(1-(1-p)^2\right) +\ldots+ (m+1)(1-p)^2\left(1-(1-p)^2\right)^m$$

$$+ \left[(m+2)(1-p) + (m+3)p(1-p) +\ldots+ (m+n+1)p^{n-1}\right]\left(1-(1-p)^2\right)^{m+1}$$

$$= \left\{1-\left(1-(1-p)^2\right)^{m+1}\left[1 + (m+1)(1-p)^2\right]\right\}/(1-p)^2$$

$$+ \theta_3\left(1-(1-p)^2\right)^{m+1}\left[m + 1 + (1-p^n)/(1-p)\right], \qquad (14)$$

where $\theta_3 = \begin{cases} 0, & n = 1, \\ 1, & n > 1. \end{cases}$

$$E(3^1,2^{m-1},1^n) = (1-p)^3 + \left(1-(1-p)^3\right)\left\{2(1-p)^2 + 3(1-p)^2\left(1-(1-p)^2\right) +\ldots\right.$$

$$\left. + m(1-p)^2\left(1-(1-p)^2\right)^{m-2}\right\}$$

$$+ \left(1-(1-p)^3\right)\left(1-(1-p)^2\right)^{m-1}\left\{(m+1)(1-p)\right.$$

$$\left. + (m+2)p(1-p) +\ldots+ (m+n+1)p^n\right\}$$

$$= (1-p)^3 + \theta_2\left(1-(1-p)^3\right)\left\{\left[1-\left(1-(1-p)^2\right)^{m-1}\left(1 + m(1-p)^2\right)\right.\right.$$

$$\left.\left. + (1-p)^2\right]/(1-p)^2\right\}$$

$$+ \left(1-(1-p)^3\right)\left(1-(1-p)^2\right)^{m-1}\left[m + (1-p^{n+1})/(1-p)\right], \qquad (15)$$

where $\theta_2 = \begin{cases} 0 , & m = 1, \\ 1 , & m > 1. \end{cases}$

Simpler formulas are available when n or m is greater than one.

$$E(2^{m+1},1^{n-1}) = (1-p)^{-2}\Big\{1-\Big(1-(1-p)^2\Big)^{m+1}\Big(1-(1-p^n)(1-p)\Big)\Big\}, \quad n > 1. \tag{16}$$

$$E(3^1,2^{m-1},1^n) = (1-p)^3 + (1-p)^{-2}\Big(1-(1-p)^3\Big)$$
$$\Big\{1-\Big(1-(1-p)^2\Big)^{m-1}\Big(1-(1-p^{n+1})(1-p)\Big) + (1-p)^2\Big\}, \quad m > 1. \tag{17}$$

Also, when m = 1, $E(2^{m+1},1^{n-1}) = E(3^1,2^{m-1},1^n)$ yield, for all n,

$$p^n + p^2 - 3p + 1 = 0. \tag{18}$$

Since in no case does $E(2^{m+1},1^{n-1}) = E(3^1,2^{m-1},1^n)$ yield a simple result such as equation (13), we have resorted to a numerical study which is summarized in Table 3. However we postpone these results until we have considered equations for comparing the allocation $(3^{\ell+1},2^{m-1},1^n)$ with the allocation $(4^1,3^{\ell-1},2^m,1^n)$.

$$E(3^{\ell+1},2^{m-1},1^n) = (1-p)^{-3}\Big[1-\big(1-(1-p)^3\big)^{\ell+1}\big(1+(\ell+1)(1-p)^3\big)\Big]$$

$$+ \theta_2\big(1-(1-p)^3\big)^{\ell+1}(1-p)^{-2}\Big[1-\big(1-(1-p)^2\big)^{m-1}$$

$$\big(1 + (m+\ell)(1-p)^2\big) + (\ell+1)(1-p)^2\Big]$$

$$+ \big(1-(1-p)^3\big)^{\ell+1}\big(1-(1-p)^2\big)^{m-1}\big(\ell+m+(1-p^{n+1})(1-p)^{-1}\big). \tag{19}$$

$$E(4^1,3^{\ell-1},2^m,1^n) = (1-p)^4 + \theta_1(1-p)^{-3}\Big[1-\big(1-(1-p)^3\big)^{\ell-1}\big(1+\ell(1-p)^3\big)$$

$$+ (1-p)^3\Big]\big(1-(1-p)^4\big)$$

$$+ \big(1-(1-p)^4\big)\big(1-(1-p)^3\big)^{\ell-1}(1-p)^{-2}\Big[1-\big(1-(1-p)^2\big)^m$$

$$\big(1 + (m+\ell)(1-p)^2\big)+ \ell(1-p)^2\Big]$$

$$+ \big(1-(1-p)^4\big)\big(1-(1-p)^3\big)^{\ell-1}\big(1-(1-p)^2\big)^m\big(\ell + m$$

$$+ (1-p^{n+1})(1-p)^{-1}\big), \tag{20}$$

where $\theta_1 = \begin{cases} 0, & \ell = 1, \\ 1, & \ell > 1. \end{cases}$

Again these equations reduce to a simpler form when $m > 1$ or $\ell > 1$.

$$E(3^{\ell+1},2^{m-1},1^n) = (1-p)^{-3}\left\{1-\left(1-(1-p)^3\right)^{\ell+1}\left[p + (1-p)\left(1 - (1-p)^2\right)^{m-1}\left(1-(1-p)(1-p^{n+1})\right)\right]\right\}, \quad m > 1. \qquad (21)$$

$$E(4^1,3^{\ell-1},2^m,1^n) = (1-p)^4 + (1-p)^{-3}\left(1-(1-p)^4\right)\left\{1 + (1-p)^3 - \left(1-(1-p)^3\right)^{\ell-1}\left[p + (1-p)\left(1-(1-p)^2\right)^m\left(1 - (1-p)^{n+1}\right)(1-p)\right]\right\}, \quad \ell > 1. \qquad (22)$$

Table 3 gives the number of offensive missiles to be engaged by one or two defensive missiles apiece, for various defensive missile kill probabilities p. To show how this table is used, assume that we have a stockpile of 39 defensive missiles, each with kill probability 0.841. We allocate 3 defensive missiles apiece to the nearest 3 offensive missiles, 2 defensive missiles apiece to the next 13 offensive missiles, and 1 defensive missile to each of the next 4 offensive missiles. The 21^{st} closest offensive missile (and all more distant ones) are not engaged.

To determine the expected rank of the nearest offensive missile penetrating, calculate $\log_e(1-p)^k$, enter the third column of Table 1 with this value, and read off the corresponding expected rank in the fourth column. Although this is only an upper bound, it will typically be only 2 or 3 per cent too high. In the above example, $\log_e(.159)^{39} = 71.71$, and the corresponding expected rank is bounded above by 17.3.

Table 3

Number of Offensive Missiles Engaged by One or Two Defensive Missiles

i	Defensive Kill Probability p $= (1/2)^{1/i}$	Offensive Missiles Engaged by One Missile	Offensive Missiles Engaged by Two Missiles
1	0.500	1	1
2	0.707	2	3
3	0.794	3	7
4	0.841	4	13
5	0.871	5	21
6	0.891	6	31
7	0.906	7	42
8	0.917	8	54
9	0.926	9	69
10	0.933	10	85
12	0.944	12	122
14	0.952	14	166
16	0.958	16	217
18	0.962	18	275
20	0.966	20	339

C. Acknowledgment

I am indebted to A. R. Eckler for providing the original ideas contained in this paper and for many helpful discussions. Also I would like to express my appreciation to Miss D. Kriechbaum for carrying out the programming for the numerical studies made.

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Att.
Appendices I and II

APPENDIX I

Suppose

$$E_n = \sum_{i=1}^{n} X_i a_i \prod_{j=1}^{i-1} (1-a_j) + X_{n+1} \prod_{j=1}^{n} (1-a_j),$$

where a_i is a function of y_i such that $\frac{da_i}{dy_i} = c\ a_i$ for $i = 1,2,\ldots,n$. Then it is easy to substitute $A = \prod_{j=1}^{n} a_j$ and form the system

$$\frac{\partial E_n}{\partial y_i} = 0 \qquad (i = 1,2,\ldots,n-1).$$

We have (replacing a_n by $A(a_1 \cdots a_{n-1})^{-1}$ in E_n above)

$$- c^{-1} \frac{\partial E_n}{\partial y_k} = - X_k a_k \prod_{i=1}^{k-1} (1-a_i) + X_{k+1}(a_{k+1})(a_k) \prod_{i=1}^{k-1} (1-a_i) + \ldots$$

$$+ X_n A(a_1 \cdots a_{n-1})^{-1}\left(\prod_{i=1}^{k-1} (1-a_i)\right)\left(\prod_{j=k+1}^{n-1} (1-a_j)\right)$$

$$+ X_{n+1}\left(a_k - A(a_1 \cdots a_{n-1})^{-1}\right)\left(\prod_{i=1}^{k-1} (1-a_i)\right)\left(\prod_{j=k+1}^{n-1} (1-a_j)\right).$$

If we now let $A = a_1 \cdots a_{n-1}$, we have

$$- C^{-1} \frac{\partial E_n}{\partial y_k} = - X_k a_k \prod_{i=1}^{k-1} (1-a_i) + X_{k+1}(a_{k+1})(a_k) \prod_{i=1}^{k-1} (1-a_i) + \ldots$$

$$+ X_n \left(\prod_{i=1}^{k-1} (1-a_i) \right) \left(\prod_{j=k+1}^{n-1} (1-a_j) \right) - X_{n+1} \prod_{i=1}^{n-1} (1-a_i).$$

The equations $\frac{\partial E}{\partial y_i} = 0$ reduce to

$$- X_1 a_1 + X_2 a_1 a_2 + X_3 a_1 a_3 (1-a_2) + \ldots + X_n \prod_{i=2}^{n-1} (1-a_i) - X_{n+1} \prod_{i=1}^{n-1} (1-a_i) = 0;$$

. . .

$$- X_k a_k + X_{k+1} a_k a_{k+1} + \ldots + X_n \prod_{i=k+1}^{n-1} (1-a_i) - X_{n+1} \prod_{i=k}^{n-1} (1-a_i) = 0;$$

. . .

$$- X_{n-3} a_{n-3} + X_{n-2} a_{n-3} a_{n-2} + X_{n-1} a_{n-3} a_{n-1} (1-a_{n-2})$$

$$+ X_n (1-a_{n-2})(1-a_{n-1}) - X_{n+1} (1-a_{n-2})(1-a_{n-1})(1-a_{n-3}) = 0;$$

$$- X_{n-2} a_{n-2} + X_{n-1} a_{n-2} a_{n-1} + X_n (1-a_{n-1}) - X_{n+1} (1-a_{n-1})(1-a_{n-2}) = 0;$$

$$- X_{n-1} a_{n-1} + X_n - X_{n+1} (1-a_{n-1}) = 0.$$

We use the last 3 equations to find a_{n-1}, a_{n-2}, and a_{n-3}. The last equation easily yields

$$a_{n-1} = (X_{n+1}-X_n)/(X_{n+1}-X_{n-1}).$$

Using this value for a_{n-1} we solve the next equation to find a_{n-2}.

$$a_{n-2} = \left[(X_{n+1}-X_n)(X_n-X_{n-1})\right]/\left[(X_n-X_{n-2})(X_{n+1}-X_{n-1})\right].$$

Substituting for a_{n-1} and a_{n-2}, we obtain

$$a_{n-3}\left[- X_{n-3} + X_{n-2}a_{n-2} + X_{n-1}a_{n-1}(1-a_{n-2}) + X_{n+1}(1-a_{n-2})(1-a_{n-1})\right]$$

$$= (X_{n+1}-X_n)(1-a_{n-2})(1-a_{n-1}).$$

This implies

$$a_{n-3} = a_{n-2}\left[(X_n-X_{n-2})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)\right]/ \left[(X_n-X_{n-3})(X_{n+1}-X_{n-1}) - (X_n-X_{n-1})(X_{n+1}-X_n)\right].$$

Formulas for $a_{n-4}, a_{n-5}, \ldots, a_1$ can be similarly derived although the algebra required to express the denominator in simple form gets increasingly complex. It is very easy to see the general form of the numerator which for a_k is determined by $(X_{n+1}-X_n) \prod_{i=k+1}^{n-1} (1-a_i)$.

APPENDIX II

Assume that the defense has a stockpile of k missiles of kill probability p. Also suppose the optimal allocation is δ_1 defensive missiles on the nearest offensive missile and $\delta_2 = k - \delta_1$ defensive missiles on the second closest offensive missile. We define $k_1 = [\delta_1]$, $k_2 = k - k_1$, and $\varepsilon = \delta_1 - k_1$. Below we derive upper bounds between the two possible integer allocations and the optimal one.

$$E = E(\text{closest missile}) = X_1(1-p)^{k_1+\varepsilon} + X_2(1-p)^{k_2-\varepsilon}\left(1-(1-p)^{k_1+\varepsilon}\right)$$

$$+ X_3\left(1-(1-p)^{k_1+\varepsilon}\right)\left(1-(1-p)^{k_2-\varepsilon}\right).$$

$$\hat{E}_1 = X_1(1-p)^{k_1} + X_2(1-p)^{k_2}\left(1-(1-p)^{k_1}\right) + X_3\left(1-(1-p)^{k_1}\right)\left(1-(1-p)^{k_2}\right).$$

$$\hat{E}_2 = X_1(1-p)^{k_1+1} + X_2(1-p)^{k_2-1}\left(1-(1-p)^{k_1+1}\right)$$

$$+ X_3\left(1-(1-p)^{k_1+1}\right)\left(1-(1-p)^{k_2-1}\right).$$

$$0 < E - \hat{E}_1 = \left[(X_3-X_1)(1-p)^{k_1} + (X_2-X_3)(1-p)^{k_2-\varepsilon}\right]\left(1-(1-p)^{\varepsilon}\right)$$

$$\leq (X_2-X_1)(1-p)^{k_1}\left(1-(1-p)^{\varepsilon}\right) \leq (X_2-X_1)(1-p)^{k_1}\, p.$$

(cont'd)

$$0 < E - \hat{E}_2 = \left[(X_1-X_3)(1-p)^{k_1+\varepsilon} + (X_3-X_2)(1-p)^{k_2-1}\right]\left(1-(1-p)^{\varepsilon}\right)$$

$$\leq (X_3-X_2)(1-p)^{k_2-1}\left(1-(1-p)^{\varepsilon}\right) \leq (X_3-X_2)(1-p)^{k_2-1}$$

Thus a bound for the difference between E and the practical allocation is $0 < E - \hat{E} < (X_3-X_1)(1-p)^{k_2-1}p$.