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On a Class of Variational Problems Arising in Mathematical Economics*

MICHAEL J. P. MAGILL

University of Southern California, Los Angeles, California 90007

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1. INTRODUCTION

This paper studies a class of variational problems that arise in the analysis of resource allocation over an infinite horizon. Such problems are characterised by an underlying technology that generates the basic set of feasible programs and by a preference ordering that selects from among these programs an optimal one. The problem of establishing the existence of an optimal program differs from the standard variational problems on a closed finite interval in that certain inequalities on parameters characterising the asymptotic properties of the underlying technology and preferences play a crucial role in the basic existence condition.

This paper generalises the earlier results of Koopmans [7] and is closely related to the abstract approach developed by Bewley [2]. The basic existence result is applied in Section 4 to the model of an expanding economy introduced by von Neumann. The definition of impatience adopted in Section 5 is motivated by the results of Brown and Lewis [3].

2. CANONICAL RESOURCE ALLOCATION PROBLEM

We are concerned with a class of resource allocation problems over the half open interval $I = [0, \infty)$. Let $(I, \mathcal{T})$ denote the measurable space induced by I, $\mathcal{T}$ the Lebesgue measurable sets and let $\mathcal{M}^n = \mathcal{M}^n(I, \mathcal{T})$ denote the space of all R^n-valued $(n \geqslant 1)$ $\mathcal{T}$-measurable functions defined on $(I, \mathcal{T})$. Let $\xi \in \mathcal{M}^n$ satisfy

$$\int_0^t \|\xi(\tau)\|\, d\tau < \infty, \qquad \forall t \in I \tag{2.1}$$

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and let

$$x(t) = x_0 + \int_0^t \xi(\tau)\, d\tau, \qquad t \in I, \tag{2.2}$$

where $x_0 \in R^n_+$. $\xi(t)$ denotes a flow of output of n goods at time t, $x(t)$ the accumulated stocks. A *technology set* $\Gamma(t) \subset R^n_+ \times R^n_+$, $t \in I$, gives the set of flow outputs $\xi(t)$ producible with the existing stocks $x(t)$. We require that

$$(x(t), \xi(t)) \in \Gamma(t) \text{ a.e.} \tag{2.3}$$

2.1 DEFINITION. $\Gamma(t): I \to R^n_+ \times R^n_+$ will be called *regular* if it is a closed, convex-valued measurable correspondence.

2.2 DEFINITION. $\xi \in \mathscr{M}^n$ satisfying (2.1)–(2.3) is called a *feasible production program.* Let $\mathscr{S}$ denote the set of all such programs. We say I satisfies a *growth condition* if there exist γ, $\underline{\gamma} \in \mathscr{M}$ and $k \in R^n_+$ such that

$$0 < \underline{\gamma}(t) \leqslant \gamma(t) \text{ a.e.}, \qquad \int_0^t \gamma(\tau)\, d\tau < \infty, \qquad \forall t \in I, \tag{2.4}$$

$$\|\xi(t)\| \leqslant \gamma(t) \text{ a.e.} \qquad \forall \xi \in \mathscr{S}, \tag{2.5}$$

$$\underline{\xi}(t) \geqslant k\underline{\gamma}(t) \text{ a.e.} \qquad \text{for some } \underline{\xi} \in \mathscr{S}, \tag{2.6}$$

Let

$$\begin{aligned} \psi_\Gamma(\chi, \zeta, t) &= 0 && \text{if} \quad (\chi, \zeta) \in \Gamma(t) \text{ a.e.} \\ &= -\infty && \text{if} \quad (\chi, \zeta) \notin \Gamma(t) \text{ a.e.} \end{aligned}$$

If Γ is regular then ψ_Γ is a *normal* integrand in the sense of Rockafellar [9, p. 173 and Proposition 2H, p. 177]. Thus for $(x(t), \xi(t)) \in \mathscr{M}^{2n}$, $\psi_\Gamma(x(t), \xi(t), t) \in \mathscr{M}$ [9, Corollary 2B, p. 174]. Thus

$$\mathscr{S} = \text{dom } \Psi(\xi) = \text{dom} \int_I \psi_\Gamma(x(t), \xi(t), t)\, dt.$$

For $\gamma \in \mathscr{M}$ satisfying (2.4) define the Banach spaces

$$\mathscr{V} = \mathscr{L}^n_{\gamma,\infty} = \left\{ \xi \in \mathscr{M}^n \mid \|\xi\|_{\gamma,\infty} = \text{ess sup} \left\| \frac{\xi(t)}{\gamma(t)} \right\| < \infty \right\},$$

$$\mathscr{V}' = \mathscr{L}^n_{\gamma,1} = \left\{ \eta \in \mathscr{M}^n \mid \|\eta\|_{\gamma,1} = \int_I \|\eta(t)\, \gamma(t)\|\, dt < \infty \right\}.$$

The family of seminorms

$$\nu_\eta(\xi) = \left| \int_I \eta(t)\, \xi(t)\, dt \right|, \qquad \eta \in \mathscr{Y}', \xi \in \mathscr{Y}$$

induces a locally convex topology on $\mathscr{Y}$, denoted by $\sigma(\mathscr{Y}, \mathscr{Y}')$.

A *preference ordering* on programs is induced by an integral functional

$$U(\xi) = \int_I u(\xi(t), t)\, \Delta(t)\, dt,$$

where $\Delta \in \mathscr{M}$, $0 < \Delta(t) < \infty$ a.e.

2.3 Definition. $u(\zeta, t): R^n_+ \times I \to [-\infty, \infty)$ will be called *regular* if $u(\cdot, t)$ is upper semicontinuous and concave, dom $u(\cdot, t)$ has a nonempty interior and $u(\zeta, \cdot) \in \mathscr{M}$, for all $t \in I$ and $\zeta \in R^n$, respectively.

If u is regular then u is a normal integrand [9, Corollary 2E, p. 176] so that $\xi(t) \in \mathscr{M}^n$ implies $u(\xi(t), t) \in \mathscr{M}$.

2.4 Definition. We say that u satisfies a *growth condition* if there exist upper semicontinuous non-decreasing functions $\underline{\phi}$, $\bar{\phi}: R_+ \to [-\infty, \infty)$ and an upper semicontinuous concave function $h: R^n_+ \to R_+$ satisfying

$$h(\zeta) \geqslant 0,\ h(\zeta) > 0 \text{ for } \zeta \in K \subset R^n_+,\ h(\lambda\zeta) = \lambda h(\zeta),\ \lambda > 0$$

such that

$$\underline{\phi}(h(\zeta)) \leqslant u(\zeta, t) \leqslant \bar{\phi}(h(\zeta)) \qquad \forall (\zeta, t) \in R^n_+ \times I. \tag{2.7}$$

2.5 Definition. Let $\underline{\lambda} = h(k)$, $\bar{\lambda} = \sup_{\zeta \geqslant 0, \|\zeta\| = 1} h(\zeta)$. We say (Γ, u) satisfy *compatible* growth conditions if

$$-\infty < \int_I \underline{\phi}(\gamma(t)\underline{\lambda})\, \Delta(t)\, dt, \qquad \int_I \bar{\phi}(\gamma(t)\bar{\lambda})\, \Delta(t)\, dt < \infty. \tag{2.8}$$

2.6 Definition. Let $f = u + \psi_\Gamma$, $F(\xi) = \int_I f(x(t), \xi(t), t)\, \Delta(t)\, dt$. $\xi \in \mathscr{Y}$ is an *optimal program* if

$$F(\xi) = \sup_{\xi' \in \mathscr{Y}} F(\xi'), \qquad -\infty < F(\xi) < \infty.$$

3. Existence of Optimal Program

3.1 Proposition. *If (Γ, u) are regular and satisfy compatible growth conditions, then there exists an optimal program.*

Proof. Equation (2.5) and the homogeneity of $h(\cdot)$ imply $h(\xi(t)) \leqslant \gamma(t)\bar{\lambda}$ a.e. $\forall \xi \in \mathscr{S}$. By (2.7), $u(\xi(t), t) \leqslant \bar{\phi}(\gamma(t)\bar{\lambda}) = \alpha(t)$ a.e. $\forall \xi \in \mathscr{S}$. Thus $g(x(t), \xi(t), t) = \alpha(t) - f(x(t), \xi(t), t) \geqslant 0$ a.e. $\forall \xi \in \mathscr{V}$ and $G(\xi) = \int_I g(x(t), \xi(t), t)\,\Delta(t)\,dt = \alpha^* - F(\xi)$, where $-\infty < \alpha^* = \int_I \alpha(t)\,\Delta(t)\,dt < \infty$ by (2.8). Let $\{\xi_n\}_{n=1}^{\infty} \subset \mathscr{V}$ such that $\xi_n \to \xi^*$, $\sigma(\mathscr{V}, \mathscr{V}')$ and let $G_n = G(\xi_n)$, $\underline{G} = \underline{\lim}_{n\to\infty} G_n$, $G^* = G(\xi^*)$. Since $G(\xi) \geqslant 0$, $\forall \xi \in \mathscr{V}$, $0 \leqslant \underline{G} \leqslant \infty$. If $\underline{G} = \infty$, then $G^* \leqslant \underline{G}$. Suppose $0 \leqslant \underline{G} < \infty$ and select a subsequence $\{\xi_m\}_{m=1}^{\infty} \subset \{\xi_n\}_{n=1}^{\infty}$ such that $G(\xi_m) = G_m \to \underline{G}$. Let $\chi_{I_t}(\tau)$ denote the characteristic function of $I_t = [0, t]$, $v_i = (0,..., 1,..., 0) \in R_+^n$ and $v_i(t, \tau) = \chi_{I_t}(\tau)\, v_i$, $i = 1,..., n$. For each $t \in I$, let $\eta(\tau) = v_i(t, \tau)$, then $\eta \in \mathscr{V}'$ by (2.4), so that

$$\lim_{m\to\infty} \int_I v_i(t, \tau)\, \xi_m(\tau)\, d\tau = \int_I v_i(t, \tau)\, \xi^*(\tau)\, d\tau, \qquad t \in I, \quad i = 1,..., n$$

implies

$$\lim_{m\to\infty} x_m(t) = \lim_{m\to\infty} \left(x_0 + \int_0^t \xi_m(\tau)\, d\tau\right) = x_0 + \int_0^t \xi^*(\tau)\, d\tau = x^*(t), \quad \forall t \in I.$$

Let $Z_k = \{\xi_m\}_{m=k}^{\infty}$, $k = 1, 2,...$ and let $\bar{Z}_k$, $\operatorname{co} Z_k$ denote the $\sigma(\mathscr{V}, \mathscr{V}')$ closure and convex hull of Z_k. Since $\xi^* \in \bar{Z}_k \subseteq \operatorname{co} \bar{Z}_k = \overline{\operatorname{co} Z_k}$ and since $\overline{\operatorname{co} Z_k} = \overline{\operatorname{co} Z_k}^s$, where the latter denotes closure in the norm topology [4, Theorem V.3.13, p. 422], $\xi^* \in \overline{\operatorname{co} Z_k}^s$. Thus if we let $[\xi_m] = \{\xi_m\}_{m=1}^{\infty}$, there exist $A_k[\xi_m] = \sum_{m=k}^{\bar{k}} \alpha_{km} \xi_m \in \operatorname{co} Z_k$, $k = 1, 2,...$, where $\alpha_{km} \geqslant 0$, $\sum_{m=k}^{\bar{k}} \alpha_{km} = 1$, $k \leqslant \bar{k} < \infty$ such that

$$\lim_{k\to\infty} \|A_k[\xi_m] - \xi^*\|_{\gamma,\infty} = 0$$

so that

$$\lim_{k\to\infty} A_k[\xi_m(t)] = \xi^*(t), \qquad \lim_{k\to\infty} A_k[x_m(t)] = x^*(t) \text{ a.e.}$$

Since $g(\cdot, t)$ is lower semicontinuous

$$G(\xi^*) = \int_I g(x^*(t), \xi^*(t), t)\, \Delta(t)\, dt$$

$$\leqslant \int_I \underline{\lim}_{k\to\infty} g(A_k[x_m(t)], A_k[\xi_m(t)], t)\, \Delta(t)\, dt.$$

By Fatou's Lemma

$$\int_I \underline{\lim}_{k\to\infty} g(A_k[x_m(t)], A_k[\xi_m(t)], t)\, \Delta(t)\, dt$$

$$\leqslant \underline{\lim}_{k\to\infty} \int_I g(A_k[x_m(t)], A_k[\xi_m(t)], t)\, \Delta(t)\, dt.$$

Since $g(\cdot, t)$ is convex

$$\varliminf_{k\to\infty} \int_I g(A_k[x_m(t)], A_k[\xi_m(t)], t)\,\Delta(t)\,dt$$

$$\leqslant \varliminf_{k\to\infty} A_k\left[\int_I g(x_m(t), \xi_m(t), t)\,\Delta(t)\,dt\right]$$

so that $G^* \leqslant \varliminf_{k\to\infty} A_k[G_m] = G$ and $G(\xi)$ is $\sigma(\mathscr{Y}, \mathscr{Y}')$ lower semicontinuous. Since (2.5) implies $\mathscr{F}$ is a norm bounded subset of $\mathscr{Y}$, by the theorem of Alaoglu [4, p. 424], $\mathscr{F}$ is relatively $\sigma(\mathscr{Y}, \mathscr{Y}')$ compact. Since $\Psi(\xi)$ is $\sigma(\mathscr{Y}, \mathscr{Y}')$ upper semicontinuous (let $F = \Psi$ in the above argument), it follows that $\mathscr{F}$ is $\sigma(\mathscr{Y}, \mathscr{Y}')$ closed. By (2.5), (2.6), (2.8), $-\infty < \bar{\mu} = \sup_{\xi\in\mathscr{F}} F(\xi) < \infty$. Thus $\{\xi \in \mathscr{F} \mid F(\xi) \geqslant \mu\}_{\mu<\bar{\mu}}$ is a family of nonempty $\sigma(\mathscr{Y}, \mathscr{Y}')$ closed subsets of the $\sigma(\mathscr{Y}, \mathscr{Y}')$ compact set $\mathscr{F}$, with the finite intersection property. $\xi \in \bigcap_{\mu<\bar{\mu}} \{\xi \in \mathscr{F} \mid F(\xi) \geqslant \mu\} \neq \varnothing$ is an optimal program.

4. Application to von Neumann Model

We consider an economy with $m \geqslant 1$ goods in which $\xi = (c, v) \in \mathscr{M}^m \times \mathscr{M}^m$ denotes the consumption–investment program over the interval $I = [0, \infty)$. Output devoted to investment accumulates to form a productive stock of capital, output devoted to consumption is immediately consumed so that $x = (0, z)$ where $z(t) = z_0 + \int_0^t v(\tau)\,d\tau$.

4.1 Definition. A *reduced form* production set $\Pi(t) \subset R_+^m \times R_+^m$ is a set of pairs (a, b) such that b is the output producible with the stocks a. The technology set $\Gamma(t)$ is representable in reduced form if there exists a production set $\Pi(t)$ such that for $(\chi, \zeta) = (\chi_1, \chi_2, \zeta_1, \zeta_2)$

$$\Gamma(t) = \{(\chi, \zeta) \in R_+^{2m} \times R_+^{2m} \mid (\chi_2, \zeta_1 + \zeta_2) \in \Pi(t), (\zeta_1, \zeta_2) \geqslant 0\} \text{ a.e.}$$

For $\chi \in R^n$ let $\chi > 0$ denote $\chi \geqslant 0$, $\chi \neq 0$ and let $\chi \gg 0$ with $\chi = (\chi_1, \ldots, \chi_n)$ denote $\chi_i > 0$, $i = 1, \ldots, n$.

4.2 Definition. A production set $\Pi \subset R_+^m \times R_+^m$ is *standard* if (i) Π is a *closed convex cone*, (ii) $(a, b) \in \Pi$ with $a = 0$ implies $b = 0$, (iii) there exists $(a, b) \in \Pi$ with $b \gg 0$, (iv) $(a, b) \in \Pi$, $a' \geqslant a$, $0 \leqslant b' \leqslant b$ implies $(a', b') \in \Pi$.

4.3 Example. The production set Π in von Neumann's model [10] is the convex polyhedral cone

$$\Pi = \{(a, b) \mid (-a, b) \leqslant (-A, B)\eta, \eta \geqslant 0, b \geqslant 0\}, \tag{4.1}$$

where (A, B) is a pair of $m \times s$ matrices $(m \geqslant 1, s \geqslant 1)$ satisfying (i) $a_{ij}, b_{ij} \geqslant 0$, (ii) for any j there exists i such that $a_{ij} > 0$, (iii) for any i there exists j such that $b_{ij} > 0$. (i)–(iii) imply (4.1) is standard.

4.4 EXAMPLE. Let $g: R_+^m \to R_+^m$, $g = (g^1, ..., g^m)$ satisfy (i) $g^i(\cdot)$ is upper semicontinuous and concave, (ii) $g^i(\cdot)$ is homogeneous of degree 1, (iii) $a' \geqslant a$ implies $g(a') \geqslant g(a)$ and $g^i(a') > g^i(a)$ for some $i \notin \{j \mid a'_j > a_j\}$ if $a' \neq a$. (i)–(iii) imply that the production set

$$\Pi = \{(a, b) \mid b \leqslant g(a), (a, b) \geqslant 0\} \tag{4.2}$$

is standard. As a special case let $g(a) = Aa$ where A is a non-negative *irreducible* $m \times m$ matrix. In this case ρ^* in Proposition 4.6 is the *Frobenius* root of A [6, Theorem 2, p. 53].

4.5 DEFINITION. For any $(a, b) \in \Pi$, $\rho(a, b) = \sup\{\rho \in R \mid b \geqslant \rho a\}$ is called the *expansion rate* of the process (a, b). $\rho^* = \sup_{(a,b) \in \Pi} \rho(a, b)$ is called the *maximal expansion rate* for Π.

The following result is due to von Neumann [10, p. 36] and Gale [5, p. 290].

4.6 PROPOSITION. *If Π is a standard production set, then there exist $p^* \in R^m$, $a^* \in R^m$, $\rho^* \in R$ such that*

(i) $(\rho^* p^*, p^*)(-a, b) \leqslant 0$ *for all* $(a, b) \in \Pi$,

(ii) $\rho^* = \rho(a^*, b^*) = \sup_{(a,b) \in \Pi} \rho(a, b)$, $b^* = \rho^* a^*$,

(iii) $p^* > 0$, $a^* > 0$, $\rho^* > 0$.

4.7 DEFINITION. Π is *productive* if $(a, b) \in \Pi$ with $\rho(a, b) = \rho^*$ implies $a \gg 0$.

4.8 EXAMPLE. Consider (4.1). $M' \subset M = \{1, 2, ..., m\}$ is an *independent* subset of goods if there exists $S' \subset S = \{1, 2, ..., s\}$ such that for each $i \in S \backslash M'$ and $j \in S'$, $a_{ij} = 0$, while for each $i \in M'$, $b_{ij} > 0$ for some $j \in S'$. The pair (A, B) in (4.1) is *irreducible* if the set M has no nontrivial independent subsets. *If (A, B) is irreducible, then Π in* (4.1) *is productive* [5, p. 295]. A simple argument shows that Π in Example 4.4 is productive.

4.9 PROPOSITION. *Let $z_0 \gg 0$. If Γ is representable in reduced form by a standard, productive set Π with maximal expansion rate ρ^*, if the discount* *factor is exponential* $\Delta(t)=e^{-\delta t}$, $-\infty<\delta<\infty$, *u is regular and satisfies a growth condition with* $K=\{\zeta=(\zeta_1,\zeta_2)\in R_+^{2m} \mid \zeta_1 \gg 0\}$

$$\underline{\phi}(s)=\bar{\phi}(s)=\frac{s^\beta}{\beta}, \qquad 0\neq\beta\leqslant 1, \\ =\ln s, \qquad \beta=0 \tag{4.3}$$

and if

$$\delta>\beta\rho^* \tag{4.4}$$

then there exists an optimal program.

Proof. We use Proposition 4.6 to show that Γ satisfies a growth condition (2.4)–(2.6). For any $\varepsilon>0$ let $r_\epsilon(p;a,b)=pb-(\rho^*+\varepsilon)\,pa$. Since Π is standard and productive, Proposition 4.6(i) and (iii) imply $\rho^*p^*a^*>0$, $r_\epsilon(p^*;a,b)\leqslant r_\epsilon(p^*;a^*,b^*)=r_\epsilon^*<0$, $\forall(a,b)\in\Pi$, $(a,b)\neq 0$. Let $\psi(p)=\sup_{(a,b)\in\Pi\cap\Sigma} r_\epsilon(p;a,b)$, where $\Sigma=\{(a,b)\in R_+^{2m} \mid \sum_{i=1}^m (a_i+b_i)=1\}$. Since $\zeta_\epsilon(\cdot;\cdot)$ is continuous on $R^m\times R^{2m}$ and since $\Pi\cap\Sigma$ is compact, $\psi(p)$ is upper semicontinuous [1, Theorem 2, p. 116]. Thus for any $\nu>0$ such that $r_\epsilon^*+\nu<0$ there exists $\mu>0$ such that if $\eta\in R^m$, $\eta\gg 0$ satisfies $\|\eta\|<\mu$ then $\psi(p^*+\eta)<r_\epsilon^*+\nu<0$. But then $r_\epsilon(p^*+\eta;a,b)<0$, $\forall(a,b)\in\Pi$, $(a,b)\neq 0$ and $p^*+\eta\gg 0$. Thus if we set $\hat{p}=p^*+\eta$

$$\hat{p}b<(\rho^*+\varepsilon)\,\hat{p}a, \qquad \forall(a,b)\in\Pi, \quad (a,b)\neq 0. \tag{4.5}$$

Consider $\bar{\xi}=(\bar{c},\bar{v})\in\mathscr{S}$ for which $\bar{c}(t)=0$ a.e. Then $(z_0+\int_0^t \bar{v}(\tau)\,d\tau,\bar{v}(t))=(\bar{z}(t),\dot{\bar{z}}(t))\in\Pi$ a.e. By (4.5) $\hat{p}\dot{\bar{z}}(t)<(\rho^*+\varepsilon)\,\hat{p}\bar{z}(t)$ a.e. so that

$$\hat{p}\bar{z}(t)<(\hat{p}z_0)\,e^{(\rho^*+\epsilon)t} \text{ a.e.} \tag{4.6}$$

Thus for any $\xi=(c,v)\in\mathscr{S}$, $(z_0+\int_0^t v(\tau)\,d\tau, c(t)+v(t))=(z(t),q(t))\in\Pi$ a.e., since $c(t)\geqslant 0$ a.e., (4.5) and (4.6) imply

$$\hat{p}q(t)<(\rho^*+\varepsilon)\,\hat{p}\bar{z}(t)<(\rho^*+\varepsilon)(\hat{p}z_0)\,e^{(\rho^*+\epsilon)t} \text{ a.e.}$$

Thus since $\hat{p}\gg 0$, for any $\varepsilon>0$ there exists $\gamma_\epsilon>0$ such that

$$\gamma(t)=\gamma_\epsilon e^{(\rho^*+\epsilon)t} \text{ a.e.} \tag{4.7}$$

satisfies (2.4) and (2.5). Since $z_0\gg 0$ there exists $\theta>0$ such that $\theta a^*\leqslant z_0$. Since Π is standard and productive, by Definition 4.2(iv) and Proposition 4.6(ii), for any $0<\varepsilon<1$ the path

$$\underline{\xi}(t)=(\underline{c}(t),\underline{v}(t))=(\varepsilon\rho^*z(t),(1-\varepsilon)\,\rho^*z(t)) \text{ a.e.,} \qquad z(0)=\theta a^*\gg 0$$

is feasible. Thus for any $0 < \varepsilon < 1$ there exists $k_\epsilon \in K$ such that

$$k\gamma(t) = k_\epsilon e^{(1-\epsilon)\rho^* t} \text{ a.e.} \tag{4.8}$$

satisfies (2.6). Since $\lambda = h(k_\epsilon) > 0$ and since (4.7) and (4.8) are valid for any $0 < \varepsilon < 1$, it follows from (4.3) and (4.4), that (2.8) is satisfied. The result follows from Proposition 3.1.

4.10 *Remark.* The definition of K in Proposition 4.9 implies $U(\xi) = U(c)$. The problem is to maximise the benefits arising from the stream of consumption c over the interval $[0, \infty)$. The problem of choice is the problem of balancing the benefits of the *present* (arising from consumption) against those of the *future* (arising from investment). When (4.4) is satisfied this problem of resource allocation has a solution. Let $m = 1$, a simple argument shows that if

$$\delta < \beta\rho^* \tag{4.9}$$

then *there is no optimal program.* If $c_T(t)$ denotes consumption optimal at time t for the problem on $[0, T]$, then

$$c_T(t) \to 0 \qquad \forall t < T, \qquad c_T(T) \to \infty \qquad \text{as} \quad T \to \infty$$

The limit of the finite horizon optimal paths is, in a real sense, the worst possible path. Let $\beta = 1 - \eta$, then it can be shown that $\delta + \eta\rho^*$ is a measure of the *rate of impatience* on a path of consumption growing exponentially at the rate ρ^*. Since (4.9) reduces to

$$\delta + \eta\rho^* < \rho^* \tag{4.10}$$

if the rate of impatience is less than the maximal rate of growth ρ^*, then the problem of resource allocation has no solution. If (4.10) holds and if agents mimick the limit of a sequence of finite horizon problems in attempting to arrive at a solution, then they are led to a program of fruitless over-accumulation of capital.

5. Upper Semicontinuity and Impatience

The essential economic content of the existence condition (4.4) or (2.8) may be uncovered if we recognise that these conditions arise from the requirement that the preference function $U(\cdot)$ be upper semicontinuous in the $\sigma(\mathscr{Y}, \mathscr{Y}')$ topology on $\mathscr{Y} = \mathscr{L}^n_{\gamma,\infty}$. If for any ξ, $\xi' \in \mathscr{S}$ such that $U(\xi) < U(\xi')$ and for any $\zeta \in \mathscr{Y}$ there exists $\tau \in I$ such that $U(\xi + \zeta_T) < U(\xi') \; \forall T > \tau$, where $\zeta_T = [\zeta\chi_{[T,\infty)}$, then $U(\cdot)$ is said to exhibit *impatience* on

$\mathscr{S}$. It is easy to show that for any $\zeta \in \mathscr{V}$, $\zeta \chi_{[T,\infty)} \to 0$ in the $\sigma(\mathscr{V}, \mathscr{V}')$ topology as $T \to \infty$. Thus *if* $U(\cdot)$ *is upper semicontinuous in the* $\sigma(\mathscr{V}, \mathscr{V}')$ *topology on* $\mathscr{S}$, *then* $U(\cdot)$ *exhibits impatience.* The real force of the existence condition (2.8) thus lies in the requirement that the preference ordering represented by $U(\cdot)$ exhibit impatience in the topology of growth generated by the technology.

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