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An Equilibrium Model of Risk and Investment*

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1. INTRODUCTION

The object of this paper is to outline a simple equilibrium theory of investment. The firms in the economy operate in an environment of uncertainty and draw up plans extending over an uncertain future horizon. Firms' plans consist of choices of rates of investment and degrees of risk for their capital projects. The firms purchase new capital equipment on one set of markets and sell their output on another. To simplify the equilibrium aspect of the model, to focus attention on the investment behaviour of firms, and to avoid entering into an extended discussion of the behaviour of consumers and their lifetime budget constraints, I use a device which reduces the analysis of equilibrium to the analysis of a simple maximum problem. This device is a natural generalisation of the single market consumers' surplus approach used by Lucas–Prescott [7], Brock–Magill [3], Magill [8], and Scheinkman [12] in the study of a theory of investment. This reduction of the equilibrium problem to a maximum problem can be justified by a more extensive analysis which I shall not enter into in this paper.

The paper is arranged as follows. Section 2 introduces the basic class of random processes on which the subsequent analysis is based. The analysis here draws on the framework of Bismut [2] and Brock–Magill [3]. Given an underlying Brownian motion process, once velocity and variance (risk) processes are given, lying in a suitable space of integrable functions, the state becomes a well-defined process. A class of concave maximum problems over a random finite horizon is introduced in which the integrand of the basic variational problem depends on the current state, velocity and variance. A class of dual imputed price processes is introduced. I recall a generalisation of the classical sufficiency theorem: a random process which is *imputed price supported* and satisfies a transversality condition is optimal.

Section 3 introduces two models of the underlying economy. The first I call an *extensive form* model, the second a *reduced form* model. The

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extensive form model examines explicitly the maximising behaviour of all agents. If in this model each consumer's preference ordering is representable by a utility function which is *additively separable* in time and across the states of nature, then the market demand (supply) of each consumer for the commodities of the state-time pair (ω, t) depends only on the price vector $p(\omega, t)$. If producers of new capital goods use a *fixed stock of capital* in conjunction with services of consumers to produce a flow of new capital goods, then their supply at the state-time (ω, t) depends only on $p(\omega, t)$. The reduced form model takes these demands and supplies of consumers and new capital goods producers as given data and summarises their activity on the markets by a *demand–supply correspondence* ϕ. It seems likely that this type of reduced form model may be of great utility in an empirical analysis of the investment behaviour of firms, since it provides the simplest equilibrium framework for the analysis of investment.

Section 4 analyses the demand–supply correspondence ϕ. Borrowing a result of Rockafellar [10] which generalises the classical theorem on the existence of a potential function, I introduce an assumption on ϕ which expresses in abstract form the fact that demand (supply) is a decreasing (increasing) function of the current price $p(\omega, t)$. Under these conditions the demand–supply correspondence ϕ is the supergradient of a concave function Φ. I form its concave *conjugate* Φ^* which I call the *price potential.* Equilibrium prices will turn out to be supergradients of Φ^*.

In section 5 I outline the basic model of the firm which may be viewed as a generalisation to the case of uncertainty of the Lucas [6] adjustment-cost model. The firm's output of final goods is random and depends on its current effective capital stock, its current investment and the current degree of risk of its capital projects. The degree of risk, which the firm is free to choose, is modelled by the extent to which the firm allows itself to be influenced by a finite number of exogenous sources of uncertainty, which are common to all firms and are represented, albeit in idealised form, by an s-dimensional Brownian motion process. The firm's activity on markets consists of the purchase of new capital goods and the sale of final goods, and it chooses an output-investment process that maximises its expected profit.

Since the effective capital stock, investment and degree of risk may be regarded as inputs in the firm's production function, it seems natural that certain prices should be imputed to these factors in calculating an output-investment process that maximises expected profit. Section 6 shows that if such imputed prices, representing the firm's demand price for capital, investment, and risk, are introduced, and if a general profit criterion involving these prices and inputs is maximised, then the firm's investment-risk process is optimal. These conditions generalise the conditions familiar in the Lucas adjustment-cost theory. A firm invests up to the point where the purchase price of new capital goods plus the adjustment cost equal its demand price (a condition closely related to Tobin's [14] ratio of market value to replacement cost). In addition the firm undertakes risk up to the point where the value of the marginal product of risk equals its demand price, which I call the firm's internal *insurance cost.*

In Section 7 I show how the price potential Φ^* of Section 4 can be used to generate an equilibrium output-investment process for all the firms in the economy in the reduced form model of Section 3. It is shown that a process for which the integral price potential is maximised generates, via the supergradients of Φ^*, an equilibrium for the reduced form model.

Since an important emphasis in Tobin's formulation of the theory of investment lies in its focus on the relation between the security market and the markets for real commodities, a few remarks are in order concerning the relation of the above analysis to an analysis of the role of the security market in the determination of investment. Under certain conditions it may be possible to relate the risk prices (ρ_i, π_i) introduced in Section 6 to the gradient and the Jacobian matrix of the *value function* representing the market value of the ith firm. It is known that ρ_i equals the gradient of the value function in the special case of the above model where uncertainty is eliminated. It may be argued that information contained in the value function can be conveyed to the firm by the security market. This would accord with Keynes' [5, p. 151] view that the investment behaviour of firms is in many instances determined by the expectations of agents dealing on the securities markets as revealed in the price of shares. This role of the security market would complement the role of exchanging risk among agents in the economy brought out in the analysis of Arrow [1].

2. Dual Random Processes

Let $(\Omega, \mathscr{F}, P)$ denote a complete probability space, $\mathscr{F}$ a σ-field on Ω and P a probability measure on $\mathscr{F}$. In the framework of Arrow [1] each element $\omega \in \Omega$ constitutes a state of nature and Ω constitutes the totality of all states of nature.

Assumption 1. For each state of nature $\omega \in \Omega$ there is an uncertain horizon $[0, T(\omega)]$, where $0 < T(\omega) < T < \infty$ almost surely and $\{\omega \mid T(\omega) \leqq t \subset \mathscr{F}_t, \quad t \in I = [0, T]$, where $\mathscr{F}_t \subset \mathscr{F}_\tau \subset \mathscr{F}, \quad t < \tau$, is an increasing family of sub-σ-fields of $\mathscr{F}$, denoting an increasing system of information. Let $I_\omega = [0, T(\omega)]$.

Let $(I, \mathscr{M}, \mu)$ denote the complete measure space of Lebesgue measurable sets $\mathscr{M}$, with Lebesgue measure μ. Let $(\Omega \times I, \mathscr{H}, P \times \mu)$ denote the associated complete product measure space, $\mathscr{H} \supset \mathscr{F} \times \mathscr{M}$. Let $(R^m, \mathscr{M}^m)$, with $m \geqq 1$, denote the measurable space formed from the m-dimensional real Euclidean space R^m with σ-field of Lebesgue measurable sets $\mathscr{M}^m$. Let

$$x(\omega, t) : (\Omega \times I, \mathscr{H}) \to (R^m, \mathscr{M}^m)$$

denote an $\mathscr{H}$-measurable function (*random process*) defined by the equation

$$x(\omega, t) = x_0 + \int_0^t \dot{x}(\omega, \tau)\, d\tau + \int_0^t \sigma(\omega, \tau)\, d\beta(\omega, \tau), \quad (\omega, t) \in (\Omega, I_\omega), \tag{1}$$

where $x_0 \in R^m$ is a nonrandom initial condition, $\beta(\omega, t) \in R^s$, $s \geqq 1$, is a ***Brownian motion process***, and where $\dot{x}(\omega, \tau)$ and $\sigma(\omega, \tau)$ are $\mathscr{H}$-measurable random processes with values in $(R^m, \mathscr{M}^m)$ and $(R^{ms}, \mathscr{M}^{ms})$ satisfying

$$\int_\Omega \left(\int_{I_\omega} \|\dot{x}(\omega, \tau)\|^2\, d\tau + \int_{I_\omega} \|\sigma(\omega, \tau)\|^2\, d\tau \right) dP(\omega) < \infty \tag{1'}$$

We assume $\mathscr{F}_t \supset \mathscr{I}_t = \mathscr{I}(\beta(\omega, \tau),\ \tau \in [0, t])$ the σ-field generated by the Brownian motion process. $\dot{x}(\omega, t)$ and $\sigma(\omega, t)$ are assumed to be nonanticipating, that is, they are $\mathscr{I}_t$-measurable for all $t \in I$. Let $(x, \dot{x}, \sigma) = (x(\omega, t), \dot{x}(\omega, t), \sigma(\omega, t))$ denote an $\mathscr{H}$-measurable random process satisfying conditions (1) and (1′) and let $\mathscr{P}$ denote the set of all such random processes.

In preparation for Section 3 and 4, we introduce a random real valued instantaneous *profit* (*potential*) *function*

$$L(\omega, t, \xi) : \Omega \times I \times R^{2m+ms} \to [-\infty, \infty) \tag{2}$$

and a real valued terminal *bequest function*

$$\ell(\chi) : R^m \to [-\infty, \infty). \tag{2'}$$

ASSUMPTION 2. (i) $L(\omega, t, \cdot)$ and $\ell(\cdot)$ are upper semicontinuous, proper concave functions,[1] (ii) dom $L(\omega, t, \cdot)$ has a nonempty interior, (iii) $L(\cdot, \xi)$ is $\mathscr{H}$-measurable.

The pair of functions (L, ℓ) in conjunction with the random processes $\mathscr{P}$ lead to the following:

MAXIMUM PROBLEM. Find an $\mathscr{H}$-measurable random process $(x, \dot{x}, \sigma) \in \mathscr{P}$, with $x_0 \in R^m$ a fixed initial condition, such that

$$\sup_{(x, \dot{x}, \sigma) \in \mathscr{P}} \mathscr{L}(x, \dot{x}, \sigma), \tag{$\mathscr{A}$}$$

[1] A concave function $f(\xi) : R^n \to [-\infty, \infty)$ is *proper* if $-\infty < f(\xi)$ for some $\xi \in R^n$.

where $\mathscr{L}(x, \dot{x}, \sigma) = \int_\Omega \int_{I_\omega} L(\omega, t, x(\omega, t), \dot{x}(\omega, t), \sigma(\omega, t))\, dt\, dP(\omega) + \ell(x(\omega, T(\omega)))$.

We impose the following *implicit growth condition* on the pair (L, ℓ) relative to the processes $\mathscr{P}$:

ASSUMPTION 3. There exist constants $-\infty < \underline{\alpha} < \bar{\alpha} < \infty$ such that

(i) $\underline{\alpha} < \mathscr{L}(\underline{x}, \dot{\underline{x}}, \sigma)$ for some $(\underline{x}, \dot{\underline{x}}, \sigma) \in \mathscr{P}$

(ii) $\mathscr{L}(x, \dot{x}, \sigma) < \bar{\alpha}$ for all $(x, \dot{x}, \sigma) \in \mathscr{P}$.

In order to express a sufficient condition for a random process $(x, \dot{x}, \sigma)$ to be a solution of $(\mathscr{A})$, we introduce a class of *dual price processes* constructed in the following way. Let

$$\rho(\omega, t) : (\Omega \times I, \mathscr{H}) \to (R^m, \mathscr{M}^m)$$

denote an $\mathscr{H}$-measurable random process defined by the equation

$$\rho(\omega, t) = \rho_0 + \int_0^t \dot{\rho}(\omega, \tau)\, d\tau + \int_0^t \pi(\omega, \tau)\, d\beta(\omega, \tau), \quad (\omega, t) \in (\Omega, I_\omega), \tag{3}$$

where $\rho_0 \in R^m$ is a nonrandom initial condition, $\beta(\omega, t) \in R^s$ is the same Brownian motion process as in (1), and where $\dot{\rho}(\omega, \tau)$ and $\pi(\omega, \tau)$ are $\mathscr{H}$-measurable random processes with values in $(R^m, \mathscr{M}^m)$ and $(R^{ms}, \mathscr{M}^{ms})$ satisfying

$$\int_\Omega \left(\int_{I_\omega} \|\dot{\rho}(\omega, \tau)\|^2\, d\tau + \int_{I_\omega} \|\pi(\omega, \tau)\|^2\, d\tau \right) dP(\omega) < \infty, \tag{3'}$$

$\dot{\rho}(\omega, t)$ and $\pi(\omega, t)$ being nonanticipating. We let $(\rho, \dot{\rho}, \pi) = (\rho(\omega, t), \dot{\rho}(\omega, t), \pi(\omega, t))$ denote an $\mathscr{H}$-measurable random process satisfying conditions (3) and (3′). We are thus requiring that $(\rho, \dot{\rho}, \pi) \in \mathscr{P}$.

DEFINITION. A random process $(x, \dot{x}, \sigma) \in \mathscr{P}$ is *imputed price supported* if there exists a dual random price process $(\dot{\rho}, \rho, \pi) \in \mathscr{P}$ such that

$$\begin{aligned} &\dot{\rho}x + \rho\dot{x} + \operatorname{tr}(\pi\sigma') + L(\omega, t, x, \dot{x}, \sigma) \\ &\qquad \geqq \dot{\rho}\chi + \rho\xi + \operatorname{tr}(\pi\zeta') + L(\omega, t, \chi, \xi, \zeta) \end{aligned} \tag{4}$$

for all $(\chi, \xi, \zeta) \in R^m \times R^m \times R^{ms}$, for almost all $(\omega, t) \in (\Omega, I_\omega)$.

DEFINITION. Let $f: R^k \to [-\infty, \infty)$ be a proper concave function, then $p \in R^k$ is a *supergradient* of f at $x \in R^k$ if

$$f(\xi) \leqq f(x) + p(\xi - x) \qquad \text{for all} \quad \xi \in R^k. \tag{5}$$

The *set of supergradients* of f at x is written as $\partial f(x)$ or $f_\xi(x)$.

Lemma 1 is immediate from the definition of a supergradient.

LEMMA 1. *Under Assumption 2 a random process $(x, \dot{x}, \sigma) \in \mathscr{P}$ is imputed price supported if and only if*

$$(\dot{p}(\omega, t), \rho(\omega, t), \pi(\omega, t)) \in -\,\partial L(\omega, t, x(\omega, t), \dot{x}(\omega, t), \sigma(\omega, t)) \tag{6}$$

for almost all $(\omega, t) \in (\Omega, I_\omega)$.

Equation (6) is a generalisation of the standard Euler–Lagrange equation. We make use of the following result (Bismut [2, Theorem IV-2]).

PROPOSITION 1. *Let Assumptions 1–3 be satisfied. If a random process $(x, \dot{x}, \sigma) \in \mathscr{P}$ is imputed price supported by a dual process $(\rho, \dot{\rho}, \pi) \in \mathscr{P}$ for which*

$$\rho(\omega, T(\omega)) \in \partial \ell(x(\omega, T(\omega))) \qquad \text{a.s.,} \tag{7}$$

then $(x, \dot{x}, \sigma)$ is a solution of $(\mathscr{A})$.

3. THE REDUCED FORM MODEL

We begin with a description of two models of an economy, an *extensive form* model and a *reduced form* model. The extensive form model makes clear the assumptions implicit in the working of the reduced form model.

The economy consists of two sectors, a consumer sector and a producer sector. The production sector is in turn divided into two sectors, one producing a flow of final goods and the other producing a flow of new capital goods. Consumers own endowments of services which they supply in flow amounts to producers in exchange for a flow of final goods. Each consumer has a preference ordering over the commodity space of final goods and services and chooses a most preferred bundle in his budget set. I assume that the preference ordering of each consumer is representable by a utility function which is concave and additively separable in time and across the states of nature. Under these conditions the demand (supply) for goods at time t and in state ω by each consumer will depend only on the price $p(\omega, t)$.

It follows that aggregate consumer demand (supply) for goods in the state-time (ω, t) depends only on the price $p(\omega, t)$.

The producers of new capital goods are assumed to use a fixed stock of capital in conjunction with the services of consumers to produce, under nondecreasing costs, a flow of new capital goods. Since the capital stock of each producer in this sector is fixed, the profit maximising problem reduces to a myopic, instantaneous profit maximising problem. The individual and hence aggregate supply of new capital goods in the state-time (ω, t) thus depends only on the price vector $p(\omega, t)$.

Final goods producers use the services of consumers and a stock of capital to produce a flow of final goods. Since these producers may also accumulate capital by purchasing it from capital goods producers, their profit maximising problem leads to an intertemporal problem in which the demand for capital goods and supply of final goods in the state-time (ω, t) no longer depends in a simple way on the price $p(\omega, t)$.

An equilibrium in the model in extensive form involves a market clearing vector of prices $p(\omega, t)$ for final goods, new capital goods and services at each state-time $(\omega, t) \in (\Omega, I_\omega)$.

The *reduced form* model differs from the extensive form model in two ways: first, it omits the explicit maximising behaviour of individual consumers and capital goods producers instead summarising their aggregate behaviour on the markets in the state-time (ω, t) in a demand–supply correspondence which depends only on the price $p(\omega, t)$. Second, in seeking to simplify the analysis and to focus attention on the investment process, it omits the flow of services provided by consumers.[2] An equilibrium in the model in reduced form thus involves a market clearing vector of prices for final goods and new capital goods for each state-time $(\omega, t) \in (\Omega, I_\omega)$. The reduced form model forms the basis for the analysis that follows. It provides a simple equilibrium framework in which attention is focused on the intertemporal investment behaviour of firms.

4. The Price Potential

In this section, we introduce the demand–supply correspondence which summarises the aggregate market behaviour of consumers and capital goods producers. At each state-time $(\omega, t) \in (\Omega, I_\omega)$, $m \geqq 1$ final goods and $k \geqq 1$ new capital goods are traded. Let

$$\phi(\omega, t, p) : \Omega \times I \times R^m_+ \times R^k_+ \to R^m_+ \times R^k_- \tag{8}$$

[2] It is however useful to retain consumer services as a device for explaining the relation between internal and external adjustment costs, see Mussa [9].

denote the random aggregate *demand–supply correspondence* of consumers and capital goods producers. Each element of $\phi(\omega, t, p)$ gives a total number of units of each of the final goods $y(\omega, t)$, and each of the new capital goods, $-v(\omega, t)$ after change of sign, that consumers will purchase and capital goods producers will supply, when the unit price of the vector of commodities is p.

Since ϕ is viewed as being derived from underlying maximising behaviour it is natural to assume that ϕ has a *monotonicity property* expressing in general terms that *quantity demanded* (*supplied*) *is a decreasing* (*increasing*) *function of price.*

ASSUMPTION 4. (i) For each $(\omega, t) \in \Omega \times I$, $\phi(\omega, t, \cdot)$ is *cyclically monotone* so that for any finite sequence of pairs $(p^0, z^0), \ldots, (p^n, z^n)$, $n \geqq 1$, satisfying $z^i \in \phi(\omega, t, p^i)$ we have

$$(p^1 - p^0)z^0 + (p^2 - p^1)z^1 + \cdots + (p^0 - p^n)z^n \geqq 0. \tag{9}$$

Furthermore, ϕ has a *maximal graph* so that there does not exist a correspondence ϕ' satisfying (9) such that

$$\{(p, z) \mid z \in \phi'(p)\} \supsetneqq \{(p, z) \mid z \in \phi(p)\}.$$

(ii) $\phi(\cdot, p)$ is $\mathscr{H}$ measurable.

(iii) The set of final goods prices

$$Q(\omega, t) = \{q \mid \text{there exists } z \in \phi(\omega, t, p),\ p = (q, r) \in R^m_+ \times R^k_+\}$$

is uniformly bounded.

Maximal cyclical monotonicity of ϕ replaces the assumption of *symmetry* of the Jacobian matrix of ϕ (when ϕ is a C^1 vector field) in the classical theorem on the existence of a potential function (Spivak [13, p. 94]) as shown by

PROPOSITION 2. *If the demand–supply correspondence* (8) *satisfies Assumption* 4(i) *and* (ii), *then there exists a function*

$$\Phi(\omega, t, p) : \Omega \times I \times R^m \times R^k \to [-\infty, \infty) \tag{10}$$

such that:

(i) $\phi(\omega, t, p) = \partial\Phi(\omega, t, p)$, $p \in R^m_+ \times R^k_+$, $(\omega, t) \in \Omega \times I$,

(ii) $\Phi(\omega, t, \cdot)$ *is an upper semicontinuous, proper concave function such that* dom $\Phi(\omega, t, \cdot) \subset R^m_+ \times R^k_+$,

(iii) $\Phi(\cdot, p)$ *is* $\mathscr{H}$*-measurable.*

Proof. (i) and (ii) follow from a theorem of Rockafellar [10, p. 239] replacing subgradients by supergradients as defined in (5). (iii) is immediate. ∎

DEFINITION. The *conjugate* of $\Phi(\omega, t, p)$ with respect to p

$$\Phi^*(\omega, t, z) = \inf_{p \in R^n} \{pz - \Phi(\omega, t, p)\} \tag{11}$$

will be called the *price potential* of the markets.

LEMMA 2. *If the demand–supply correspondence* (8) *satisfies Assumption* 4(i) *and* (ii), *then the price potential* (11)

$$\Phi^*(\omega, t, z) : \Omega \times I \times R^m \times R^k \to [-\infty, \infty)$$

has the following properties:

(i) $z \in \partial\Phi(\omega, t, p)$ *if and only if* $p \in \partial\Phi^*(\omega, t, z)$,

(ii) $\Phi^*(\omega, t, \cdot)$ *is an upper semicontinuous, proper concave function such that* dom $\Phi^*(\omega, t, \cdot) \subset R^m_+ \times R^k_-$,

(iii) *if* z_1, $z_2 \in$ dom $\Phi^*(\omega, t, \cdot)$, $z_1 \geqq z_2$, *then* $\Phi^*(\omega, t, z_1) \geqq \Phi^*(\omega, t, z_2)$,

(iv) $\Phi^*(\cdot, z)$ *is* $\mathscr{H}$*-measurable.*

Proof. (i) In view of definitions (5) and (11), $z \in \partial\Phi(\omega, t, p) \Leftrightarrow \Phi(\omega, t, p) + \Phi^*(\omega, t, z) = pz \Leftrightarrow p \in \partial\Phi^*(\omega, t, z)$ since $\Phi^{**}(\omega, t, p) = \Phi(\omega, t, p)$ by the upper semicontinuity of $\Phi(\omega, t, \cdot)$ (Proposition 2(ii)). (ii) follows from Rockafellar [10, Theorem 12.2, p. 104]. (iii) follows from $\partial\Phi^*(\omega, t, z) \subset$ dom $\Phi(\omega, t, \cdot) \subset R^m_+ \times R^k_+$, for all $z \in$ dom $\Phi^*(\omega, t, \cdot)$. (iv) is immediate. ∎

Let $\phi^*(\omega, t, z) = \partial\Phi^*(\omega, t, z)$. By Lemma 2(i), the graphs of ϕ and ϕ^* correspond to the same subset of $(R^m_+ \times R^k_+) \times (R^m_+ \times R^k_-)$. Thus the aggregate market behaviour summarised in correspondence (8) can also be introduced by the demand–supply *price correspondence*

$$\phi^*(\omega, t, z) : \Omega \times I \times R^m_+ \times R^k_- \to R^m_+ \times R^k_+ \tag{12}$$

requiring that it satisfy Assumption 4.

EXAMPLE. Let $\phi^*(\omega, t, \cdot)$ induce a positive C^1 vector field with negative definite Jacobian on $R^m_+ \times R^k_-$. The price potential $\Phi^*(\omega, t, \cdot)$ can then be discovered directly from the vector field $\phi^*(\omega, t, \cdot)$ by taking the line integral of $\phi^*(\omega, t, \cdot)$ from an arbitrary[3] fixed point $\bar{z} \in R^m_+ \times R^k_-$ to the point $z \in R^m_+ \times R^k_-$ along the line segment $\gamma(\bar{z}, z)$ joining these two points

$$\Phi^*(\omega, t, z) = \int_{\gamma(\bar{z}, z)} \phi^*(\omega, t, \xi)\, d\xi. \tag{13}$$

Since $\phi^*(\omega, t, \cdot) > 0$, $\Phi^*(\omega, t, \cdot)$ is a strictly increasing function.

5. Behaviour of Firms

In this section, we consider the investment behaviour of firms in the final goods producing sector. The ith firm ($i = 1, \ldots, n$) produces a flow output of at least one of the $m \geqq 1$ final goods with the aid of stocks of the $k \geqq 1$ capital goods. The firm's output of final goods

$$y_i(\omega, t) : \Omega \times I \to R^m_+$$

is an $\mathscr{H}$-measurable, nonanticipating process. The actual output at any given state-time (ω, t) depends on the current *effective capital stock* $x_i(\omega, t)$, on the current *rate of investment* $\dot{x}_i(\omega, t)$ and on the current *degree of risk* $\sigma_i(\omega, t)$ of existing capital projects. Let $f^i = (f^i_1, \ldots, f^i_m)$ denote a vector valued random function, where

$$f^i_j(\omega, t, \xi) : \Omega \times I \times R^{2m+ms} \to [-\infty, \infty), \qquad j = 1, \ldots, m \tag{14}$$

satisfy Assumption 2. The firm's output is given by

$$y_i(\omega, t) = f^i(\omega, t, x_i(\omega, t), \dot{x}_i(\omega, t), \sigma_i(\omega, t)).$$

We assume in addition that output increases when the effective capital stock x_i or degree of risk σ_i are increased, at least for small nonnegative values of (x_i, σ_i), while output decreases when investment $\dot{x}_i$ increases, to reflect costs of adjustment resulting from the installation of new capital equipment.

The rate of current investment $\dot{x}_i$ and the degree of risk σ_i in turn determine the accumulation of the firm's effective capital stock by an equation of the form (1) in Section 2

$$x_i(\omega, t) = x_{i0} + \int_0^t \dot{x}_i(\omega, \tau)\, d\tau + \int_0^t \sigma_i(\omega, \tau)\, d\beta(\omega, \tau), \quad (\omega, t) \in (\Omega, I_\omega), \tag{15}$$

[3] In some cases it is necessary to choose $\bar{z} \in \operatorname{int}(R^m_+ \times R^k_-)$ to ensure the finiteness of the line integral (13).

where $x_{i0} \in R^m_+$ is the firm's nonrandom initial capital stock, $\beta(\omega, t)$ is an s-dimensional Brownian motion process and where $\dot{x}(\omega, t)$ and $\sigma(\omega, t)$ are $\mathscr{H}$-measurable, nonanticipating random processes satisfying (1′). We are thus requiring that $(x_i, \dot{x}_i, \sigma_i) \in \mathscr{P}$. The s components of the Brownian motion process represent s basic *sources of uncertainty* in the economic environment to which all n firms are subjected in varying degrees and which constitute the common environment of uncertainty in which all firms must make their production decisions. These s sources of uncertainty arise from natural causes such as the weather, strikes, the unpredictability of technological developments, the breakdown of instruments of production and so on. By virtue of its mathematical properties, $\beta(\omega, t)$ represents such types of uncertainty in a highly idealised, continuous manner, but I shall not enter into a discussion of these issues here. Suffice it to say that not all exogenous sources of uncertainty can be satisfactorily modelled by a Brownian motion process.

Constraints on the feasible choices of processes $(x_i, \dot{x}_i, \sigma_i) \in \mathscr{P}$, such as $(x_i(\omega, t), \dot{x}_i(\omega, t), \sigma_i(\omega, t)) \geqq 0$ for almost every $(\omega, t) \in (\Omega, I_\omega)$ can be incorporated into the function $f^i(\omega, t, \chi, \xi, \zeta)$, without violating Assumption 2, by letting $f^i(\omega, t, \chi, \xi, \zeta) = -\infty$ whenever $(\chi, \xi, \zeta) \ngeqq 0$. Using this convention and letting the $\mathscr{H}$-measurable *output-investment process* be denoted by

$$z_i(\omega, t) = (y_i(\omega, t), -\dot{x}_i(\omega, t)) : \Omega \times I \to R^m_+ \times R^k_-$$

we may define the *technology set* of the ith firm

$$Z_i = \left\{ z_i = (y_i, -\dot{x}_i) \;\middle|\; \begin{array}{l} y_i(\omega, t) \leqq f^i(\omega, t, x_i(\omega, t), \dot{x}_i(\omega, t), \sigma_i(\omega, t)) \\ y_i(\omega, t) \neq -\infty, (x_i, \dot{x}_i, \sigma_i) \in \mathscr{P} \end{array} \right\}. \quad (16)$$

We assume that $0 \in Z_i$ so that the firm has the option of not participating on the markets. Let

$$p(\omega, t) : \Omega \times I \to R^m_+ \times R^k_+$$

be an $\mathscr{H}$-measurable, nonanticipating random process denoting the vector of *prices* for the m final goods and k capital goods at each state-time pair (ω, t). Given its expectations concerning the price process $p(\omega, t)$, the ith firm selects an output-investment process $z_i \in Z_i$ so as to maximise its *expected profit* (*market value*)

$$\int_\Omega \int_{I_\omega} p(\omega, t)\, z_i(\omega, t)\, dt dP(\omega) \quad (17)$$

subject to the natural boundary condition at the random terminal time $T(\omega)$

$$x_i(\omega, T(\omega)) \geqq 0 \qquad \text{a.s.} \quad (18)$$

6. Optimal Investment-Risk Process

It is clear from the definition of the ith firm's technology set Z_i in Eq. (16) that the choice of an otimal output-investment process $z_i \in Z_i$, reduces to the choice of an optimal *investment-risk process* $(\dot{x}_i, \sigma_i)$ satisfying (1′). The results of Section 2 may thus be used to throw light on the procedure by which a firm arrives at its choice of an optimal investment-risk process.

We introduce a dual *imputed price process*

$$\rho_i(\omega, t) = \rho_{i0} + \int_0^t \dot{\rho}_i(\omega, \tau)\, d\tau + \int_0^t \pi_i(\omega, \tau)\, d\beta(\omega, \tau), \qquad (\omega, t) \in (\Omega, I_\omega), \quad (19)$$

where $\rho_{i0} \in R_+^m$ is nonrandom, $\beta(\omega, t)$ is the Brownian motion process appearing in (15) and where $\dot{\rho}_i(\omega, t)$ and $\pi_i(\omega, t)$ are $\mathscr{H}$-measurable nonanticipating processes satisfying (3′). We are thus requiring that $(\rho_i, \dot{\rho}_i, \pi_i) \in \mathscr{P}$.

Proposition 3. *Let $T(\omega)$ satisfy Assumption* 1, *let $f_j^i(\omega, t, \xi)$ in* (14) *satisfy Assumption* 2, *$f^i = (f_1^i, ..., f_m^i)$ with $f^i(\cdot) < \gamma_i$ for some $\gamma_i \in R_+^m$. Let $p(\omega, t) = (q(\omega, t), r(\omega, t))$ be an $\mathscr{H}$-measurable nonanticipating price process such that*

$$\sup_{z_i \in Z_i} \int_\Omega \int_{I_\omega} p(\omega, t) z_i(\omega, t)\, dt dP(\omega) < \infty. \quad (20)$$

If $(x_i, \dot{x}_i, \sigma_i) \in \mathscr{P}$ generates a process $z_i \in Z_i$ for which (18) *holds and if there exists an imputed price process $(\rho_i, \dot{\rho}_i, \pi_i) \in \mathscr{P}$ such that*

(i) $(\dot{\rho}_i(\omega, t),\ \rho_i(\omega, t),\ \pi_i(\omega, t)) \in -(q(\omega, t) f_{x_i}^i,\ q(\omega, t) f_{\dot{x}_i}^i - r(\omega, t),\ q(\omega, t) f_{\sigma_i}^i)$ *a.e.*

(ii) $\rho_i(\omega, T(\omega))\, x_i(\omega, T(\omega)) = 0$ *a.s.*,

where $(f_{x_i}^i, f_{\dot{x}_i}^i, f_{\sigma_i}^i) = \partial f^i(\omega, t, x_i(\omega, t), \dot{x}_i(\omega, t), \sigma_i(\omega, t))$, then $(\dot{x}_i, \sigma_i)$ is an optimal investment-risk process.

Proof. Consider the problem $(\mathscr{A})$ in Section 2 with

$$L(\omega, t, \chi, \xi, \zeta) = q(\omega, t) f^i(\omega, t, \chi, \xi, \zeta) - r(\omega, t)\, \xi, \quad (21)$$

$$\ell(\chi) = \psi(\chi) = 0, \qquad \text{if } \chi \in R_+^n,$$
$$= -\infty, \qquad \text{if } \chi \notin R_+^n.$$

Assumption 2 is satisfied. Assumption 3(i) holds with $\underline{\alpha} = 0$ since $0 \in Z_i$, (ii) holds by (20). (i) follows from Lemma 1 and Proposition 1. For $A \subset R^n$, let

$$N(x, A) = \{ p \in R^n \mid p(\xi - x) \geqq 0,\ \forall \xi \in A \}$$

denote the *normal cone* to A at x. Since $\partial\psi(x) = N(x, R^n_+)$ and $x_i(\omega, T(\omega)) \geqq 0$ a.s. by (18), $\rho_i(\omega, T(\omega)) \in N(x_i(\omega, T(\omega)), R^n_+)$ a.s. is equivalent to (ii). ▌

When $f^i(\omega, t, \cdot) \in C^1$, (i) reduces to three simple conditions:

$$-\dot{\rho}_i = qf^i_{x_i} \tag{22}$$

$$\rho_i = r - qf^i_{\dot{x}_i} \tag{23}$$

$$-\pi_i = qf^i_{\sigma_i}. \tag{24}$$

Equation (22) requires that *the imputed rental cost of capital* $(-\dot{\rho}_i)$ *equals the value of the marginal product of capital* $(qf^i_{x_i})$. Equation (23) requires that the *imputed value of an increment to the firm's capital stock* (ρ_i) *equals the marginal cost of acquiring an additional unit* (the supply price r of an additional unit, plus the marginal cost of installing it, $-qf^i_{\dot{x}_i}$). When (23) is written in the form

$$r = \rho_i + qf^i_{\dot{x}_i}$$

it gives the Marshallian demand function for investment $\dot{x}_i$ by the ith firm. Equation (24) is the new element that arises from the presence of uncertainty. $-\pi_i$ is the *imputed cost of risk* or the internal *insurance cost* for each unit of risk. Equation (24) requires that *the degree of risk* σ_i *be chosen in such a way that the insurance cost for an additional unit of risk* $(-\pi_i)$ *equals the value of the marginal product of risk* $(qf^i_{\sigma_i})$.

By Lemma 1, condition (i) is just the supergradient condition (6) associated with the maximisation of (4). In view of (21), (4) becomes

$$[\dot{\rho}_i x_i + \rho_i \dot{x}_i + tr(\pi_i \sigma_i')] + [qf^i(\omega, t, x_i, \dot{x}_i, \sigma_i) - r\dot{x}_i] \tag{25}$$

which may be called the *generalised current profit* of the ith firm. Finding an optimal investment-risk process thus reduces to the maximisation of generalised current profit at each state-time (ω, t) under an appropriate imputed price process $(\rho_i, \dot{\rho}_i, \pi_i)$.

The expression in the second square bracket of (25) is the familiar current profit of the ith firm. In the standard deterministic adjustment cost theory of the firm we add the first two components of the first bracket: these represent the imputed rental costs for the services of the current capital stock plus the imputed value of newly acquired capital. In an environment of uncertainty an additional imputed insurance cost $\text{tr}(-\pi_i \sigma_i')$ must be deducted depending on the imputed insurance cost $-\pi_i$ and the degree of risk σ_i of existing capital projects. When insurance premiums are paid for insurable risks such as fire and theft, the insurance premiums are included as a component of costs. Not all business risks, however, can be insured against in the marketplace, indeed, perhaps only a small proportion of business risks is insurable in this way. Proposition 3 suggests that in arriving at an optimal investment-risk process the ith firm costs out internally those risks which cannot be insured against in the marketplace.

7. Equilibrium in Reduced Form Model

In this section, we show how the price potential Φ^* of Section 4 can be used to determine an equilibrium output-investment process for the reduced form model of Section 3. For $z_i \in Z_i$, let $z = \sum_{i=1}^n z_i$ and $Z = \sum_{i=1}^n Z_i$.

Definition. An *equilibrium* for the reduced form model is an $\mathscr{H}$-measurable, nonanticipating process $(z_1(\omega, t),..., z_n(\omega, t), p(\omega, t))$ such that:

(i) $z_i \in Z_i$, $x_i(\omega, \mathrm{T}(\omega)) \geqq 0$ a.s., $i = 1,..., n$, and

$$\int_\Omega \int_{I_\omega} p(\omega, t)\, z_i(\omega, t)\, dtdP(\omega) \geqq \int_\Omega \int_{I_\omega} p(\omega, t)\, \bar{z}_i(\omega, t)\, dtdP(\omega)$$

for all $\bar{z}_i \in Z_i$ such that $\bar{x}_i(\omega, \mathrm{T}(\omega)) \geqq 0$ a.s.

(ii) $z(\omega, t) \in \phi(\omega, t, p(\omega, t))$ for almost all $(\omega, t) \in (\Omega, I_\omega)$.

Let us introduce the following simplified notation:

$$F(\omega, t, \alpha, \beta, \gamma) = \sum_{i=1}^n (f^i(\omega, t, \alpha_i, \beta_i, \gamma_i), -\beta_i),$$

$$\xi = (x_1,..., x_n, \dot{x}_1,..., \dot{x}_n, \sigma_1,..., \sigma_n),$$

$$\eta = (\rho_1,..., \rho_n, \dot{\rho}_1,..., \dot{\rho}_n, \pi_1,..., \pi_n),$$

$$\hat{\eta} = (\dot{\rho}_1,..., \dot{\rho}_n, \rho_1,..., \rho_n, \pi_1,..., \pi_n).$$

Proposition 4. *If* (i) $T(\omega)$ *satisfies Assumption* 1, (ii) $f_j^i(\cdot)$ *in* (14) *satisfy Assumption* 2, $f^i = (f_1^i,..., f_m^i)$, *with* $f^i(\cdot) < \gamma_i$ *for some* $\gamma_i \in R_+^n$, $i = 1,..., n$, (iii) *the correspondence* $\phi(\omega, t, p)$ *in* (8) *satisfies Assumption* 4, (iv) *there exist constants*[4] $-\infty < \underline{\beta} < \bar{\beta} < \infty$ *such that* $\underline{\beta} \leqq \Phi^*(\omega, t, 0)$, $\Phi^*(\omega, t, c) \leqq \bar{\beta}$ *a.e., where* $c = (0, \sum_{i=1}^n \gamma_i)$, (v) *there exist* $\xi \in \mathscr{S}$ *and an imputed price process* $\eta \in \mathscr{S}$ *such that* (18) *holds*, $i = 1,..., n$, *and*

(a) $\hat{\eta}(\omega, t) \in -\partial\Phi^*(\omega, t, F(\omega, t, \xi(\omega, t)))$ *a.e.*,

[4] This condition is automatically satisfied when $\phi(\omega, t, p) = \phi(p)$ in (8) is nonrandom and time independent.

(b) $p_i(\omega, T(\omega))\, x_i(\omega, T(\omega)) = 0$ *a.s.*, $i = 1,\dots, n$,

then $\xi \in \mathscr{P}$ *maximises the integral price potential*

$$\int_\Omega \int_{I_\omega} \Phi^*(\omega, t, F(\omega, t, \xi(\omega, t)))\, dtdP(\omega) \qquad (\mathscr{B})$$

subject to (18), $i = 1,\dots, n$, *and the associated output-investment-price process*

$$(z_1(\omega, t),\dots, z_n(\omega, t), p(\omega, t)), \qquad p(\omega, t) \in \partial\Phi^*(\omega, t, z(\omega, t)) \qquad (\mathscr{E})$$

is an equilibrium for the reduced form model.

Proof. Since ϕ satisfies Assumption 4, by Lemma 2(ii) and (iii), $\Phi^*(\omega, t, \cdot)$ is an increasing, upper semicontinuous proper concave function. Since $f_j^i(\cdot)$ satisfy Assumption 2, $j = 1,\dots, m$, $i = 1,\dots, n$, $F(\omega, t, \cdot)$ is an upper semicontinuous proper concave function. Thus the composite function $L(\omega, t, \cdot) = \Phi^*(\omega, t, F(\omega, t, \cdot))$ is an upper semicontinuous proper concave function and $\operatorname{int}\operatorname{dom} L(\omega, t, \cdot) \neq \varnothing$. Consider the problem ($\mathscr{A}$) with L as defined and $\ell(x_1,\dots, x_n) = \sum_{i=1}^n \psi(x_i)$. Since $T(\omega) < \mathrm{T} < \infty$ a.s. by Assumption 1, (iv) implies Assumption 3(i) holds with $\underline{\alpha} = \underline{\beta} T$. Since for any $z \in Z$, $z(\omega, t) \leqq c$ a.e. and since $\Phi^*(\omega, t, \cdot)$ is increasing $\Phi^*(\omega, t, z(\omega, t)) \leqq \Phi^*(\omega, t, c) \leqq \bar{\beta}$ a.e. so that

$$\sup_{z\in Z} \int_\Omega \int_{I_\omega} \Phi^*(\omega, t, z(\omega, t))\, dtdP(\omega) \leqq \bar{\beta}\mathrm{T} < \infty$$

and Assumption 3(ii) holds with $\bar{\alpha} = \bar{\beta}T$. Conditions (a), (b), and Proposition 1 imply ξ maximises ($\mathscr{B}$). By the theorem on the set of supergradients of a composite function (Ioffe–Levin [4, p. 44]), if we let $z(\omega, t) = F(\omega, t, \xi(\omega, t))$ then $\partial\Phi^*(\omega, t, F(\omega, t, \xi(\omega, t))) = \partial\Phi^*(\omega, t, z(\omega, t))\, \partial F(\omega, t, \xi(\omega, t))$. Condition (a) implies that there exists an $\mathscr{H}$-measurable function $p(\omega, t) \in \partial\Phi^*(\omega, t, z(\omega, t))$ such that $\hat{\eta}(\omega, t) \in -p(\omega, t)\, \partial F(\omega, t, \xi(\omega, t))$. Thus (a) implies that Proposition 3(i) holds. By Lemma 2(i), $p(\omega, t) \in \partial\Phi^*(\omega, t, z(\omega, t))$ if and only if $z(\omega, t) \in \partial\Phi(\omega, t, p(\omega, t)) = \phi(\omega, t, p(\omega, t))$ a.e. by Proposition 2(i). Since $z \in Z$ implies $z(\omega, t) \leqq c$ a.e. and since $p(\omega, t) = (q(\omega, t), r(\omega, t)) \in \partial\Phi^*(\omega, t, z(\omega, t))$ implies $\|q(\omega, t)\| \leqq \delta$ a.e. by Assumption 4(iii) and $r(\omega, t) \geqq 0$ a.e.,

$$\sup_{z\in Z} \int_\Omega \int_{I_\omega} p(\omega, t)\, z(\omega, t)\, dtdP(\omega) < \infty. \qquad (26)$$

Since $0 \in Z_i$, for each firm (20) is nonnegative. But then (26) implies (20).

By Proposition 3, (i) in the definition of equilibrium holds and since (ii) has been shown, ($\mathscr{E}$) is an equilibrium for the reduced form model. ▮

Each sample path of the equilibrium process ($\mathscr{E}$) has a simple geometric interpretation in $R^m_+ \times R^k_-$ when Φ^* is independent of time, $\Phi^*(\omega, t, \cdot) = \Phi^*(\omega, \cdot)$. Since $\Phi^*(\omega, \cdot)$ is an increasing upper semicontinuous concave function, it has upper contour sets[5]

$$\Phi^*_+(\chi) = \{\chi' \in R^m_+ \times R^k_- \mid \Phi^*(\chi') \geqq \Phi^*(\chi)\}, \qquad \chi \in R^m_+ \times R^k_-$$

which form a nested sequence of closed convex subsets of $R^m_+ \times R^k_-$

$$\Phi^*_+(\chi_1) \supset \Phi^*_+(\chi_2) \qquad \text{if} \quad \chi_1 < \chi_2.$$

For each $\omega \in \Omega$ the realisation $\{z(\omega, t), t \in I_\omega\}$ of the process $z \in Z$ which maximises ($\mathscr{B}$) traces a continuous trajectory in $R^m_+ \times R^k_-$. The equilibrium price $\{p(\omega, t), t \in I_\omega\}$ lies in the normal cone $N(z(\omega, t), \Phi^*_+(z(\omega, t)))$ to the upper contour set $\Phi^*_+(z(\omega, t))$ at each point along the trajectory $\{z(\omega, t), t \in I_\omega\}$.

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[5] The dependence of Φ^* on $\omega \in \Omega$ is omitted since ω is assumed to be fixed.

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