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Real effects of money in general equilibrium*

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This paper studies a simple stochastic general equilibrium model with money and nominal assets. We examine the role of money as a medium of exchange and as a store of value and give conditions under which local changes in the money supply lead to local changes in the equilibrium allocation.

1. Introduction

Debreu (1970, 1976) has emphasized the importance of establishing that generically an economy $\mathscr{E}$ has *locally unique* or equivalently a *finite* set of equilibria. Without this property serious questions are raised about the adequacy and explanatory power of the equilibrium model. This property has been shown to be dramatically absent in certain recent studies of equilibrium in economies with incomplete markets and nominal assets [Balasko and Cass (1989), Geanakoplos and Mas-Colell (1989)]. These authors have shown that generically such economies generate a high-dimensional submanifold of equilibria. The attempt to resolve this problem of indeterminacy provided the original motivation for this paper.

The basic economic reason for the indeterminacy can be explained as follows. In these equilibrium models with uncertainty and nominal assets, different profiles of price levels across the states of nature lead to different purchasing power of the nominal asset returns across the states. When the financial markets are incomplete changes in purchasing power of the nominal returns lead to changes in the subspace of income transfers achievable by

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trading in the assets. It is the freedom to vary the price levels and hence to tilt the subspace of income transfers which leads to the high-dimensional submanifold within the set of equilibrium allocations.

Price levels, however, should not be a free variable of an economy, rather they should be endogenously determined in equilibrium. The fact that in an economy with Arrow–Debreu markets price levels play no role has perhaps blinded general equilibrium theorists to the importance of endogenously determining price levels. Since price levels are essentially determined by the quantity of money in the economy, all this reflects the absence of a theory of money in the standard general equilibrium model. In this paper we explore a prototype model of money in a general equilibrium model with two time periods. [For a critique of alternative theories of money in general equilibrium models, see Hahn (1983).]

A key characteristic of a modern economy is the extensive degree of specialisation and trade. The large array of goods and services produced is made possible by the high degree of specialisation in the tasks performed by agents. Specialisation necessitates exchange and the complexity of exchange in such an economy necessitates the use of money as a medium of exchange to facilitate transactions. The problem is to find a way of modelling the way money flows through the economy inducing the complex process of marketing goods without abandoning too much of the market clearing description characteristic of a general equilibrium model.

The simplest way of modelling the transactions role of money is to separate the moments in time at which goods can be sold and purchased: in our model this is achieved by dividing each period into two subperiods: an initial subperiod in which goods are sold in exchange for money and a second subperiod in which money is used to purchase goods [see Clower (1967)]. This introduction of separate subperiods for the sale and purchase of commodities allows us to distinguish between the role of money as a *medium of exchange* and its role as a *store of value*. For money acquired in the initial period which is not used for transactions purposes can be held as a store of value for use in the second period. In this latter role money must compete with other financial assets: thus an agent's behavior with respect to money is integrated into his overall *portfolio decision*. The importance of such an integration has frequently been stressed in the literature [see Tobin (1961)]. We shall find that when agents hold money as a store of value they affect the *velocity of circulation* of money and this in turn has important effects on the equilibrium.

By a scheme that we shall explain later, the government injects a certain amount of money into the private sector at date 0 and in each state $(s=1,\dots,S)$ at date 1. We define a concept of a *monetary equilibrium* in which each agent chooses the amount of goods to purchase on the spot markets (at date 0 and in each state $s=1,\dots,S$ at date 1), the portfolio of financial assets and the amount of money to be carried forward into period 1. Spot and asset markets clear and a system of monetary equations asserts that the level of spot prices is determined by the amount of money injected by the government and the transactions balances retained by individuals for the purchase of commodities on the spot markets.

It can be argued that this way of modelling money involves a certain amount of brute force. On this we would agree. However, it is our conviction that the basic message that we draw from this model should be robust to alternative specifications of the way money enters the economy.

We show that generically in endowments and money supply *an economy* $\mathscr{E}$ *has a finite number of locally unique monetary equilibria* (Theorem 2). *Thus introducing money eliminates all the indeterminacy of the model without money studied by Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989) and restores the property of local uniqueness demanded by Debreu for a satisfactory equilibrium model.* We show that monetary equilibria are of two kinds: those in which money is used purely as a medium of exchange and those in which money is used in addition as a store of value. *For an equilibrium of the former type, if the asset markets are complete, local changes in the money supply have no real effects* [Theorem 3(a)]; *if the asset markets are incomplete, then local changes in the money supply translate into an* $S-1$ *dimensional submanifold of real allocations* [Theorem 3(b)]. Thus the indeterminacy of the model without money becomes parameterized by the monetary policy.

Theorem 3 can be regarded as a result on the neutrality or non-neutrality of monetary policy. It is thus closely related to the *policy effectiveness* debate of Sargent and Wallace (1975) and Fischer (1977). Theorem 3(a) may be viewed as a general equilibrium version of the Sargent–Wallace neutrality proposition: with rational expectations monetary policy is neutral if (i) asset markets are complete and (ii) the velocity of circulation of money is independent of M (the money supply). Theorem 3(b) can be viewed as a general equilibrium version of the Fischer critique: with rational expectations if (i) asset markets are incomplete and (ii) nominal asset returns and the velocity of circulation are independent of M, then generically monetary policy has real effects. Of course, for some types of contracts it may not be realistic to assume that nominal returns are fixed independently of anticipated monetary policy.

Equilibria in which money serves not only as a medium of exchange but also as a store of value, arise when monetary policy gives rise to anticipated deflation. For such equilibria local non-proportional changes in the money supply have real effects regardless of whether the asset markets are complete or incomplete. In this case changes in M induce changes in the velocity of circulation which have redistributive effects – they alter the present value of each agent's income. *More formally, for such equilibria local changes in the* *money supply translate into an S-dimensional submanifold of real allocations in both the case of complete and incomplete markets* (Theorem 4).

An important condition required for the validity of Theorems 3(b) and 4 is that there be sufficient diversity among agents in the economy. This diversity is twofold. First, there must be enough agents – the exact condition is given in the theorems – but in each case we must have at least two agents. Second, the agents must be distinct – more precisely genericity conditions are made to ensure that the agents have distinct endowments and hence distinct income profiles. The fact that the arguments underlying Theorems 3(b) and 4 depend in an essential way on diversity among the agents places these results in sharp contrast with an important strand of modern macroeconomics which is based on models of equilibrium with a single representative agent. The redistributive income effects that lie behind the real effects of money supply changes are necessarily absent in all such economies.

Two comments are in order regarding our modelling of money: first, we follow a tradition in the literature by using *helicopter* money as opposed to the more realistic *open market operations* money. To properly handle the latter type of money requires more than two time periods – however, our principal interest lies in making a precise connexion with the earlier result of Balasko and Cass and Geanakoplos and Mas-Colell which takes place in a two-period model. Second, our money is *fiat money* and not *deposit* money: to model the latter would require the introduction of a *banking system* – an important topic which we leave for later research.

2. The monetary exchange economy

This section presents our model of a monetary exchange economy. The basic characteristics (endowments, preferences) of the economy are set in a two-period model in which there is uncertainty about the state of nature which occurs in the second period. The subdivision of each period into a subperiod for the sale and a subperiod for the purchase of commodities leads to a transactions demand for money as a medium of exchange. The presence of money permits the introduction of *nominal contracts*, that is contracts promising to pay contingent amounts of money across the states. By trading such contracts agents can redistribute their income across time and across the states. If a known supply of money is injected into the economy, then the transactions demand for money creates a well-defined purchasing power (price) for money. This leads to a well-defined equilibrium for the monetary exchange economy. The phenomenon we propose to analyse is the way changes in the money supply lead to changes in the purchasing power of the nominal asset returns which in turn alter the equilibrium allocation. To formalise these intuitive ideas we need to make precise how money is injected into the economy, permits transactions on the markets to take place and circulates between the various transacting parties.

Consider therefore an economy with $I \geqq 1$ $(i=1,\ldots,I)$ consumers which is set in a two-period model $(t=0,1)$ in which one of $S \geqq 1$ states of nature $(s=1,\ldots,S)$ occurs at date 1. For convenience we include $t=0$ as a state and writes $s=0,1,\ldots,S$. In each state s there are L goods $(l=1,\ldots,L)$: we let $n=L(S+1)$ denote the total number of goods so that $\mathbb{R}^n$ is the basic commodity space. The markets are of two kinds: *spot markets* for the real goods, on which transactions take place through the use of money, and *financial markets* for contracts denominated in money. To describe the way money circulates through the economy and is used to carry out transactions, we decompose each state s $(s=0,1,\ldots,S)$ into three subperiods (s_1,s_2,s_3): in the first subperiod (s_1) agents sell their endowment of goods in exchange for money, in the second (s_2) agents transact on the financial markets – an activity which leads to a redistribution of the money balances – and in the last subperiod (s_3) agents use their transactions balances to purchase goods.

Let $w_s^i \in \mathbb{R}_+^L$ $(s=0,\ldots,S)$ denote the 'endowment' of agent i in subperiod (s_1). There is an institution which we call the Central Exchange which performs the basic function of marketing the agents endowments (w_s^i): we have in mind the idea that agents' endowments are not directly consumable but need to be 'processed' before they are suitable for consumption and this processing or marketing function is performed by the Central Exchange.

Assumption 1. In subperiod s_1 $(s=0,\ldots,S)$ each agent sells the full amount of his endowment w_s^i to the Central Exchange.

Assumption 1 is present in most models of a monetary economy in one form or another. It can be viewed as the assertion that agents have no direct utility for their own initial endowment or in the more direct language of Diamond (1984) that there is a 'taboo' on consuming ones own initial endowment. Behind it lies the idea that money permits the high degree of specialisation characteristic of a modern (production) economy in which consumers offer specialised labor services to firms which process and transform these into consumption goods. In the simplified framework of our exchange model, the processing of goods by the Central Exchange can be viewed as a surrogate for this more roundabout process of transforming specialised inputs into consumption goods.

In the first subperiod 0_1 each agent i receives the money income $m_0^i = p_0' w_0^i$ where $p_0' = (p_{01}', \ldots, p_{0L}')$ is the vector of *selling prices* for the goods at date 0. In subperiod 0_2, J assets are available for trading, where the jth asset is a contract which promises to pay the stream $N^j = (N_1^j, \ldots, N_S^j)^{\mathrm{T}}$ of dollars at date 1 (written as a column vector) and costs q_j dollars (payable at date 0). These assets, which we call *nominal* assets are in zero net supply and represent a generalized form of borrowing and lending between agents. For simplicity we assume that buyers and sellers meet directly on the financial markets so that the exchange of contracts for money is carried out directly without the intervention of the Central Exchange. If in subperiod 0_2 agent i purchases the portfolio $z^i=(z_1^i,\dots,z_J^i)\in\mathbb{R}^J$ and if $q=(q_1,\dots,q_J)$ is the vector of asset prices, then the money balances he has available are $p_0'w_0^i-qz^i$. These can be divided between *transactions balances* for use in purchasing goods in subperiod 0_3 and *precautionary balances* $z_0^i\in\mathbb{R}_+$ laid aside to finance expenditures at date 1. In subperiod 0_3 agent i uses his transactions balances to buy the consumption bundle $x_0^i=(x_{01}^i,\dots,x_{0L}^i)\in\mathbb{R}_+^L$ from the Central Exchange at the purchase prices $p_0=(p_{01},\dots,p_{0L})$. The transactions activities of agent i in the three subperiods of date 0 can thus be summarized in the budget equation

$$p_0x_0^i=p_0'w_0^i-qz^i-z_0^i. \tag{1}$$

At date 1, one of the states s ($s=1,\dots,S$) occurs. In the first subperiod (s_1) agent i receives the money income $m_s^i=p_s'w_s^i$ in exchange for his endowment w_s^i which is sold to the Central Exchange at the *selling prices* $p_s'=(p_{s1}',\dots,p_{sL}')$. In the second subperiod (s_2) agent i receives the dividend income $\sum_{j=1}^J N_s^jz_j^i$. Thus in the third subperiod (s_3) he has available the transactions balances $m_s^i+\sum_{j=1}^J N_s^jz_j^i+z_0^i$ for purchasing the consumption bundle $x_s^i=(x_{s1}^i,\dots,x_{sL}^i)\in\mathbb{R}_+^L$ from the Central Exchange at the *purchase prices* $p_s=(p_{s1},\dots,p_{sL})$. The transactions in state s can thus be summarized in the budget equation

$$p_sx_s^i=p_s'w_s^i+\textstyle\sum_{j=1}^J N_s^jz_j^i+z_0^i,\quad s=1,\dots,S. \tag{2}$$

The market transactions of agents and the twin roles of money as a medium of exchange and as a store of value can be illustrated as in fig. 1.

Let $N=[N^1\dots N^J]$ denote the $S\times J$ matrix of date 1 returns from the nominal assets and let

$$W(q,N)=\begin{bmatrix}-q\\ N\end{bmatrix}=\begin{bmatrix}-q_1 & \cdots & -q_J\\ N_1^1 & \cdots & N_1^J\\ \vdots & & \vdots\\ N_S^1 & \cdots & N_S^J\end{bmatrix} \tag{3}$$

denote the full matrix of returns (date 0 and date 1) from the assets. If for

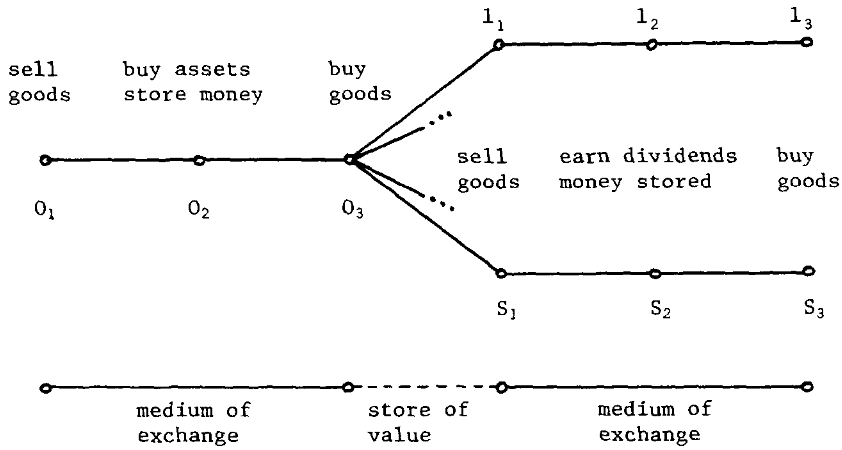

Fig. 1. Transactions and role of money.

$x^i=(x_0^i, x_1^i, \ldots, x_S^i) \in \mathbb{R}^{L(S+1)}$ and $p=(p_0, p_1, \ldots, p_S) \in \mathbb{R}^{L(S+1)}$ we define the *box product*[1]

$$p \,\square\, x^i=(p_0 x_0^i, p_1 x_1^i, \ldots, p_S x_S^i)$$

and let $\varepsilon=(-1,1,\ldots,1)$, then the budget set of agent i can be written as

$$B(p', p, q; w^i)=\{x^i \in \mathbb{R}_+^n \mid \exists\,(z^i, z_0^i) \in \mathbb{R}^J \times \mathbb{R}_+ \text{ such that } p \,\square\, x^i = p' \,\square\, w^i + W(q, N)z^i + \varepsilon z_0^i\}, \tag{4}$$

where $n=L(S+1)$ is the total number of commodities.

Agent i's preference ordering among consumption bundles $x^i=(x_0^i, x_1^i, \ldots, x_S^i) \in \mathbb{R}^n$ is represented by a utility function $u^i\colon \mathbb{R}_+^n \mapsto \mathbb{R}$ and his characteristics (u^i, w^i) satisfy the following assumption.

Assumption 2 (*Agent Characteristics*). (a) $u^i\colon \mathbb{R}_+^n \to \mathbb{R}$ is continuous on $\mathbb{R}_+^n$ and $\mathscr{C}^2$ on $\mathbb{R}_{++}^n$. (b) If $U^i(\xi)=\{x \in \mathbb{R}_+^n \mid u^i(x) \geq u^i(\xi)\}$, then $U^i(\xi) \subset \mathbb{R}_{++}^n$, $\forall\, \xi \in \mathbb{R}_{++}^n$. (c) For each $x \in \mathbb{R}_{++}^n$, $Du^i(x) \in \mathbb{R}_{++}^n$ and $h^T D^2 u^i(x) h < 0$ for all $h \neq 0$ such that $Du^i(x)h=0$. (d) $w^i \in \mathbb{R}_{++}^n$.

[1]More generally we use the box product to extend both the *inner product* of two vectors and *scalar multiplication* of a vector in a Cartesian product of Euclidean spaces. Thus for $a \in \mathbb{R}^{km}$, $b \in \mathbb{R}^{km}$ define $a \,\square\, b=(a_1 \cdot b_1, \ldots, a_m \cdot b_m)$ where $a_i \cdot b_i$ is the inner product on $\mathbb{R}^k$ and for $\alpha \in \mathbb{R}^m$, $b \in \mathbb{R}^{km}$ define $\alpha \,\square\, b=(\alpha_1 b_1, \ldots, \alpha_m b_m)$.

Remark. (a)–(c) is the standard assumption of *smooth preferences* which lead to smooth demand functions which are needed for generic arguments. (d) can be weakened to

$$w^i \in \mathbb{R}^n_+, \quad w^i_s \neq 0, \quad s=0,1,\ldots,S \quad \text{and} \quad \sum_{i=1}^{I} w^i \gg 0. \tag{d'}$$

(d′) is more consistent with the idea underlying Assumption 1, namely that agents have specialised endowments. However, working with (d′) needlessly complicates the proofs and the statements of the generic results.

In addition to its role of processing (marketing) goods, the Central Exchange determines the money supply

$$M_s = \sum_{i=1}^{I} m^i_s, \quad s=0,1,\ldots,S,$$

that is injected into the economy in each state s. We call the vector $M=(M_0, M_1, \ldots, M_s) \in \mathbb{R}^{S+1}_{++}$ the *monetary policy.* The Central Exchange injects the money supply $M_0 = \sum_{i=1}^{I} m^i_0$ in subperiod 0_1 and as a result of the consumers transactions the amount $M_0 - \sum_{i=1}^{I} z^i_0$ is returned to it in subperiod 0_3. Similarly in subperiod s_1 it injects the money supply $M_s = \sum_{i=1}^{I} m^i_s$ and the amount $M_s + \sum_{i=1}^{I} z^i_0$ returns to it in subperiod s_3. When $\sum_{i=1}^{I} z^i_0 > 0$, so that some agents are hoarding money, the amount of money offered in exchange for goods at the selling and buying stages is different. Since the same amount of goods is exchanged at each stage the selling prices p'_s must differ from the buying prices p_s.

We need to explain how the buying (selling) prices are formed and how they are influenced by the money supply M_s. In a more fully articulated model the two functions of the Central Exchange – the marketing of goods and the determination of the money supply – would be performed by two separate agencies, the maximising behaviour of stores (the firms which market goods) leading to price formation and the monetary authority fixing the money supply. A complete description of such a model seems difficult to achieve, especially in a two-period model. To simplify the analysis we thus make the following hypothesis regarding price formation.

Assumption 3. In each state s $(s=0,1,\ldots,S)$ the selling and buying prices are proportional, i.e. there exists $v=(v_0, v_1, \ldots, v_S) \in \mathbb{R}^{S+1}_{++}$ such that

$$p_s = v_s p'_s, \quad s=0,1,\ldots,S.$$

Thus, if agents do not use money as a store of value ($\sum_{i=1}^{I} z_0^i=0$), then price formation is based on the usual zero-profit condition for an activity with constant returns $p_s=p'_s$. When agents hold precautionary balances ($\sum_{i=1}^{I} z_0^i>0$), then $v_0<1$ and $v_s>1$, $s=1,\ldots,S$: there are losses from the activity of marketing at date 0 which are compensated by gains at date 1. The assumption of proportional prices ensures that the losses and gains from marketing each commodity are the same.

We can now define the concept of a monetary equilibrium which forms the basis for the analysis that follows. The actions of the agents consist of the consumption, portfolio and precautionary balances for each of the I agents

$$(x, z, z_0)=(x^1,\ldots,x^I, z^1,\ldots,z^I, z_0^1,\ldots,z_0^I)$$

and equilibrium prices (p', p, q) which clear the spot, financial and money markets.

Definition 1. A *monetary equilibrium* is a pair of actions and prices $((\bar{x},\bar{z},\bar{z}_0), (\bar{p}',\bar{p},\bar{q},\bar{v}))$ such that

(i) $(\bar{x}^i,\bar{z}^i,\bar{z}_0^i)$, $i=1,\ldots,I$ satisfy

$$\bar{x}_i=\operatorname*{arg\,max}_{x^i\in B(\bar{p}',\bar{p},\bar{q};w^i)} u^i(x^i),\quad \bar{p}\,\Box\,\bar{x}^i-\bar{p}'\,\Box\,w^i=W\bar{z}^i+\varepsilon\bar{z}_0^i,$$

(ii) $$\sum_{i=1}^{I}(\bar{x}^i-w^i)=0,$$

(iii) $$\sum_{i=1}^{I}\bar{z}^i=0,$$

(iv) $$\bar{p}^0\sum_{i=1}^{I}\bar{x}_0^i+\sum_{i=1}^{I}\bar{z}_0^i=M_0,$$

$$\bar{p}_s\sum_{i=1}^{I}\bar{x}_s^i-\sum_{i=1}^{I}\bar{z}_0^i=M_s,\quad s=1,\ldots,S,$$

(v) $$\bar{p}_s=\bar{v}_s\bar{p}'_s,\quad s=0,\ldots,S.$$

Remark. In this concept of equilibrium the government and all agents in the private sector have *common information.* The private sector agents take as given and correctly anticipate the price $(\bar{p}',\bar{p},\bar{q})$ and hence implicitly the monetary policy M. (ii) and (iii) express market clearing on the goods and financial markets. (iv) asserts that the demand for money, which consists of the transactions and precautionary demand, is equal to the money supply. (v) acts as a substitute in our model for the first-order conditions which would be associated with maximizing behaviour in the activity of marketing goods.

In the above model money can be viewed as performing its three roles of unit of account, medium of exchange and store of value. The latter two roles can be suppressed by:

(a) eliminating the monetary equations (iv), expressing equality of the demand and supply of money;
(b) replacing (v) by $\bar{p}=\bar{p}'$, since in an idealised barter economy goods can be exchanged directly for goods;
(c) setting $z_0^i=0$ in the budget set of each agent $i=1,\ldots,I$, to eliminate the possibility of using money as a store of value.

We call an equilibrium satisfying assumptions (a)–(c) a *nominal asset equilibrium.* This is the concept of equilibrium studied by Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989).

3. Abstract equilibrium

Our object is to study the qualitative properties of a monetary equilibrium. We shall find it convenient to do this by transforming a monetary equilibrium into an equivalent form which we call an *abstract equilibrium.* The qualitative analysis of this latter concept is then reduced to the study of a parametric system of equations in an appropriate number of unknowns.

The monetary exchange economy of the previous section can be summarised by the characteristics of the agents $(u,\omega)=(u^1,\ldots,u^I,w^1,\ldots,w^I)$, the nominal asset structure N and the monetary policy M. If we fix a profile of preferences $u=(u^1,\ldots,u^I)$ where each u^i satisfies Assumption 2 and the nominal asset structure N, then we obtain an economy $\mathscr{E}_{u,N}(\omega,M)$ parametrized by the endowments and money supply

$$(\omega,M)\in\Omega\times\mathscr{M},\quad \Omega=\mathbb{R}^{nI}_{++},\quad \mathscr{M}=\mathbb{R}^{S+1}_{++},$$

where Ω is the *endowment space* and $\mathscr{M}$ is the *monetary policy space.* The transformation to an abstract equilibrium is achieved by transforming the price variables in a sequence of three steps which can be summarized as follows:

Step 1. Reduce the prices determining an equilibrium to (v_0,p,q).

Step 2. Use a *no-arbitrage argument* to replace the vector of asset prices q by a vector of state prices β.

Step 3. Replace the spot prices p by the present value prices $P=\beta \square p$ leading to the price variables (v_0, P, β).

Step 1. Condition (v) in Definition 1 can be used to eliminate the vector of sales prices p' so that the vector of prices reduces to (v, p, q). If we define

$$\frac{1}{v}=\left(\frac{1}{v_0}, \frac{1}{v_1}, \ldots, \frac{1}{v_S}\right),$$

then in terms of these new variables the budget set of agent i can be written as

$$B(v, p, q; w^i)=\{x^i \in \mathbb{R}^n_+ \mid \exists (z^i, z_0^i) \in \mathbb{R}^J \times \mathbb{R}_+ \text{ such that}$$
$$p \square (x^i-(1/v) \square w^i)=Wz^i+\varepsilon z_0^i\}. \tag{5}$$

Let us establish some relations that must be satisfied in equilibrium by the vector $v=(v_0, v_1, \ldots, v_S)$. Summing the budget equations in (5) over i and using the market clearing conditions (ii) and (iii) in Definition 1 gives the equation

$$p \square \sum_{i=1}^{I} x^i \square \left(e-\frac{1}{v}\right)=\varepsilon \sum_{i=1}^{I} z_0^i \quad \text{with} \quad e=(1, 1, \ldots, 1). \tag{6}$$

Since (6) must be satisfied in equilibrium, the monetary equations (iv) which can be written as

$$p \square \sum_{i=1}^{I} x^i-\varepsilon \sum_{i=1}^{I} z_0^i=M$$

are satisfied if and only if the equation

$$\text{(iv}') \qquad p \square \sum_{i=1}^{I} x^i=M \square v$$

holds. A necessary condition for the monetary equations (iv) and (iv′) to hold is that

$$M \square (v-e)=\varepsilon \sum_{i=1}^{I} z_0^i, \tag{7}$$

which is equivalent to

$$v_0=\left(M_0-\sum_{i=1}^{I} z_0^i\right)\bigg/M_0, \tag{7'}$$

$$v_s=\left(M_s+\sum_{i=1}^{I} z_0^i\right)\bigg/M_s, \quad s=1,\dots,S. \tag{7''}$$

Thus in equilibrium v_0 (which satisfies $0\leqq v_0\leqq 1$) is the proportion of the money supply M_0 used by the agents to carry out their transactions on the commodity markets in subperiod 0_3. We call v_0 the *velocity of circulation of money at date* 0. Similarly in equilibrium v_s (which satisfies $v_s\geqq 1$) is the proportion of the money supply M_s used by the agents to finance commodity transactions in subperiods s_3 and we call v_s the *velocity of circulation of money in state s.* Notice that the velocities of circulation of money are endogenously determined by the total precautionary demand for money $\sum_{i=1}^{I} z_0^i$. The monetary equations (iv′) can be interpreted as the *quantity theory equations* in the context of our model.

Eqs. (7′) and (7″) imply

$$v_s=1+\frac{M_0(1-v_0)}{M_s}, \quad s=1,\dots,S, \tag{8}$$

which simply asserts that money withdrawn from circulation at date 0 is returned at date 1. If we use eqs. (8) to define v_s $(s=1,\dots,S)$ as a function of v_0, then the vector of prices (v,p,q) is determined by (v_0,p,q).

Step 2. We now use the fact that agent i's maximum problem (i) in Definition 1 has a solution if and only if there are no arbitrage possibilities on the financial markets, i.e. there does not exist $z\in\mathbb{R}^J$ such that $W(q,N)z>0$ (where $y\in\mathbb{R}^{S+1}$, $y>0$ means $y_s\geqq 0$ for all s and $y\neq 0$). This in turn implies by a standard separation theorem that there exists a vector of state prices $\bar{\beta}\in\mathbb{R}^{S+1}_{++}$, which can be normalized to $\bar{\beta}=(1,\beta)$, such that

$$\bar{\beta}W(q,N)=0\Leftrightarrow q=\beta N. \tag{9}$$

Replacing q by βN in the budget set (5) and using (8) to define v_s $(s=1,\dots,S)$ leads to a budget set $B(v_0,p,\beta N;w^i,M)$. In view of Assumption 2, the Kuhn–Tucker conditions for agent i's maximum problem lead to a vector of marginal utilities of income for agent i, $\lambda^i=(\lambda_0^i,\lambda_1^i,\dots,\lambda_S^i)=(\lambda_0^i,\lambda_1^i)$ satisfying

$$Du^i(x^i)=\lambda^i \square p, \quad \left(\frac{1}{\lambda_0^i}\right)\lambda_1^i N=\beta N, \tag{10}$$

$$\lambda_0^i-\sum_{s=1}^{S}\lambda_s^i \geqq 0, \quad \left(\lambda_0^i-\sum_{s=1}^{S}\lambda_s^i\right)z_0^i=0. \tag{11}$$

Define the *present value vector* of agent i by $\pi^i=(\pi_1^i,\dots,\pi_S^i)=(1/\lambda_0^i)(\lambda_1^i,\dots,\lambda_S^i)$, π_s^i being the rate of substitution between income in state s and income at date 0. Then $\sum_{s=1}^{S}\pi_s^i=1/(1+\rho^i)$ defines the *income rate of impatience* ρ^i of agent i; $1+\rho^i$ is the number of dollars that agent i needs to receive in each state s at date 1 in order to give up one dollar at date 0. Then (11) is equivalent to

$$\rho^i \geqq 0 \text{ and if } \rho^i>0 \text{ then } z_0^i=0.$$

Since agents can hold money as a store of value the income rate of impatience is always non-negative and when it is positive agent i does not hold money as store of value.

The analysis of an agent's precautionary demand for money is greatly simplified if there is a riskless bond which permits an agent to transfer income between date 0 and date 1.

Assumption 4. Riskless bond. $N^1=(1,\dots,1)^{\mathrm{T}}$.

If q_1 is the price of the riskless bond then the *riskfree rate of interest* r is defined by $q_1=1/(1+r)$. The presence of this asset leads to equality in the agents' income rates of impatience. The no-arbitrage condition (9) for asset 1 implies $\sum_{s=1}^{S}\beta_s=1/(1+r)$ and the individuals' first-order conditions (10) for asset 1 imply

$$\sum_{s=1}^{S}\pi_s^i=\sum_{s=1}^{S}\beta_s, \quad i=1,\dots,I \Leftrightarrow \rho^i=r, \quad i=1,\dots,I. \tag{12}$$

The presence of the riskless bond implies that each agent's income rate of impatience is equal to the riskfree rate of interest. Eqs. (11) and (12) imply

$$\sum_{s=1}^{S}\beta_s \leqq 1 \text{ and if } \sum_{s=1}^{S}\beta_s<1 \text{ then } z_0^i=0, \quad i=1,\dots,I, \tag{13}$$

which is equivalent to

$$r \geqq 0 \text{ and if } r > 0 \text{ then } z_0^i = 0, \quad i = 1, \ldots, I. \tag{13'}$$

The presence of the riskless bond allows us to define the riskfree rate of interest r. *The fact that money can be held as a store of value forces the riskfree rate of interest to be non-negative. If the riskless bond offers a positive rate of return* ($r > 0$), *then no agent holds money as a store of value.*

Remark. The fact that when there is a *riskless asset* agents will only hoard money when the rate of interest is zero may seem to eliminate the relevance of holding money as a store of value in this model. In interpreting this case it should be recalled that *transactions costs* are in practice involved in the purchase and sale of all assets. If such costs were introduced, then agents would hold money as a store of value as soon as r falls below some positive level depending on the transactions costs. In this more general framework money would be held as a store of value whenever the rate of interest is sufficiently low.

In a world of uncertainty where each consumer faces some probability of bankruptcy and hence where lenders cannot be sure to be repaid in each state, the riskless bond does not exist [more generally $(1, \ldots, 1)^T$ does not lie in the span of the columns of the matrix N]. Consumers then have an additional uncertainty based reason for holding money as a store of value since money may provide a better hedge against uncertainty than the other financial assets, provided that the level of prices is not too variable. This case is important, but the analysis of the equilibria is technically harder without Assumption 4. To keep the analysis simple we have chosen to omit this case in this paper.

The presence of the riskless bond allows us to drop the money holdings z_0^i explicitly from the analysis. This is done by defining modified holdings of the first asset

$$\gamma_1^i = z_0^i + z_1^i, \quad i = 1, \ldots, I.$$

For if $r > 0$, then $z_0^i = 0$ and $\gamma_1^i = z_1^i$, and if $r = 0$, then the agent is indifferent between storing value via money or the riskless asset. Of course, at the *aggregate* level we need to be sure that

$$\sum_{i=1}^{I} \gamma_1^i = M_0(1 - v_0).$$

The advantage of transforming to the variables γ_1^i is that they are not subject to the non-negativity constraint that must be satisfied by the money holdings

z_0^i. To keep a unified notation for each agent's portfolio we set $y_j^i = z_j^i$, $j = 2, \dots, J$.

The choice of a portfolio $y \in \mathbb{R}^J$ amounts to choosing a linear combination of the columns of the nominal asset returns matrix

$$N = \begin{bmatrix} N_1^1 & \dots & N_1^J \\ \vdots & & \vdots \\ N_S^1 & \dots & N_S^J \end{bmatrix}.$$

This leads to a vector $\tau = Ny \in \mathbb{R}^S$ of income transfers. Let $\langle N \rangle = \{\tau \in \mathbb{R}^S | \tau = Ny \text{ for some } y \in \mathbb{R}^J\}$ denote the *substance of income transfers* generated by the columns of N. It is also convenient to let $N_s = (N_s^1, \dots, N_s^J)$ denote row s of N.

Recalling that we have substituted the no-arbitrage relation $q = \beta N$ into agent i's budget set, the date 0 constraint becomes

$$p_0\left(x_0^i - \frac{1}{v_0} w_0^i\right) = -qy^i = -\beta N y^i = -\sum_{s=1}^{S} \beta_s p_s\left(x_s^i - \frac{1}{v_s} w_s^i\right), \tag{14}$$

since $\beta N y^i = \sum_{s=1}^{S} \beta_s N_s y^i$ which equals the right-hand side of (14). If we write the vectors p, x^i, w^i and v in terms of their date 0 and date 1 components

$$p = (p_0, p_1), \quad x^i = (x_0^i, x_1^i), \quad w^i = (w_0^i, w_1^i), \quad v = (v_0, v_1),$$

then the date 1 budget constraint which asserts that there exits $y^i \in \mathbb{R}^J$ such that $p_1 \square (x_1^i - (1/v_1) \square w_1^i) = Ny^i$ is equivalent to

$$p_1 \square \left(x_1^i - \frac{1}{v_1} \square w_1^i\right) \in \langle N \rangle. \tag{15}$$

To simplify notation define the variables

$$\underset{\sim}{w}^i = \frac{1}{v} \square w^i, \quad \tilde{\beta} = (\beta_0, \beta) = (1, \beta),$$

then (14) and (15) imply that the budget set of agent i has been transformed into the set

$$\mathscr{B}(v_0, p, \beta; w^i, M) = \left\{ x^i \in \mathbb{R}_+^n \,\middle|\, \begin{matrix} \sum_{s=0}^{S} \beta_s p_s (x_s^i - \underset{\sim}{w}_s^i) = 0 \\ p_1 \square (x_1^i - \underset{\sim}{w}_1^i) \in \langle N \rangle \end{matrix} \right\}. \tag{16}$$

In transforming from the price variables (v_0, p, q) to the price variables (v_0, p, β), whenever the budget sets are the same, the equilibrium allocation will be the same. Th budget sets (16) are the same for all those β and β' such that

$$\beta N = \beta' N = q.$$

For each equilibrium it suffices therefore to select one β satisfying $\beta N = q$. Since (9) and (10) imply that each agent's present value vector satisfies $\pi^i N = q$, a convenient procedure for selecting a single β is to choose $\beta = \pi^1$, the present value vector of agent 1. With this choice of β, utility maximization for agent 1 over the budget set (16) is equivalent to utility maximization over the budget set

$$\mathbb{B}(v_0, p, \beta; w^1, M) = \left\{ x^1 \in \mathbb{R}^n_+ \,\middle|\, \sum_{s=0}^{S} \beta_s p_s (x_s^1 - \underset{\sim}{w}_s^1) = 0 \right\}. \tag{17}$$

Step 3. The final transformation is straightforward and natural. Since $\beta = \pi^1$ is a present value vector which translates dollar amounts in each state at date 1 into dollar amounts at date 0, the vector of prices

$$P = \tilde{\beta} \,\square\, p, \quad \tilde{\beta} = (1, \beta), \tag{18}$$

is a vector of *present value prices* with focal date at time 0. The budget sets (16) and (17) written in terms of the date 0 price vector P become

$$\mathscr{B}(v_0, P, \beta; w^i, M) = \left\{ x^i \in \mathbb{R}^n_+ \,\middle|\, \begin{array}{l} P(x^i - \underset{\sim}{w}^i) = 0 \\ P_1 \,\square\, (x_1^i - \underset{\sim}{w}_1^i) \in \langle [\beta] N \rangle \end{array} \right\}, \tag{19}$$

$$\mathbb{B}(v_0, P, \beta; w^1, M) = \{ x^1 \in \mathbb{R}^n_+ \,|\, P(x^1 - \underset{\sim}{w}^1) = 0 \}, \tag{20}$$

where

$$[\beta] = \begin{bmatrix} \beta_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \beta_S \end{bmatrix}.$$

Definition 2. An *abstract equilibrium* for the economy $\mathscr{E}_{u,N}(\omega, M)$ is a pair of actions and prices $(\bar{x}, (\bar{v}_0, \bar{P}, \bar{\beta}))$ such that:

(i) $\bar{x}^i$, $i = 1, \dots, I$ satisfy

$$\bar{x}^1 = \underset{x^1 \in \mathbb{B}(\bar{v}_0, \bar{P}, \bar{\beta}; w^1, M)}{\arg\max} \; u^1(x^1),$$

$$x^i = \underset{x^i \in \mathscr{B}(\bar{v}_0, \bar{P}, \bar{\beta}; w^i, M)}{\arg\max} \; u^i(x^i), \quad i = 2, \ldots, I;$$

(ii) $$\sum_{i=1}^{I} (\bar{x}^i - w^i) = 0;$$

(iii) $$\bar{P} \square \sum_{i=1}^{I} \bar{x}^i = \tilde{\bar{\beta}} \square (M \square \bar{v}).$$

Proposition 1. *(i) If $((x, z, z_0), (p', p, q, v))$ is a monetary equilibrium, then there exists $\beta \in \mathbb{R}^S_{++}$ such that if $P = \tilde{\beta} \square p$, with $\tilde{\beta} = (1, \beta)$, then $(x, (v_0, P, \beta))$ is an abstract equilibrium.*
(ii) If $(x, (v_0, P, \beta))$ is an abstract equilibrium with $v_0 \leqq 1$ and $\sum_{s=1}^{S} \beta_s \leqq 1$, then there exist (z, z_0) such that if $P = \tilde{\beta} \square p$, $p = v \square p'$ with $(v_1, \ldots, v_S)$ given by (8), $q = \beta N$, then $((x, z, z_0), (v_0, p, q))$ is a monetary equilibrium.

Proof. (i) follows from the above analysis; (ii) is left to the reader. □

We have thus reduced the analysis of a monetary equilibrium to the analysis of an abstract equilibrium.

We let the prices (v_0, P, β) lie in the domain

$$\mathscr{P} = \mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^S_{++}.$$

Under Assumption 2 the solution of each agent i's utility maximizing problem in Definition 2(i) exists for every $(v_0, P, \beta, w^i, M) \in \mathscr{P} \times \mathbb{R}^n_{++} \times \mathscr{M}$, is unique and leads to the demand functions

$$f^i(v_0, P, \beta; w^i, M) = \underset{x^i \in \mathscr{B}(v_0, P, \beta; w^i, M)}{\arg\max} \; u^i(x^i), \quad i = 2, \ldots, I, \tag{21}$$

$$f^1(v_0, P, \beta; w^1, M) = \underset{x^1 \in \mathbb{B}(v_0, P, \beta; w^1, M)}{\arg\max} \; u^1(x^1). \tag{22}$$

By standard arguments the functions f^i can be shown to be $\mathscr{C}^1$ functions. If we define the *aggregate excess demand for goods* $F: \mathscr{P} \times \Omega \times \mathscr{M} \to \mathbb{R}^n$,

$$F(v_0, P, \beta; \omega, M) = \sum_{i=1}^{I} (f^i(v_0, P, \beta; w^i, M) - w^i),$$

and the *aggregate excess demand for money* $G:\mathscr{P}\times\Omega\times\mathscr{M}\to\mathbb{R}^{S+1}$,

$$G(v_0, P, \beta, \omega, M) = P \,\square \sum_{i=1}^{I} w^i - \tilde{\beta} \,\square\, (M \,\square\, v),$$

then an *equilibrium price vector* $(v_0, P, \beta)\in\mathscr{P}$ for the economy with parameter value $(\omega, M)\in\Omega\times\mathscr{M}$ is a solution of the system of equations

$$F(v_0, P, \beta; \omega, M) = 0, \quad G(v_0, P, \beta; \omega, M) = 0. \tag{23}$$

We have thus shown how the analysis of an abstract equilibrium can be reduced to the analysis of a parametric system of eqs. (23).

4. Determinacy of equilibrium

Our first result asserts that a monetary equilibrium always exists. This is important and the result does not follow directly from known existence theorems. Since the proof does not involve essentially new ideas and since the object of this paper is to focus on qualitative properties of a monetary equilibrium, the proof is briefly sketched in the appendix.

Theorem 1 (Existence). Under Assumptions 1–4 the economy $\mathscr{E}_{u,N}(\omega, M)$ has a monetary equilibrium for all $(\omega, M)\in\Omega\times\mathscr{M}$.

Proof. See appendix.

In the remark following Definition 1 we pointed out that if the role of money as a medium of exchange and as a store of value is removed then the concept of equilibrium becomes that of a *nominal asset equilibrium* studied by Cass (1985), Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989). These authors have essentially shown that such an equilibrium is *indeterminate* in the sense that the set of equilibrium allocations contains a manifold of positive dimension. The next theorem shows that introducing explicitly the three roles of money eliminates this indeterminacy. The resulting monetary equilibrium is determinate in that generically an economy has a finite number of monetary equilibria.

Theorem 2 (Finiteness and regularity). Under Assumptions 1–4 there exists an open set of full measure in the space of parameters, $\Delta\subset\Omega\times\mathscr{M}$ such that

(i) an economy $\mathscr{E}_{u,N}(\omega, M)$ with $(\omega, M)\in\Delta$ has a finite number of monetary equilibria,

(*ii*) *for each* $(\bar{\omega}, \bar{M}) \in \Delta$ *there exists a neighborhood* $\mathcal{N}_{\bar{\omega},\bar{M}}$ *such that each monetary equilibrium is a smooth function of* (ω, M) *for all* $(\omega, M) \in \mathcal{N}_{(\bar{\omega},\bar{M})}$.

Remark. The theorem asserts in particular that the equilibria are locally unique. This determinacy of the monetary equilibria in our model is based on the fact that a well-defined amount of money $(v_s M_s)$ is being used to purchase a known amount of goods $(\sum_{i=1}^{I} x_s^i)$ at the current prices (p_s). In particular, Assumption 1 ensures that each agent sells the full amount of his initial endowment (w_s^i) in subperiod (s_1) and purchases the full amount of his consumption (x_s^i) in subperiod (s_3). If the model is interpreted more in the spirit of a traditional 'real' exchange economy, then agents would be permitted to consume directly any desired portion of their initial endowment. In this case when the buying and selling prices are equal $(p_s = p_s')$ agents would be indifferent between selling the full amount of their initial endowment (w_s^i) or only the net trade vector $(x_s^i - w_s^i)^-$ or any intermediate vector. In such a model where there is no hypothesis such as Assumption 1 which determines the quantity of goods passing through the markets, the transactions demand for money and hence the equilibrium would not be determinate.

Proof. (i) Let $H: \mathcal{P} \times \Omega \times \mathcal{M} \to \mathbb{R}^n \times \mathbb{R}^{S+1}$ be defined by $H = (F, G)$, then the equations for an abstract equilibrium (23) become

$$H(v_0, P, \beta; \omega, M) = 0. \tag{23'}$$

Summing the date 0 budget constraint $P(x^i - \underset{\sim}{w}^i) = 0$ in (19) and (20) over i, gives $P(\sum_{i=1}^{I} x^i - (1/v) \square \sum_{i=1}^{I} w^i) = 0$. Substituting $P \square \sum_{i=1}^{I} x^i = \tilde{\beta} \square (M \square v)$ gives

$$\sum_{s=0}^{S} \tilde{\beta}_s M_s (v_s - 1) = 0 \Leftrightarrow (v_0 - 1)\left(1 - \sum_{s=1}^{S} \beta_s\right) = 0 \tag{24}$$

in view of (8). By Proposition 1 the *equilibrium set* is given by

$$E = \left\{ (v_0, P, \beta, \omega, M) \in \mathcal{P} \times \Omega \times \mathcal{M} \,\middle|\, \begin{array}{l} v_0 \leqq 1, \sum_{s=1}^{S} \beta_s \leqq 1 \\ H(v_0, P, \beta, \omega, M) = 0 \end{array} \right\}. \tag{25}$$

Define

$$\Lambda = \mathcal{P} \times \Omega \times \mathcal{M}, \quad \xi = (v_0, P, \beta, \omega, M),$$

and the submanifolds of Λ

$$\Lambda_{v_0}=\left\{\xi\in\Lambda \middle| v_0=1,\ \sum_{s=1}^{S}\beta_s<1\right\},$$

$$\Lambda_{\beta}=\left\{\xi\in\Lambda \middle| v_0<1,\ \sum_{s=1}^{S}\beta_s=1\right\},$$

$$\Lambda_{v_0,\beta}=\left\{\xi\in\Lambda \middle| v_0=1,\ \sum_{s=1}^{S}\beta_s=1\right\}.$$

Note that $\dim\Lambda_{v_0}=\dim\Lambda_\beta=\dim(\Lambda)-1$ and $\dim\Lambda_{v_0,\beta}=\dim(\Lambda)-2$. Let $F=(F_0,F_1,\dots,F_S)$ and $\hat{F}=(\hat{F}_0,F_1,\dots,F_S)$ where $\hat{F}_0=(F_{02},\dots,F_{0L})$ so that excess demand F_{01} is omitted. Define $\hat{H}=(\hat{F},G)$. Then on the sets $\Lambda_{v_0},\Lambda_\beta,\Lambda_{v_0,\beta}$, $\hat{H}(\xi)=0$ implies $H(\xi)=0$. The point here is that on the sets $\Lambda_{v_0},\Lambda_\beta$ and $\Lambda_{v_0,\beta}$, (24) is satisfied, so that one of the excess demand equations $F(\xi)=0$ must be removed to ensure that $\hat{H}(\xi)=0$ is an independent system of equations.

Let $\hat{H}_\alpha$ denote the restriction of $\hat{H}$ to Λ_α where $\alpha=v_0,\beta$ or (v_0,β). Then

$$E=E_{v_0}\cup E_\beta\cup E_{v_0,\beta}, \tag{26}$$

where $E_\alpha=\{\xi\in\Lambda_\alpha|\hat{H}_\alpha(\xi)=0\}=\hat{H}_\alpha^{-1}(0)$.

Lemma 1. *0 is a regular value of* $\hat{H}_\alpha:\Lambda_\alpha\to\mathbb{R}^{n-1}\times\mathbb{R}^{S+1}$ *for* $\alpha=v_0,\beta$ *and* (v_0,β).

Proof. See appendix.

By the Preimage theorem [Hirsch (1976, Theorem 3.3, p. 22)] E_α is a submanifold of Λ_α and $\dim E_\alpha=\dim(\Lambda_\alpha)-(n-1+S+1)=nI+S+1$ for $\alpha=v_0$ and $\alpha=\beta$, while $\dim E_{v_0,\beta}=nI+S$. Let

$$\pi_\alpha:E_\alpha\to\Omega\times\mathcal{M},\quad \alpha=v_0,\beta\ \text{or}\ (v_0,\beta),$$

denote the projection $\pi_\alpha(v_0,P,\beta,\omega,M)=(\omega,M)$.

Lemma 2. *The projection* $\pi_\alpha:E_\alpha\to\Omega\times\mathcal{M}$ *is proper for* $\alpha=v_0,\beta$ *and* (v_0,β).

Proof. See appendix.

Let $\mathcal{R}_\alpha$ denote the set of regular values of π_α. By Sard's theorem [Hirsch (1976, Theorem 1.3, p. 69)] $\mathcal{R}_\alpha$ is of full measure in $\Omega\times\mathcal{M}$ (i.e. its complement is of measure zero). Since π_α is proper, $\mathscr{R}_\alpha$ is open. Since $\dim E_\alpha = \dim(\Omega \times \mathscr{M})$ for $\alpha = v_0$ and $\alpha = \beta$, $\pi_\alpha^{-1}(\omega, M)$ is a compact manifold of dimension 0, and hence consists of a finite number of points, for all $(\omega, M) \in \mathscr{R}_\alpha$ with $\alpha = v_0$ or $\alpha = \beta$. Since $\dim E_{v_0,\beta} < \dim(\Omega \times \mathscr{M})$, $\pi_{v_0,\beta}^{-1}(\omega, M) = \varnothing$ for all $(\omega, M) \in \mathscr{R}_{v_0,\beta}$. Define

$$\Delta = \mathscr{R}_{v_0} \cap \mathscr{R}_\beta \cap \mathscr{R}_{v_0,\beta},$$

then $\#\pi_\alpha^{-1}(\omega, M) < \infty$ for all $(\omega, M) \in \Delta$ (where $\#$ denotes the number of elements).

(ii) Let $\mathscr{P}_{v_0} = \{(v_0, P, \beta) \in \mathscr{P} \mid v_0 = 1\}$ and $\mathscr{P}_\beta = \{(v_0, P, \beta) \in \mathscr{P} \mid \sum_{s=1}^{S} \beta_s = 1\}$. Let $T_{\bar{P},\bar{\beta}}\mathscr{P}_{v_0}$ denote the tangent space to $\mathscr{P}_{v_0}$ at $(\bar{P}, \bar{\beta})$. If $(\bar{\omega}, \bar{M}) \in \Delta$, then $(\bar{\omega}, \bar{M}) \in \mathscr{R}_{v_0}$ so that for each solution $(1, \bar{P}, \bar{\beta}, \bar{\omega}, \bar{M}) \in \pi_{v_0}^{-1}(\bar{\omega}, \bar{M})$ the derivative map

$$D_{\bar{P},\bar{\beta}} \hat{H}_{v_0} : T_{\bar{P},\bar{\beta}} \mathscr{P}_{v_0} \to \mathbb{R}^{n+S}$$

is an isomorphism. It follows from the Implicit Function Theorem that there exists a neighborhood $\mathscr{N}_{(\bar{\omega},\bar{M})}$, an integer k_{v_0} and $\mathscr{C}^1$ functions $\phi_{v_0}^k : \mathscr{N}_{(\bar{\omega},\bar{M})} \to \mathscr{P}_{v_0}$, $k = 1, \ldots, k_{v_0}$ such that

$$\pi_{v_0}^{-1}(\omega, M) = \{\phi_{v_0}^1(\omega, M), \ldots, \phi_{v_0}^{k_{v_0}}(\omega, M), (\omega, M)\}, \quad (\omega, M) \in \mathscr{N}_{(\bar{\omega},\bar{M})}. \tag{27}$$

By a similar argument if $(\bar{v}_0, \bar{P}, \bar{\beta}, \bar{\omega}, \bar{M}) \in \pi_\beta^{-1}(\bar{\omega}, \bar{M})$, then there exists a neighborhood $\mathscr{N}_{(\bar{\omega},\bar{M})}$, an integer k_β and $\mathscr{C}^1$ functions $\phi_\beta^k : \mathscr{N}_{(\bar{\omega},\bar{M})} \to \mathscr{P}_\beta$, $k = 1, \ldots, k_\beta$ such that

$$\pi_\beta^{-1}(\omega, M) = \{\phi_\beta^1(\omega, M), \ldots, \phi_\beta^{k_\beta}(\omega, M), (\omega, M)\}, \quad (\omega, M) \in \mathscr{N}_{(\bar{\omega},\bar{M})}, \tag{28}$$

which completes the proof. □

In the proof of the above theorem we have established an important property of the *equilibrium set E* defined by (25). In view of (26) and the fact that $\pi_{v_0,\beta}^{-1}(\omega, M) = \varnothing$ for all $(\omega, M) \in \Delta$, *that part of the equilibrium set E which lies above Δ can be written as a disjoint union of two manifolds*

$$\pi_{v_0}^{-1}(\Delta) \cup \pi_\beta^{-1}(\Delta), \quad \pi_{v_0}^{-1}(\Delta) \cap \pi_\beta^{-1}(\Delta) = \varnothing.$$

Monetary equilibria are thus generically of two distinct types summarized by the following proposition.

Proposition 2 (Types of equilibria). For each $(\omega, M) \in \Delta$ each abstract equilibrium, and hence each monetary equilibrium is one of two types:

(*a*) *a positive interest rate equilibrium* $r>0$ $(v_0=1, \sum_{s=1}^{S}\beta_s<1)$ *in which money serves only as a medium of exchange*

(*b*) *a zero interest rate equilibrium* $r=0$ $(v_0<1, \sum_{s=1}^{S}\beta_s=1)$ *in which money serves both as a medium of exchange and as a store of value.*

Remark. The content of (27) and (28) can be expressed more geometrically as follows. For every $(\bar{\omega}, \bar{M}) \in \Delta$ such that $\pi_\alpha^{-1}(\bar{\omega}, \bar{M}) \neq \varnothing$ there exists an open neighborhood $\mathcal{N}_{(\bar{\omega}, \bar{M})} = \mathcal{N} \subset \Delta$ such that

$$\pi_\alpha^{-1}(\mathcal{N}) = V_\alpha^1 \cup \cdots \cup V_\alpha^{k_\alpha}, \qquad V_\alpha^i \cap V_\alpha^j = \varnothing, \quad i \neq j,$$

and $\pi_\alpha : V_\alpha^k \to \mathcal{N}$ is a diffeomorphism, for $k=1,\dots,k_\alpha$, $\alpha=v_0$ and β. *Thus the sets* V_α^k *and* $\mathcal{N}$ *are diffeomorphic for* $k=1,\dots,k_\alpha$, $\alpha=v_0$ *and* β. It follows that if we fix $\bar{\omega}$ and let M vary in the set

$$\mathcal{N}_{\bar{M}} = \{M \in \mathcal{M} \mid (\bar{\omega}, M) \in \mathcal{N}_{(\bar{\omega}, \bar{M})}\},$$

since $\dim \mathcal{M} = S+1$, $\pi_\alpha^{-1}(\mathcal{N}_{\bar{M}})$ *generates a submanifold of* V_α^k *for* $k=1,\dots,k_\alpha$ *of dimension* $S+1$. We are thus led to ask *to what extent the changes in prices induced by varying the money supply* M *in* $\mathcal{N}_{\bar{M}}$ *create real changes in the equilibrium allocation.* The next section provides a precise qualitative answer to this question.

5. Real effects of monetary policy

In the previous section it was shown that there is a generic subset $\Delta \subset \Omega \times \mathcal{M}$ such that whenever $(\bar{\omega}, \bar{M}) \in \Delta$ the equilibria can be written locally as smooth functions of the parameters: we can thus carry out a local comparative static analysis of equilibria. We have also shown that if we fix the economy by fixing the endowment vector at $\bar{\omega}$, then varying the money supply in the neighborhood $\mathcal{N}_{\bar{M}}$ traces out an $S+1$ dimensional manifold of prices.

Not all such price changes, however, can be expected to have real effects. Changes in the money supply are neutral in one important sense in this model. *If the money supply is changed by the same factor in all states and dates, then the prices in all states and dates are changed by the same factor, leaving the equilibrium allocation unchanged.* More precisely, if $(v_0, P, \beta, \omega, M) \in E$ and if $\lambda>0$, then $(v_0, \lambda P, \beta, \omega, \lambda M) \in E$ and

$$f^i(v_0, \lambda P, \beta, w^i, \lambda M) = f^i(v_0, P, \beta, w^i, M), \quad i=1,\dots,I.$$

This property holds because agents are assumed not to have initial endowments of nominal assets. If at date 0 such contracts were inherited from the past, then even this type of neutrality would in general disappear. Since we are interested in the real effects of money we will restrict M to lie in the space of *normalized money supplies*

$$\mathscr{M}^* = \{M \in \mathscr{M} \mid M_0 = 1\}.$$

In Proposition 2 we distinguished two types of equilibria depending on whether the interest rate is *positive* or *zero* or equivalently on whether money is not or is used as a store of value. Money supply changes affect these equilibria in distinct ways.

For *positive* interest rate equilibria changing the money supply only affects an agent's ability to redistribute income across the states at date 1. Changing the money supply at date 1 alters the purchasing power of the nominal asset returns at date 1. Provided this change is non-proportional and provided the asset markets are *incomplete*, it tilts the subspace of income transfers achievable by trading the nominal assets. Such changes in the money supply lead to $S-1$ dimensional changes in the equilibrium allocation. If the asset markets are *complete*, then such money supply changes cannot tilt the space of income transfers since it coincides with the whole space $\mathbb{R}^S$. Thus for positive interest rate equilibria if the markets are complete, then changing the money supply has no real effects. The equilibria in this case coincide with the Arrow–Debreu equilibria.

For *zero* interest rate equilibria money is used as a store of value. Thus altering the money supply alters the velocity of circulation which in turn changes the *present value of the income stream* of each agent. This income effect, induced by the use of money as a store of value, ensures that in zero interest rate equilibria changing the money supply has real effects regardless of whether the asset markets are complete or incomplete. When the asset markets are incomplete the effect of the change in the subspace of income transfers needs to be added to the basic income effect. Thus for zero interest rate equilibria changes in the money supply lead to S dimensional changes in the equilibrium allocation.

Let us define

$$\mathscr{A} = \mathbb{R}^{nI}_{++}$$

as the *space of allocations* and the *vector of demand functions* of the agents

$$f: \mathscr{P} \times \Omega \times \mathscr{M} \to \mathscr{A}, \quad f = (f^1, \dots, f^I),$$

where f^i are defined by (21) and (22). To study the effect of a change in M on the equilibrium allocations, in a neighborhood of $(\bar{\omega}, \bar{M}) \in \Delta$ with the endowment fixed at $\bar{\omega}$, we need to compose the map f with the *equilibrium price functions*

$$\phi_\alpha^k: \mathcal{N}_{(\bar{\omega},\bar{M})} \to \mathcal{P}_\alpha, \qquad k=1,\ldots,k_\alpha, \quad \alpha = v_0 \text{ or } \beta,$$

in (27) and (28). This leads to the following family of maps.

Definition 3. Let $(\bar{\omega},\bar{M}) \in \Delta$ and let $\mathcal{N}_{(\bar{\omega},\bar{M})}$ denote the neighborhood in (27) and (28), If $\mathcal{N}_{\bar{M}} = \{M \in \mathcal{M}^* \mid (\bar{\omega}, M) \in \mathcal{N}_{(\bar{\omega},\bar{M})}\}$, *then the maps* $\psi_\alpha^k: \mathcal{N}_{\bar{M}} \to \mathcal{A}$ defined by

$$\psi_\alpha^k(M) = f(\phi_\alpha^k(\bar{\omega}, M), \bar{\omega}, M), \qquad k=1,\ldots,k_\alpha, \quad \alpha = v_0 \text{ or } \beta,$$

are called the *equilibrium allocation maps* at $(\bar{\omega},\bar{M}) \in \Delta$.

The first theorem describes how equilibrium allocations vary with M for positive interest rate equilibria. To obtain the simplest statement in the case of incomplete markets, we use the following condition on N [see Geanakoplos and Mas-Colell (1989)].

Definition 4. The $S \times J$ *asset returns matrix* N *is in general position* if every $J \times J$ submatrix of N is of rank J.

As an example suppose $J=1$ and the single asset is the riskless bond, then N is in general position. Recall that a parameter value $(\bar{\omega},\bar{M}) \in \Delta$ for which $\pi_{v_0}^{-1}(\bar{\omega},\bar{M}) \neq \emptyset$ has at least one positive interest rate equilibrium.

Theorem 3 (Real effects of money when $v_0 = 1$). Consider an economy $\mathcal{E}_{u,N}$ for which Assumptions 1–4 hold.

(*a*) *If* rank $N = S$, *then for all* $(\bar{\omega},\bar{M}) \in \Delta$ *with* $\pi_{v_0}^{-1}(\bar{\omega},\bar{M}) \neq \emptyset$ *the equilibrium allocation* $\psi_{v_0}^k(M)$ *is independent of M for all* $M \in \mathcal{N}_{\bar{M}}$, *for* $k=1,\ldots,k_{v_0}$.
(*b*) *If (i)* $J < S$, *(ii)* $I > J$, *(iii) N is in general position, then there exists an open subset* $\Delta' \subset \Delta$ *of full measure such that if* $(\bar{\omega},\bar{M}) \in \Delta'$ *with* $\pi_{v_0}^{-1}(\bar{\omega},\bar{M}) \neq \emptyset$, *the image of the equilibrium allocation map* $\psi_{v_0}^k$ *is a submanifold of* $\mathcal{A}$ *of dimension* $S-1$ *for* $k=1,\ldots,k_{v_0}$.

Proof. (a) When $\langle N \rangle = \mathbb{R}^S$ and $v_0 = 1$ the budget sets (19) and (20) in an abstract equilibrium reduce to the budget set

$$\mathbb{B}(P, w^i) = \{x^i \in \mathcal{R}_+^n \mid P(x^i - w^i) = 0\}$$

which agent i faces in a contingent market equilibrium. Thus each solution of conditions (i) and (ii) in Definition 2 is a contingent market equilibrium and such an equilibrium depends only on the parameter ω. Local variations in M with $\bar{\omega}$ fixed, affect only the monetary equations (iii).

Remark. When the date 1 money supply $(M_1,\ldots,M_S)$ is changed, the date 0 vector of (contingent market) prices P remains unchanged, but the spot and state price vectors (p,β) adjust so as to maintain the equalities

$$p_s \sum_{i=1}^{I} w_s^i = M_s \Leftrightarrow P_s \sum_{i=1}^{I} w_s^i = \beta_s M_s, \quad s=0,\ldots,S.$$

Since

$$q_1 = \frac{1}{1+r} = \sum_{s=1}^{S} \beta_s, \quad q_j = \sum_{s=1}^{S} \beta_s N_s^j, \qquad j=2,\ldots,J,$$

the change in β leads to a change in the rate of interest and the security prices. Thus when markets are complete and $v_0=1$, changes in M lead to substantial nominal changes in prices (p,q) but to no real change in the allocation x.

(b) We need the following lemma in the case $\alpha=v_0$: the case $\alpha=\beta$ is used in the proof of Theorem 4.

Lemma 3. If $I>J$, then there exists an open set of full measure $\Delta'\subset\Delta$ such that if $(v_0, P, \beta)=\phi_\alpha^k(\omega, M)$, $x=\psi_\alpha^k(\omega, M)$ with $(\omega, M)\in\Delta'$, then $\{P_1 \,\square\, (x_1^i - w_1^i),$ $i=2,\ldots,J+1\}$ *are linearly independent vectors for $k=1,\ldots,k_\alpha$, $\alpha=v_0$ or β.*

Proof. See appendix.

Note that the money supply change $(1, M_1)\to(1, \lambda M_1)$ leads to the nominal change in the equilibrium $\beta\to\beta/\lambda$; P and $f=(f^1,\ldots,f^I)$ remain unchanged, since $f^i(1, P, \beta/\lambda, w^i, 1, \lambda\mathcal{M}_1)$ is constant for all $\lambda>0$. To factor out such purely nominal changes we introduce one more normalization. Let $\mathcal{N}'_{(\bar{\omega},\bar{M})}$ denote the neighborhood obtained in the proof of Theorem 2(ii) when $(\bar{\omega},\bar{M})$ is restricted to lie in Δ', then we let

$$\mathcal{N}'_{\bar{M}} = \left\{ M\in\mathcal{M}^* \,\middle|\, (\bar{\omega}, M)\in\mathcal{N}'_{(\bar{\omega},\bar{M})}, \sum_{s=1}^{S} M_s = \sum_{s=1}^{S} \bar{M}_s \right\},$$

where $\dim(\mathcal{N}'_{\bar{M}})=S-1$. To prove the theorem we show that the restriction $\psi_{v_0}^k:\mathcal{N}'_{\bar{M}}\to\mathcal{A}$ is an *embedding*.

Step 1. $\psi_{v_0}^k$ is an immersion: This means that the map $D_M\psi_{v_0}^k: T_M\mathcal{N}'_{\bar{M}}\to\mathbb{R}^{nI}$ is injective for all $M\in\mathcal{N}'_{\bar{M}}$. This is equivalent to showing $\ker D_M\psi_{v_0}^k=0$ which is equivalent to $\mathrm{d}\psi_{v_0}^k=0$ implies $\mathrm{d}M=0$ where $\mathrm{d}\psi_{v_0}^k=D_M\psi_{v_0}^k\mathrm{d}M$. Since the equilibrium under consideration lies in the component E_{v_0} of E, $dv_0=0$ for all $dM \in T_M \mathcal{N}'_{\bar{M}} = \{dM \in R^{S+1} \mid dM_0=0, \sum_{s=1}^{S} dM_s=0\}$. Thus the demand functions $f^i(v_0, P, \beta, w^i, M)$ do not depend directly on M. Since $(v_0, w^i) = (1, \bar{w}^i)$ are fixed we may write $f^i = f^i(P, \beta)$, $i=1,\ldots,I$ and $\psi^k_{v0}(M) = f(\phi^k_{v0}(\bar{\omega}, M)) = f(P(M), \beta(M))$ for fixed k. We need to compute $df = (df^1, \ldots, df^I)$ arising from a local change in the money supply dM. We do this using the *Slutsky equation* for each agent. Differentiating the first-order conditions for agent 1 gives as usual

$$J^1 \begin{bmatrix} df^1 \\ d\lambda_0^1 \end{bmatrix} = \begin{bmatrix} \lambda_0^1 \, dP^{\mathrm{T}} \\ dP(x^1 - w^1) \end{bmatrix} \quad \text{where} \quad J^1 = \begin{bmatrix} D^2_{x^1} u^1 & -P^{\mathrm{T}} \\ -P & 0 \end{bmatrix}. \tag{29}$$

The positive Gaussian curvature of the indifference curves in Assumption 2 implies that $(J^1)^{-1}$ exists: let

$$(J^1)^{-1} = \begin{bmatrix} A_1^1 & A_2^1 \\ (A_2^1)^{\mathrm{T}} & A_3^1 \end{bmatrix}, \tag{30}$$

where A_1^1 is $n \times n$ and A_2^1 is $n \times 1$, then

$$df^1 = A_1^1 \lambda_0^1 \, dP^{\mathrm{T}} + A_2^1 \, dP(x^1 - w^1). \tag{31}$$

From standard consumer demand theory we know that (i) the Slutsky matrix A_1^1 is symmetric, of rank $n-1$ and $\ker A_1^1 = \langle P^{\mathrm{T}} \rangle$, (ii) the vector of income effects A_2^1 satisfies $-PA_2^1 = 1$. From (i) and (ii) we deduce that any change $dM \in T_M \mathcal{N}'_{\bar{M}}$ such that $df^1 = 0$ implies $dP = 0$: $df^1 = 0 \Rightarrow P\,df^1 = 0 \Leftrightarrow PA_1^1 \lambda_0^1 \, dP^{\mathrm{T}} + PA_2^1 \, dP(x^1 - w^1) = 0 \Rightarrow dP(x^1 - w^1) = 0$. This in turn implies $A_1^1 \, dP^{\mathrm{T}} = 0$ which by (i) implies $dP^{\mathrm{T}} \in \langle P^{\mathrm{T}} \rangle$. The monetary equation $P_0 \sum_{i=1}^{I} w_0^i = M_0$ and the normalization $M_0 = 1$ thus implies $dP^{\mathrm{T}} = 0$.

To obtain an expression similar to (31) for agents $i = 2, \ldots, I$ we need to write their budget constraints in (19) as a system of equations. Let

$$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{S-J} \end{bmatrix}$$

be an $(S-J) \times S$ matrix whose rows form a basis for $\langle N \rangle^{\perp}$. Then

$$P_1 \,\square\, (x_1^i - w_1^i) \in \langle [\beta] N \rangle \Leftrightarrow Q[\beta]^{-1}(P_1 \,\square\, (x_1^i - w_1^i)) = 0.$$

If we define

$$\Phi(P,\beta)=\begin{bmatrix} P_0 & P_1 & \cdots & P_S \\ 0 & & & \\ \vdots & & Q[\beta]^{-1}[P_1] & \\ 0 & & & \end{bmatrix},$$

where

$$[P_1]=\begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & P_S \end{bmatrix}.$$

then the system of date 0 and date 1 budget constraints in (19) can be written as

$$\Phi(P,\beta)(x^i-w^i)=0, \quad i=2,\ldots,I. \tag{32}$$

The first-order conditions for utility maximization subject to (32) become

$$\begin{aligned} &D_{x^i}u^i-\Phi(P,\beta)^{\mathrm{T}}v^i=0, \quad i=2,\ldots,I, \\ &-\Phi(P,\beta)(x^i-w^i)=0, \end{aligned} \tag{33}$$

where $v^i\in\mathbb{R}^{S-J+1}_{++}$ is the vector of Lagrange multipliers for agent i. Differentiating (33) gives

$$J^i\begin{bmatrix} \mathrm{d}f^i \\ \mathrm{d}v^i \end{bmatrix}=\begin{bmatrix} \mathrm{d}\Phi^{\mathrm{T}}v^i \\ \mathrm{d}\Phi(x^i-w^i) \end{bmatrix},$$

where

$$J^i=\begin{bmatrix} D^2_{x^i}u^i & -\Phi^{\mathrm{T}} \\ -\Phi & 0 \end{bmatrix}, \quad i=2,\ldots,I.$$

The classical properties of the matrix J^1 and its inverse generalise to the case of multiple budget constraints when Φ replaces P as follows [see Balasko and Cass (1987) for the proofs]. First J^i is invertible. Writing the inverse in block form similar to that in (30),

$$(J^i)^{-1}=\begin{bmatrix} A^i_1 & A^i_2 \\ (A^i_2)^{\mathrm{T}} & A^i_3 \end{bmatrix}, \quad i=2,\ldots,I.$$

A_1^i is $n\times n$ and A_2^i is $n\times(S+1)$. The following properties analogous to those in the standard theory can be derived. (i′) the Slutsky matrix A_1^i, is symmetric, of rank $n-(S-J+1)$ and $\ker A_1^i=\langle\Phi^{\mathrm{T}}\rangle$, (ii′) the matrix of income effects A_2^i satisfies $-\Phi A_2^i=I_{S-J+1}$. The analogue of (31) is

$$\mathrm{d}f^i=A_1^i\,\mathrm{d}\Phi^{\mathrm{T}}v^i+A_2^i\,\mathrm{d}\Phi(x^i-w^i),\quad i=2,\dots,I,$$

where

$$\mathrm{d}\Phi=\begin{bmatrix}\mathrm{d}P_0 & \mathrm{d}P_1 & \cdots & \mathrm{d}P_S\\ 0 & & & \\ \vdots & & Q\,\mathrm{d}([\beta]^{-1})[P_1]+Q[\beta]^{-1}[\mathrm{d}P_1] & \\ 0 & & & \end{bmatrix}. \tag{34}$$

A change $\mathrm{d}M\in T_M\mathcal{N}'_{\bar{M}}$ such that $\mathrm{d}f=0$ implies $\mathrm{d}f^1=0$, which from the analysis for agent 1 implies $\mathrm{d}P=0$. $\mathrm{d}f^i=0$ then implies $\Phi\,\mathrm{d}f^i=0\Rightarrow \mathrm{d}\Phi(x^i-w^i)=0$ since by (i′) and (ii′), $\Phi A_1^i=0$, $-\Phi A_2^i=I_{S-J+1}$. Since $\mathrm{d}P=0$ using (34) gives

$$Q\,\mathrm{d}([\beta]^{-1})P_1\,\square\,(x_1^i-w_1^i)=0,\quad i=2,\dots,I.$$

Since $(\bar{\omega},\bar{\mathcal{M}})\in\Delta'$, by Lemma 3 the vectors $\{P_1\,\square\,(x_1^i-w_1^i)\}_{i=2}^{J+1}$ are linearly independent. Thus

$$Q\,\mathrm{d}([\beta]^{-1})[\beta]N=0\Leftrightarrow Q[\beta]^{-1}[\mathrm{d}\beta]N=0.$$

Since $Q[\beta]^{-1}[\beta]N=0$ this implies

$$Q[\beta^{-1}][\beta+\mathrm{d}\beta]N=0. \tag{35}$$

For $\mathrm{d}M$ sufficiently small $\beta+\mathrm{d}\beta\gg 0$ so that (35) implies

$$\langle[\beta+\mathrm{d}\beta]N\rangle=\langle[\beta]N\rangle.$$

Since N is in general position Lemma 4 in Geanakoplos and Mas-Colell (1989) implies $\mathrm{d}\beta=(\mathrm{d}\mu)\beta$, $\mathrm{d}\mu\in\mathbb{R}$. The date 1 monetary equation $P_s\sum_{i=1}^I\bar{w}_s^i=\beta_sM_s$ implies $\mathrm{d}\beta_sM_s+\beta_s\,\mathrm{d}M_s=0\Rightarrow\mathrm{d}M_s=-\mathrm{d}\mu M_s$, for $s=1,\dots,S$. The normalization $\sum_{s=1}^S M_s=\sum_{s=1}^S\bar{M}_s$ then implies $\mathrm{d}M_s=0$, $s=1,\dots,S$, which together with the normalization at date 0 implies $\mathrm{d}M=0$. Thus ψ_{vo}^k is an immersion for $k=1,\dots,k_{vo}$.

Step 2. $\psi_{vo}^k:\mathcal{N}'_{\bar{M}}\to\mathcal{A}$ *is injective*: this means that if $M,M'\in\mathcal{N}'_{\bar{M}}$, $M\neq M'$, then $\psi^k_{v0}(M) \neq \psi^k_{v0}(M') \Leftrightarrow f(P(M), \beta(M)) \neq f(P(M'), \beta(M'))$. Suppose not. Let $(P, \beta) = (P(M), \beta(M))$, $(P', \beta') = (P(M'), \beta(M'))$. Then by agent 1's demand function $f^1(P) = f^1(P')$ and $P_0 \sum_{i=1}^I w_0^i = P'_0 \sum_{i=1}^I w_0^i = 1$ imply $P = P'$. $f^i(P, \beta) = f^i(P, \beta')$, $i = 2, \ldots, I$, and $\{P_1 \square (f_1^i - w_1^i),\ i = 2, \ldots, J+1\}$ linearly independent by Lemma 3, imply $\langle [\beta] N \rangle = \langle [\beta'] N \rangle$. Since N is in general position there exists $\mu \in \mathbb{R}$ such that $\beta' = \mu \beta$. From the date 1 monetary equations $M'_1 = (1/\mu) M_1$. The date 1 normalization $\sum_{s=1}^S M_s = \sum_{j=1}^S \bar{M}_s$ implies $M'_1 = M_1$. Thus by the date 0 normalization $M' = M$ and the contradiction completes Step 2.

Step 3. *$\psi^k_{v0} : \mathcal{N}'_{\bar{M}} \to \mathcal{A}$ is proper*: this means that for each compact set $K \subset \mathcal{A}$, $(\psi^k_{v0})^{-1}(K)$ is compact. It suffices to show that there exist bounds $\underline{m}, \bar{m} \in \mathbb{R}^{S+1}_{++}$ such that $\underline{m} \leqq M \leqq \bar{m}$ for all $M \in (\psi^k_{v0})^{-1}(K)$, since the continuity of ψ^k_{v0} then implies $(\psi^k_{v0})^{-1}(K)$ is compact. Note that $\underline{m}_0 = \bar{m}_0 = 1$. $f^1(P)$ in a compact set and $P_0 \sum_{i=1}^I w_0^i = 1$ imply that P lies in a compact set. $\sum_{s=1}^S \beta_s \leqq 1$ and $M_s = (P_s \sum_{i=1}^I w_s^i)/\beta_s$, $s = 1, \ldots, S$ gives the lower bound $\underline{m}$. $\sum_{s=1}^S M_s = \sum_{s=1}^S \bar{M}_s$ gives the upper bound, which completes the proof. □

Remark. The proof of the above theorem in case (b) can be adapted to the case where N *is not in general position*, using the result of Geanakoplos and Mas-Colell (1989, Theorem 1′) for the model without money. The dimension of the image $\psi^k_{v0}(\mathcal{N}_{\bar{M}})$ is then in general less than $S-1$.

Remark. Step 1 establishes a local property, namely that ψ^k_{v0} is an immersion. Intuitively this means that non-trivial local changes in M lead to non-trivial local changes in the allocation. More precisely, because of the linearity of the local approximation to ψ^k_{v0} it implies that the $S-1$ dimensional tangent plane to the money supply space translates into an $S-1$ dimensional hyperplane in the allocation space. It is this property which ensures that the image under ψ^k_{v0} of a sufficiently small neighborhood of $\bar{M}$ is a submanifold of the allocation space of dimension $S-1$. Steps 2 and 3 extend the result to a global property over $\mathcal{N}'_{\bar{M}}$ – so that the image under ψ^k_{v0} of $\mathcal{N}_{\bar{M}}$ is a submanifold of $\mathcal{A}$ of dimension $S-1$ – by establishing the additional properties of injectivity and properness. For the case where $v_0 < 1$ we have only been able to establish the first property, namely that ψ^k_{β} is an immersion.

Theorem 4 (*Real effect of money when* $v_0 < 1$). *Consider an economy for which Assumptions 1–4 hold. If either* (*a*) $\operatorname{rank} N = S$ *and* $I > S$, *or* (*b*) $\operatorname{rank} N < S$, $I > S - J$ *and N is in general position, then there exists an open set* $\Delta'' \subset \Delta$ *of full measure such that if* $(\bar{\omega}, \bar{M}) \in \Delta''$ *with* $\pi_{\beta}^{-1}(\bar{\omega}, \bar{M}) \neq \emptyset$, *then the equilibrium allocation map* ψ^k_{v0} *restricted to a subset of dimension* S, $\mathcal{N}''_{\bar{M}} \subset \mathcal{N}_{\bar{M}}$ *is an immersion,* $k = 1, \ldots, k_{\beta}$.

Proof. Let $\Delta'' \subset \Delta$ denote an open set of full measure which will be defined by Lemma 4 for the case of complete markets and by Lemmas 5 and 6 for the case of incomplete markets. For $(\bar{\omega}, M) \in \Delta''$ define

$$\mathcal{N}''_{\bar{M}} = \{M \in \mathcal{M}^* \,|\, (\bar{\omega}, M) \in \mathcal{N}_{(\bar{\omega}, \bar{M})} \cap \Delta''\},$$

where $\mathcal{N}_{(\bar{\omega}, \bar{M})}$ is the neighborhood given in Definition 3. Since $\dim \mathcal{M}^* = S$, $\dim \mathcal{N}''_{\bar{M}} = S$. *We show that $\psi^k_\beta : \mathcal{N}''_{\bar{M}} \to \mathcal{A}$ is an immersion.* This is equivalent to proving $D_M \psi^k_\beta \, \mathrm{d}M = 0$ with $\mathrm{d}M_0 = 0$ implies $\mathrm{d}M_s = 0$, $s = 1, \ldots, S$ or in terms of the notation in Step 1 of the proof of Theorem 3(b), $\mathrm{d}f = 0 \Rightarrow \mathrm{d}M = 0$. In short, there is no non-trivial local change in the money supply which does not affect the equilibrium allocation.

Case 1 (Complete markets). Let rank $N = S$, We need to calculate the change in the equilibrium allocation $\mathrm{d}f = (\mathrm{d}f^1, \ldots, \mathrm{d}f^I)$ arising from a local change in the money supply $\mathrm{d}M$. Since each agent has a single budget constraint $P(x^i - (1/v) \,\square\, w^i) = 0$, which depends on the velocity of circulation v, which can now vary, the *Slutsky equation* becomes

$$\mathrm{d}f^i = A^i_1 \lambda^i_0 \, \mathrm{d}P^T + A^i_2 \, \mathrm{d}P\left(x^i - \frac{1}{v} \,\square\, w^i\right) + A^i_2 P\left(\frac{\mathrm{d}v}{v^2} \,\square\, w^i\right),$$

$$i = 1, \ldots, I,$$

where A^i_1 is the $n \times n$ matrix of substitution effects, A^i_2 is the $n \times 1$ vector of income effects and $\mathrm{d}v/v^2 = (\mathrm{d}v_0/v_0^2, \ldots, \mathrm{d}v_S/v_S^2)$. $\mathrm{d}f^i = 0 \Rightarrow P\,\mathrm{d}f^i = 0 \Rightarrow \mathrm{d}P(x^i - (1/v) \,\square\, w^i) + P((\mathrm{d}v/v^2) \,\square\, w^i) = 0 \Rightarrow A^i_1 \, \mathrm{d}P^{\mathrm{T}} = 0 \Rightarrow \mathrm{d}P^{\mathrm{T}} \in \ker A^i_1 = \langle P^{\mathrm{T}} \rangle$. Thus there exists $\mathrm{d}\alpha \in \mathbb{R}$ such that $\mathrm{d}P = \mathrm{d}\alpha P$. By the budget constraint $\mathrm{d}\alpha P(x^i - (1/v) \,\square\, w^i) = 0$, thus

$$P\left(\frac{\mathrm{d}v}{v^2} \,\square\, w^i\right) = 0, \quad i = 1, \ldots, I. \tag{36}$$

Since $\mathrm{d}P = \mathrm{d}\alpha P$, differentiating the monetary equations and the condition $\sum_{s=1}^S \beta_s = 1$ gives the linear equations for $(\mathrm{d}v_0, \mathrm{d}\alpha, \mathrm{d}\beta)$:

$$\mathrm{d}\alpha P_0 \sum_{i=1}^I w^i_0 = M_0 \, \mathrm{d}v_0,$$

$$\mathrm{d}\alpha P_s \sum_{i=1}^I w^i_s = \mathrm{d}\beta_s M_s v_s + \beta_s \, \mathrm{d}(M_s v_s), \tag{37}$$

$$\sum_{s=1}^{S} \mathrm{d}\beta_s = 0.$$

Since $M_s v_s = M_s + M_0(1-v_0)$ implies $\mathrm{d}(M_s v_s) = \mathrm{d}M_s - M_0\,\mathrm{d}v_0$,

$$\mathrm{d}v_0 = \frac{v_0}{\rho}\sum_{s=1}^{S}\beta_s\frac{\mathrm{d}M_s}{M_s v_s}, \quad \rho = 1 + M_0 v_0 \sum_{s=1}^{S}\frac{\beta_s}{M_s v_s}, \tag{38}$$

$$\mathrm{d}v_s = \frac{1}{\rho M_s}\left(\rho(1-v_s)\,\mathrm{d}M_s - M_0 v_0 \sum_{\sigma=1}^{S}\beta_\sigma\frac{\mathrm{d}M_\sigma}{M_\sigma v_\sigma}\right), \quad s=1,\dots,S. \tag{39}$$

Substituting (38) and (39) into (36) and introducing the new variables $\mathrm{d}\mu_s = (\beta s/M_s v_s)\,\mathrm{d}M_s$, $s=1,\dots,S$, gives the system of linear equations

$$\sum_{s=1}^{S} a_s^i\,\mathrm{d}\mu_s = 0, \quad i=1,\dots,I, \tag{40}$$

with

$$a_s^i = \frac{1}{v_0}(\alpha_0^i - \alpha_s^i) + \sum_{\sigma=1}^{S}\frac{\beta_\sigma}{v_\sigma}(\alpha_s^i - \alpha_\sigma^i), \qquad s=1,\dots,S, \quad i=1,\dots,I, \tag{41}$$

where $\alpha_s^i = P_s w_s^i/\beta_s M_s v_s$, $s=0,1,\dots,S$, with $\beta_0 = 1$. If there are at least as many agents as states of nature and if S of the vectors a^i are linearly independent, then the only solution of (40) is $\mathrm{d}\mu = 0$ which implies $\mathrm{d}M = 0$. The proof of Case 1 thus follows from Lemma 4.

Lemma 4. Let $\{a^i,\ i=1,\dots,I\}$ *denote the vectors defined by (41). If* $\operatorname{rank} N = S$ *and* $I > S$ *then there exists an open set of full measure* $\Delta'' \subset \Delta$ *such that if* $(v_0, P, \beta) = \phi_\beta^k(\omega, M)$ *with* $(\omega, M) \in \Delta''$, *then the vectors* $\{a^i,\ i=1,\dots,S\}$ *are linearly independent, for* $k=1,\dots,k_\beta$.

Proof. See appendix.

Case 2 (Incomplete Markets). Let $\operatorname{rank} N < S$. We proceed as before. For agent $i=2,\dots,I$ the budget constraints in (19) contain a non-trivial date 1 component. Thus with the notation used in the proof of Theorem 3(b) (Step 1) the Slutsky equations become

$$\mathrm{d}f^1 = A_1^1\,\mathrm{d}P^{\mathrm{T}}\lambda_0^1 + A_2^1\,\mathrm{d}P\left(x^1 - \frac{1}{v}\,\square\, w^1\right) + A_2^1 P\left(\frac{\mathrm{d}v}{v^2}\,\square\, w^1\right),$$

$$\mathrm{d}f^i = A_1^i \,\mathrm{d}\Phi^{\mathrm{T}} v^i + A_2^i \,\mathrm{d}\Phi\left(x^i - \frac{1}{v} \,\square\, w^i\right) + A_2^i \Phi\left(\frac{\mathrm{d}v}{v^2} \,\square\, w^i\right),$$

$$i = 2, \ldots, I.$$

Suppose $\mathrm{d}M \in \{0\} \times \mathbb{R}^S$ implies $\mathrm{d}f = (\mathrm{d}f^1, \ldots, \mathrm{d}f^I) = 0$. By the same argument as in Case 1, $\mathrm{d}f^1 = 0$ implies that there exists $\mathrm{d}\alpha \in \mathbb{R}$ such that $\mathrm{d}P = \mathrm{d}\alpha P$ and $P((\mathrm{d}v/v^2) \,\square\, w^1) = 0$. $\mathrm{d}f^i = 0 \Rightarrow \Phi\,\mathrm{d}f^i = 0 \Rightarrow \mathrm{d}\Phi(x^i - (1/v) \,\square\, w^i) + \Phi(\mathrm{d}v/v^2) \,\square\, w^i) = 0 \Rightarrow A_1^i \,\mathrm{d}\Phi^{\mathrm{T}} v^i = 0 \Rightarrow \mathrm{d}\Phi^{\mathrm{T}} v^i \in \langle \Phi^{\mathrm{T}} \rangle$, $i = 2, \ldots, I$. Let $v^i = (\lambda_0^i, v_1^i)$ and $\tilde{v}^i = (0, v_1^i)$, then by (34)

$$\begin{aligned}\mathrm{d}\Phi^{\mathrm{T}} v^i &= \lambda_0^i \,\mathrm{d}P^{\mathrm{T}} + [0, Q[\beta]^{-1}[\mathrm{d}P_1]]^{\mathrm{T}} \tilde{v}^i + [0, Q\,\mathrm{d}([\beta]^{-1})[P_1]]^{\mathrm{T}} \tilde{v}^i \\ &= \mathrm{d}\alpha \Phi^{\mathrm{T}} v^i + [0, Q\,\mathrm{d}([\beta]^{-1})[P_1]]^{\mathrm{T}} \tilde{v}^i\end{aligned}$$

since $\mathrm{d}P = \mathrm{d}\alpha P$. Since $\Phi^{\mathrm{T}} v^i \in \langle \Phi^{\mathrm{T}} \rangle$, and since $[0, Q\,\mathrm{d}([\beta]^{-1})[P_1]]$ has date 0 coordinates equal to zero $(Q\,\mathrm{d}([\beta]^{-1})[P_1])^{\mathrm{T}} v_1^i \in \langle (Q[\beta]^{-1}[P_1])^{\mathrm{T}} \rangle$, $i = 2, \ldots, I$. If there exist $S - J$ agents such that their date 1 constraint multipliers $v_1^2, \ldots, v_1^{S-J+1}$ are linearly independent, then

$$\langle (Q\,\mathrm{d}([\beta]^{-1})[P_1])^{\mathrm{T}} \rangle \subset \langle (Q[\beta]^{-1}[P_1])^{\mathrm{T}} \rangle. \tag{42}$$

Since rank $[P_1]^{\mathrm{T}} = S$, (42) implies $\langle (Q\,\mathrm{d}([\beta]^{-1}))^{\mathrm{T}} \rangle \subset \langle (Q[\beta]^{-1})^{\mathrm{T}} \rangle$. Thus $Q\,\mathrm{d}([\beta]^{-1})[\beta] N = 0$ so that eq. (35) holds. By the same argument as before, since N is in general position, there exists $\mathrm{d}\mu \in \mathbb{R}$ such that $\mathrm{d}\beta = (\mathrm{d}\mu)\beta$. Since $\sum_{s=1}^{S} \mathrm{d}\beta_s = 0$, $\mathrm{d}\beta = 0$. The validity of (42) follows from Lemma 5.

Lemma 5. If rank $N < S$ *and* $I > S - J$, *then there exists an open set of full measure* $\hat{\Delta} \subset \Delta$ *such that if* $(\omega, M) \in \hat{\Delta}$, *then the date 1 constraint multipliers* $\{v_1^2, \ldots, v_1^{S-J+1}\}$ *are linearly independent, in every equilibrium of* $\mathscr{E}(\omega, M)$.

Proof. See appendix.

Since $\mathrm{d}P = \mathrm{d}\alpha P$ and $\mathrm{d}\beta = 0$ the linearized monetary equations (37) have a solution only if

$$\mathrm{d}M_s = (M_s v_s + M_0 v_0)\,\mathrm{d}\alpha, \quad s = 1, \ldots, S. \tag{43}$$

In this case (38) and (39) become

$$\mathrm{d}v_0 = v_0 \,\mathrm{d}\alpha, \quad \mathrm{d}v_s = -\frac{M_0 v_s}{M_s}\,\mathrm{d}\alpha, \quad s = 1, \ldots, S.$$

Eq. (36) then implies

$$\left(\frac{P_0 w_0^i}{M_0 v_0} - \sum_{s=1}^{S} \frac{P_s w_s^i}{M_s v_s}\right) \mathrm{d}\alpha = 0, \quad i = 1, \ldots, I. \tag{44}$$

Lemma 6 proves that generically (43) implies $\mathrm{d}\alpha = 0$.

Lemma 6. If $I > 1$ then there exists an open set of full measure $\hat{\hat{\Delta}} \subset \Delta$ such that if $(\omega, M) \in \hat{\hat{\Delta}}$, then

$$\frac{P_0 w_0^1}{M_0 v_0} - \sum_{s=1}^{S} \frac{P_s w_s^1}{M_s v_s} \neq 0,$$

in every equilibrium of $\mathscr{E}(\omega, M)$.

Proof. See appendix.

Let $\Delta'' = \hat{\Delta} \cap \hat{\hat{\Delta}}$. If $(\bar{w}, \bar{M}) \in \Delta''$, then by Lemma 6 and (43), $\mathrm{d}f = 0$ implies $\mathrm{d}M = 0$ and the proof of Case 2 is complete. □

Appendix

Proof of Theorem 1

By Proposition 1(ii), it suffices to prove the existence of a solution (v_0, P, β) to the system of eqs. (23) satisfying $v_0 \leqq 1$, $\sum_{s=1}^{S} \beta_s \leqq 1$.

Step 1. Establishing bounds. The price vector P can be decomposed into a product $P = \alpha P'$ with $\alpha \in [\alpha_0, \alpha_1]$ and $P' \in \Sigma' = \{P' \in \mathbb{R}_+^n \mid \sum_{j=1}^{n} P_j' = 1\}$. Let us exhibit the bounds α_0, α_1. Let $w_s = \sum_{i=1}^{I} w_s^i$, $s = 0, \ldots, S$. The equalities $\alpha P_0' w_0 = M_0 v_0$, $\alpha P_s' w_s = \beta_s M_s v_s$, $M_s v_s = M_s + M_0(1 - v_0)$ and $v_0 \leqq 1$ imply $\alpha P' w \leqq (S+1) M_0 + \sum_{s=1}^{S} M_s$ so that $\alpha_1 = (1/\rho)((S+1) M_0 + \sum_{s=1}^{S} M_s)$, $\rho = \inf_{P' \in \Sigma'} P' w$. Similarly $v_s \geqq 1$ and $(\sum_{s=1}^{S} \beta_s - 1)(1 - v_0) = 0$ imply $\alpha_0 = \min(M_0 / \sum_{l=1}^{L} w_{0l}, \min_s \{M_s\} / \sum_{s=1}^{S} \Sigma_{l=1}^{L} w_{sl}) > 0$. Let $\Sigma = \{\beta \in \mathbb{R}_+^S \mid \sum_{s=1}^{S} \beta_s \leqq 1\}$. Consider the compact convex set

$$X = [0, 1] \times [\alpha_0, \alpha_1] \times \Sigma' \times \Sigma.$$

We look for an equilibrium price vector $\zeta = (v_0, \alpha, P', \beta) \in X$. To avoid discontinuity in the budget correspondences of agents at $v_0 = 0$ and $\beta_s = 0$ we introduce the modified budget sets

$$\mathbb{B}(v_0^\varepsilon, \alpha P', \beta^\varepsilon; w^1, M), \qquad \mathscr{B}(v_0^\varepsilon, \alpha P', \beta^\varepsilon; w^i, M), \quad i = 2, \ldots, I,$$

where $v_0^\varepsilon = \max(\varepsilon, v_0)$, $\beta^\varepsilon = (\beta_1^\varepsilon, \ldots, \beta_S^\varepsilon)$, $\beta_s^\varepsilon = \max(\varepsilon, \beta_s)$. Let $f_\varepsilon^i(v_0, \alpha, P', \beta; w^i, M)$ denote the associated demand function. We show that there exists $\bar{\varepsilon} > 0$ such that the following *property* $\bar{\varepsilon}$ holds: if $\alpha P_0' w_0 = M_0 v_0$, $\alpha P_s' w_s = \beta_s M_s v_s$, $s = 1, \ldots, S$, $(\sum_{s=1}^S \beta_s - 1)(1 - v_0) = 0$ and $v_0 < \bar{\varepsilon}$ or $\beta_s < \bar{\varepsilon}$ for some s, then $\sum_{i=1}^I f_\varepsilon^i(\zeta; w^i, M) \nleqq \sum_{i=1}^I w^i$.

The budget constraint of agent 1 can be written as

$$P'x^1 = P_0' \frac{w_0^1}{v_0^\varepsilon} + \sum_{s=1}^S P_s' \frac{w_s^1}{v_s^\varepsilon} \geqq \inf_{P' \in \Sigma} \left\{ P_0' w_0^1 + \sum_{s=1}^S \frac{P_s' w_s^1}{1 + \max_\sigma (M_0/M_\sigma)} \right\} > 0.$$

Since $\alpha \geqq \alpha_0$ the equations $\alpha P_0' w_0 = M_0 v_0$, $\alpha P_s' w_s = \beta_s M_s v_s$ imply that whenever $v_0 \to 0$, then $P_0' \to 0$ and whenever $\beta_s \to 0$, then $P_s' \to 0$, so that by standard arguments $\|f_\varepsilon^1\| \to \infty$, for all $\varepsilon > 0$. Thus we may choose $\bar{\varepsilon} > 0$ satisfying the *property* $\bar{\varepsilon}$ stated above.

Step 2. Construction of correspondence $\psi: X \to X$. We define the components of the correspondence

$$\psi(\zeta) = (v_0(\zeta), \alpha(\zeta), P'(\zeta), \beta(\zeta))$$

as follows. Let

$$x^i(\bar{\zeta}) = \underset{x^i \in \mathscr{B}(\bar{v}_0^\varepsilon, \bar{\alpha}\bar{P}', \bar{\beta}^\varepsilon; w^i, M) \cap C}{\arg\max} u^i(x^i), \quad i = 2, \ldots, I,$$

with $x^i(\bar{\zeta})$ denoting the truncated demand of agent 1, where $C = \{x \in \mathbb{R}_+^n \mid x_{sl} \leqq k,\ s = 0, \ldots, S,\ l = 1, \ldots, L\}$ is a truncation of the consumption set with k sufficiently large so that *property* $\bar{\varepsilon}$ holds for the truncated demands $x^i(\zeta)$. Define $P'(\zeta)$ by

$$P'(\bar{\zeta}) = \underset{P' \in \Sigma'}{\arg\max}\, P' \sum_{i=1}^I (x^i(\bar{\zeta}) - w^i).$$

To define the remaining components we consider two cases, $\bar{P}_0' \neq 0$, and $\bar{P}_0' = 0$. If $\bar{P}_0' \neq 0$, let $a(v_0) = \sum_{s=1}^S [\bar{P}_s' w_s / (M_s + M_0(1 - v_0))]$, $b(v_0) = \bar{P}_0' w_0 / M_0 v_0$ and let $\hat{v}_0 = \min_{1 \leqq s \leqq S} \{1 + M_s/M_0\}$. Note that $a(0) > 0$ and $a(v_0) \to \infty$ as $v_0 \to \hat{v}_0$, while $b(\hat{v}_0) > 0$ and $b(v_0) \to \infty$ as $v_0 \to 0$. Since $a(\cdot)$ and $b(\cdot)$ are monotone, there is a unique solution $v_0^* > 0$ of the equation $a(v_0) - b(v_0) = 0$. Note that $v_0^* = 1$ if and only if $\sum_{s=1}^S (\bar{P}_s' w_s / M_s) \leqq \bar{P}_0' w_0 / M_0$ and that if $\bar{P} \to 0$ then $v_0^* \to 0$. Thus we can define

$$v_0(\bar{\zeta})=\begin{cases}\max(1, v_0^*) & \text{if } \bar{P}'_0 \neq 0\\ 0 & \text{if } \bar{P}'_0 = 0\end{cases},$$

$$\alpha(\bar{\zeta})=\begin{cases}M_0 v_0(\bar{\zeta})/\bar{P}'_0 w_0 & \text{if } \bar{P}'_0 \neq 0\\ 1/a(0) & \text{if } \bar{P}'_0 = 0\end{cases},$$

$$\beta_s(\bar{\zeta})=\begin{cases}\alpha(\bar{\zeta})\bar{P}'_s w_s/[M_s+M_0(1-v_0(\bar{\zeta}))] & \text{if } \bar{P}'_0 \neq 0\\ \alpha(\bar{\zeta})\bar{P}'_s w_s/(M_s+M_0) & \text{if } \bar{P}'_0 = 0\end{cases}.$$

Note that $v_0(\bar{\zeta}) \leqq 1$, $\sum_{s=1}^{S} \beta_s(\bar{\zeta}) \leqq 1$ and if $\bar{P}'_0 \neq 0$ then $(\sum_{s=1}^{S} \beta_s(\bar{\zeta})-1)$ $(1-v_0(\bar{\zeta}))=0$. It is easy to check that the correspondence ψ is an upper semicontinuous convex valued correspondence such that $\psi(\zeta) \neq \varnothing$ for all $\zeta \in X$. Thus the Kakutani Fixed Point Theorem implies that ψ *has a fixed point*, $\bar{\zeta} \in \psi(\bar{\zeta})$.

Step 3. A fixed point $\bar{\zeta} \in \psi(\zeta)$ is an equilibrium price. It suffices to check that $v_0 \geqq \bar{\varepsilon}$, $\beta_s \geqq \bar{\varepsilon}$, $s=1,\dots,S$ and $\sum_{i=1}^{I}(x^i(\bar{\zeta})-w^i)=0$. Summing the date 0 budget constraints gives

$$\sum_{s=0}^{S} \bar{P}'_s\left(\sum_{i=1}^{I}(x_s^i(\bar{\zeta})-w_s^i)\right)+\sum_{s=0}^{S} \bar{P}'_s\left(\sum_{i=1}^{I} w_s^i\left(1-\frac{1}{\bar{v}_s^{\bar{\varepsilon}}}\right)\right)=0.$$

We show that $A=\sum_{s=0}^{S} \bar{P}'_s w_s(1-1/\bar{v}_s^{\bar{\varepsilon}}) \geqq 0$. Note that $A=M_0\bar{v}_0((\bar{v}_0^{\bar{\varepsilon}}-1)/\bar{v}_0^{\bar{\varepsilon}})+\sum_{s=1}^{S}\beta_s M_s \bar{v}_s((\bar{v}_s^{\bar{\varepsilon}}-1)/\bar{v}_s^{\bar{\varepsilon}})$. Thus

$$M_s(\bar{v}_s^{\bar{\varepsilon}}-1)=M_0(1-\bar{v}_0^{\bar{\varepsilon}}) \Rightarrow A=M_0(1-\bar{v}_0^{\bar{\varepsilon}})\left[-\frac{\bar{v}_0}{\bar{v}_0^{\bar{\varepsilon}}}+\sum_{s=1}^{S}\beta_s\frac{\bar{v}_s}{\bar{v}_s^{\bar{\varepsilon}}}\right].$$

If $\bar{v}_0=1$, $\bar{v}_0^{\bar{\varepsilon}}=1$ and $A=0$. If $\bar{v}_0<1$ then $\sum_{s=1}^{S}\bar{\beta}_s=1$. The term in the square bracket is a decreasing function of $\bar{v}_0$ which is 0 at $\bar{v}_0=\bar{v}_0^{\bar{\varepsilon}}$. Since $\bar{v}_0 \leqq \bar{v}_0^{\bar{\varepsilon}}$ this term is non-negative, so that $A \geqq 0$.

Since $A \geqq 0$, $\bar{P}'(\sum_{i=1}^{I}(x^i(\bar{\zeta})-w^i)) \leqq 0$. By *property* $\bar{\varepsilon}$ this implies $\bar{v}_0 \geqq \bar{\varepsilon}$, $\bar{\beta}_s \geqq \bar{\varepsilon}$, $s=1,\dots,S$, so that $\bar{v}_0^{\bar{\varepsilon}}=\bar{v}_0$, $\bar{\beta}_s^{\bar{\varepsilon}}=\beta_s$, $s=1,\dots,S$ and $A=0$. Thus $\bar{P}(\sum_{i=1}^{I}(x^i(\bar{\zeta})-w^i))=0$ which by standard arguments implies $\sum_{i=1}^{I}(x^i(\bar{\zeta})-w^i)=0$. □

Proof of Lemma 1

We need to show that $D_\xi \hat{H}_\alpha : T_\xi A_\alpha \to \mathbb{R}^{n-1}\times\mathbb{R}^{S+1}$ is surjective for all $\xi \in \hat{H}_\alpha^{-1}(0)$. Let $e=\{e^j\}_{j=1}^{n-1+S+1}$ denote the standard basis for $\mathbb{R}^{n-1+S+1}$ with

$e_i^j=0$, $i\neq j$, $e_i^j=1$, $i=j$, for $i,j=1,\ldots,n-1+S+1$. It suffices to show that for $\xi\in\hat{H}_\alpha^{-1}(0)$, for each $e^j\in e$ there exists $\mathrm{d}\xi^j\in T_\xi\Lambda_\alpha$ such that $(D_\xi\hat{H}_\alpha)\,\mathrm{d}\xi^j=e^j$, for $\alpha=v_0,\beta$ or (v_0,β). This amounts to showing that we can marginally control each equation j (for $j=1,\ldots,n-1+S+1$) without affecting any other equation $k\neq j$, by an appropriate marginal change $\mathrm{d}\xi^j\in T_\xi\Lambda_\alpha$. We chose marginal changes of the form $\mathrm{d}\xi=(\mathrm{d}v_0,0,0,\mathrm{d}w,\mathrm{d}M)$. All cases can be covered by considering the following system of equations for appropriate choices of $(a,b,c)\in\mathbb{R}^3$, for all (s,ℓ) except $(s,\ell)=(0,1)$.

$$\mathrm{d}w_{s\ell}^1+\sum_{i=2}^{I}\mathrm{d}w_{s\ell}^i=a, \tag{i}$$

$$P_{01}\sum_{i=1}^{I}\mathrm{d}w_{01}-M_0\,\mathrm{d}v_0-v_0\,\mathrm{d}M_0=b, \tag{ii}$$

$$P_{s\ell}\sum_{i=1}^{I}\mathrm{d}w_{s\ell}-\tilde{\beta}_s\,\mathrm{d}(M_sv_s)=c, \tag{iii}$$

$$\mathrm{d}w_{s'\ell'}^i=0,\quad i=1,\ldots,I,\qquad (s',\ell')\neq(0,1)\ \text{or}\ (s,\ell), \tag{i$'$}$$

$$(1-v_0)\,\mathrm{d}M_0-M_0\,\mathrm{d}v_0=0, \tag{ii$'$}$$

$$\mathrm{d}M_{s'}=0,\quad s'\neq 0\ \text{or}\ s, \tag{iii$'$}$$

$$P_{01}\frac{\mathrm{d}w_{01}^1}{v_0}-P_0w_0^1\frac{\mathrm{d}v_0}{v_0^2}+P_{s\ell}\frac{\mathrm{d}w_{s\ell}^1}{v_s}-P_sw_s^1\frac{\mathrm{d}v_s}{v_s^2}=0, \tag{c$_1$}$$

$$P_{01}\frac{\mathrm{d}w_{01}^i}{v_0}-P_0w_0^i\frac{\mathrm{d}v_0}{v_0^2}=0,\quad i=2,\ldots,I, \tag{c$_i$}$$

$$P_{s\ell}\frac{\mathrm{d}w_{s\ell}^i}{v_s}-P_sw_s^i\frac{\mathrm{d}v_s}{\mathrm{d}v_s^2}=0,\quad i=2,\ldots,I. \tag{c$_i'$}$$

$(a,b,c)=(1,0,0)$ controls the market clearing equation for good (s,ℓ), provided that the agent's demands are unchanged; $(a,b,c)=(0,1,0)$ and $(0,0,1)$ control the monetary equations at date 0 and in state s at date 1. The primed equations ensure that the changes are confined to the market clearing equation (s,ℓ) and the monetary equations 0 and s. (ii′) ensures that when $v_0=1$, $\mathrm{d}v_0=0$ and when $v_0<1$, $\mathrm{d}v_{s'}=0$ for $s'\neq 0$ or s, since $M_0(1-v_0)$ is unchanged and $v_{s'}=1+(M_0/M_{s'})(1-v_0)$. The eqs. (c) are the *income compensation equations* which ensure that in controlling the eqs. (i)–(iii) we do not change the demands of the agents. If $v_0=1$ then $dw_{01}^i=dw_{s\ell}^i=0$, $i=2,\dots,I$: in this case agent 1 is used in the standard way to control the market clearing equation (s,ℓ). If $v_0<1$, then the control of eqs. (i)–(iii) induces changes in v_0 and v_s which must be compensated for by the changes (c_1), (c_i) and (c_i').

The system of linear equations (i)–(c_i') can be solved by successive substitutions, giving $d\xi=(dv_0,0,0,dw,dM)$ where

$$dv_0=\frac{1-v_0}{M_0}\left(-aP_{s\ell}-\frac{b}{v_0}+c\frac{(v_s-1)}{M_sv_s}\right),\quad dv_s=\frac{(1-v_s)}{\beta_sM_s}(aP_{s\ell}-c),$$

$$dv_{s'}=0,\quad s'=0 \text{ or } s,$$

$$d_{01}^1=-a\frac{P_{s\ell}}{P_{01}}\left(v_0+\frac{(1-v_0)P_0w_0^1}{M_0v_0}\right)$$

$$-b\frac{P_0w_0^1(1-v_0)}{P_{01}M_0v_0^2}-c\frac{(v_s-1)(1-v_0)}{P_{01}M_sv_s}\left(v_0M_0-\frac{P_0w_0^1}{M_0v_0}\right),$$

$$dw_{s\ell}^1=av_s+\frac{1-v_s}{\beta_sM_sv_s}\left(aP_sw_s^1+c\left[\left(\sum_{i=2}^{I}P_sw_s^i\right)\Big/P_{s\ell}\right]\right),$$

$$dw_{s'\ell'}^1=0,\quad (s',\ell')\neq(0,1) \text{ or } (s,\ell),$$

$$dw_{01}^i=\frac{P_0w_0^i(1-v_0)}{P_{01}M_0v_0}\left(-aP_{s\ell}-\frac{b}{v_0}+c\frac{(v_s-1)}{M_sv_s}\right),$$

$$dw_{s\ell}^i=\frac{(1-v_s)P_sw_s^i}{\beta_sM_sv_s}\left(a-\frac{c}{P_{s\ell}}\right),$$

$$dw_{s'\ell'}^i=0,\quad (s',\ell')\neq(0,1) \text{ or } (s,\ell),$$

$$dM_0=-aP_{s\ell}-\frac{b}{v_0}-c(v_s-1)\frac{M_0v_0}{M_sv_s},$$

$$\mathrm{d}M_s = \frac{1}{\beta_s}(aP_{s\ell} - c),$$

$$\mathrm{d}M_{s'} = 0, \quad s' = 0 \text{ or } s.$$

Proof of Lemma 2

Let K be a compact subset of $\Omega \times \mathcal{M}$. We need to show that $\pi_\alpha^{-1}(K)$ is compact. Let $(v_0, P, \beta, \omega, M) \in \pi_\alpha^{-1}(K)$. Recall that $v_0 \leqq 1$, $\sum_{s=1}^S \beta_s \leqq 1$. Arguments similar to those in the proof of Theorem 1 show that there exist $\bar{\varepsilon} > 0$, $0 < \alpha_0 < \alpha_1$, such that $v_0 \geqq \bar{\varepsilon}$, $\beta_s \geqq \bar{\varepsilon}$, $s = 1, \ldots, S$, and if $P = \alpha P'$ with $P' \in \Sigma'$ then $\alpha \in [\alpha_0, \alpha_1]$. It remains to show that no component of P' can converge to zero: this is achieved by the standard argument on the demand function of agent 1.

Proof of Lemma 3

The proof is a modification of Geanakoplos and Mas-Colell (1989). Let $\gamma(\beta, v)$ denote the function which associates with each vector $v \in \langle [\beta] N \rangle$ the vector $\gamma \in \mathbb{R}^J$ such that $v = [\beta] N \gamma$. For $\delta \in \mathcal{S}^{J-1}$ (the $J-1$ dimensional unit sphere) and $\xi \in \Lambda_\alpha$ ($\alpha = v_0$ or β) define

$$K(\xi, \delta) = \sum_{i=2}^{J+1} \delta_i \gamma(\beta, P_1 \,\square\, (f^i(\xi) - \underset{\sim}{w}^i)).$$

The vectors $\{P_1 \,\square\, (x_1^i - \underset{\sim}{w}_1^i),\ i = 2, \ldots, J+1\}$ are linearly dependent if and only if there exists $\delta \in \mathcal{S}^{J-1}$ such that $K(\xi, \delta) = 0$. Define $L(\xi, \delta) = (\hat{H}_\alpha(\xi), K(\xi, \delta))$ and note that the system of equations $L(\xi, \delta) = 0$ for fixed (ω, M) has more equations than unknowns. Thus if we show that 0 is a regular value of L, then repeating the standard argument gives the result. It suffices to show that equation $K_j = 0$ can be controlled without affecting the remaining equations. Pick any agent i such that $\delta^i \neq 0$. To induce the asset demand change $\mathrm{d}\gamma_j^i$, consider the endowment change $\mathrm{d}w^i$ such that

$$P_s \,\mathrm{d}\underset{\sim}{w}_s^i = -\beta_s N_s^j \,\mathrm{d}\gamma_j^i,\ P_0 \,\mathrm{d}\underset{\sim}{w}_0^i = -\textstyle\sum_{s=1}^S P_s \,\mathrm{d}\underset{\sim}{w}_s^i.$$

It is easy to check that agent i's demand is unchanged. To maintain the equality $\hat{H}_\alpha(\xi) = 0$ change the endowment of agent 1 by $\mathrm{d}w^1 = -\mathrm{d}w^i$ leaving the endowments of other agents unchanged.

Proof of Lemma 4

We show that adding the condition that S of the vectors a^i are linearly dependent to the equations of equilibrium, gives a system of equations which generically in (ω, M) has no solution. Let $\mathscr{S}^{S-1}$ denote the $S-1$ dimensional unit sphere. For $\xi \in \Lambda_\beta$, $\delta \in \mathscr{S}^{S-1}$ define $K(\xi,\delta)=\sum_{i=1}^{S}\delta_i a^i(\xi)$ where $a^i(\xi)$ is defined by (41). Let $L:\Lambda_\beta \times \mathscr{S}^{S-1} \to \mathbb{R}^{n-1} \times \mathbb{R}^{S+1} \times \mathbb{R}^S$ be defined by $L(\xi,\delta)=(\hat{H}_\beta(\xi), K(\xi,\delta))$. Note that for fixed (ω, M) the system of equations $L(\xi,\delta)=0$ has more equations than unknowns. It suffices to prove that 0 is a regular value of L. To do this we show that we can control the equations $K=0$ without affecting the equilibrium equations $\hat{H}_\beta=0$, Pick any agent i such that $\delta_i \neq 0$. We show that for any $\mathrm{d}k \in \mathbb{R}^S$, there exists $\mathrm{d}w^i \in \mathbb{R}^n$ such that

$$\delta_i(D_{w^i}a^i)\,\mathrm{d}w^i=\mathrm{d}k, \quad P \,\square\, \frac{\mathrm{d}w^i}{v}=0. \tag{A.1}$$

Letting $\mathrm{d}w^{S+1}=-\mathrm{d}w^i$ and $\mathrm{d}w^j=0$, $j\neq i$ or $S+1$ ensures that the equations of equilibrium are unchanged. Using the notation in (44), the system (A.1) can be written as

$$\frac{1}{v_0}(\mathrm{d}\alpha_0^i-\mathrm{d}\alpha_s^i)+\sum_{\sigma=1}^{S}\frac{\beta_\sigma}{v_\sigma}(\mathrm{d}\alpha_s^i-\mathrm{d}\alpha_\sigma^i)=\frac{\mathrm{d}k_s}{\delta_i}, \quad s=1,\dots,S,$$

$$M_0\,\mathrm{d}\alpha_0^i+\sum_{\sigma=1}^{S}\beta_\sigma M_\sigma\,\mathrm{d}\alpha_\sigma^i=0,$$

where $\mathrm{d}\alpha_s^i=P_s\,\mathrm{d}w_s^i/\beta_s M_s v_s$. This system can be written in matrix form $A\,\mathrm{d}\alpha=b$ with $b=\mathrm{d}k/\delta_i$ and

$$A=\begin{bmatrix} \dfrac{1}{v_0} & \sum\limits_{\sigma\neq 1}\dfrac{\beta_\sigma}{v_\sigma}-\dfrac{1}{v_0} & -\dfrac{\beta_2}{v_2} & \cdots & -\dfrac{\beta_S}{v_S} \\ \dfrac{1}{v_0} & -\dfrac{\beta_1}{v_1} & \sum\limits_{\sigma\neq 2}\dfrac{\beta_\sigma}{v_\sigma}-\dfrac{1}{v_0} & \cdots & -\dfrac{\beta_S}{v_S} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{v_0} & -\dfrac{\beta_1}{v_1} & -\dfrac{\beta_2}{v_2} & \cdots & \sum\limits_{\sigma\neq S}\dfrac{\beta_\sigma}{v_\sigma}-\dfrac{1}{v_0} \\ M_0 & \beta_1 M_1 & \beta_2 M_2 & \cdots & \beta_S M_S \end{bmatrix}.$$

The matrix A is nonsingular, since with $v_0<1$, $v_s>1$, $\sum_{s=1}^{S}\beta_s=1$,

$$\det A = \frac{1}{v_0}\left(\frac{1}{v_0} - \sum_{\sigma=1}^{S} \frac{\beta_\sigma}{v_\sigma}\right)^{S-1}\left(M_0 + \sum_{\sigma=1}^{S} \beta_\sigma M_\sigma\right) > 0.$$

Thus there exists $\mathrm{d}w^i \in \mathbb{R}^n$ satisfying (A.1) and the proof is complete.

Proof of Lemma 5

For $\xi \in \Lambda_\beta$, $\delta \in \mathscr{S}^{S-J-1}$ define $K(\xi,\delta) = \sum_{i=2}^{S-J+1} \delta_i v_1^i(\xi)$. Let $L: \Lambda_\beta \times \mathscr{S}^{S-J-1} \to \mathbb{R}^{n-1} \times \mathbb{R}^{S+1} \times \mathbb{R}^{S-J}$ be defined by $L(\xi,\delta) = (\hat{H}_\beta(\xi), K(\xi,\delta))$. We show that 0 is a regular value of L. Pick any agent i such that $\delta_i \neq 0$. To obtain the change $\mathrm{d}v^i = (0, \mathrm{d}k/\delta^i)$, without affecting the prices (P,β), we must ensure that the first-order conditions (33) remain satisfied. By a result of Aumann (1975), a preference relation representable by a utility function u^i satisfying Assumption 2 can be represented on every compact convex set $K \subset \mathbb{R}^n_{++}$ by a differentiably strictly concave function U^i i.e. $D^2_{x^i} U^i$ is negative definite). Thus there exists $\mathrm{d}x^i \in \mathbb{R}^n$ such that

$$D^2_{x^i} U^i \,\mathrm{d}x^i = \Phi^{\mathrm{T}} \mathrm{d}v^i, \quad \mathrm{d}v^i = \left(0, \frac{\mathrm{d}k}{\delta_i}\right).$$

To ensure $\mathrm{d}x^i$ is affordable we change $\mathrm{d}w^i$ such that $\Phi((1/v) \,\square\, \mathrm{d}w^i - (\mathrm{d}v/v_2) \,\square\, w^i) = \Phi\, \mathrm{d}x^i$. For this it suffices to choose $\mathrm{d}w^i$ such that

$$P \,\square \left(\frac{1}{v} \,\square\, \mathrm{d}w^i - \frac{\mathrm{d}v}{v^2}\right) = P \,\square\, \mathrm{d}x^i. \tag{A.2}$$

To ensure that the incomes of agents $j \neq i$ or 1 are unchanged, choose $\mathrm{d}w^j$ such that

$$P \,\square \left(\frac{1}{v} \,\square\, \mathrm{d}w^j - \frac{\mathrm{d}v}{v^2} \,\square\, w^j\right) = 0. \tag{A.3}$$

To assure equality of demand and supply on the goods markets, choose $\mathrm{d}w^1$ such that

$$\mathrm{d}w^1 = \mathrm{d}x^i - \sum_{j \neq 1} \mathrm{d}w^j. \tag{A.4}$$

Finally $(\mathrm{d}M, \mathrm{d}v_0)$ are chosen to ensure that the monetary equations remain satisfied

$$P_0\,\mathrm{d}x_0^i = \mathrm{d}(M_0 v_0), \qquad P_s\,\mathrm{d}x_s^i = \beta_s\,\mathrm{d}(M_s, v_s), \quad s = 1, \ldots, S. \tag{A.5}$$

The reader can check that the system of eqs. (A.2)–(A.5) has solutions and that any such solution implies

$$P\left(\frac{1}{v} \square \mathrm{d}w^1 - \frac{\mathrm{d}v}{v^2} \square w^1\right) = 0$$

so that the income and hence demand of agent 1 is unchanged. The system of equations $K = 0$ is thus controllable without affecting $\hat{H}_\beta = 0$, and the proof is complete.

Proof of Lemma 6

For $\xi \in \Lambda_\beta$ define $K(\xi) = P_0 w_0^1 / M_0 v_0 - \sum_{s=1}^S (P_s w_s^1 / M_s v_s)$ and $L: \Lambda_\beta \to \mathbb{R}^{n-1} \times \mathbb{R}^{S+1} \times \mathbb{R}$ by $L(\xi) = (\hat{H}_\beta(\xi), K(\xi))$. We show that 0 is a regular value of L by proving that the equation $K = 0$ can be controlled without affecting the equations $\hat{H}_\beta = 0$. To perturb $K = 0$ by $\mathrm{d}k \in \mathbb{R}$, we choose $\mathrm{d}w^1$ such that

$$P_0 \frac{\mathrm{d}w_0^1}{v_0} = \frac{\mathrm{d}k}{1/M_0 + \sum_{s=1}^S (\beta_s / M_s)},$$

$$P_1 \square \left(\frac{1}{v_1} \square \mathrm{d}w_1^1\right) = \frac{-\mathrm{d}k}{1/M_0 + \sum_{s=1}^S (\beta_s / M_s)} \beta.$$

$\sum_{s=1}^S \beta_s = 1 \Rightarrow P((1/v) \square \mathrm{d}w^1) = 0$ so that the demand of agent 1 is unchanged. Select any agent i and set $\mathrm{d}w^i = -\mathrm{d}w^1$, then $Q[\beta]^{-1} P_1 \square ((1/v) \square \mathrm{d}w^i) = 0$, since $P_1 \square ((1/v) \square \mathrm{d}w^i)$ is collinear to $\beta \in \langle [\beta] N \rangle$. Thus the demand of agent i is unchanged. Let $\mathrm{d}w^j = 0$, $j \neq 1$ or i, then the equilibrium is unchanged and we have shown that $K = 0$ is controllable without affecting $\hat{H}_\beta = 0$.

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