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# On the Qualitative Properties of Futures Market Equilibrium

By

**Michael J. P. Magill**, Los Angeles, California, U. S. A., and
**Manfred Nermuth**, Bielefeld, F. R. G.*

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## 1. Introduction

Broadly speaking futures markets play two roles. The first is that of sharing risk between agents, the second is that of disseminating information regarding future supply and demand conditions. The recent papers of Grossman (1977), Danthine (1978) and Bray (1981) have studied the informational role of futures markets. Our object is to study their risk sharing role and in particular how the nature of the risks influences the qualitative properties of the equilibrium.

We consider a model of futures market equilibrium similar to that analysed by Danthine (1978). The approach involves a partial equilibrium model in which an individually owned firm makes a production decision before the price of its output is known. In addition to selling his output on the spot market the producer can hedge against price risks by trading on the futures market. We assume that price fluctuations on the spot market are generated by demand fluctuations. It was observed by Danthine (1978) and Holthausen (1979) that in such a framework the production decision depends only on the futures price. Holthausen studied how the production and hedging decision depends on the relation between the futures price and the expected spot price. Our object is to extend Holthausen's framework to an equilibrium model. This is done in the simplest possible way by introducing a random de-

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mand function on the spot market and many identical speculators on the futures market. When discussing the typical speculator's demand for futures contracts we take his returns from investments on all other markets as given — these are represented by a single random variable. This approach while not entirely satisfactory enables us to show in a particularly simple way how risks in the rest of the economy impinge on this market.

In a futures market there are no natural a priori bounds that can be placed on the positions taken by traders. We do not impose any such bounds but use an argument related to a concept of asymptotic risk aversion to delimit the price region in which the agents do not wish to take unbounded futures positions and show that a (not necessarily unique) equilibrium always exists in this region (theorem 1). Our main interest lies in exploring the qualitative properties of this equilibrium. In addition to the results of Danthine and Holthausen cited above our analysis is based on two key ideas. First, the investment decisions of speculators in the futures market depend not only on the returns of this market (the distribution of spot prices and the futures price) but also on the nature of the stochastic dependence between these returns and the returns speculators get on other markets. Second, if the number of speculators becomes large, so that each of them trades only very little on the futures market under consideration, then in the limit the idiosyncratic risk is completely diversified away and only the covariance risk remains. Theorem 3 states that the futures market equilibrium converges to a unique limit when the number of speculators becomes arbitrarily large and characterises its asymptotic properties: if the spot prices on this market and the speculators' returns on other markets are positively (negatively) dependent, then the equilibrium futures price $q$ will lie below (above) the expected spot price $Ep$ — the futures price thus exhibits backwardation (contango). The economic intuition behind the result is simply that in the positively dependent case (for instance) a speculator increases the overall variability of his portfolio by taking a long position in the futures market and requires compensation for this added risk in the form of an expected profit (backwardation). In the independent case no such return is required and the futures price is unbiased. In this case producers are fully hedged and all price risk is carried costlessly by speculators. The nature of the stochastic dependence between the spot price and the returns of speculators in the rest of the economy also influences the firm's production decision through its influence on the equilibrium prices. Equilibrium output is greater than, equal to or less than what it would be in the absence of risk according as the stochastic dependence is positive, zero or negative. A correspondingly complete characterisation of equilibrium is not available when the number of speculators is finite but we do provide certain partial characterisations in lemmas 4 and 5 and theorems 2 and 4.

## 2. The Model

We consider the market for a single homogeneous good ("wheat") produced by a fixed finite number $m$ of identical competitive producers ("farmers"). Without loss of generality we put $m=1$, the case of $m>1$ requiring only trivial modifications. The *technology* is given by a cost function $c(y)$ satisfying, for all output levels, $y \geqq 0$,

*Assumption 1.* $c(0)=0,\ c'>0,\ c''>0,\ \lim_{y\to\infty} c'(y)=\infty.$

The production decision is made in "spring" and the output is harvested in "autumn". After the harvest there is a *spot market* on which the entire output is sold (the producer cannot store the output). The price on the spot market is a random variable $p=p(\omega)$, defined on a probability space $(\Omega, F, P)$. Here $\Omega$ is the set of states of the world, $F$ is a $\sigma$-field of subsets of $\Omega$, and $P$ is a probability measure on $F$. The spot price $p(\omega)$ is not revealed until autumn; in spring only the distribution of $p(\cdot)$ is known. We shall restrict attention to nonnegative prices which are genuinely uncertain in the sense that their variance is positive, and which are bounded above by some fixed constant $K>0$; i. e., consider only prices in the set

$$L=\{p:\Omega\to R \mid \operatorname{var} p>0, \text{ and } 0\leqq p(\omega)<K \text{ for all } \omega\in\Omega\}.$$

Convergence in $L$ will always be convergence in *probability*, i. e., $p^n\to p$ for $n\to\infty$ if and only if for all $\varepsilon>0$

$$P(\omega \mid |p^n(\omega)-p(\omega)|\geqq\varepsilon)\to 0 \quad \text{for } n\to\infty.$$

In addition to the spot market in autumn, there is a *futures market* held in the spring. We write $q\in\mathbb{R}$ for the price on the futures market, and $z\in\mathbb{R}$ for the amount of the good the producer sells on the futures market (if $z<0$ he purchases futures). We do not restrict $z$ in any way, i. e., allow arbitrarily large long or short positions. Given a production $y\geqq 0$ and a futures trade $z\in\mathbb{R}$, the producer's *profit* in state $\omega\in\Omega$ is

$$\pi(y,z,\omega)=p(\omega)\,y-c(y)+z\,(q-p(\omega)).$$

The first two terms on the right hand side represent the producer's profit from his production activities, and the last term is his gain (loss) from futures trading (all prices and costs are discounted to the same date).

The producer's *preferences* are represented by a strictly increasing, strictly concave von Neumann-Morgenstern utility function $u=u(\pi)$. Given prices $(p, q) \in L \times \mathbb{R}$ he chooses $(y, z) \in \mathbb{R}_+ \times \mathbb{R}$ so as to maximise his expected utility[1]

$$U(y, z)=Eu(\pi(y, z, \omega))=\int_{\Omega} u(\pi(y, z, \omega))\, dP(\omega).$$

It is easy to check that $U(y, z)$ is strictly concave. The producer's optimal *supply* decision $(y^{\mathrm{opt}}(p, q), z^{\mathrm{opt}}(p, q))$, if it exists, is therefore unique.

The total *demand* for the good *on the spot market* is exogenous and given by a random inverse demand function $\phi(y, \omega)$. This function is assumed to be (i) downward sloping and such that the demand price exceeds marginal cost at zero output and (ii) genuinely random and continuous. Formally:

*Assumption* 2. (i) *For all* $\omega \in \Omega$, $\phi(y, \omega)$ *is nonincreasing in* $y$ *and* $\phi(0, \omega)>c'(0)$. (ii) *For all* $y \geqq 0, \phi(y, \cdot) \in L$ *and* $y^n \to y$ *implies* $\phi(y^n, \cdot) \to \phi(y, \cdot)$.

The *demand for futures contracts* comes from a number of identical speculators, indexed $i=1, \ldots, s$. A typical speculator is endowed with a random variable $r=r(\omega)$ which represents his profits from his investments in other markets (taken as exogenous). We make the "limited liability" assumption

*Assumption* 3. $r(\cdot)$ *is bounded below.*

If a speculator buys $\zeta$ units of "wheat" on the futures market[2], then his total profit in state $\omega$ is

$$\Pi(\zeta, \omega)=\zeta(p(\omega)-q)+r(\omega).$$

His *preferences* are represented by a strictly increasing, strictly concave von Neumann-Morgenstern utility function $w=w(\Pi)$.

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[1] To justify this form for the firm's objective function it is best to think of the representative firm as being individually owned (a farm).

[2] For speculators we use the opposite sign convention as for producers, i. e., $\zeta>0$ means that the speculator *buys* futures.

Given prices $(p, q) \in L \times \mathbb{R}$ he chooses $\zeta \in \mathbb{R}$ so as to maximise his expected utility

$$W(\zeta) = Ew(\Pi(\zeta, \omega)).$$

Again, it is easy to check that $W(\zeta)$ is strictly concave, and hence the speculator's optimal trade $\zeta^{opt}(p, q)$, if it exists, is unique. Since all speculators are identical, their total demand for futures contracts is $s\,\zeta^{opt}(p, q)$.

A *futures market equilibrium* is now a price system $(p, q)$ and a set of production $(y)$ and futures trading $(z, \zeta)$ decisions such that all agents maximise their expected utility given the prices, and both the spot and futures market clear. Formally:

*Definition 1.* A *futures market equilibrium* is given by $(p, q, y, z, \zeta) \in L \times \mathbb{R}^4$ such that $q \geqq 0$ and (i) $y = y^{opt}(p, q)$, $z = z^{opt}(p, q)$, $\zeta = \zeta^{opt}(p, q)$; (ii) $p(\omega) = \phi(y, \omega)\ \forall\, \omega \in \Omega$; (iii) $z = s\zeta$.

## 3. Futures Market Equilibrium

Our main objective is to relate the qualitative properties of a futures market equilibrium to the underlying data of the model. To this end we begin by establishing some simple properties of the agent's optimal production and futures trading decision. These results lead to an elementary proof of the existence of a futures market equilibrium.

Note that in a futures market there are no natural a priori bounds on the positions that can be taken by traders. In a general equilibrium context it is well-known that this can lead to non-existence of equilibrium [see Hart (1975)]. In the present partial equilibrium context we admit "infinite" trades but show that the demand and supply schedules of producers and speculators always intersect for a price system at which all agents trade finite amounts.

*Theorem 1. Under assumptions 1—3 there exists a futures market equilibrium.*

The proof depends on the following three lemmas. The first observation (originally due to Danthine (1978) and Holthausen (1979)) is that the producer's optimal output $y^{opt}(p, q)$ does not depend on the spot prices $p$, but is determined simply by equating marginal cost to the futures price $q$.

*Lemma 1. At equilibrium $y > 0$ and $c'(y) = q$.*

*Proof.* (See appendix.)

The idea is that if $c'(y) \neq q$, for instance $c'(y) < q$, then the producer could increase his profit (with certainty) by raising both his output $y$ and his futures trade $z$ by the same small amount ("selling the extra output on the futures market").

By lemma 1 and definition 1, when looking for an equilibrium, we need only consider price systems of the form

$$(p_y, q_y) = (\phi(y, \cdot), c'(y)) \quad \text{for } y \geqq 0.$$

Let $z(y)$ denote the producer's optimal supply of futures, given prices $(p_y, q_y)$ and given that his output is $y$, and let $\zeta(y) = \zeta^{\text{opt}}(p_y, q_y)$ denote the demand for futures of the typical speculator. If we admit the values $\pm\infty$, these optimal trades always exist. To obtain a more precise characterisation we define the following quantities[3]

$$f(y) = \frac{B(y)}{A(y)},\ A(y) = \int_{p_y > q_y} (p_y(\omega) - q_y),\ B(y) = \int_{p_y < q_y} (q_y - p_y(\omega))$$

$$u'(\infty) = \lim_{y \to \infty} u'(y), \text{ etc., } \alpha = \frac{u'(-\infty)}{u'(\infty)} > 1,\ \beta = \frac{w'(-\infty)}{w'(\infty)} > 1 \quad (3.1)$$

*Lemma* 2.

(i) $$z(y) = \begin{cases} \infty & \text{if } \alpha \leqq f(y) \\ \text{finite} & \text{if } \frac{1}{\alpha} < f(y) < \alpha \\ -\infty & \text{if } \quad f(y) \leqq \frac{1}{\alpha} \end{cases}$$

(ii) $$\zeta(y) = \begin{cases} -\infty & \text{if } \beta \leqq f(y) \\ \text{finite} & \text{if } \frac{1}{\beta} < f(y) < \beta \\ \infty & \text{if } \quad f(y) \leqq \frac{1}{\beta} \end{cases}$$

*Moreover, $z(y)$, $\zeta(y)$ are continuous*[4] *in $y$ and when $z(y)$ (respectively $\zeta(y)$) is finite, it is the unique solution of $U_z(y, z) = 0$ (respectively $W'(\zeta) = 0$).*

*Proof.* (See appendix.)

Note that the optimal futures trades $z(y)$ and $\zeta(y)$ will always be finite if the marginal utility of wealth of both types of agents

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[3] We admit "infinite" values and use the obvious conventions, e. g., $\alpha = \infty$ whenever $u'(-\infty) = \infty$ or $u'(\infty) = 0$.

[4] The concept of continuity is extended to infinite values in the natural way by requiring $\lim_{y \to \bar{y}} z(y) = z(\bar{y})$ even if $z(\bar{y}) = \infty$.

either tends to zero for very large limits of wealth or tends to infinity for very low levels of wealth (or both)[5]. Even if this is not the case, the producers and speculators will never want to trade infinitely with each other (i. e., $z(y) = \zeta(y) = \infty$ or $-\infty$) since $\frac{1}{\alpha} < \beta$ and $\frac{1}{\beta} < \alpha$ by Eq. (3.1).

Intuitively lemma 2 can be understood as follows. $B(y)$ (resp. $A(y)$) represents the expected money gain (resp. loss) associated with the sale of one futures contract when the prices are $(p_y, q_y)$. Using the strict concavity of $u$, the expected utility gain $\Delta U$ for the producer (say) from selling an extra unit on the futures market is therefore always greater than

$$B(y)\, u'(\infty) - A(y)\, u'(-\infty)$$

and tends to this expression if his futures position $z$ goes to infinity. This implies the first line in the lemma, and the rest can be proved similarly.

One can also say that the quotient $f(y) = \frac{B(y)}{A(y)}$ is a measure of how good the "odds" are for a seller of futures contracts. If $y$ increases, then $p_y(\omega) = \phi(y, \omega)$ decreases and $q_y = c'(y)$ increases, i. e., selling of futures becomes more attractive (the odds improve). Formally, it follows easily from the definition resp. assumptions 1, 2 that $f(y)$ satisfies

*Lemma 3. $f(y)$ is continuous in $y$ and there exist numbers $0 \leqq y_1 < y_2$ such that*

$$f(y) = \begin{cases} 0 & \text{for } 0 \leqq y \leqq y_1 \\ \text{finite and strictly increasing from 0 to } \infty & \text{for } y_1 < y < y_2 \\ \infty & \text{for } y \geqq y_2 \end{cases}$$

*Proof.* (Immediate.)

The proof of theorem 1 can now be completed as follows. By lemma 3 there exist uniquely defined numbers (see Fig. 1)

$$y^c, y_u, y^u, y_w, y^w \in [y_1, y_2]$$

such that (3.2)

$$f(y^c) = 1,\ f(y_u) = \frac{1}{\alpha},\ f(y^u) = \alpha,\ f(y_w) = \frac{1}{\beta},\ f(y^w) = \beta$$

[5] In the terminology introduced in the next section, the asymptotic risk aversion of both types of traders is infinite.

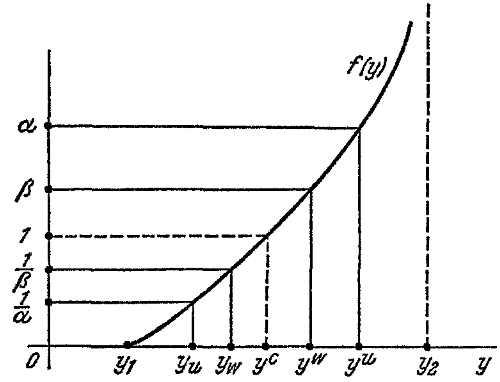


Fig. 1

By construction $y_u < y^c < y^u$, $y_w < y^c < y^w$ and of the two intervals $[y_u, y^u]$, $[y_w, y^w]$ one is contained in the other. Lemma 2 implies (see Fig. 2)

$$z(y) = \begin{cases} -\infty & \text{for } y \leqq y_u \\ \text{finite} & \text{for } y_u < y < y^u \\ \infty & \text{for } y^u \leqq y \end{cases} \qquad \zeta(y) = \begin{cases} \infty & \text{for } y \leqq y_w \\ \text{finite} & \text{for } y_w < y < y^w \\ -\infty & \text{for } y^w \leqq y \end{cases}$$

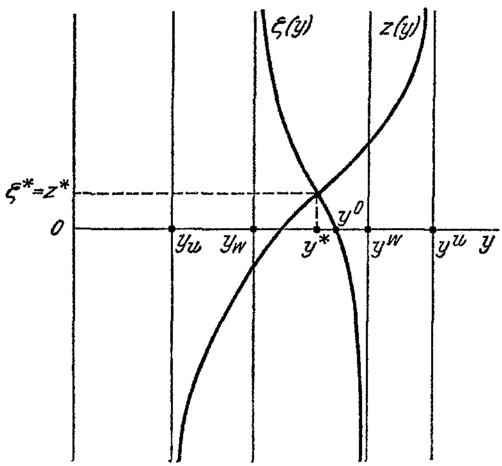


Fig. 2

(The equilibrium is drawn for $s=1$, only one speculator. The point $y^0$ is defined below, cf. Lemma 6.)

The curves $z(y)$ and $\zeta(y)$ can be shown to be continuous, and hence there must exist at least one point $y^*$ in the interior of

$[y_u, y^u] \cap [y_w, y^w]$ such that $z(y^*) = s\zeta(y^*)$. It is easily checked that $(p^*, q^*, y^*, z^*, \zeta^*)$, where

$$p^* = \phi(y^*, \cdot), \quad q^* = c'(y^*), \quad z^* = z(y^*), \quad \zeta^* = \zeta(y^*),$$

constitutes an equilibrium. □

## 4. Properties of Equilibrium

In this section we study certain qualitative properties of the futures market equilibrium $(p, q, y, z, \zeta)$, in particular the relationship between the futures price $q$ and the expected spot price $Ep$ ("backwardation" $q < Ep$ or "contango" $q > Ep$), and the extent to which the output $y$ is hedged $(z)$ in the futures market.

Define the *asymptotic risk aversion*[6] of the producer by

$$\varrho_u = \frac{u'(-\infty)}{u'(\infty)} - 1.$$

By definition $0 < \varrho_u \leqq \infty$ and $\varrho_u = 0$ if and only if the producer is risk-neutral ($u'(\pi)$ = constant). Define similarly $\varrho_w$ for the speculators.

*Lemma 4. At equilibrium*

$$\frac{|Ep - q|}{\sigma_p} < \frac{1}{2} \min\{\varrho_u, \varrho_w\}$$

*where $\sigma_p = E\,|p - Ep|$ is the mean absolute deviation of the spot price from its expected value.*

*Proof.* (See appendix.)

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[6] $\varrho_u$ can be interpreted as follows: $\varrho_u + 1$ gives the minimum odds at which the agent would be willing to accept arbitrarily large bets. To see this consider an agent who faces a lottery (bet) which pays him the amount $L > 0$ with probability $p$ and $-L$ with probability $1-p$. The expected utility of this bet is $U(L) = pu(L) + (1-p)\,u(-L)$, so that $U'(L) = pu'(L) - (1-p)\,u'(-L)$. Thus $U'(L) \geqq 0$ if and only if $\frac{p}{1-p} \geqq \frac{u'(-L)}{u'(L)}$ and hence by the concavity of $u(\cdot)$, $U'(L) \geqq 0$ for all $L > 0$ if and only if $\frac{p}{1-p} \geqq \frac{u'(-\infty)}{u'(\infty)} = \varrho_u + 1$. Thus the agent is willing to accept arbitrarily large bets ($L \to \infty$) if and only if the winning odds $\frac{p}{1-p}$ are at least $\varrho_u + 1$. Note that many standard utility functions have $\varrho_u = \infty$ in which case the agent will never accept infinite bets.

Lemma 4 shows that the asymptotic risk aversion of the less (asymptotically) risk-averse side of the market provides an upper bound for the extent to which the futures price $q$ can differ from the expected spot price $Ep$ after normalisation by the risk factor $\sigma_p$. In particular if one side of the market becomes asymptotically risk-neutral, then $q \to Ep$.

Next define for arbitrary (not necessarily equilibrium) values $(y, z, \zeta) \in \mathbb{R}_+ \times \mathbb{R}^2$ the *risk-premium* $\Delta^u(y, z)$ of the producer and the risk-premium $\Delta^w(\zeta)$ of speculators by

$$\begin{aligned} Eu(\pi(y, z, \omega)) &\equiv u(E(\pi(y, z, \omega)) - \Delta^u(y, z)) \\ Ew(\Pi(\zeta, \omega)) &\equiv w(E(\Pi(\zeta, \omega)) - \Delta^w(\zeta)). \end{aligned} \tag{4.1}$$

The following lemma says that, at equilibrium, $Ep - q$ is the *marginal risk-premium*[7] for both speculators and producers.

*Lemma 5. At equilibrium*

(i) $$\Delta_y{}^u = -\Delta_z{}^u = Ep - q = -\frac{\operatorname{cov}\{u'(\pi), p\}}{Eu'(\pi)}$$

(ii) $$\Delta_\zeta{}^w = Ep - q = -\frac{\operatorname{cov}\{w'(\Pi), p\}}{Ew'(\Pi)}$$

*Proof.* (See appendix.)

Define $\bar{y}$, the *certainty output*, as that output that would prevail at equilibrium if the market clearing price on the spot market was not random, but fixed at its expected value $\bar{\phi}(y) = E\phi(y, \cdot)$. In this case the producer would choose his output $y$ so as to equate marginal cost $c'(y)$ with the (certain) price $p$, and the market clearing price would be $p = \bar{\phi}(y)$. By assumptions 1 and 2 the equation $c'(y) = \bar{\phi}(y)$ has exactly one solution $\bar{y} > 0$. It is easy to check that $\bar{y} = y^c$ where the output[8] $y^c$ is defined in (3.2).

It turns out that there are only three possible types of equilibria — these are described in theorem 2 below. We note that the first equivalence in theorem 2 is really only a statement about the producer's optimal production-hedging decision and is true for arbitrary price profiles — not necessarily equilibrium ones, as was ob-

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[7] $\Delta_\zeta{}^w = \frac{d}{d\zeta}\Delta^w(\zeta)$, $\Delta_y{}^u = \frac{\partial}{\partial y}\Delta^u(y, z)$, etc.

[8] Note that $y^c$ is efficient in the sense that it maximises the expected surplus

$$S(y) = E\int_0^y \phi(t, \omega)\, dt - c(y).$$

served by Holthausen (1979) in his study of producer behaviour under price uncertainty.

*Theorem 2. Let $(p, q, y, z, \zeta)$ be an equilibrium, then*

$$Ep \gtreqless q \Leftrightarrow y \gtreqless z \Leftrightarrow y \lesseqgtr y^c.$$

*Thus if the futures price is below the expected spot price $(q < Ep)$, the output is only partly hedged $(z < y)$ and is less than the "certainty output" $y^c$. The converse is true when $q > Ep$.*

*Proof.* Since $u'(\pi) = u'(p(\omega)(y-z) - c(y) + zq)$ and since $u'$ is a decreasing function, the sign of $y-z$ is the same as the sign of $-\operatorname{cov}(u', p)$, which is the same as the sign of $Ep - q$, by lemma 5 (i). This proves the first equivalence. Moreover, the function

$$g(y) = \bar{\phi}(y) - c'(y) \tag{4.2}$$

is downward sloping, and $g(y^c) = 0$. If $y$ is the equilibrium output on the futures market, then $g(y) = \bar{\phi}(y) - c'(y) = Ep - q$. This implies the second equivalence. □

In general it is hard to determine which of the three cases in theorem 2 will obtain in a given market. Indeed, in view of the possible non-uniqueness of the futures equilibria, different cases may be consistent with the same underlying data. As we shall see below, however, when the number of speculators becomes large $(s \to \infty)$ this non-uniqueness vanishes and we can give sufficient conditions for the various cases in terms of the form of the stochastic dependence between the spot price on our market $(\phi)$ and the speculators' returns on other markets $(r)$.

Following Lehmann (1966) we say that a pair of random variables $\psi, \chi : \Omega \to \mathbb{R}$ are *positively* (*negatively*) *dependent* if for all $(\alpha, \beta) \in \mathbb{R}^2$:

$$P\{\psi \leqq \alpha, \chi \leqq \beta\} \geqq (\leqq)\, P\{\psi \leqq \alpha\}\, P\{\chi \leqq \beta\} \tag{4.3}$$

with strict inequality for some $(\alpha, \beta)$. It is readily shown that if $(\psi, \chi)$ are positively dependent (negatively dependent), then $(\psi, f(\chi))$ are negatively dependent (positively dependent) if $f$ is a decreasing function. Also if $(\psi, \chi)$ are positively dependent (negatively dependent), then $\operatorname{cov}(\psi, \chi) > 0$ $(<0)$. Of course $(\psi, \chi)$ are independent if and only if there is equality in (4.3) for all $(\alpha, \beta) \in \mathbb{R}^2$.

In our model we say that $(\phi, r)$ are positively dependent (etc.) if and only if $(\phi(y, \cdot), r(\cdot))$ are positively dependent (etc.) for all $y \geqq 0$. We need the following lemma.

*Lemma 6. There exists a unique output* $y^0 > 0$ *such that*

$$\zeta^{\text{opt}}(\phi(y^0, \cdot), c'(y^0)) = 0.$$

*Thus the speculator's demand curve* $\zeta(y)$ *intersects the y-axis in exactly one point* $y^0$ (see Fig. 2).

*Proof.* (See appendix.)

Now let $e^s = (p^s, q^s, y^s, z^s, \zeta^s)$ denote an equilibrium on the futures market when there are $s \geqq 1$ speculators (all endowed with the same $r(\omega)$, which is assumed not to depend on $s$. Such equilibria exist for all $s$, by theorem 1. To exclude some degenerate limits of little economic interest, the following assumption will be useful.

*Assumption 4.* (i) $z(y_w) < 0 < z(y^w)$, (ii) $\zeta(y_u) > 0 > \zeta(y^u)$.

Assumption 4 is not restrictive. It says simply that if the prices are so favorable for a seller of futures contracts that one side of the market is already willing to sell an infinite amount, then the other side will also want to sell futures (possibly only in finite amount), rather than buy. A sufficient condition for assumption 4 (i) is that $\varrho_u \leqq \varrho_w$, i. e., the producer is asymptotically less risk-averse, and a sufficient condition for assumption 4 (ii) is that $\varrho_w < \varrho_u$, i. e., the speculators are asymptotically less risk-averse. Of course assumption 4 is always satisfied in the special case where both types of traders have infinite asymptotic risk aversion. In remark 1 below we indicate briefly the limiting behaviour of the equilibrium when assumption 4 is violated. Our main result is the following.

*Theorem 3. Let assumptions 1, 2, 3, 4 (ii) be satisfied and let* $(e^s)$, $s = 1, 2, \ldots$, *be a sequence of futures market equilibria. Then for* $s \to \infty$, *this sequence converges to the unique limit*

$$e = (p, q, y, z, \zeta) = (\phi(y^0, \cdot), c'(y^0), y^0, z^{\text{opt}}(\phi(y^0, \cdot), c'(y^0)), 0)$$

*where* $y^0$ *is as in lemma 6. Moreover,*

(i) $Ep = q$, $z = y = y^c$ iff. $\text{cov}[w'(r), p] = 0$,

(ii) $Ep > q$, $z < y < y^c$ iff. $\text{cov}[w'(r), p] < 0$,

(iii) $Ep < q$, $y^c < y < z$ iff. $\text{cov}[w'(r), p] > 0$.

A sufficient condition for Case (i) [resp. (ii), resp. (iii)] is that $(\phi, r)$ are independent [resp. positively dependent, resp. negatively dependent].

*Proof*. From the proof of theorem 1, for each $s$, an equilibrium $e^s = (p^s, q^s, y^s, z^s, \zeta^s)$ is given by the intersection $y^s$ of the curves $s\zeta(y)$ and $z(y)$, as shown in Fig. 3. More precisely, $p^s = \phi(y^s, \cdot)$, $q^s = c'(y^s)$, $z^s = z^{\mathrm{opt}}(p^s, q^s)$, $\zeta^s = \frac{1}{s} z^s$. By lemma 6 each curve $s\zeta(y)$ intersects the $y$-axis only once at $y^0$. Assumption 4 (ii) implies that

$$y_u < y^0 < y^u.$$

Therefore $z(y)$ remains bounded in a neighborhood of $y^0$. As $s \to \infty$ the curves $s\zeta(y)$ approach the vertical line through $y^0$, so that $y^s \to y^0$. This proves the first part of the theorem.

To verify the rest, note that for $s \to \infty$: $\zeta^s \to 0$, hence $\Pi(\zeta^s, \omega) = \zeta^s(p(\omega) - q) + r(\omega) \to r(\omega)$, hence (by lemma 5)

$$Ep^s - q^s \to -\frac{\operatorname{cov}\{w'(r(\omega)), p(\omega)\}}{E w'(r(\omega))}$$

The function $w'$ is always positive and decreasing, hence $\operatorname{cov}\{w'(r), p\} > 0$ ($< 0$) if $(p, r)$ are negatively (positively) dependent. The desired result now follows from theorem 2. □

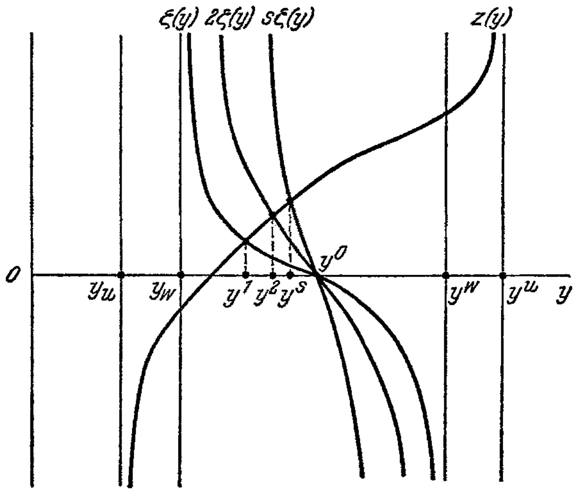


Fig. 3

Intuitively, when the number of speculators is large, market clearing implies that each one of them trades only little on the futures market. The "idiosyncratic risk" (due to the spot price variability per se) is then diversified away (it vanishes to the second order), and the speculators are concerned only with the "covariance risk" stemming from the correlation between the spot price $p$ and their other returns $r$. If this correlation is positive, then buying futures increases the variability of the speculator's total portfolio. To compensate for this he requires a positive expected profit from buying futures, i. e., backwardation ($q < Ep$). Conversely if the correlation is negative, then buying futures decreases the variability of the speculator's total portfolio. The speculator is prepared to pay for this risk reduction by accepting an expected loss ($q > Ep$). When the spot price and other returns $r$ are independent, then the futures price is *unbiased*. In this case the spot market price risk is carried by the speculators at no charge. The bias in the futures price $q$ is thus independent of the agents' attitudes towards risk.

If a speculator adds a small investment $\zeta$ to his portfolio $r$, his utility in state $\omega$ changes by (up to a first-order approximation)

$$w(\zeta(p(\omega)-q)+r(\omega))-w(r(\omega))=\zeta(p(\omega)-q)\,w'(r(\omega))$$

and his expected utility changes by

$$\zeta Ep[p(\omega)-q]\,w'(r(\omega))=\zeta[\operatorname{cov}\{p, w'(r)\}+(Ep-q)\,Ew'(r)].$$

The term in square brackets is the expected utility gain from a small unit investment, which must be zero at equilibrium. For example if $(p, r)$ are positively dependent, then $\operatorname{cov}\{p, w'(r)\}<0$, and $Ep-q$ must be positive.

In a different context the results of theorem 3 for futures markets may be compared to the Arrow-Lind theorem (see Arrow-Lind (1970)) and its generalisation (see Magill (1984)) for the valuation of public goods. The independent case (theorem 3 (i)) corresponds to the original Arrow-Lind result by which a risky public project spread amongst many individuals is valued at its expected value.

*Remark 1:* Theorem 3 implies, that the futures equilibrium for large numbers of speculators is *unique*, even though for small numbers of speculators there may be more than one futures equilibrium (cf. also Th. 4 below). The reason for this is twofold: first, in the limit, each individual speculator's optimal trade $\zeta$ must go to zero; second, by Lemma 6, there is only *one* output level $y=y^0$ at which the associated prices $\varphi(y)$, $c'(y)$ induce the speculator not to trade in futures.

*Remark 2:* To see what happens if Assumption 4 (ii) is violated imagine for example that $\zeta(y^u)>0$. Then the speculators are so eager to buy (rather then sell) futures contracts — due presumably to very strong negative correlation between their other income $r(\omega)$ and the price $p(\omega)$ — that they are willing to do so even at terms sufficiently unfavorable to them so as to induce the other side (the producer) to supply an infinite amount of such contracts, namely at the prices $(p^u, q^u) = (\phi(y^u, \cdot), c'(y^u))$. Of course $Ep^u < q^u$. In such an implausible case, as the number of speculators increases, prices would converge to $(p^u, q^u)$, output to $y^u$ and the producer's supply of futures contracts would go to infinity. Assumption 4 (ii) excludes such a possibility.

As a final exercise let us ask what happens when the number of speculators, $s$, becomes very small. Noting that nothing in the preceding analysis depended on $s$ being an integer, we now treat it as a positive real number and let $s \to 0$. This means simply that at all prices $(p, q)$ the aggregate demand of speculators for futures contracts, $s\zeta^{opt}(p, q)$ will become smaller and smaller (provided it is finite). Intuitively, the speculators will then "disappear" from the market and one would expect the futures market equilibrium to converge to a pure spot-market equilibrium as defined below.

*Definition 2.* $(p, y) \in L \times \mathbb{R}_+$ is a *spot-market equilibrium* (without futures trading) if (i) $Eu(p(\omega)y - c(y)) = \max$, given $p$, (ii) $p(\omega) = \phi(y, \omega)\ \forall\ \omega \in \Omega$.

Thus (i) the producer chooses his output $y \geqq 0$ so as to maximise his expected utility, given the prices $p(\cdot)$ and (ii) the spot market clears.

*Lemma 7. Under assumptions 1 and 2 (i) a spot market equilibrium $(p, y)$ exists (ii) $0 < y < y^c$ for any spot market equilibrium.*

The spot-market equilibrium need not be unique; the set of all spot-market equilibrium outputs is denoted by

$$Y = \{y \geqq 0 \mid (p, y) \text{ is a spot-market equilibrium for some } p \in L\}$$

*Lemma 8.* $y \in Y \Leftrightarrow z(y) = 0.$

Thus the spot market equilibrium outputs $y$ are characterised by the property that at the associated prices $(p_y, q_y) = (\phi(y, \cdot), c'(y))$ the producer would not want to trade on the futures market. We make the "generic" assumption

*Assumption 5.* The curve $z(y)$ intersects the horizontal axis transversally at all $y \in Y$.

By theorem 1 for any number of speculators $s > 0$ at least one futures equilibrium $e^s(p^s, q^s, y^s, z^s, \zeta^s)$ exists. Denote the set of all equilibria by $E^s$, and the corresponding set of equilibrium outputs by

$$Y^s = \text{proj}_3\, E^s \quad (s>0).$$

*Theorem 4. Let assumptions 1, 2, 3, 4 (ii), 5 be satisfied and let* $Y^s$ *be the set of futures equilibrium outputs for* $s>0$. *Then*

$$\lim_{s \to 0} Y^s = Y \cap (y_w, y^w).$$

*Proof.* By lemma 8 the set $Y$ of spot market equilibria is given by the intersection points of the $z(y)$-curve with the horizontal axis. By assumption 5 these intersection points are discretely spaced, and by assumption 4 (i) $Y \cap (y_w, y^w) \neq 0$, $y_w, y^w \notin Y$. From the proof of theorem 1 for all $s>0$, the set $Y^s$ of futures equilibria is given by the intersection points between the curves $z(y)$ and $s\zeta(y)$. As $s \to 0$ the curve $s\zeta(y)$ approaches the horizontal axis in the interval $(y_w, y^w)$ (remaining "infinite" elsewhere). It is geometrically obvious that the set $Y^s$ converges to $Y \cap (y_w, y^w)$, cf. Fig. 4. □

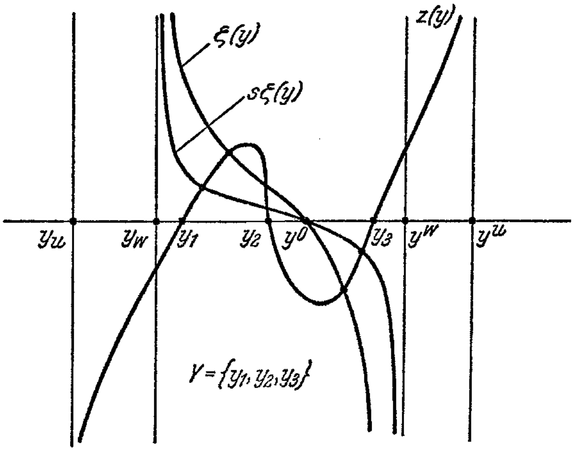


Fig. 4

## 5. Appendix

*Proof of Lemma 1.*

By assumption 2 (i) for $y=0$, the spot price is higher than the marginal production cost with certainty, hence $y=0$ cannot be an equilibrium. Let $y>0$ be an equilibrium output and define

$$f(\varepsilon) = U(y+\varepsilon, z+\varepsilon).$$

For $\varepsilon$ sufficiently small this is well-defined; moreover $f'(0)=0$. We have

$$f'(\varepsilon)=Eu'(\cdot)\,(p(\omega)-c'(y+\varepsilon)+q-p(\omega))$$
$$\Rightarrow f'(0)=Eu'(\cdot)\,(q-c'(y))=0.$$

Since $u'>0$, this implies $q=c'(y)$, as asserted. □

*Proof of Lemma 2.*

We prove first (ii). Let $y>0$, $(p,q)=(p_y,q_y)=(\phi(y,\cdot),\ c'(y))$, $A=\int_{p>q}(p(\omega)-q)$, $B=\int_{p<q}(q-p(\omega))$, $f(y)=\frac{B}{A}$, and denote by $\zeta(y)$ the speculator's optimal trade given $(p,q)=(p_y,q_y)$. Clearly if $A=0$ then $\zeta(y)=-\infty$ and if $B=0$, then $\zeta(y)=\infty$. Assume therefore $A>0, B>0$ and consider $W'(\zeta)=Ew'[\zeta(p(\omega)-q)+r(\omega)]\,(p(\omega)-q)$. Since $W'(\zeta)$ is continuous and strictly decreasing in $\zeta$, it suffices to show that

$$W'(\zeta)\to -Bw'(-\infty)+Aw'(\infty)\quad \text{for } \zeta\to\infty, \tag{A.1}$$
$$W'(\zeta)\to -Bw'(\infty)+Aw'(-\infty)\quad \text{for } \zeta\to-\infty. \tag{A.2}$$

To prove (A.1) write

$$W'(\zeta)=-B_1(\zeta)+A_1(\zeta)$$

where

$$B_1(\zeta)=\int_{p<q} w'[\zeta(p(\omega)-q)+r(\omega)]\,(q-p(\omega)),$$
$$A_1(\zeta)=\int_{p>q} w'[\zeta(p(\omega)-q)+r(\omega)]\,(p(\omega)-q).$$

Using the strict concavity of $w$ we have, for all $\zeta\in\mathbb{R}$, $\omega\in\Omega$,

$$w'(-\infty)>w'[\zeta(p(\omega)-q)+r(\omega)]>w'(\infty)\geqq 0.$$

Therefore $B_1(\zeta)<w'(-\infty)\,B$. Moreover,

$$B_1'(\zeta)=-\int_{p<q} w''(\cdot)\,(q-p(\omega))^2>0.$$

Now, using the boundedness assumptions on $p$, $r$, it is not difficult to see that

$$B_1(\zeta)\to w'(-\infty)\,B\quad \text{for } \zeta\to\infty$$

(even when $w'(-\infty)=\infty$. Similarly,

$$A_1(\zeta)\to w'(\infty)\,A\quad \text{for } \zeta\to\infty.$$

This proves (A.1), (A.2) can be proved similarly, which establishes assertion (ii). An analogous argument proves (i). The remaining assertions are easy to check; for example

$$z(y) \to \infty \quad \text{for } f(y) \to \alpha, \text{ etc.} \qquad \square$$

*Proof of Lemma 4.*

Let $(p, q, y, z, \zeta)$ be an equilibrium and define $A = A(y)$, $B = B(y)$ as in Lemma 2. By definition

$$\left.\begin{array}{l} a = A - B = Ep - q \\ b = A + B = E|p-q| \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2A = a + b \\ 2B = b - a \end{array}\right\} \Rightarrow \frac{B}{A} = \frac{b-a}{b+a} \tag{A.3}$$

By lemma 2

$$\max\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) < \frac{B}{A} < \min(\alpha, \beta).$$

Putting $\zeta = \min(\varrho_u, \varrho_w)$ and noting that $\varrho_u = \alpha - 1$, $\varrho_w = \beta - 1$, we can rewrite this as

$$\frac{1}{\varrho+1} < \frac{B}{A} < \varrho + 1$$

or using (A.3)

$$b + a < (\varrho+1)(b-a) < (\varrho+1)^2 (b+a).$$

This implies

$$\varrho b > (\varrho+2) a > -\varrho b \text{ or } |a| < \frac{\varrho}{\varrho+2} b.$$

From this the assertion of the lemma follows easily, using the triangle inequality

$$b = E|p-q| < E|p - Ep| + |Ep - q| = \sigma_p + |a|. \qquad \square$$

*Proof of Lemma 5.*

We prove only (ii), the proof of (i) being analogous. The first-order condition for utility maximisation for a speculator can be written

$$0 = W'(\zeta) = E w'(\Pi)(p(\omega) - q) = E w'(\Pi)(p(\omega) -$$
$$- Ep + Ep - q) = \operatorname{cov}(w', p) + (Ep - q) E w'.$$

This implies the second equality in (ii); for the first use (4.1) to write the first-order condition as follows

$$0 = w'(E\Pi - \Delta^w)(Ep - q - \Delta_\zeta^w).$$

Since $w' > 0$ by assumption, this implies $Ep - q = \Delta_\zeta^w$. $\square$

*Proof of Lemma 6.*

For any $y \in (y_w, y^w)$, $\zeta = \zeta(y) = \zeta^{\text{opt}}(\phi(y, \cdot), c'(y))$ is the unique solution of the equation

$$f(\zeta, y) = Ew'[\zeta(\phi(y, \omega) - c'(y)) + r(\omega)](\phi(y, \omega) - c'(y)) = 0.$$

We have

$$f_\zeta(\zeta, y) = Ew''(\cdot)(\phi - c')^2 < 0,$$
$$f_y(\zeta, y) = Ew''(\cdot)\zeta(\phi' - c'')(\phi - c') + w'(\cdot)(\phi' - c'').$$

In particular for $\zeta = 0$, $f_y(0, y) = Ew'(\cdot)(\phi' - c'') < 0$ by assumptions 1, 2. Therefore whenever $\zeta = \zeta(y) = 0$,

$$\frac{d\zeta}{dy} = -\frac{f_y}{f_\zeta} < 0, \text{ i. e.}$$

the curve $\zeta = \zeta(y)$ intersects the horizontal axis always from above, hence in exactly one point $y^0$, $y_w < y^0 < y^w$ (cf. lemma 2). □

*Proof of Lemma 7.*

(i) For $y \geqq 0$ define

$$h(y) = Eu'(\phi(y, \omega)y - c(y))(\phi(y, \omega) - c'(y)).$$

By assumptions 1, 2, $h(0) > 0$ and $h(y) < 0$ for $y$ sufficiently large. Therefore there exists $y > 0$ such that $h(y) = 0$. With $p = \phi(y, \cdot)$ this is a spot market equilibrium.

(ii) Let $(p, y)$ be a spot market equilibrium. Then

$$0 = h(y) < Eu'(py - c)(p - c') = \text{cov}\{u', p\} + (Ep - c')Eu'. \quad (A.5)$$

Recalling the function $g(y) = \bar{\phi}(y) - c'(y)$ defined in (4.2) we also have

$$Ep - c' = g(y).$$

By (A.5) this implies $g(y) = -\frac{\text{cov}\{u', p\}}{Eu'} > 0$ since $u'$ is downward sloping. Since by definition $g(y^c) = 0$ and $g$ is also downward sloping, we must have $y < y^c$. □

*Proof of Lemma 8.*

The producer's expected utility is (cf. (2.3))

$$U(y, z) = Eu[p(\omega)y - c(y) + z(q - p(\omega))]$$

with partial derivatives

$$U_y = Eu'(\cdot)\,(p(\omega) - c'(y)), \quad U_z = Eu'(\cdot)\,(q - p(\omega)).$$

Therefore $U_y = -U_z$ for $q = c'(y)$. Now if $y \in Y$, we must have

$$U_y(y, 0) = 0 \quad \text{where } p = \phi(y, \cdot).$$

With $q = c'(y)$ this implies $U_z(y, 0) = 0$, i. e., $z(y) = 0$. Conversely, if $z(y) = 0$, then $U_z(y, 0) = 0$, where $(p, q) = (\phi(y, \cdot), c'(y))$. Again this implies $U_y(y, 0) = 0$, i. e., $y \in Y$. □

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Addresses of authors: Professor Michael J. P. Magill, Department of Economics, University of Southern California, Los Angeles, CA 90089-0152, U. S. A.; and Professor Manfred Nermuth, Faculty of Economics, University of Bielefeld, P. O. Box 86 40, D-4800 Bielefeld 1, Federal Republic of Germany.