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# Price Relations on Futures Markets for Storable Commodities

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## 1. INTRODUCTION

This paper presents a simple theoretical model of the spot and futures markets for a storable commodity. We focus our attention in particular on the classical futures markets, those for harvested storable commodities. For these commodities which include grains such as wheat, corn, and soybeans, while there is active trading on the spot and futures markets at each instant, the output of the production process does not occur continuously in time but appears rather at the end of each crop year (production period) at harvest time. For this class of commodities we present a simple partial equilibrium model for which the predicted relations between the spot and futures prices are broadly consistent with those familiar from empirical data.

The analysis is divided into two parts. We first introduce spot markets and establish the existence of a spot market equilibrium (Sections 3–6). We then introduce futures markets and by an arbitrage argument determine equilibrium futures prices (Section 6). This subdivision of the problem of determining a simultaneous spot and futures market equilibrium is clearly artificial in a more general analysis but is the appropriate first approximation under the assumptions of this paper.

Section 3 sets up the model of intertemporal equilibrium on the spot market. A single representative firm produces, sells, and stores the commodity. An important component in the description of the firm's profit maximising activity is summarised in a *stock-out cost function.* This function

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measures the expected costs incurred by the firm when it fails to meet demand on certain related markets, the demand on these markets being subject to continual random fluctuations. These expected costs are a decreasing function of the level of working stocks held (Assumption 1). The activity of consumers at each instant is summarised in an aggregate demand function (Assumption 3). A spot market equilibrium is then a pair consisting of a spot price and a current rate of sale of inventory at each instant $t \in [0, \infty)$ such that the firm maximises (expected) discounted profit and the spot market is cleared at each instant $t \in [0, \infty)$ (Definition 1).

The problem of establishing the existence of a spot market equilibrium is reduced to the problem of showing that a certain maximum problem ($\mathscr{P}$) has a solution which can be supported by a system of prices. This problem is in turn decomposed into two subproblems: first, a sequence of problems of allocating a given amount of inventory within a crop year ($\mathscr{Q}_t$, $t = 0, 1,...$) and second, the problem of allocating the inventory across the crop years, the so-called carry-over problem ($\mathscr{R}$). Sections 4 and 5 establish the existence of a solution and a system of prices for these two subproblems. The analysis of these two sections draws extensively on some recent results of Rockafellar [14, 15] and Magill [12] on convex variational problems as well as more traditional results from the theory of differential equations (differentiable dependence of a solution on initial conditions [3]) and difference equations (the stable manifold theorem [7]).

In Section 6 after establishing a preliminary result (Proposition 5), we use the results of Sections 4 and 5 to establish the existence of a spot market equilibrium (Corollary). We then introduce futures markets and the basic assumption concerning the pricing of futures contracts[1]: trading on the futures market leads the price of each futures contract into equality with the (expected) future spot price at the time of its maturity (Assumption 4). Our objective, in the language of futures markets, is to determine the behaviour of *basis*, that is, the spread between spot and futures prices. In Proposition 6 we show how basis depends on the level of initial inventory: if there is deficient (surplus) inventory at date zero, then the futures market has an inverted (carrying charge) structure. We also show that if initial inventory is at its average long-run level, then basis exhibits a regular seasonal cycle (see Figs. 2 and 3). The profile of spot–futures price spreads reflects two basic arbitrage conditions: those arising from profit opportunities connecting adjoining instants within a crop year and those that arise from profit opportunities connecting adjoining crop years. The stock-dependent structure of the spot–futures price spreads predicted by our equilibrium model conforms

[1] The problem of bias in futures prices is discussed in Section 2. This is a long standing controversy for which the reader is also referred to the papers of Cootner, Telser, Gray, Houthakker, and Rockwell [13, pp. 41–100, 155–189].

well with the empirical evidence [19, Chaps. III, IX; 20; 21][2], provided the spot–futures price spreads over an interval of time are interpreted as the average spreads over many such intervals, each interval starting with the same level of inventory. These results are a natural generalisation to an explicit model of intertemporal equilibrium of the earlier results of Kaldor [8] and Working [22, pp. 3–31]. They reveal, within the admittedly idealised framework of this model, the role that futures prices play in inducing a (Pareto) optimal allocation of inventory over time.

## 2. Extensions and Earlier Work

An early attempt to develop a theory of price formation on futures markets was presented by Keynes [9][3]. He viewed futures markets as insurance markets in which speculators insure hedgers against the risk of price changes. As compensation for this role, hedgers pay speculators a risk premium. The price of a futures contract maturing in a given future period lies below the expected spot price of that period by the amount of this risk premium. His theory of *normal backwardation* then asserts that under normal conditions, under which today's spot price coincides with the expected spot price of the future period, the futures price will lie below today's spot price by the amount of the risk premium.

A wealth of empirical data regarding the structure of spot and futures prices on grain markets was published in the *Report of the Federal Trade Commission on the Grain Trade* [19]. This was followed by the careful and extensive investigations of Working [20, 21]. This analysis revealed that in the case of harvested storable commodities, the spread between spot and futures prices varies in a systematic way with the level of existing inventories of the commodity, the spread being negative (positive) when the stocks are relatively abundant (scarce). It is easy to see that a theory of price formation based solely on the risk premium paid to speculators leads to results that contradict these empirical findings if, as Keynes assumed, the risk premium is an increasing function of the level of inventories.

In his *Treatise on Money* [10, pp. 127–129] Keynes realised this difficulty and introduced at least implicitly the idea that inventories offer producers a convenience yield. It remained for Kaldor [8] to clarify these ideas and to develop a theory of price formation based on the risk premium and the convenience yield of inventories. Various modifications and extensions of this theory were subsequently studied by Working [22], Brennan [2], Telser [18], and Samuelson [17]. These papers were largely confined to an intuitive

[2] See also Section 2.

[3] This is the theory of futures markets presented by Hicks [6, pp. 137–139].

analysis of the spot-futures price spread implied by a single period equilibrium on the spot market.

The earlier theories fail to distinguish between the forces that determine futures prices within a crop year (*old* crop futures) and those that determine futures prices in a subsequent crop year (*new* crop futures). At any given time both types of futures contract are traded and the distinction is not unimportant, since the Keynesian theory of normal backwardation must in general refer to new crop futures, while the traditional theory [19, Chap. IX] that emphasises the regular seasonal cycle in the spot-futures price spread (as an average over years) refers to old crop futures. In our framework the forces that lead to the pricing of old and new crop futures fall out in a simple and natural way.

Empirical evidence does not support Keynes' contention [9; 10, pp. 127–129] that futures prices for harvested storable commodities exhibit in an appreciable degree the property of normal backwardation. It is clear from empirical evidence that the convenience yield is of much greater importance in determining price relations on these commodity markets [22, pp. 9–10]. It is thus appropriate in the first approximation developed in this paper to omit consideration of the risk premium in analysing the spot-futures price spreads.

It will be clear from the analysis of Sections 3–6 that the inventory model of this paper does much to explain the basic relations between spot and futures prices. To proceed beyond this first approximation we need to introduce uncertainty explicitly. There are three basic directions in which extensions can be sought: random harvests, random demand, and information.

If the harvest sequence of Section 3 is replaced by a random harvest sequence, then the spot market equilibrium becomes a random process; it may then be reasonable to use the Keynesian hypothesis allowing futures prices to be equal to the expected spot price less a risk premium. If the harvest sequence has a stationary probability distribution, then it may be possible to obtain an analogue of Proposition 6. This would amount to a bona fide intertemporal model of the Kaldor–Keynes theory [8].

Of more basic theoretical importance is the need to obtain an explicit model of the process that generates the stock-out cost function (convenience yield). One approach would be to introduce random demand and to model more precisely the process by which stocks are transformed into saleable output: holding stocks somewhat in excess of the average annual requirement enables the producer to meet random surges in demand.

If speculators carry risk for hedgers (the representative firm in the present paper), speculators are also typically equipped with advance information about future random events (harvests, demand). Grossman [5], Danthine [4], and Bray [1] have analysed the role of speculators in disseminating advance information through the futures price to uninformed agents (hedgers) in simple two-period models. Their equilibrium, when it exists, is a bona fide joint equilibrium on the spot and futures markets. This is perhaps the most important direction in which the analysis of this paper needs to be generalised.[4]

## 3. Spot Market Equilibrium

We consider the problem of generating a spot market equilibrium for a harvested storable commodity like wheat. There is a certain supply and demand activity at each instant for both flow and stock amounts of the commodity. We view the trading activity as taking place in continuous time. However, at certain discretely spaced intervals of time (the end of each crop year) a harvest is produced. Since the harvest arrives but once each year, the sequence of harvests needs to be allocated not only within each crop year, but also across the crop years.

We model the activity of producers by the activity of a single representative firm. This firm produces, stores, and sells the commodity. For simplicity the production is costless: the output arrives as a *harvest* at the end of each crop year. The firm can store the commodity and in so doing incurs interest charges arising from the foregone interest on the funds tied up in inventory. The firm can also sell flow amounts of the commodity at each instant to consumers, thereby depleting its inventory. In addition the firm is involved in production activities for certain related markets on which demand is subject to continual random fluctuations. To meet these fluctuations the firm needs to keep on hand working stocks of the commodity. Since the failure to meet demand on these markets incurs costs (the lost revenue) and since the probability of such a failure decreases as the level of working stocks is increased, we assume that the (expected) *stock-out* costs incurred in this way are a decreasing function of the level of working stocks. For simplicity we do not distinguish between these working stocks and the stocks held as inventory for direct subsequent sale to consumers.

This description of production, storage, and sales activity of the firm thus combines within a single firm activities which, in the case of a commodity like wheat, would be undertaken by separate agents, namely farmers (producers), millers (holders of working stocks), and elevator operators (storers and sellers of inventory). The firm seeks to allocate its production, storage, and sales activity over time so as to maximise its (expected) discounted profits.

The activity of consumers is modelled in a highly simplified way. Their

[4] The Grossman–Danthine–Bray informational arguments cannot be used to explain the observed spot-futures price spreads that are, however, explained by the model of this paper.

aggregate demand for flow amounts of the commodity at each instant depends only on the current spot price and is a strictly decreasing function of this price.

Let $\Psi(\chi)$ denote the (expected) stock-out costs at any instant when the firm holds an amount of inventory $\chi \geqq 0$. Let $\mathscr{C}^2(D)$ denote the collection of real-valued functions defined on $D \subset R^n$, $n \geqq 1$, which are twice continuously differentiable on $D$ and let $\Psi'(\chi) = (d/d\chi)\, \Psi(\chi)$.

ASSUMPTION 1 (Properties of stock-out cost function).

(i) $\Psi(\chi)$: $[0, \infty) \to [0, \infty]$, $\Psi \in \mathscr{C}^2(0, \infty)$

(ii) *there exists* $\hat{s} \in (0, \infty)$ *such that*

$$\Psi(\chi) > 0, \quad \Psi'(\chi) < 0, \quad \Psi''(\chi) > 0, \quad \chi \in (0, \hat{s}), \quad \Psi(\chi) \equiv 0, \quad \chi \in [\hat{s}, \infty)$$

(iii) *there exists* $\tilde{s} \in (0, \hat{s})$, $\alpha \geqq 1$ *such that*

$$\chi^{-\alpha} \leqq \Psi(\chi) < \infty, \qquad \chi \in (0, \tilde{s}).$$

(ii) implies that stock-outs are not expected when stocks are at or above the level $\hat{s}$. (iii) implies that the firm must hold some positive level of working stocks if it is to stay in business: this condition is used to ensure that the steady state level of carryover is positive.

Let $h_t$ denote the output (harvest) produced by the firm during the production period $[t-1, t]$ (the $(t-1)$th crop year): the output $h_t$ appears at the instant $t = 0, 1, 2,\ldots$. Let $x(\tau)$ denote the level of stocks (inventory) held by the firm at time $\tau \in [0, \infty)$ and let $\dot{x}(\tau) = (d/d\tau)(x(\tau))$, $\tau \in [0, \infty)$. The sale of inventory $-\dot{x}(\tau)$ at (almost every) instant $\tau \in [0, \infty)$ in conjunction with the sequence of outputs $(h_0, h_1,\ldots)$ and the initial carryover $(\sigma_0 > 0)$ determine the path of the stocks $x(\tau)$, $\tau \in [0, \infty)$. Thus if we let

$$x(t^+) = \lim_{\tau \downarrow t} x(\tau), \qquad x(t^-) = \lim_{\tau \uparrow t} x(\tau)$$

denote the right- (left-) hand limits of $x(\tau)$ at time $t$ for $t = 0, 1,\ldots$, then

$$x(\tau) = x(t^+) + \int_t^\tau \dot{x}(v)\, dv, \qquad \tau \in (t, t+1) \tag{1}$$

$$x(t^+) = x(t^-) + h_t, \qquad t = 0, 1,\ldots. \tag{2}$$

Thus $x(t^-)$ denotes the *carryover* of stocks from the production period $[t-1, t]$ into the period $[t, t+1]$, with $x(0^-) = \sigma_0$, while $x(t^+)$ denotes the initial inventory at the beginning of period $[t, t+1]$. The points of discontinuity of $x(\tau)$, $\tau \in [0, \infty)$ are thus the instants at the end of each production period where $h_t$ arrives.

To avoid awkward problems of existence we assume that there is a maximal physical rate $m > 0$ (independent of $\tau$) at which the firm can unload and hence sell its inventory at each instant $\tau \in [0, \infty)$. Since we do not wish $m$ to play a crucial economic role in the analysis that follows, we take $m$ to be large relative to $\hat{s} + h$.

ASSUMPTION 2. (i) $h_t = h > 0$, $t = 0, 1, \ldots$. (ii) $\hat{s} + h \ll m$.

Let $\mathscr{M}$ denote the space of real-valued Lebesgue measurable functions defined on $[0, \infty)$. We are interested in the two subsets of $\mathscr{M}$ defined by

$$\mathscr{L}_\infty[0, \infty) = \{v \in \mathscr{M} \mid \operatorname*{ess\,sup}_{\tau \in [0,\infty)} |v(\tau)| < \infty\},$$

$$\mathscr{L}_1[0, \infty) = \left\{ q \in \mathscr{M} \;\middle|\; \int_0^\infty |q(\tau)| \, d\tau < \infty \right\}.$$

Let $\mathscr{F}$ denote the set of feasible paths for the firm's disposition (sale) of inventory. We take $\mathscr{F}$ to be a subset of $\mathscr{L}_\infty[0, \infty)$,

$$\mathscr{F} = \left\{ \dot{x} \in \mathscr{L}_\infty[0, \infty) \;\middle|\; \begin{array}{l} (\dot{x}(t), x(t)) \in [-m, 0] \times [0, \infty) \text{ a.e.} \\ (1) \text{ and } (2) \text{ hold, } x(0^-) = \sigma_0 \end{array} \right\}.$$

Let $q(\tau)$ denote the price at date zero for one unit of the commodity deliverable at time $\tau \in [0, \infty)$. If $q \in \mathscr{L}_1[0, \infty)$, then the firm's (expected) discounted profit is well defined for each $\dot{x} \in \mathscr{F}$ and is given by[5]

$$\pi(\dot{x}) = \sum_{t=0}^{\infty} \int_t^{t+1} (q(\tau)(-\dot{x}(\tau)) - \Psi(x(\tau)) \, e^{-\delta\tau}) \, d\tau,$$

where $\delta > 0$ denotes the instantaneous interest rate.

It only remains to specify the aggregate demand behaviour of consumers. Let $\phi(c(\tau))$ denote the *spot price* that consumers are prepared to pay at time $\tau$ for the aggregate flow amount $c(\tau)$ deliverable at time $\tau \in [0, \infty)$.

ASSUMPTION 3 (Properties of demand function).

(i) $\phi\colon (0, m] \to (0, \infty)$

(ii) $\phi(\xi) \to \infty$ *as* $\xi \to 0$

(iii) $\phi \in \mathscr{C}^1(0, m]$, $\phi'(\xi) < 0$, $\xi \in (0, m]$

(iv) $-\infty < \Phi(\xi) = \int_m^\xi \phi(s) \, ds$, $\xi \in (0, m]$.

[5] A proportional storage cost can also be included, but is omitted here to simplify the analysis.

DEFINITION 1. Let $q^*(\tau) = p^*(\tau)\, e^{-\delta\tau}$ a.e. A *spot market equilibrium* is a pair $(q^*, -\dot{x}^*) \in \mathscr{L}_1[0, \infty) \times \mathscr{L}_\infty[0, \infty)$ satisfying

(i) $\dot{x}^* \in \mathscr{F}$ and $\pi(\dot{x}^*) \geqq \pi(\dot{x})\ \forall \dot{x} \in \mathscr{F}$

(ii) $$-\dot{x}^*(\tau) = \begin{cases} \phi^{-1}(p^*(\tau)) & \text{if } p^*(\tau) \geqq \phi(m) \\ m & \text{if } p^*(\tau) < \phi(m) \end{cases} \quad \text{a.e.}$$

(i) is the standard profit maximisation condition. (ii) requires that the spot market clear at almost every instant $\tau \in [0, \infty)$ with the proviso that whenever the spot price falls below $\phi(m)$ the amount traded be equal to $m$, the maximum flow amount that can be delivered at each instant.

Our objective is not only to establish the existence of a spot market equilibrium, but also to show that this equilibrium has an important convergence property. In Section 6 we introduce futures prices and show that this convergence property leads to certain basic relations between spot and futures prices which conform closely with the known empirical relations between spot and futures prices on markets for harvested storable commodities.

The existence of a spot market equilibrium will be established by showing that the problem

$$\sup_{\dot{x} \in \mathscr{F}} \sum_{t=0}^{\infty} \int_t^{t+1} (\Phi(-\dot{x}(\tau)) - \Psi(x(\tau)))\, e^{-\delta\tau}\, d\tau \qquad (\mathscr{P})$$

has a solution and that this solution can be supported by (is value maximised by) a system of prices. As it stands the problem $(\mathscr{P})$ is rather complex. The analysis is made much simpler if it is decomposed as follows: Let

$$x(t^-) = s_t, \qquad x(t^+) = s_t + h, \qquad t = 0, 1, \ldots,$$

so that $\{s_t\}_{t=0}^{\infty}$ denotes the sequence of carryovers of inventory from period $t$ to period $t+1$ and $\{s_t + h, s_{t+1}\}_{t=0}^{\infty}$ denotes the sequence of initial and terminal inventory levels for the $t$th crop year for $t = 0, 1, \ldots$. Introduce the feasible sets

$$\mathscr{S} = \{(s_0, s_1, \ldots) \mid s_0 = \sigma_0, 0 \leqslant s_t, 0 \leqslant s_t + h - s_{t+1} \leqslant m, t = 0, 1, \ldots\} \quad (3)$$

and for $\alpha \in (0, m)$, $t = 0, 1, \ldots$,

$$\mathscr{F}_t(\alpha) = \left\{ v \in \mathscr{L}_\infty[t, t+1] \mid v(\tau) \in [-m, 0] \text{ a.e.}, \int_t^{t+1} |v(\tau)|\, d\tau = \alpha \right\}. \quad (3')$$

For a function $y(\cdot)$ defined on $[0, \infty)$ let $y_t(\cdot)$ denote the restriction of $y(\cdot)$ to the interval $[t, t+1]$

$$y_t(\tau) = y(\tau), \qquad \tau \in [t, t+1], \quad t = 0, 1, \ldots.$$

With this notation establishing the existence of a solution and a system of prices for $(\mathscr{P})$ is equivalent to establishing the existence of a solution and a system of prices for the *carryover problem*

$$\sup_{(s_0, s_1, \ldots) \in \mathscr{S}} \sum_{t=0}^{\infty} \beta^t v(s_t + h, s_{t+1}) \qquad (\mathscr{R})$$

(where $\beta = e^{-\delta}$) and the sequence of *crop year problems*

$$v(s_t + h, s_{t+1}) = \sup_{\dot{x}_t \in \mathscr{I}_t} \int_t^{t+1} (\Phi(-\dot{x}_t(\tau)) - \Psi(x_t(\tau)))\, e^{-\delta(\tau - t)}\, d\tau, \qquad t = 0, 1, \ldots, \quad (\mathscr{Q}_t)$$

where $\mathscr{I}_t = \mathscr{F}_t(s_t + h - s_{t+1})$, $x_t(\tau) = s_t + h + \int_t^\tau \dot{x}_t(v)\, dv$, $\tau \in (t, t+1)$. In $(\mathscr{R})$ the function $v(\cdot, \cdot)$ is defined by $(\mathscr{Q}_t)$, and in particular by $(\mathscr{Q}_0)$ since it is independent of time. In $(\mathscr{Q}_t)$ the endpoints $(s_t + h, s_{t+1})$ are obtained from the solution of $(\mathscr{R})$.

In the next section we establish the existence of a solution and a system of prices for each crop year problem $(\mathscr{Q}_t)$, and in Section 5 a similar result is established for the carryover problem $(\mathscr{R})$. In the latter case the result is obtained by establishing an important convergence property for the solution of the problem $(\mathscr{R})$.

## 4. Crop Year Problem

In this section we establish the existence of a solution and a system of prices for the typical crop year problem $(\mathscr{Q}_t)$. It suffices to consider the problem $(\mathscr{Q}_0)$ on the interval $[0, 1]$. To further simplify notation we let $(s_0 + h, s_1) = (x_0, x_1)$.

Definition 2. We say that the pair of inventory endpoints $(x_0, x_1)$ is *attainable* if $(x_0, x_1) \in \mathscr{N} = \{(x_0, x_1) \mid 0 < x_1, 0 < x_0 - x_1 < m\}$.

This requires that the crop year begin and end with a positive level of inventory and that a positive amount be allocated to consumption within the period, this amount being less than the maximal amount $m$ that can physically be delivered over a unit interval of time.

Proposition 1. *If Assumptions* 1 *and* 3 *are satisfied and the pair of inventory endpoints* $(x_0, x_1)$ *is attainable, then the crop year problem* $(\mathscr{Q}_0)$ *has a unique solution.*

*Proof.* We embed the problem $(\mathscr{Q}_0)$ into the framework of Magill [12].

Let $F(\dot{x}) = \int_0^1 (\Phi(-\dot{x}(t)) - \Psi(x_0 + \int_0^t \dot{x}(\tau)\,d\tau))\,e^{-\delta t}\,dt$ and recall the definition of $\mathscr{F}_0(x_0 - x_1)$ in (3′). There exist constants $-\infty < \underline{a} < \bar{a} < \infty$ such that

$$\Phi(-\dot{x}(t)) - \Psi\left(x_0 + \int_0^t \dot{x}(\tau)\,d\tau\right) \leqq \bar{a} \qquad \text{a.e.} \quad \forall \dot{x} \in \mathscr{F}_0(x_0 - x_1),$$

$$\underline{a} \leqq \Phi(-\underline{\dot{x}}(t)) - \Psi\left(x_0 + \int_0^t \underline{\dot{x}}(\tau)\,d\tau\right) \qquad \text{a.e.} \quad \text{for some} \quad \underline{\dot{x}} \in \mathscr{F}_0(x_0 - x_1).$$

Since it follows from Assumptions 1 and 3 that $-\Psi(\chi) \leqq 0$, $\chi \in [0, \infty)$, $\Phi(\xi) \leqq 0$, $\xi \in [0, m]$, we may set $\bar{a} = 0$. Consider the path $-\underline{\dot{x}}(t) = x_0 - x_1 > 0$, $t \in [0, 1]$. Since $\underline{x}(t) \geqq x_1 > 0$, $t \in [0, 1]$ by Assumptions 1(iii) and 3(iv), $\Psi(x_1) < \infty$, $-\infty < \Phi(x_0 - x_1)$ so that we may set $\underline{a} = \Phi(x_0 - x_1) - \Psi(x_1)$. By [12, Proposition 7.1(i)] with $I = [0, 1]$ it follows that $F(\dot{x})$ is upper semicontinuous on $\mathscr{F}_0(x_0 - x_1)$ in the $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$ topology. It follows from [12, Proposition 7.5(i)] that $\mathscr{F}_0(x_0 - x_1)$ is $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$ compact. Thus a solution to $(\mathscr{Q}_0)$ exists. By a standard argument the strict concavity of the integrand $\Phi(\cdot) - \Psi(\cdot)$ implies that the solution is unique. ∎

PROPOSITION 2. *Let Assumptions* 1 *and* 3 *be satisfied, and let the pair of inventory endpoints* $(x_0, x_1)$ *be attainable. If* $\dot{x}^* \in \mathscr{F}_0(x_0 - x_1)$ *is the optimal solution to* $(\mathscr{Q}_0)$, *then there exists an absolutely continuous function* $p^*(\cdot)$ *on* $[0, 1]$ *such that*

$$\begin{aligned} \dot{p}^*(t) &= \delta p^*(t) + \Psi'(x^*(t)) && \text{a.e.} && \text{(i)} \\ \dot{x}^*(t) &= -\phi^{-1}(p^*(t)) \quad \text{if } p^*(t) \geqq \phi(m) && \text{a.e.} && \\ &= -m \quad \text{if } p^*(t) < \phi(m) && \text{a.e.} && \text{(ii)} \end{aligned} \qquad (\mathscr{H})$$

*Proof.* Since $\dot{x}^*(t) \leqq 0$ a.e., $x^*(t) \geqq x_1$, $t \in [0, 1]$. Thus we may, without altering the optimal solution, redefine $\Psi(\cdot)$ as

$$\begin{aligned} \Psi_0(\chi) &= \Psi(\chi) && \text{if} \quad \chi \geqq x_1 \\ &= \Psi(x_1) + \Psi'(x_1)(\chi - x_1) && \text{if} \quad \chi < x_1. \end{aligned}$$

Let $\Delta_m(\cdot)$ and $f(\cdot)$ be defined by

$$\begin{aligned} \Delta_m(\xi) &= 0 && \text{if} \quad \xi \in [-m, 0], \\ &= -\infty && \text{if} \quad \xi \notin [-m, 0], \end{aligned} \qquad f(\chi, \xi, t) = (\Phi(-\xi) + \Delta_m(\xi) - \Psi_0(\chi))\,e^{-\delta t}$$

and let $B = [0, x_0] \times [x_1, \infty)$; then the problem $(\mathscr{Q}_0)$ is equivalent to finding an absolutely continuous function $x(\cdot)$ on $[0, 1]$ such that $(x(0), x(1)) \in B$ and $\int_0^1 f(x(t), \dot{x}(t), t)\,dt$ is maximised. This is equivalent to [16, p. 3, Problem (1.1)]. We note that since the pair of endpoints $(x_0, x_1)$ is attainable, $\text{int}(\mathcal{N} \cap B) \neq \emptyset$. Furthermore, since $-\infty < \Psi_0(x)$, $\forall x \in R$, condition $D_0$ of Rockafellar [16, p. 5] is satisfied. It follows from [16, p. 7, Corollary 1] and [14, p. 208, Eq. (9.3)] that there exists an absolutely continuous function $q^*(\cdot)$ on [0, 1] such that

$$\begin{aligned} -(\dot{q}^*(t), q^*(t)) &\in \partial f(x^*(t), \dot{x}^*(t); t) \\ &= (-\Psi_0'(x^*(t)), -\Phi'(-\dot{x}^*(t)) + \partial\Delta_m(\dot{x}^*(t))) e^{-\delta t} \qquad \text{a.e.} \end{aligned} \tag{4}$$

where $\partial f(z^*; t)$ denotes the superdifferential[6] of $f(\cdot\,; t)$ at $z^*$. Let $q^*(t) = p^*(t) e^{-\delta t}$ a.e.; then $\dot{q}^*(t) = (\dot{p}^*(t) - \delta p^*(t)) e^{-\delta t}$ a.e. It then follows from the definition of a supergradient that (4) is equivalent to

$$\begin{aligned} &p^*(t) \dot{x}^*(t) + (\dot{p}^*(t) - \delta p^*(t)) x^*(t) + \Phi(-\dot{x}^*(t)) - \Psi_0(x^*(t)) \\ &\quad \geqq p^*(t)\xi + (\dot{p}^*(t) - \delta p^*(t))\chi \\ &\qquad + \Phi(-\xi) - \Psi_0(\chi) \qquad \forall (\chi, \xi) \in R \times [-m, 0] \quad \text{a.e.} \end{aligned} \tag{5}$$

Since $q^*$ is absolutely continuous on [0, 1], $p^*(t) = q^*(t) e^{\delta t} < \infty$, $t \in [0, 1]$. Since by Assumption 3(ii) $\Phi'(\xi) \to \infty$ as $\xi \to 0$, it follows from (5) that $-\dot{x}^*(t) > 0$ a.e. But then we may replace the region $R \times [-m, 0]$ in (5) by $R \times [-m, \infty)$. In view of the concavity of $\Phi(\cdot) - \Psi(\cdot)$ and the fact that $x^*(t) \geqq x_1$, $t \in [0, 1]$, (5) is then equivalent to the conditions

$$\begin{aligned} p^*(t) - \Phi'(-\dot{x}^*(t)) &= 0 \qquad \text{if} \quad p^*(t) \geqq \Phi(m) \quad \text{a.e.} \\ \dot{x}^*(t) &= -m \qquad \text{if} \quad p^*(t) < \Phi'(m) \quad \text{a.e.} \\ \dot{p}^*(t) - \delta p^*(t) - \Psi'(x^*(t)) &= 0 \qquad \text{a.e.} \end{aligned}$$

which is precisely $(\mathcal{H})$. ∎

Propositions 1 and 2 allow us to deduce the following result which is basic for the analysis of Section 5.

LEMMA 1. *If Assumptions* 1 *and* 3 *are satisfied, then the function* $v(\cdot, \cdot)$ *defined by* $(\mathcal{Q}_0)$ *is strictly concave and* $v(\cdot, \cdot) \in \mathscr{C}^1(\mathcal{N})$.[7]

[6] Let $f(z; t)$: $R^n \times R \to \bar{R}$; then the superdifferential of $f(\cdot\,; t)$ at $z^* \in R^n$ is defined as the subset of the dual space $R^n$

$$\partial f(z^*; t) = \{r \in R^n \mid f(z; t) - f(z^*; t) \leqq r(z - z^*), \forall z \in R^n\}$$

$r \in \partial f(z^*; t)$ is called a *supergradient* of $f(\cdot\,; t)$ at $z^*$. When $f(\cdot\,; t)$ is differentiable at $z^*$, $\partial f(z^*; t)$ consists of a single element, the gradient of $f(\cdot\,; t)$ at $z^*$. See [15] for a full discussion.

[7] If $x_0 - x_1 = m$ and $x_1 > \hat{s}$, since $\Psi'(x^*(t)) \equiv 0$, $t \in [0, 1]$, the price paths $p(t) = p_0 e^{\delta t}$, $t \in [0, 1]$, for $p_0 \in (0, \phi(m) e^{-\delta})$ and $\dot{x}^*(t) = -m$, $t \in [0, 1]$, satisfy $(\mathcal{H})$. In this case, $v(\cdot, \cdot)$ is not differentiable.

*Proof.* It is straightforward to show that $v(\cdot,\cdot)$ is strictly concave. By [16, p. 35, Theorem 4(b)[8]], to show that $v(\cdot,\cdot)$ is differentiable on $\mathscr{N}$, it suffices to show that if $(x_0, x_1) \in \mathscr{N}$ and if $x^*$ denotes the optimal solution of $(\mathscr{Q}_0)$ satisfying $(x^*(0), x^*(1)) = (x_0, x_1)$, then there is at most one absolutely continuous function $p^*(\cdot)$ on $[0, 1]$ such that $(p^*, x^*)$ satisfies $(\mathscr{H})$. Since $(x_0, x_1) \in \mathscr{N}$ implies $x_0 - x_1 < m$, there is a subset $A \subset [0, 1]$ such that $-\dot{x}^*(t) < m$ and hence $p^*(t) = \phi(-\dot{x}^*(t))$ for $t \in A$. Since $-\dot{x}^*(t) > 0$, $t \in [0, 1]$, it follows from the properties of $\Psi'(\cdot)$ that $A$ must be an interval $[\underline{\tau}, \bar{\tau}]$ for some $\underline{\tau} < \bar{\tau}$. Since $p^*$ is continuous, $\dot{x}^*$ is continuous on $[\underline{\tau}, \bar{\tau}]$. Since $\dot{x}^*$ is unique, any absolutely continuous function $p$ satisfying $(\mathscr{H})$ must coincide with $p^*$ on the interval $[\underline{\tau}, \bar{\tau}]$. Since there is at most one solution of $\dot{p}(t) = \delta p(t) + \Psi'(x^*(t))$ on the interval $[0, \underline{\tau}]$ satisfying $p(\underline{\tau}) = p^*(\underline{\tau})$ and on the interval $[\bar{\tau}, 1]$ satisfying $p(\bar{\tau}) = p^*(\bar{\tau})$, $p(t) = p^*(t)$, $t \in [0, 1]$. But then $(p(0), p(1)) = (p^*(0), p^*(1))$ so that $v(\cdot,\cdot)$ is differentiable at $(x_0, x_1)$. Since $v(\cdot,\cdot)$ is concave and differentiable on an open convex set $\mathscr{N}$, it follows by a standard result [15, p. 246, Corollary 25.5.1] that $v(\cdot,\cdot)$ is continuously differentiable on $\mathscr{N}$. ∎

## 5. Carryover Problem

Consider the carryover problem $(\mathscr{R})$ stated at the end of Section 3. The object of this section is to establish the existence of a sequence $(s_0, s_1, \ldots) \in \mathscr{S}$ satisfying the first order conditions

$$v_2(s_t + h, s_{t+1}) + \beta v_1(s_{t+1} + h, s_{t+2}) = 0, \qquad t = 0, 1, \ldots \qquad (\mathscr{E})$$

the transversality condition

$$\beta^t v_1(s_t + h, s_{t+1})(s_t + h) \to 0 \quad \text{as} \quad t \to \infty \qquad (T)$$

and the positivity and interiority conditions

$$0 < s_t, \qquad 0 < s_t + h - s_{t+1} < m, \qquad 0 < v_1(s_t + h, s_{t+1}), \qquad t = 0, 1, \ldots \quad (6)$$

provided a certain condition is imposed on the initial carryover $\sigma_0$. Our method consists of establishing the existence of a solution to $(\mathscr{E})$, $(T)$, and (6) which also has an important convergence property.

Definition 3. A sequence $s_t = \sigma > 0$, $t = 0, 1, \ldots$ satisfying $(\mathscr{E})$ and $v_1(\sigma + h, \sigma) > 0$ will be called a *steady state sequence* for the carryover problem $(\mathscr{R})$.

[8] See also the remarks in [16, Sec. 5, p. 36].

PROPOSITION 3. *If Assumptions* 1–3 *are satisfied, then there exists a unique steady state sequence for the carryover problem* ($\mathscr{R}$)

$$s_t = \sigma, \qquad t = 0, 1, \ldots, \quad \sigma < \hat{s}$$

*Proof.* The attainable set $\mathscr{N}$ induces the following set of attainable pairs of initial inventory and consumption:

$$D = \{(y, c) \mid (y, y - c) \in \mathscr{N}\}$$

By Lemma 1 the function $u(\cdot, \cdot)$ defined by

$$u(y, c) = v(y, y - c), \qquad (y, c) \in D \tag{7}$$

is continuously differentiable on $D$. Thus the function

$$r(s) = u_1(s + h, h) - \mu u_2(s + h, h), \qquad s \in (0, \infty), \quad \beta = 1/(1 + \mu)$$

is continuous on $(0, \infty)$. Our object is to show that there exist $0 < \underset{\sim}{s} < \hat{s}$ such that $r(\underset{\sim}{s}) > 0 > r(\hat{s})$. By the intermediate value theorem, there then exists $\underset{\sim}{s} < \sigma < \hat{s}$ such that $r(\sigma) = 0$. In view of (7), $\sigma$ satisfies ($\mathscr{E}$).

Let $(x_0, x_1) \in \mathscr{N}$ with $\hat{s} \leqq x_1$; then the solution $x^*(\cdot)$ of ($\mathscr{Q}_0$) satisfies $x^*(t) \geqq \hat{s}$, $t \in [0, 1]$ so that $\Psi(x^*(t)) = 0$, $t \in [0, 1]$ by Assumption 1(ii). Thus $u_1(\hat{s} + h, h) = 0$. If we let

$$w(h) = u(\hat{s} + h, h) = \sup_{\dot{x} \in \mathscr{F}_0(h)} \int_0^1 \Phi(-\dot{x}(t))\, e^{-\delta t}\, dt$$

then $w'(h) > 0$ since by Assumption 3(i) $\Phi'(\xi) = \phi(\xi) > 0$, $\xi \in (0, m]$, and $0 < h < m$ by Assumption 2. Thus $w'(h) = u_1(\hat{s} + h, h) + u_2(\hat{s} + h, h) = u_2(\hat{s} + h, h) > 0$ so that $r(\hat{s}) = -\rho u_2(\hat{s} + h, h) < 0$.

Consider the crop year problem ($\mathscr{Q}_0$) with $(x_0, x_1) = (c, 0)$, $0 < c < m$. Since $-\dot{x}(t) \leqq m$ a.e. for any $\dot{x} \in \mathscr{F}_0(c)$, there exists $\theta > 0$ such that on the terminal segment, $x(t) \leqq m(1 - t)$, $t \in [1 - \theta, 1]$. By Assumption 1(iii) since $\alpha \geqq 1$,

$$\lim_{\varepsilon \to 0} \int_{1-\theta}^{1-\varepsilon} \Psi(x(t))\, e^{-\delta t}\, dt \geqq \lim_{\varepsilon \to 0} \int_{1-\theta}^{1-\varepsilon} (m(1 - t))^{-\alpha}\, dt = \infty$$

Since $\Phi(\xi) \leqq 0$, $\xi \in [0, m]$, $u(c, c) = -\infty$. Thus $\lim_{c \uparrow y} u(y, c) = \lim_{y \downarrow c} u(y, c) = -\infty$. Let $f(y) = u(y, h)$, $y \in (h, \infty)$. For any $\hat{y} > h$, $f(y) = f(\hat{y}) + \int_{\hat{y}}^{y} f'(s)\, ds$, $y \in (h, \infty)$, where $-\infty < f(\hat{y}) \leqq 0$. Since $f(\cdot)$ is concave, $f'(\cdot)$ is monotone, and since $f(y) \to -\infty$ as $y \downarrow h$, there exists $h < \underset{\sim}{y} < \hat{y}$ such that

$$f'(y) > 0, \qquad y \in (h, \underset{\sim}{y}) \tag{8}$$

Let $g(c) = u(y, c)$, $c \in (0, y)$. For any $\hat{c} \in (0, y)$, $g(c) = g(\hat{c}) + \int_{\hat{c}}^{c} g'(s)\, ds$, $c \in (0, y)$, where $-\infty < g(\hat{c}) \leqq 0$. Since $g(\cdot)$ is concave, $g'(\cdot)$ is monotone and since $g(c) \to -\infty$ as $c \uparrow y$, there exists $\underset{\sim}{c} \in (0, y)$ such that $g'(c) < 0$, $c \in [\underset{\sim}{c}, y]$. In particlar if $y = h$, there exists $\underset{\sim}{c} \in (0, h)$ such that $u_2(h, \underset{\sim}{c}) < 0$. Since $u_2(\cdot, \underset{\sim}{c})$ is continuous, there exists $\bar{s} > 0$ such that $u_2(h + s, \underset{\sim}{c}) < 0$, $s \in [0, \bar{s}]$. But then $u_2(h + s, c) < 0$, $c \in [\underset{\sim}{c}, h + s)$, $s \in [0, \bar{s}]$. In view of (8) there exists $\underline{s} > 0$ such that $f'(h + \underline{s}) = u_1(h + \underline{s}, h) > 0$, $u_2(h + \underline{s}, h) < 0$, so that $r(\underline{s}) > 0$. Thus there exists $\underline{s} < \sigma < \hat{s}$ such that $r(\sigma) = 0$. Since $u_1(s + h, h) > 0$, $s \in (0, \hat{s})$, $v_1(\sigma + h, \sigma) = ((1 + \mu)/\mu)\, u_1(\sigma + h, h) > 0$. Uniqueness of the steady state sequence follows from Lemmas 2 and 3 below which imply that $r'(\sigma) = u_{11}(\sigma + h, h) - \mu u_{21}(\sigma + h, h) < 0$ for any steady state sequence $s_t = \sigma$. ∎

For $\alpha > 0$, $z \in R^n$, let $N_\alpha(z) = \{\xi \in R^n \mid \|\xi - z\| < \alpha\}$ and let

$$v_{ij}(x_0, x_1) = \frac{\partial^2}{\partial x_i\, \partial x_j} v(x_0, x_1), \qquad i, j = 1, 2.$$

The following two lemmas are crucial to the analysis of this section.

LEMMA 2. *Let Assumptions* 1–3 *be satisfied, and let* $v(\cdot)$ *denote the function defined by* $(\mathscr{Q}_0)$; *then there exists* $\gamma > 0$ *such that* $v \in \mathscr{C}^2(N_\gamma(\sigma + h, \sigma))$. *Furthermore,*

$$v_{12}(\sigma + h, \sigma) = v_{21}(\sigma + h, \sigma) > 0 \tag{9}$$

*Proof.* We make use of the following result.[9] If $(\bar{x}_0, \bar{x}_1) \in \mathscr{N}$, then $(\bar{p}_0, -\beta \bar{p}_1) = \nabla v(\bar{x}_0, \bar{x}_1)$ if and only if there exists a solution $(p, x)$ of $(\mathscr{H})$ satisfying $(p_0, x_0) = (\bar{p}_0, \bar{x}_0)$ and $(p_1, x_1) = (\bar{p}_1, \bar{x}_1)$. Since $(\sigma + h, \sigma) \in \mathscr{N}$, by Propositions 1 and 2 there exists a solution $(p^*, x^*)$ of $(\mathscr{H})$ with $(x_0^*, x_1^*) = (\sigma + h, \sigma)$. Let

$$\rho = v_1(\sigma + h, \sigma)$$

then $p_0^* = \rho = -(1/\beta)\, v_2(\sigma + h, \sigma) = p_1^*$. The solution $(p^*, x^*)$ is shown as the curve $E'E$ in Fig. 1. The proof will be based on the following result [3, p. 22, Theorem 7.1; p. 25, Theorem 7.2]: Let $H\colon U \subset R^n \to R^n$, $H \in \mathscr{C}^1(U)$, $U$ an open connected set. If $z^*(t) \in U$, $t \in [a, b]$, is a known solution of the differential equation

$$\dot{z}(t) = H(z(t)), \qquad t \in [a, b] \tag{10}$$

then there exists $\alpha > 0$ such that for all $z_\tau \in N_\alpha(z^*(\tau))$ for some $\tau \in (a, b)$

[9] See Rockafellar [14, p. 209, Theorem 5; p. 212, Theorem 6; 16, p. 35, Theorem 4].

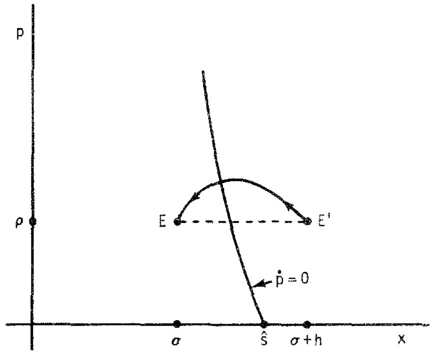


FIGURE 1

there is a unique solution $z(t, z_\tau)$ of (10) satisfying $z(\tau, z_\tau) = z_\tau$; furthermore $z(t, \cdot) \in \mathscr{C}^1(N_\alpha(z^*(\tau)))$, $t \in (a, b)$. Let $n = 2$, $z = (p, x)$, $U = (0, \infty) \times (0, \infty)$, and let $H = (H^1, H^2)$ denote the right-hand side of $(\mathscr{H})$. Then by Assumptions 1(i) and 3(iii), $H \in \mathscr{C}^1(U)$. Let $z^* = (p^*, x^*)$ denote the known solution of $(\mathscr{H})$. It is easy to see that this solution can be continued to an interval $[a, b] = [-\varepsilon, 1 + \varepsilon]$ containing the interval $[0, 1]$, for some $\varepsilon > 0$. This allows us to set $\tau = 1$, $t = 0$ and to conclude that there exist $\alpha > 0$ and a function $F(\cdot)$ such that

$$x_0 - F(p_1, x_1) = 0, \qquad (p_1, x_1) \in N_\alpha(\rho, \sigma) \tag{11}$$

where $F \in \mathscr{C}^1(N_\alpha(\rho, \sigma))$. If $F_1(\rho, \sigma) = 0$, then if we let the terminal price be increased $p_1 \to \rho + \Delta p_1$, while $x_1$ remains unchanged at $x_1 = \sigma$, then we must have $\Delta x_0 = 0$. Since the only point $(\pi, \chi)$ satisfying $H^1(\pi, \chi) = H^2(\pi, \chi) = 0$ is $(\pi, \chi) = (\infty, 0)$ and since $(\infty, 0) \notin U$, the solution curves of $(\mathscr{H})$ do not intersect in the domain $U$. Since the new solution curve $(p', x')$ cannot intersect the curve $(p^*, x^*)$ and since $\Delta p_1 > 0$, the curve $(p', x')$ must lie entirely above the curve $E'E$ in Fig. 1. But if $(p', x')$ lies above the solution curve $(p^*, x^*)$, then at each stock level spot prices are higher, implying by Assumption 3(iii) that consumption and hence the rate of depletion is smaller, contradicting the assumption that the total depletion of stock on the two paths is the same ($\Delta x_0 = 0$, since $x_1' = x_1 = \sigma$). Thus $\Delta p_1 > 0$ must imply $\Delta x_0 < 0$ so that

$$F_1(\rho, \sigma) < 0 \tag{12}$$

Equation (11) thus satisfies the conditions of the implicit function theorem [11, p. 357, Theorem 6] and we can assert the existence of $\alpha_1 > 0$ and a function $f(\cdot)$ such that

$$p_1 = f(x_0, x_1), \qquad (x_0, x_1) \in N_{\alpha_1}(\sigma + h, \sigma)$$

where $f \in \mathscr{C}^1(N_{\alpha_1}(\sigma + h, \sigma))$. But then $-v_2(x_0, x_1) = f(x_0, x_1) \in \mathscr{C}^1(N_{\alpha_1}(\sigma + h, \sigma))$. Furthermore by (12)

$$v_{21}(\sigma + h, \sigma) = -f_1(\sigma + h, \sigma) = 1/F_1(\rho, \sigma) > 0 \tag{13}$$

Similarly if we set $\tau = 0$, $t = 1$, we may conclude from the cited theorem that there exists $\beta > 0$ and a function $G(\cdot)$ such that

$$x_1 - G(p_0, x_0) = 0, \qquad (p_0, x_0) \in N_\beta(\rho, \sigma + h) \tag{14}$$

where $G \in \mathscr{C}^1(N_\beta(\rho, \sigma + h))$. An argument similar to that given above shows that if the initial price is increased $p_0 \to \rho + \Delta p_0$, while $x_0$ remains unchanged at $x_0 = \sigma + h$, then we must have $\Delta x_1 > 0$ so that

$$G_1(\rho, \sigma + h) > 0 \tag{15}$$

Equation (14) thus satisfies the conditions of the implicit function theorem and we can assert the existence of $\beta_1 > 0$ and a function $g(\cdot)$ such that

$$p_0 = g(x_0, x_1), \qquad (x_0, x_1) \in N_{\beta_1}(\sigma + h, \sigma)$$

where $v_1(x_0, x_1) = g(x_0, x_1) \in \mathscr{C}^1(N_{\beta_1}(\sigma + h, \sigma))$. Let $\gamma = \min(\alpha_1, \beta_1)$; then $v \in \mathscr{C}^2(N_\gamma(\sigma + h, \sigma))$. The conditions for the theorem on the interchange of order of differentiation [11, p. 318, Theorem 7] are thus satisfied. Applying this theorem and using (13) leads to (9), which completes the proof. ∎

LEMMA 3. *Under the assumptions of Lemma* 2,

(i) $v_{11}(\sigma + h, \sigma) + v_{21}(\sigma + h, \sigma) < 0$

(ii) $v_{12}(\sigma + h, \sigma) + v_{22}(\sigma + h, \sigma) < 0$.

*Proof.* Let $q^*(t) = e^{-\delta t} p^*(t)$; then the first equation in ($\mathscr{H}$) becomes

$$\dot{q}^*(t) = \Psi'(x^*(t)) \quad \text{a.e.,} \qquad t \in [0, 1]. \tag{16}$$

Let $x^*(t; x_0)$ denote the solution to ($\mathscr{Q}_0$) with endpoints $(x_0, x_1)$, the initial inventory being considered variable, and let $q^*(t; x_0)$ denote the associated price which satisfies (16). Integrating (16) gives

$$\alpha(x_0) = q^*(0; x_0) - q^*(1; x_0) = -\int_0^1 \Psi'(x^*(t; x_0))\, dt \tag{17}$$

If we set $x_1 = \sigma$ and consider the two initial inventories $x_0 = \sigma + h$ and $x_0' = x_0 + \varepsilon$ with $\varepsilon > 0$, then we must have

$$x^*(t; x_0 + \varepsilon) \geqq x^*(t; x_0), \qquad t \in [0, 1]$$

with strict inequality for $t \in [0, \eta]$ for some $\eta > 0$, in view of the continuity of $x^*(\cdot\,; x_0)$ and $x^*(\cdot\,; x_0 + \varepsilon)$. For suppose not, then the two paths $x^*(t; x_0)$ and $x^*(t; x_0 + \varepsilon)$ must intersect for some $\tau \in (0, 1)$. But this implies that there are two optimal paths on the interval $[\tau, 1]$ joining the endpoints $x^*(t; x_0)$ and $x_1$, which is impossible by the strict concavity of the integrand in $(\mathscr{Q}_0)$. Since by Assumption 1(ii) $-\Psi'(\cdot)$ is a strictly decreasing function on $(0, \hat{s})$,

$$-\int_0^1 \Psi'(x^*(t; x_0 + \varepsilon))\, dt < -\int_0^1 \Psi'(x^*(t; x_0))\, dt$$

It follows, on recalling the definition of $\alpha(\cdot)$ in (17), that

$$\alpha(x_0 + \varepsilon) = v_1(x_0 + \varepsilon, x_1) + v_2(x_0 + \varepsilon, x_1)$$

is a strictly decreasing function of $\varepsilon$. Letting $(x_0, x_1) = (\sigma + h, \sigma)$ gives

$$\frac{\partial}{\partial x_1} (v_1(\sigma + h, \sigma) + v_2(\sigma + h, \sigma)) < 0$$

which proves (i). A similar argument with the terminal inventory $x_1$ considered variable, $x_1 = \sigma + \varepsilon$, and the initial inventory held fixed, $x_0 = \sigma + h$, leads to (ii). ▮

PROPOSITION 4. *If Assumptions* 1–3 *are satisfied, then there exist* $\alpha \in (0, \sigma)$ *and a sequence* $(\sigma_0, s_1^*, s_2^*, \ldots)$ *satisfying* $(\mathscr{E})$, $(T)$, (6), and

$$|s_t^* - \sigma| \to 0 \qquad \textit{as} \quad t \to \infty \tag{18}$$

*whenever* $\sigma_0 \in [\sigma - \alpha, \sigma + \alpha]$.

*Proof.* Let $(\mathscr{E})$ be written as $G(s_t, s_{t+1}, s_{t+2}) = 0$, $t = 0, 1, \ldots$. By Proposition 3, $G(\sigma, \sigma, \sigma) = 0$; by Lemma 2, $G \in \mathscr{C}^1(N_{\gamma_1}(\sigma, \sigma, \sigma))$ for some $\gamma_1 > 0$ and $G_3(\sigma, \sigma, \sigma) = \beta v_{12}(\sigma + h, \sigma) > 0$. Thus by the implicit function theorem there exist $\gamma > 0$ and a function $g \in \mathscr{C}^1(N_\gamma(\sigma, \sigma))$ such that if we let $v_t = s_{t+1}$, $F(s_t, v_t) = (v_t, g(s_t, v_t))$, $(s_t, v_t) = z_t$, $(\sigma, \sigma) = \bar{z}$, then $(\mathscr{E})$ reduces to

$$z_{t+1} = F(z_t), \qquad z_t \in N_\gamma(\bar{z}), \quad t = 0, 1, \ldots \tag{$\mathscr{E}'$}$$

where $F \in \mathscr{C}^1(N_\gamma(\bar{z}))$. Let $z_t = \bar{z} + \xi_t$ and define

$$r(\xi) = F(\bar{z} + \xi) - F(\bar{z}) - F_z(\bar{z})\xi, \qquad \xi \in N_\gamma(0);$$

then $(\mathscr{E}')$ may be written as

$$\xi_{t+1} = F_z(\bar{z})\, \xi_t + r(\xi_t), \qquad \xi_t \in N_\gamma(0), \quad t = 0, 1, \ldots \tag{$\mathscr{E}''$}$$

If we let $v_{ij}^* = v_{ij}(\sigma + h, \sigma)$, $i, j = 1, 2$ and note by Lemma 2 that $v_{12}^* = v_{21}^*$, then

$$F_z(\bar{z}) = \begin{bmatrix} 0 & 1 \\ -1/\beta & 2b \end{bmatrix}, \qquad b = \frac{-(\beta v_{11}^* + v_{22}^*)}{2\beta v_{12}^*}$$

The linear part of $(\mathscr{E}'')$

$$\xi_{t+1} = F_z(\bar{z})\,\xi_t, \qquad \xi_t \in N_\gamma(0), \quad t = 0, 1, \ldots$$

thus has the characteristic equation and associated eigenvalues

$$\lambda^2 - 2b\lambda + 1/\beta = 0, \qquad \lambda_1, \lambda_2 = b \pm \sqrt{b^2 - (1/\beta)}$$

It follows from Lemmas 2 and 3 that

$$v_{11}^* < 0, \qquad v_{22}^* < 0, \qquad v_{12}^{*2} < v_{11}^* v_{22}^*$$

so that

$$\sqrt{\beta}\, v_{12}^* < \sqrt{(-\beta v_{11}^*)(-v_{22}^*)} \leqq -\tfrac{1}{2}(\beta v_{11}^* + v_{22}^*)$$

Thus $b > 0$ and $b^2 - (1/\beta) > 0$ so that the eigenvalues are real and positive. Since $\lambda_1 \lambda_2 = 1/\beta > 1$, only one eigenvalue can satisfy $\lambda < 1$. This holds if and only if

$$\lambda_1 = b - \sqrt{b^2 - (1/\beta)} < 1 \Leftrightarrow \beta v_{11}^* + v_{22}^* + (1 + \beta)\, v_{12}^* < 0$$

an inequality that follows at once from Lemma 3, using $v_{12}^* = v_{21}^*$.

We are now in a position to use the following result which is a special case of the stable manifold theorem [7, p. 146, Theorem 2.4]. If in the difference equation $(\mathscr{E}'')$, $r \in \mathscr{C}^1(N_\gamma(0))$ satisfies the following condition: for any $\varepsilon > 0$ there exists $\eta \in (0, \gamma)$ such that

$$\| r(\xi) - r(\xi') \| \leqq \varepsilon \, \| \xi - \xi' \|, \qquad \forall \xi, \xi' \in N_\eta(0) \tag{19}$$

and if the eigenvalues $(\lambda_1, \lambda_2)$ of the matrix $F_z(\bar{z})$ satisfy $\lambda_1 < 1$, $\lambda_2 > 1$, then there exist $\alpha > 0$ and a solution $(\xi_0, \xi_1^*, \xi_2^*, \ldots)$ of $(\mathscr{E}'')$ such that $\| \xi_t^* \| \to 0$ as $t \to \infty$ whenever $\| \xi_0 \| < \alpha$. In our case since $F \in \mathscr{C}^1(N_\gamma(\bar{z}))$ implies $r \in \mathscr{C}^1(N_\gamma(0))$, (19) is readily deduced from the mean value theorem and the fact that $r$ satisfies $r_\xi(0) = 0$. Since $(\mathscr{E}'')$ is equivalent to $(\mathscr{E}')$ under the transformation $z_t = \bar{z} + \xi_t$ and since a sequence satisfying $(\mathscr{E}')$ also satisfies $(\mathscr{E})$, it follows that there exists a sequence $(\sigma_0, s_1^*, s_2^*, \ldots)$ satisfying $(\mathscr{E})$ such that $|s_t^* - \sigma| \to 0$ as $t \to \infty$ whenever $|\sigma_0 - \sigma| < \alpha$. But then $|v_1(s_t^* + h, s_{t+1}^*) - \rho| \to 0$ as $t \to \infty$, so that $0 < \beta < 1$ implies $|\beta^t v_1(s_t^* + h, s_{t+1}^*)(s_t^* + h)| \to 0$ as $t \to \infty$, which is $(T)$. Since $\alpha$ can be chosen so that (6) is satisfied, the proof is complete. ∎

## 6. Equilibrium Spot and Futures Prices

In this section we establish the existence of a spot market equilibrium and introduce futures markets. The analysis will be based on the results of Sections 4 and 5 and

Proposition 5. *Let* $\{p_t^*(\cdot), x_t^*(\cdot)\}_{t=0}^{\infty}$ *be a sequence of pairs of absolutely continuous functions on* $[t, t+1]$ *which satisfy the Hamiltonian equations* ($\mathscr{H}$) a.e. *on* $[t, t+1]$ *for* $t = 0, 1,\ldots$ . *If in addition for* $t = 0, 1,\ldots$

(i) $p_t^*(t^+) = p_{t-1}^*(t^-)$

(ii) $x_t^*(t^+) = x_{t-1}^*(t^-) + h,\ x^*(0^-) = \sigma_0$

(iii) $p_t^*(\tau) > 0,\ x_t^*(\tau) > 0$ a.e.

(iv) $\lim_{t\to\infty} e^{-\delta t} p_t^*(t)\, x_t^*(t^+) = 0$

(v) $\int_0^\infty p^*(\tau)\, e^{-\delta\tau}\, d\tau < \infty,\ p^*(\tau) = p_t^*(\tau),\ \tau \in [t, t+1],\ t = 0, 1,\ldots$

*then* $(q^*, -\dot{x}^*) = \{(p_t^*(\tau)\, e^{-\delta\tau}, -\dot{x}^*(\tau)),\ \tau \in [t, t+1],\ t = 0, 1,\ldots\}$ *is a spot market equilibrium.*

*Proof.* ($\mathscr{H}$)(ii) implies that condition (ii) in Definition 1 is satisfied—the spot market is cleared at almost every instant $\tau \in [0, \infty)$. It remains to check Definition 1(i). It follows from ($\mathscr{H}$)(ii) and Assumption 3(ii) that $\dot{x}^*(\tau) \in [-m, 0]$ a.e. (iii) implies $x^*(\tau) > 0$ a.e. The absolute continuity of $x_t(\cdot)$ and (ii) imply that (1) and (2) hold. Thus $\dot{x}^* \in \mathscr{F}$. It remains to show that $\dot{x}^*$ maximises profit

$$\pi(\dot{x}^*) = \sup_{\dot{x}\in\mathscr{F}} \sum_{t=0}^{\infty} \int_t^{t+1} (q^*(\tau)(-\dot{x}(\tau)) - \Psi(x(\tau)))\, e^{-\delta\tau}\, d\tau \qquad (\mathscr{S}')$$

Just as the maximum problem ($\mathscr{P}$) can be decomposed into the single carryover problem ($\mathscr{R}$) and the sequence of crop year problems ($\mathscr{Q}_t$), $t = 0, 1,\ldots$, so the profit maximising problem ($\mathscr{P}'$) can be decomposed into the single problem of determining the profit maximising carryover sequence

$$\sup_{(s_0, s_1,\ldots)\in\mathscr{S}} \sum_{t=0}^{\infty} \beta^t \pi^t(s_t + h, s_{t+1}) \qquad (\mathscr{R}')$$

and the sequence of problems of maximising crop year profit

$$\pi^t(s_t + h, s_{t+1}) = \sup_{\dot{x}_t\in\mathscr{F}_t} \int_t^{t+1} (p_t^*(\tau)(-\dot{x}_t(\tau)) - \Psi(x_t(\tau))) \times e^{-\delta(\tau - t)}\, d\tau \qquad (\mathscr{Q}_t'), \quad t = 0, 1,\ldots$$

where $\mathscr{I}_t = \mathscr{F}_t(s_t + h - s_{t+1})$, $x_t(\tau) = s_t + h + \int_t^\tau \dot{x}_t(v)\,dv$, $\tau \in (t, t+1)$, and $\mathscr{S}$ and $\mathscr{F}_t$ are defined by (3) and (3′).

We start by giving sufficient conditions for a maximum for ($\mathscr{R}'$) and ($\mathscr{Q}_t'$) in terms of a system of supporting prices. The proof is completed by showing that such a system of prices can be constructed from the sequence $\{p_t(\cdot)\}_{t=0}^{\infty}$.

Let $\{\eta_t, s_t\}_{t=0}^{\infty}$ be a pair of sequences satisfying

$$(s_0, s_1, ...) \in \mathscr{S}, \qquad \eta_t \geqq 0, \qquad t = 0, 1, ..., \qquad \lim_{t \to \infty} \eta_t s_t = 0 \tag{20}$$

$$(\eta_t, -\eta_{t+1}) \in \beta^t\, \partial\pi^t(s_t + h, s_{t+1}), \qquad t = 0, 1, ... \tag{21}$$

Consider any sequence $(s_0', s_1', ...) \in \mathscr{S}$. Using the definition of the superdifferential $\partial\pi^t$ and summing the resulting inequalities over a finite number of periods $T$ gives

$$\sum_{t=0}^{T} \beta^t(\pi^t(s_t' + h, s_{t+1}') - \pi^t(s_t + h, s_{t+1})) \leqq \eta_T(s_T - s_T') \tag{22}$$

Since $\Psi(\cdot) \leqq 0$, (v) and $\mathscr{F} \subset \mathscr{L}_\infty[0, \infty)$ imply $\pi(\dot{x}) < \infty$ $\forall \dot{x} \in \mathscr{F}$, it follows from $-\eta_T s_T' \leqq 0$, $\forall T > 0$, $\lim_{T \to \infty} \eta_T s_T = 0$, and (22) that $(s_0, s_1, ...)$ is a solution of ($\mathscr{R}'$). Let $(q_t(\cdot), y_t(\cdot))$ be a pair of absolutely continuous functions defined on $[t, t+1]$ satisfying

$$\dot{y}_t \in \mathscr{I}_t, \qquad y_t(t^+) = s_t + h, \qquad y_t(t+1^-) = s_{t+1} \tag{23}$$

$$(\dot{q}_t(\tau), q_t(\tau)) = (\Psi'(y_t(\tau)), p_t^*(\tau))\, e^{-\delta(\tau - t)} \qquad \text{a.e.} \tag{24}$$

then by the standard sufficient conditions for a concave variational problem on a finite interval [14, p. 209, Theorem 5], $\dot{y}_t$ is a solution of ($\mathscr{Q}_t'$). Furthermore[10],

$$(q_t(t^+), -q_t(t+1^-)) \in \partial\pi^t(s_t + h, s_{t+1}), \qquad t = 0, 1, ... \tag{25}$$

Consider the sequence of pairs of absolutely continuous functions on $[t, t+1]$ defined by

$$(q_t(\tau), y_t(\tau)) = (p_t^*(\tau)\, e^{-\delta(\tau - t)}, x_t^*(\tau)), \qquad \tau \in [t, t+1], \quad t = 0, 1, ... \tag{26}$$

Since $\dot{q}_t(\tau) = (\dot{p}_t^*(\tau) - \delta p_t^*(\tau))\, e^{-\delta(\tau - t)}$ a.e., it follows from (26) and ($\mathscr{H}$)(i) that (24) is satisfied. Let

$$(\eta_t, s_t) = (\beta^t q_t(t^+), y_t(t^+) - h), \qquad t = 0, 1, ...$$

[10] See footnote 9.

then (20) and (23) are satisfied. (i) implies $\eta_{t+1} = \beta^{t+1} q_{t+1}(t+1^-) = \beta^t q_t(t+1^-)$. It follows from (25) that the sequence $(\eta_0, \eta_1, \ldots)$ satisfies (21). Since $x^* = (x_t^*(\tau),\ \tau \in [t, t+1],\ t = 0, 1, \ldots)$ solves $(\mathscr{R}')$ and $(\mathscr{Q}_t')$, $t = 0, 1, \ldots$, $x^*$ is a solution of $(\mathscr{P}')$ and the proof is complete. ∎

COROLLARY. *Under Assumptions* 1–3 *there exists* $\alpha \in (0, \sigma)$ *such that if* $\sigma_0 \in [\sigma - \alpha, \sigma + \alpha]$, *then there exists a spot market equilibrium.*

*Proof.* We show that the conditions of Proposition 5 are satisfied. By Proposition 4 there exist $\alpha \in (0, \sigma)$ and a sequence $(s_0^*, s_1^*, s_2^*, \ldots)$ satisfying $(\mathscr{E})$, $(T)$, (6), and (18) whenever $s_0^* = \sigma_0 \in [\sigma - \alpha, \sigma + \alpha]$. In view of (6) the endpoints $(s_t^* + h, s_{t+1}^*)$ are attainable for $t = 0, 1, \ldots$ . Thus by Propositions 1 and 2 there exists a sequence of pairs of absolutely continuous functions $\{p_t^*(\cdot), x_t^*(\cdot)\}_{t=0}^{\infty}$ satisfying $(\mathscr{H})$ a.e. on $[t, t+1]$ for $t = 0, 1, \ldots$ . Since $x_t^*(t^+) = s_t^* + h$, $x_t^*(t+1^-) = s_{t+1}^*$, and $x(0^-) = \sigma_0$, (ii) holds. Since $(p_t^*(t^+), -\beta p_t^*(t+1^-)) = \nabla v(s_t^* + h, s_{t+1}^*)$, $t = 0, 1, \ldots$, and $(\mathscr{E})$ imply

$$-\beta p_{t-1}^*(t^-) = v_2(s_{t-1}^* + h, s_t^*) = -\beta v_1(s_t^* + h, s_{t+1}^*) = -\beta p_t^*(t^+)$$

(i) holds. Equation (6) implies (iii), while $(T)$ implies (iv). In view of (18) $p_t^*(t^+) = v_1(s_t^* + h, s_{t+1}^*) \to \rho$, and since $p_t^*(\tau)$ is continuous, there exists $\gamma > 0$ such that $0 < p^*(\tau) < \gamma$ a.e. so that (v) holds. The result then follows from Proposition 5. ∎

There is a simple geometric way of constructing the spot market equilibrium. Consider the value function induced by the carryover problem $(\mathscr{R})$

$$W(\sigma_0 + h) = \sup_{(s_0, s_1, \ldots) \in \mathscr{S}} \sum_{t=0}^{\infty} \beta^t v(s_t + h, s_{t+1})$$

It is easy to see that the sequence $(\sigma_0, s_1^*, s_2^*, \ldots)$ of Proposition 4 satisfies (20) and (21) with $\pi^t(\cdot)$ replaced by $v(\cdot)$ and hence is the solution to $(\mathscr{R})$. Since the optimal sequence satisfies (6) and since by Lemma 1 $v(\cdot)$ is differentiable, it follows that $W(\cdot)$ is differentiable and satisfies

$$W'(s_t^* + h) = v_1(s_t^* + h, s_{t+1}^*) = p_t^*(t^+).$$

Let

$$\mathscr{G} = \{(x, p) \mid p = W'(x)\}, \qquad \mathscr{G}' = \{(x, p) \mid (x + h, p) \in \mathscr{G}\}$$

so that $\mathscr{G}$ is the graph of the gradient of the value function and $\mathscr{G}'$ is the translation of this set by $h$. Let $K = \{(x, p) \mid p = -\psi(x)/\delta\}$. $\dot{p}_t^*(\tau) > 0$ (<0) to the right (left) of the curve $K$, while $\dot{x}_t^*(\tau) < 0$ (see Fig. 2). The crop year inventory and spot price associated with the steady state endpoints $(\sigma + h, \sigma)$

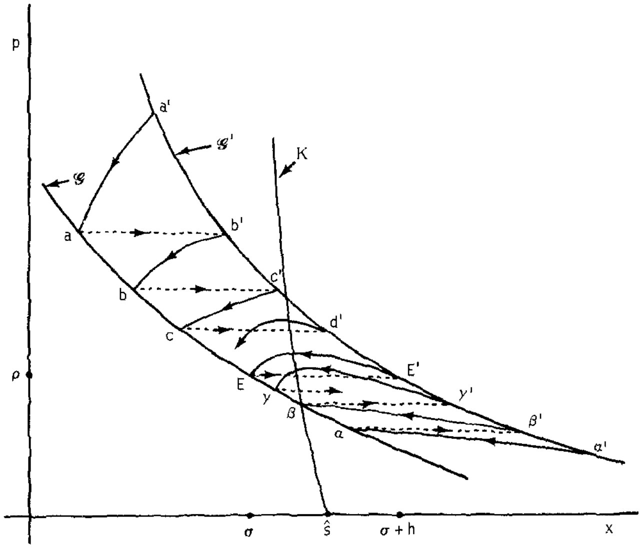


FIGURE 2

is given by the curve $E'E$. Note that the endpoints $E$, $E'$ lie on either side of the curve $K$: if this were not so, the spot price would increase (decrease) throughout the crop year, contradicting $p_{t+1}^*(t^-) = p_t^*(t^+)$. Note also that $\mathscr{G}'$ intersects the curve $K$, for otherwise there would be no initial price $p_0^*$ such that $(p_t^*(t^+), s_t^*) \to (\rho, \sigma)$. Consider two initial conditions

$$\sigma_0 \in [\sigma - \alpha, \sigma], \qquad \sigma_0' \in [\sigma, \sigma + \alpha].$$

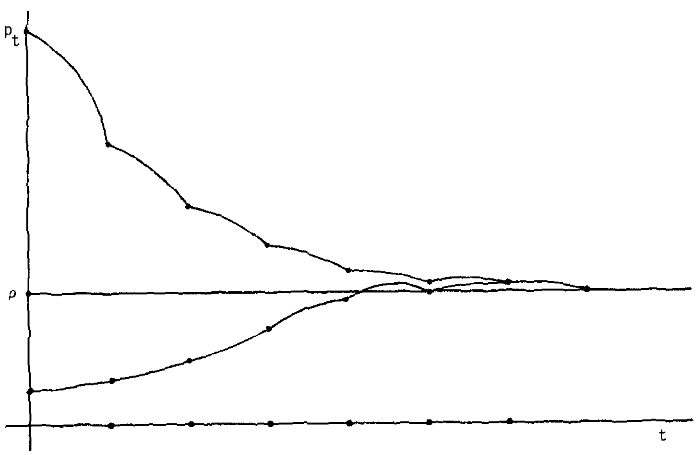


FIGURE 3

Then the curves $(a'a, b'b, c'c,...)$ and $(\alpha'\alpha, \beta'\beta, \gamma'\gamma,...)$ converging to the steady state path $E'E$ trace out the spot market equilibrium of the Corollary. The time profile of the spot price associated with these two equilibrium paths is shown in Fig. 3.

We now complete the final step of the analysis and introduce futures markets. The *θ-futures price at date* $\tau$, denoted by $p(\theta, \tau)$, is defined as the price at time $\tau$ for one unit of the commodity deliverable at time $\tau + \theta$, for $\theta \geqq 0$, $\tau \geqq 0$. Since there is no explicit uncertainty, it is reasonable to assume that the futures market has the following property.

ASSUMPTION 4 (Price formation on futures market). *Arbitrage on the futures market forces the θ-futures price at date τ into equality with the expected spot price at time τ + θ*

$$p(\theta, \tau) = p^*(\tau + \theta), \qquad \tau \geqq 0, \quad \theta \geqq 0.$$

Thus the current profile of futures prices can be read off from the future course of spot prices. Our analysis is designed in particular to throw light on the structure of futures prices at date $\tau = 0$. To this end let $\pi(\theta) = (\partial/\partial\theta)\, p(\theta, 0)/p(\theta, 0)$.

DEFINITION 3. At date $\tau = 0$, the futures market is said to exhibit

(i) a *full carrying charge* (*carrying charge*) *structure* for futures prices on the interval $[t, t+1]$ if $\pi(\theta) = \delta$ $(0 < \pi(\theta) \leqq \delta)$, $\theta \in [t, t+1]$;

(ii) an *inverted structure* for futures prices on the interval $[t, t+1]$ if $\pi(\theta) < 0$, $\theta \in [t, t+1]$;

(iii) a *normal structure* for futures prices on the interval $[t, t+1]$ if there exists $t' \in (t, t+1)$ such that $\pi(\theta) > 0$, $\theta \in [t, t')$, $\pi(\theta) < 0$, $\theta \in (t', t]$.

In view of the convergence property of the spot market equilibrium, it follows that if $\sigma_0$ is sufficiently close to $\sigma$, then at date $\tau = 0$ the futures market will exhibit a normal structure for futures prices of all future crop years. More generally we have

PROPOSITION 6. *Let* $(\xi, \eta) = K \cap \mathscr{G}$, $(\xi', \eta') = K \cap \mathscr{G}'$, *and let* $(p_t^*(\cdot), x_t^*(\cdot))$, $t = 0, 1,...$ *denote the spot market equilibrium of the Corollary.*

(i) *If there exists* $T \geqq 0$ *such that*

$$x_t^*(t^+) - \xi < 0, \qquad t \leqq T$$

*then at date* $\tau = 0$ *the futures market has an inverted structure for futures prices* $p(\theta, 0)$ *with* $\theta \leqq T$.

(ii) *If there exists* $T' \geqq 0$ *such that*

$$x_{t-1}^*(t^-) - \hat{s} > 0 \quad (x_{t-1}^*(t^-) - \xi' > 0), \qquad t \leqq T'$$

*then at date* $\tau = 0$ *the futures market exhibits a full carrying charge (carrying charge) structure for futures prices* $p(\theta, 0)$ *with* $\theta \leqq T'$.

*Proof.* When $\sigma_0 < \sigma$ $(\sigma_0 > \sigma)$ it follows from the proof of Proposition 4 that the convergence of $s_t^* = x_{t-1}^*(t^-)$ to $\sigma$ is monotone, $s_t^* < s_{t+1}^*$ $(s_t^* > s_{t+1}^*)$, $t = 0, 1, \ldots$. It follows from the phase portrait for the Hamiltonian system $(\mathscr{H})$ that $W'(\cdot)$ must be a strictly decreasing function, for otherwise the above monotonicity is contradicted. Thus in (i) $p_t^*(t) > p_{t+1}^*(t+1)$. Since $(x_t^*(\tau), p_t^*(\tau))$ lies to the left of the curve $K$ for $\tau \leqq T$, $\pi(\theta) < 0$, $\theta \in [0, T]$. In (ii) $p_t^*(t) < p_{t+1}^*(t+1)$ and since $x_{t-1}^*(t^-) > \hat{s}$, $t \leqslant T'$ implies $\Psi'(x_{t-1}^*(\tau)) = 0$, $\tau \leqq T'$, $\pi(\theta) = \delta$, $\theta \in [0, T']$. If $x_{t-1}^*(t^-) > \xi'$, $t \leqq T'$, then $(x_t^*(\tau), p_t^*(\tau))$ lies to the right of the curve $K$ for $\tau \leqq T'$ so that $0 < \pi(\theta) \leqq \delta$, $\theta \in [0, T']$ since $-\Psi'(\cdot) \geqq 0$ by Assumption 1(ii). ∎

Let $\Delta_0 = \xi - x_0^*(0^+)$ $(\Delta_1 = x_0^*(1^-) - \xi')$ be defined as *deficient* (*surplus*) *inventory* at date $\tau = 0$. Proposition 6 then leads to the following criterion: *If there is deficient* (*surplus*) *inventory at date* $\tau = 0$, *then the futures market has an inverted* (*carrying charge*) *structure for futures prices* $p(\theta, 0)$ *with* $\theta \leqq 1$, *namely, for futures within the current crop year.* When there is surplus inventory, the spot price is at a discount relative to futures prices, but the extent of the discount is limited by interest (and more generally carrying) charges. When there is deficient inventory, however, and the spot price is at a premium relative to futures prices, there is no upper bound to the extent of this premium (see Fig. 2). This unboundedness of the premium that spot prices can acquire over futures prices has its origin in the unboundedness of the stock-out cost function (Assumption 1(iii)). The asymmetry between the magnitude of the discount arising in periods of surplus inventory and the magnitude of the premium arising in periods of deficient inventory is a familiar empirical property of futures markets for storable commodities.

Futures markets are a mechanism for projecting expected spot prices into the present. When there is deficient (surplus) inventory an inverted (carrying charge) market discourages (encourages) consumption by a high (low) spot price, while simultaneously discouraging (encouraging) inventory holding by revealing through the structure of the prices on the futures market a falling (rising) course of expected spot prices. It is thus by projecting the future into the present that the futures market serves to guide the consumption and inventory accumulation behaviour of consumers and producers in the economy.

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