

## Portfolio Selection with Transactions Costs\*

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### 1. INTRODUCTION

In several recent contributions [2, 3, 4] Merton has shown that the continuous time formulation of portfolio theory provides a powerful *analytical framework* for extending the standard results of one-period mean-variance portfolio theory to the dynamical case. As is by now familiar the simplifications introduced by the continuous time theory have their origin in Samuelson's basic Approximation Theorem [5], for the mean-variance solution provides the exact solution in the limit of infinitesimal time periods.<sup>1</sup> Thus when security prices are lognormally distributed the Tobin-Cass-Stiglitz Separation Theorem as well as the Sharpe-Lintner-Mossin capital market equilibrium theory can be extended in a natural manner to the dynamical case.<sup>2</sup> The simplicity with which the earlier mean-variance results can be extended to the dynamical case is certainly a strong point in favor of the continuous time analysis.

The most important *empirical* justification for the use of continuous time analysis arises from a structural property common to most well-

\* This paper reports results of our joint work, some of which also appears in Constantinides' dissertation [1]. We are grateful to Robert C. Merton and John F. Muth for valuable discussion. Needless to say we remain responsible for all remaining errors.

<sup>1</sup> In all the lengthy discussion of the applicability of mean-variance theory the Approximation Theorem provides by far the most fundamental justification for the use of mean-variance theory. For it leads naturally to an analysis of continuous time diffusion processes—processes which are completely characterized by their instantaneous mean and variance, see [6, Chap. 8].

<sup>2</sup> See [3, 4].

developed capital markets: *trading opportunities in securities are available continuously in time*. Rational investors will then wish to avail themselves of the opportunity of trading at every instant of time. But herein lies the principal weakness of Merton's formulation of the continuous time theory.<sup>3</sup> For by combining the assumption that trading opportunities are available continuously with the assumption that the trading opportunities are available *costlessly* the investor is led to a quite unrealistic type of portfolio behavior. In the absence of any transactions costs the continuous time theory predicts that an investor faced with continually varying security prices will indulge in a completely unrealistic amount of security trading. Indeed it is for this reason that the discrete time theory<sup>4</sup> is often adhered to as a more reasonable and realistic explanation for observed investor behavior since the investor trades only at suitably spaced discrete intervals of time.

It is the object of this paper to show, however, that this weakness of Merton's continuous time theory is readily overcome by explicitly introducing into the analysis the impact of transactions costs. For when such transactions costs are introduced it will be found that the investor only seeks to make use of the available trading opportunities at randomly spaced instants of time—a behavior pattern which accords much more readily with observed investor behavior. Indeed since the discrete theory<sup>5</sup> only allows the investor the option of trading at preassigned intervals of time while in well-developed capital markets trading opportunities are available continuously, there are strong grounds for believing that the continuous time theory more accurately reflects both the trading opportunities available and the associated investor behavior that is observed on well-developed capital markets. Thus while there has been a tendency to focus more attention on the discrete time theory on the grounds that the continuous time theory is unnecessarily complex, we would argue that with the introduction of transactions costs the continuous time theory provides both on theoretical and empirical grounds the most realistic image of investor behavior that has been available so far.

In the formal solution which emerges it is found that the investor

<sup>3</sup> It should be pointed out that Merton was very well aware of this weakness of the continuous time theory, see [4, p. 869].

<sup>4</sup> For an analysis in discrete time see [7].

<sup>5</sup> The use of the discrete time theory as opposed to the continuous time theory can really only be justified when the trading interval  $h$  is not taken to be very "small" (Merton suggests  $h = 1/270$  of a year [4, p. 869]). For if the discrete theory is defined for every  $h$  and if the discrete theory converges to a well defined continuous time process as  $h \rightarrow 0$  then on theoretical grounds (the Approximation Theorem) and on empirical grounds (continuous trading opportunities) the continuous time theory is likely to be preferred.

trades in securities when the variation in the underlying security prices forces his portfolio proportions outside a certain region about the optimal proportions in the absence of transactions costs. The solution is related in an interesting way to the classic Arrow–Harris–Marschak [8] and Bellman–Glicksberg–Gross [9] analyses of the *commodity inventory problem*.<sup>6</sup> Indeed some of the earlier papers examining the impact of transactions costs have relied heavily on this analogy between the portfolio and the inventory problem. The classic analysis of Baumol [10] can be viewed as a translation of earlier results in deterministic inventory theory [11] into corresponding results on the demand for money. The extension of this analysis to an environment of uncertainty by Miller and Orr [12] in the case of fixed transactions costs and by Eppen and Fama [13] for the case of proportional costs similarly depended strongly on the earlier results [8, 9] in inventory theory. Zabel [14] who considered a discrete two-period two-asset (cash, security) model where the consumer maximizes the expected utility of his consumption explicitly considers the attitude toward risk of the investor as well as the cash–security composition of his portfolio, rather than just the stock of cash as in [10, 12, 13]. In this respect the present analysis is similar to that of Zabel. The method of analysis developed in this paper is, however, quite different from that of Zabel and enables us to obtain an exact characterization of the individual's portfolio behavior for an arbitrary number of securities and for an arbitrary time horizon.

Section 2 formalizes the portfolio problem in the presence of transactions costs making explicit the underlying assumptions about the capital market and the individual investor. In Section 3 we derive the optimal portfolio policy which is characterized first in the case where the portfolio proportions in the absence of transactions costs are small, and subsequently for the general case. The paper concludes with some observations on the effect of transactions costs on the general theory of the capital market.

## 2. THE PORTFOLIO PROBLEM WITH TRANSACTIONS COSTS

Consider an investor who faces a capital market with the following properties.

<sup>6</sup> That the portfolio theory should be related in this way to inventory theory is really not surprising, for we can view the investor's portfolio as an inventory of securities which instead of being continually depleted by a random demand is depleted or augmented at random as a result of the random fluctuations in the underlying security prices. The problem of determining when to realign the portfolio proportions is then equivalent to the problem of determining when to reorder stocks for the basic inventory

ASSUMPTION 1 (*Continuous Competitive Markets*). A fixed number  $m$  of securities can be bought and sold at current prices in unlimited amounts. A bank (security) is available to all investors which pays a constant interest rate ( $r > 0$ ) on deposits and charges the same rate on borrowing, which is available in unlimited amount. Trading takes place continuously in time.

ASSUMPTION 2 (*Securities*). Each security is perfectly divisible. The value of the bank security is unchanged (no inflation or deflation). The prices of the remaining securities are lognormally distributed, all instantaneous variances are positive and all instantaneous correlations are less than one in absolute value.

ASSUMPTION 3 (*Information*). All information concerning the underlying probability distribution of security prices as well as current quotations of security prices is perfect information that is available continuously and costlessly to all investors.

ASSUMPTION 4 (*Transactions Costs*). Transactions costs are incurred in the purchase or sale of each security. The costs are proportional to the value of each transaction.<sup>7</sup> Thus if  $v_i$  denotes the value of the  $i$ th security purchased ( $v_i > 0$ ) or sold ( $v_i < 0$ ) per unit of time the transactions cost function  $T(v_1, \dots, v_m)$  indicating the cost of buying or selling any combination of the  $m$  securities is given by

$$T(v_1, \dots, v_m) = \sum_{i=1}^m \chi_{v_i} v_i \quad \text{where} \quad \chi_{v_i} = \begin{cases} \chi^i & v_i > 0 \\ -\chi_i & v_i < 0 \end{cases} \quad (1)$$

and where  $0 \leq \chi^i < 1$ ,  $0 \leq \chi_i < 1$ ,  $i = 1, \dots, m$ .

The following assumption is made concerning the investor.

ASSUMPTION 5 (*Income and Lifespan*). The investor has an expected lifespan  $[0, T]$  during which he expects to earn a flow of contractual income  $y(t)$ , where  $y(t)$  is a continuous function on the interval  $[0, T]$ . The

<sup>7</sup> Since the cost of buying or selling a given security is attributable to two separate costs, the broker's commission and the bid-asked spread [15], a fully realistic transactions cost function should be the sum of a concave *brokerage cost* with a discontinuity at the origin depending on the number of securities transacted and a proportional *spread cost* as in (1). Assumption 4 considers the special case where transactions costs are generated solely by the spread cost. In the analysis which follows it is not necessary or advisable however to impute such a narrow or specific interpretation to the transactions costs—for they can be any costs that are associated with the purchase and sale of securities and can be interpreted to include a much more general class of costs such as information costs, taxes, and the like.

investor acts as if both  $T$  and  $y(t)$  were known with certainty in advance at  $t = 0$ .

If  $p_i(t)$  denotes the price of the  $i$ th security and  $x_i(t)$  the number of its securities held by the investor at time  $t$ ,  $i = 0, 1, \dots, m$ , then  $s_i(t) = x_i(t) p_i(t)$  is the value of his holdings of this security at time  $t$ . By Assumption 2 the value of the bank security  $p_0(t)$  is constant for all  $t$ , while at each instant  $t$  the security prices  $p_1(t), \dots, p_m(t)$  satisfy the joint diffusion process

$$dp_i(t) = \alpha_i p_i(t) dt + p_i(t) dz_i(t) \quad i = 1, \dots, m, \tag{2}$$

where  $dz(t) = (dz_1(t), \dots, dz_m(t))$  is the *increment of a Brownian motion process*, so that for any partition  $0 = t_0 < t_1 < \dots < t_k = T$  of the interval  $[0, T]$  the random variables

$$z(t_1) - z(t_0), \dots, z(t_k) - z(t_{k-1})$$

are independent and normally distributed with mean

$$E[z(t_i) - z(t_{i-1})] = 0, \quad i = 1, \dots, k$$

and covariance matrix

$$E[(z(t_i) - z(t_{i-1}))(z(t_i) - z(t_{i-1}))'] = \Sigma(t_i - t_{i-1}) \quad i = 1, \dots, k,$$

where  $\Sigma$  is positive definite.<sup>8</sup> On all those subintervals of  $[0, T]$  where  $x_i(t)$ ,  $\dot{x}_i(t) = dx_i(t)/dt$  are continuous,  $i = 1, \dots, m$ , *Ito's Lemma*<sup>9</sup> can be applied to  $s_i(t) = x_i(t) p_i(t)$  so that

$$ds_i(t) = (\alpha_i s_i(t) + v_i(t)) dt + s_i(t) dz_i(t) \quad i = 1, \dots, m, \tag{3}$$

where  $v_i(t) = \dot{x}_i(t) p_i(t)$  denotes the transaction rate for the  $i$ th security at time  $t$ . If  $c(t)$  denotes the investor's flow of consumption expenditure at time  $t$ , since income is paid in cash and since both consumption expenditure and transactions costs must be financed from his stock of cash while the purchase (sale) of securities reduces (adds to) his stock of cash, Assumptions 1, 4, and 5 imply

$$ds_0(t) = \left[ rs_0(t) + y(t) - c(t) - \sum_{i=1}^m (1 + \chi_{v_i}) v_i(t) \right] dt. \tag{4}$$

Let  $s = (s_0, \dots, s_m)$  and let the investor choose a transaction-consumption policy of the form  $(v, c) = (v(s, t), c(s, t))$ ,  $t \in [0, T]$  where

<sup>8</sup> (2) is equivalent to the  $m$  stochastic integral equations

$$p_i(t) = p_i(0) + \alpha_i \int_0^t p_i(\theta) d\theta + \int_0^t p_i(\theta) dz_i(\theta) \quad i = 1, \dots, m,$$

where the second integral is the *Ito stochastic integral* of  $p_i(t)$  see [6, Chap. 8].

<sup>9</sup> For a statement and proof of Ito's Lemma see [6, pp. 386-391].

$v = (v_1, \dots, v_m)$ . Then (3) and (4) lead to an associated conditional probability density function  $\psi^{(v,c)}(c, t | c(0))$  for the path of consumption.<sup>10</sup> We make the following crucial assumption about the investor's preferences.

**ASSUMPTION 6 (Preferences).** The investor has a *preference ordering*  $U(\psi^{(v,c)})$  among the probability distributions  $\psi^{(v,c)}$ . Furthermore there exists a *utility function*  $u(c, \tau)$  such that the preference ordering can be represented<sup>11</sup> as follows

$$U(\psi^{(v,c)}) = \int_0^T \int_{-\infty}^{\infty} u(c, \tau) \psi^{(v,c)}(c, \tau | c(0)) dc d\tau. \quad (5)$$

Under Assumption 6 a *rational investor will choose his transaction-consumption policy over his lifespan*  $[0, T]$  so as to maximize (5). This is equivalent to maximizing

$$E_{(s,0)}^{(v,c)} \int_0^T u(c, \tau) d\tau \quad (6)$$

subject to (3), (4), and the initial condition  $(s, 0)$  where  $c(s, 0) = c(0)$  and where  $E_{(s,t)}^{(v,c)}$  denotes the conditional expectation given the transaction-consumption policy  $(v, c)$  over the time interval  $[0, T]$  and that his holdings of securities are  $s$  at time  $t$ .

We will now introduce a procedure which makes it possible to solve the above problem using the stochastic theory of control.<sup>12</sup> We shall consider (3) as the limit of the following equations as  $\epsilon \rightarrow 0^+$  (implying  $\epsilon$  converges to zero through positive values)<sup>13</sup>

$$ds_i(t) = (\alpha_i s_i(t) + v_i(t)) dt + (s_i(t) + \epsilon v_i(t)) dz_i(t) \quad i = 1, \dots, m. \quad (7)$$

Then we have the following result

<sup>10</sup> Given  $(v, c) = (v(s, t), c(s, t))$  (3) and (4) lead to a well-defined diffusion process for  $ds$ . Applying Ito's Lemma we find the diffusion process  $dc$ ,  $\psi^{(v,c)}(c, t | c(0))$  is then the solution of the forward Kolmogorov (Fokker-Planck) equation associated with the diffusion process  $dc$ , see [6, Chap. 8].

<sup>11</sup> Sufficient conditions for such a representation have not yet been given. For a discussion of the static case where the Von-Neumann-Morgenstern Axioms are sufficient see [16, Chap. III]. Note that (5) is time-additive and implies no bequest motive.

<sup>12</sup> The limiting procedure which is introduced here is useful and interesting in its own right as a *general method of solving stochastic control problems in which the controls enter linearly but do not directly affect the disturbance terms*. The method has not appeared before in either the economic or the stochastic control theory literature and should prove useful in solving problems of this kind.

<sup>13</sup> It can be shown that the optimal transaction policy  $v^*$  defined by (17) has the following properties:  $\epsilon v_i^*$  is finite and bounded as  $\epsilon \rightarrow 0^+$  and  $\epsilon v_i^* \rightarrow 0$  as  $\xi$  tends to the boundary of  $\Omega_0$ . For the limiting operations employed in the proof of Proposition 1, (7) is thus a valid representation of the process (3).

**THEOREM 1.<sup>14</sup>** *If  $u(c, \tau) = e^{-\rho\tau}u(c)$ , if the maximum in (6) exists and if  $(v, c)$  maximizes (6) subject to (4) and (7) then the value function*

$$W(s, t) = \max_{(v,c)} E_{(s,t)}^{(v,c)} \int_t^T e^{-\rho(\tau-t)} u(c) d\tau$$

satisfies

$$\max_{(v,c)} \left\{ u(c) + \sum_{i=1}^m W_i(\alpha_i s_i + v_i) + W_0 \left( r s_0 + y - c - \sum_{i=1}^m (1 + \chi_{v_i}) v_i \right) + \frac{1}{2} \sum_{i,j=1}^m W_{ij}(s_i + \epsilon v_i)(s_j + \epsilon v_j) \sigma_{ij} - \rho W + W_t \right\} = 0, \quad (8)$$

$W(s, T) = 0$ , where

$$W_i = \frac{\partial W}{\partial s_i}, \quad W_{ij} = \frac{\partial^2 W}{\partial s_i \partial s_j}, \quad \Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & & \vdots \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{bmatrix}.$$

If the maximum in (8) is well defined then the maximizing  $(v^*, c^*)$  must satisfy  $u_c(c^*) - W_0 = 0$ ,

$$W_j - W_0(1 + \chi_{v_j}) + \sum_{i=1}^m W_{ij} \sigma_{ij} \epsilon (s_i + \epsilon v_i^*) = 0 \quad j = 1, \dots, m$$

implying

$$\begin{aligned} v^* &= (1/\epsilon)\{(1/\epsilon) Q^{-1}(W_0(n + \chi_v) - W_{\bar{s}}) - \bar{s}\} \\ c^* &= u_c^{-1}(W_0), \quad u_{cc} \neq 0, \end{aligned} \quad (9)$$

where

$$Q = \begin{bmatrix} W_{11}\sigma_{11} & \cdots & W_{1m}\sigma_{1m} \\ \vdots & & \vdots \\ W_{m1}\sigma_{m1} & \cdots & W_{mm}\sigma_{mm} \end{bmatrix}, \quad n = (1, \dots, 1), \quad \chi_v = (\chi_{v_1}, \dots, \chi_{v_m}) \\ W_{\bar{s}} = (W_1, \dots, W_m), \quad \bar{s} = (s_1, \dots, s_m).$$

### 3. THE OPTIMAL PORTFOLIO POLICY

We will use Theorem 1 to determine the nature of the investor's portfolio and consumption policies under the additional assumption that the utility

<sup>14</sup> For a proof of Theorem 1 see [17]. A heuristic proof is easily established by applying Bellman's Principle to the definition of the value function  $W(s, t)$  and then using Ito's Lemma. The assumption  $u(c, t) = e^{-\rho t} u(c)$  is introduced here so as to reduce equation (8) to a form that is simpler to solve in Section 3. It may be considered as part of Assumption 7.

function is a member of the following family characterizing an investor with *decreasing absolute risk aversion*.<sup>15</sup>

ASSUMPTION 7 (*Utility function*).

$$\begin{aligned} u(c, \tau) &= e^{-\rho\tau} \frac{(1-\eta)}{\eta} \left( \frac{\beta c}{1-\eta} + \gamma(\tau) \right)^\eta \\ &= e^{-\rho\tau} (1-\eta)^{1-\eta} \frac{\beta^\eta}{\eta} (c - \hat{c}(\tau))^\eta, \quad c \geq \hat{c}(\tau), \end{aligned} \quad (10)$$

where

$$\hat{c}(\tau) = -\gamma(\tau) \frac{(1-\eta)}{\beta}, \quad \begin{array}{ll} -\infty < \eta < 1, & \beta > 0, \\ -\infty < \gamma(\tau) < \infty, & \rho \geq 0. \end{array}$$

Equations (8), (9), and (10) imply that the value function  $W(s, t)$  satisfies the equation

$$\begin{aligned} &\frac{(1-\eta)^2}{\eta} \left( \frac{W_0}{\beta} \right)^{\eta/(\eta-1)} + \sum_{i=1}^m W_i \alpha_i s_i + W_0(rs_0 + y - \hat{c}) - \rho W \\ &\quad + W_t + (W_0(n + \chi_v) - W_s)' \frac{\bar{s}}{\epsilon} \\ &\quad - \frac{1}{2\epsilon^2} (W_0(n + \chi_v) - W_s)' Q^{-1} (W_0(n + \chi_v) - W_s) = 0 \end{aligned} \quad (11)$$

with boundary condition  $W(s, T) = 0$ . Equation (11) has a solution of the form

$$W(s, t) = \frac{a(t)}{\eta} \left( s_0 + \sum_{i=1}^m b_i s_i + A(t) \right)^\eta. \quad (12)$$

Equation (12) implies that

$$\frac{rs_0 + y - \hat{c} + \sum_{i=1}^m s_i [\alpha_i b_i + (1/\epsilon)(1 + \chi_{v_i} - b_i)] + \dot{A}(t)}{s_0 + \sum_{i=1}^m b_i s_i + A(t)}$$

must be constant, so that

$$b_i = \frac{1 + \chi_{v_i}}{1 - \epsilon(\alpha_i - r)} \quad i = 1, \dots, m, \quad A(t) = Y(t) - \hat{C}(t), \quad (13)$$

where

$$Y(t) = \int_t^T y(\tau) e^{-r(\tau-t)} d\tau, \quad \hat{C}(t) = \int_t^T \hat{c}(\tau) e^{-r(\tau-t)} d\tau.$$

<sup>15</sup> The Arrow-Pratt [18, 19] measure of absolute risk aversion  $-u_{cc}/u_c$  is positive and decreasing. When  $\eta > 1$  absolute risk aversion is increasing: this however seems to be an unlikely attitude toward risk (see [18, pp. 90-98]) and furthermore would give rise to perverse behavior in the analysis that follows. For a complete analysis of the properties of this family see [3,20].



$Y(t)$  and  $\hat{C}(t)$  are just the present value of his future income stream and his future minimum required consumption stream (when  $\hat{c}(\tau) \geq 0$ ) respectively. Noting that

$$Q^{-1} = \frac{1}{a(t)(\eta - 1)(s_0 + \sum_{i=1}^m b_i s_i + A(t))^{n-2}} \times \begin{bmatrix} \frac{1}{b_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{b_m} \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \frac{1}{b_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{b_m} \end{bmatrix}$$

if we let  $\alpha = (\alpha_1, \dots, \alpha_m)$ ,  $q = (\alpha - rn)' \Sigma^{-1}(\alpha - rn)/(2(1 - \eta))$ , then (11) reduces to the familiar Bernoulli equation

$$\dot{a} + [\eta(r + q) - \rho]a + (1 - \eta)^2 \left(\frac{a}{\beta}\right)^{\eta/(\eta-1)} = 0, \quad a(T) = 0. \quad (14)$$

Equations (13) and (14) imply

$$W(s, t) = (1 - \eta)^{1-\eta} \frac{\beta^\eta}{\eta} \left[ \frac{1 - e^{-(1/(1-\eta))(\rho - \eta(r+q))(T-t)}}{(1/(1-\eta))(\rho - \eta(r+q))} \right]^{1-\eta} \times \left( s_0 + \sum_{i=1}^m \frac{(1 + \chi_{v_i}) s_i}{1 - \epsilon(\alpha_i - r)} + Y(t) - \hat{C}(t) \right)^\eta \quad (15)$$

which is well defined for all  $\eta < 1$  provided that

$$\rho > \eta(r + q) \quad (16)$$

and where the dependence of  $\chi_{v_i}$  on  $s$  remains to be determined. Note that (15) coincides with Merton's solution [21] when  $\chi_{v_i} = \epsilon = 0$ . (9) and (15) imply

$$v^* = \frac{1}{\epsilon} \left\{ \begin{bmatrix} \frac{1 - \epsilon(\alpha_1 - r)}{1 + \chi_{v_1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1 - \epsilon(\alpha_m - r)}{1 + \chi_{v_m}} \end{bmatrix} \times \frac{\Sigma^{-1}(\alpha - rn)}{1 - \eta} \left( s_0 + \sum_{i=1}^m \frac{(1 + \chi_{v_i}) s_i}{1 - \epsilon(\alpha_i - r)} + Y(t) - \hat{C}(t) \right) - \bar{s} \right\}. \quad (17)$$

Equation (17) is a remarkably compact set of linear equations, which contains the basic information used to characterize the individual's transaction policy. The analysis of (17) is simplified if we divide through by

$$w(t) = \sum_{i=0}^m s_i(t) + Y(t) - \hat{C}(t),$$

which may be called the *effective wealth* of the individual.<sup>16</sup> Let  $\xi_i = s_i/w$ ,  $i = 0, \dots, m$ ,  $\xi = (\xi_1, \dots, \xi_m)$ ,  $\xi_y = (Y - \hat{C})/w$  so that  $\sum_{i=0}^m \xi_i + \xi_y = 1$ . Provided  $w \neq 0$  we can write (17) as  $v^*/w = (1/\epsilon) v^*(\xi, \chi_v, \epsilon)$ . If we define

$$\xi^0 = \frac{\Sigma^{-1}(\alpha - rn)}{1 - \eta} \tag{18}$$

since  $\epsilon(\alpha_j - r) \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ ,

$$v_j^*(\xi, \chi_v, \epsilon) \left[ \frac{1 + \chi_{v_j}}{1 - \epsilon(\alpha_j - r)} \right] \rightarrow v_j^*(\xi, \chi_v) \quad \text{as } \epsilon \rightarrow 0^+,$$

where

$$v_j^*(\xi, \chi_v) = [\chi_{v_j}(\xi_j^0 - 1) - 1] \xi_j + \xi_j^0 \left( 1 + \sum_{\substack{i=1 \\ i \neq j}}^m \chi_{v_i} \xi_i \right) \quad j = 1, \dots, m. \tag{19}$$

The functions  $v_j^*(\xi, \chi_v)$  are essentially *signal functions* which immediately signal when and how the securities are to be traded. To see how these signal functions work consider the simplest case, namely when transactions costs are zero. Since  $v_j^*(\xi, 0) = \xi_j^0 - \xi_j$ , (17) implies that whenever  $\xi_j < \xi_j^0$ ,  $v_j^*$  should be such that  $\xi_j$  approaches  $\xi_j^0$  at a rate dependent on  $\epsilon$ . As  $\epsilon \rightarrow 0^+$ ,  $v_j^* \rightarrow \infty$  in such a way that  $\xi_j$  is raised to  $\xi_j^0$  instantaneously. Similarly whenever  $\xi_j > \xi_j^0$  as  $\epsilon \rightarrow 0^+$ ,  $v_j^* \rightarrow -\infty$  in such a way that  $\xi_j$  is lowered to  $\xi_j^0$  instantaneously. Thus by taking the limit as  $\epsilon \rightarrow 0^+$  in (17) we find that Theorem 1 implies that the optimal portfolio policy in the absence of transactions costs consists in adjusting the vector of portfolio proportions  $\xi$  so that  $\xi = \xi^0$  at every instant. Since  $\xi^0$  coincides with the optimal portfolio proportions in [2, 3], Theorem 1, the signal functions  $v_j^*$ , and (17) lead to an alternative derivation of Merton's portfolio policy. Notice that the signal functions  $v_j^*$  bring out

<sup>16</sup> Wealth can be defined in a number of ways. Merton chooses to let  $w(t) = \sum_{i=0}^m s_i(t)$ . It can be argued that a natural definition should also include  $Y(t)$ . If  $\hat{c}(t) \geq 0$  then it also seems natural to subtract the preplanned consumption  $\hat{c}(t)$  from the future income  $y(t)$  so as to obtain the capital value of net income  $Y(t) - \hat{C}(t)$  which is then added to the current value of his financial assets to obtain his wealth. Ultimately the definition is a matter of convenience. In this respect this definition greatly simplifies the subsequent analysis.

very clearly the massive amount of trading that takes place over time in the absence of transactions costs.

When transactions costs are present the terms  $\xi_j^0 \sum_{i=1}^m \chi_{v_i} \xi_i$  in  $v_j^*(\xi, \chi_v)$  make the analysis considerably more complex. However when the portfolio proportions  $\xi_j^0$  are sufficiently small these terms become unimportant. This leads to

**PROPOSITION 1.<sup>17</sup>** *If Assumptions 1-7 are satisfied and if  $|\xi_j^0|$  are sufficiently small ( $|\xi_j^0| \ll 1$ )  $j = 1, \dots, m$  then there exists a region  $\Omega_0 \subset R^m$  such that the investor always confines his portfolio proportions to this region*

$$\Omega_0 = \{\xi \in R^m \mid \xi_j \in K(\xi_j^0), j = 1, \dots, m\},$$

where

$$K(\xi_j^0) = \left[ \frac{\xi_j^0}{1 + \chi^j}, \frac{\xi_j^0}{1 - \chi_j} \right] \quad \text{if } \xi_j^0 > 0,$$

$$K(\xi_j^0) = \left[ \frac{\xi_j^0}{1 - \chi_j}, \frac{\xi_j^0}{1 + \chi^j} \right] \quad \text{if } \xi_j^0 < 0.$$

*Proof.* By Theorem 1 the optimal transaction policy under Assumption 7 is given by (17). If  $\xi_j^0$  are sufficiently small the signal functions reduce to  $v_j^*(\xi, \chi_v) = \xi_j^0 - (1 + \chi_{v_j}) \xi_j$ . Suppose  $\xi_j^0 > 0$ , since  $v_j^* > 0$  implies  $\chi_{v_j} = \chi^j$ ,  $v_j^* > 0$  whenever  $\xi_j < \xi_j^0 / (1 + \chi^j)$ . Similarly since  $v_j^* < 0$  implies  $\chi_{v_j} = -\chi_j$  and since  $\chi_j < 1$  implies  $1 - \chi_j > 0$ ,  $v_j^* < 0$  whenever  $\xi_j > \xi_j^0 / (1 - \chi_j)$ . Suppose  $\xi_j < \xi_j^0 / (1 + \chi^j)$  then by (17)  $v_j^*$  is such that  $\xi_j$  approaches  $\xi_j^0 / (1 + \chi^j)$  at a rate dependent on  $\epsilon$ . As  $\epsilon \rightarrow 0^+$   $v_j^* \rightarrow \infty$  in such a way that  $\xi_j$  is raised instantly to  $\xi_j^0 / (1 + \chi^j)$ .<sup>18</sup>

<sup>17</sup> Propositions 1 and 2 determine the investor's transaction policy under the assumption that the initial portfolio proportions lie in the region  $\Omega_0$ . It appears that there are conditions under which it is not optimal to transact to the boundary of  $\Omega_0$  if  $\xi$  is not initially in  $\Omega_0$ . For example an investor with a very short lifespan facing a high transaction cost rate and starting with all holdings in cash may not find it worthwhile to purchase the risky securities. If we let  $-(c - \delta)(u_{cc}/u_c) = 1 - \eta$  be a measure of relative risk aversion then the condition that the portfolio proportions  $\xi^0$  be sufficiently small is equivalent to the condition that the investor be sufficiently risk averse. Recall that the usual measure of relative risk aversion is  $-c(u_{cc}/u_c)$ , see [18, 19].

<sup>18</sup> The limiting process involved here is the same as that involved in the definition of the *Dirac Delta function*, the integral of which is the unit step function [22, pp. 22-26]. If  $s_j(t^-)$  and  $s_j(t^+)$  denote the holdings of the  $j$ th security before and after the transaction at time  $t$ , then

$$\lim_{h \rightarrow 0} \int_{t-h}^{t+h} v_j^*(\tau) d\tau = s_j(t^+) - s_j(t^-),$$

so that  $\chi^j(s_j(t^+) - s_j(t^-))$  is the transaction cost incurred if  $v_j^* > 0$ . It should be recalled that since  $\xi$  is the solution of a diffusion process, its velocity is infinite. It is for this reason that transactions must be undertaken at an infinite rate whenever  $\xi$  attempts to penetrate the boundary of  $\Omega_0$ .

For as soon as  $\xi_j = \xi_j^0/(1 + \chi^j)$ ,  $v_j^*(\xi, \chi_v) = \xi_j^0 - (1 + \chi^j) \xi_j = 0$  implying  $v_j^* = 0$ . The rest is immediate.

Suppose  $\xi_j^0 < 0$  and suppose  $\xi_j < \xi_j^0/(1 - \chi_j)$ . Since  $v_j^*(\xi, \chi_v) = \xi_j^0 - (1 + \chi^j) \xi_j > 0$ ,  $v_j^* > 0$  is clearly optimal. Suppose the investor trades until  $v_j^*(\xi, \chi_v) = \xi_j^0 - (1 + \chi^j) \xi_j = 0$  so that  $\xi_j$  is raised to  $\xi_j = \xi_j^0/(1 + \chi^j)$ . Since at this point  $v_j^*(\xi, \chi_v) = \xi_j^0 - (1 - \chi_j) \xi_j < 0$ ,  $v_j^* < 0$  now becomes optimal and the investor trades until  $\xi_j = \xi_j^0/(1 - \chi_j)$  at which point  $v_j^*(\xi, \chi_v) = \xi_j^0 - (1 - \chi_j) \xi_j = 0$ . But then it is clearly not optimal to let  $\xi_j$  exceed  $\xi_j^0/(1 - \chi_j)$  for exceeding this point involves a redundant transactions charge since the investor always finds it optimal to return to this point. Thus when  $\xi_j < \xi_j^0/(1 - \chi_j)$  the investor trades to  $\xi_j = \xi_j^0/(1 - \chi_j)$ . Similarly when  $\xi_j > \xi_j^0/(1 + \chi^j)$  the investor trades to  $\xi_j = \xi_j^0/(1 + \chi^j)$ . Since it cannot be optimal to repeatedly trade both ways, when  $\xi_j \in [\xi_j^0/(1 - \chi_j), \xi_j^0/(1 + \chi^j)]$  the investor refrains from transacting. ■

Proposition 1 has a straightforward economic interpretation. Since the prices of the securities are continually changing according to (2) the portfolio proportions  $\xi$  are continually changing. In Merton's case, since there are no transactions costs whenever  $\xi \neq \xi^0$ , the benefit to be gained from improved diversification always induces the investor to transact so as to return  $\xi$  to  $\xi^0$ . *As soon as the investor is faced with transactions costs, however, he must match the benefits of improved diversification against the associated transactions costs.* Thus whenever the prices move  $\xi$  around but  $\xi$  still lies in the central region  $\Omega_0$  about  $\xi^0$ , the investor does not find it worthwhile to alter  $\xi$ ; in this region the transactions costs would exceed the benefits from improved diversification. But as soon as  $\xi$  pierces the boundary of  $\Omega_0$  the investor finds it worthwhile to transact so as to bring  $\xi$  back to the boundary of  $\Omega_0$ . In this case the benefits of improved diversification outweigh the transactions costs.

It is interesting to note that the transaction policy of Proposition 1 is of exactly the same form as the Bellman–Glicksberg–Gross ordering policy for the infinite horizon multicommodity inventory problem with proportional ordering costs, stated as [9, Theorem 3]. Indeed the portfolio policy of Proposition 1 also has an important simplifying independence property akin to the property that Bellman–Glicksberg–Gross refer to as *sub-optimality*. This independence property only holds in the present context however when the portfolio proportions are small, as assumed in Proposition 1.

**COROLLARY 1.** *The interval to which the transaction policy confines the portfolio proportion  $\xi_j$  of the  $j$ th security is independent of the proportions  $\xi_i$  and the transaction cost rates  $\chi^i, \chi_i$  for all other securities  $i \neq j$ .*

This property is of great importance on purely empirical grounds. For it is the essential property that is required if the portfolio policy is to have a reasonable and manageable form in the presence of transactions costs. As is shown in the proof of Proposition 2, in the general case where the portfolio proportions are not necessarily small the interval to which  $\xi_j$  is confined depends in a very complex way on  $\xi_i$  and  $\chi^i, \chi_i, i \neq j$ . The complexity of the region in the general case makes it highly unlikely that even the most rational of investors would involve himself with such calculations.

*COROLLARY 2. The region  $\Omega_0$  is independent of the investor's wealth and independent of the length of his remaining lifespan.*<sup>19</sup>

Both of these properties which hold independently of the magnitude of the portfolio proportions, arise from the homogeneity property characteristic of the HARA (hyperbolic absolute risk aversion) family of utility functions [3]. These properties generalize to the case of transactions costs two results whose importance was first stressed by Samuelson [23]. The first is that contrary to the advice of much investment literature the fact that the businessman is more wealthy than the widow does not imply that their portfolios should differ with respect to the risk that they carry, the businessman for example accepting a higher risk portfolio for the sake of obtaining a better yield. Secondly the fact that the businessman has a longer life ahead of him than the widow does not imply that the businessman should be prepared to invest more heavily in the risky securities. For the HARA family the proportion of his wealth that an investor carries in the risky securities is independent of his age. This time independence property of the portfolio policy is one important respect in which Proposition 1 differs from the Bellman–Glicksberg–Gross Theorem 3. For in [9] the critical levels at which stocks are reordered depend in the case of a finite planning horizon on the number of years left to the end of the plan and are constant only when the horizon is infinite. For a more general family of utility functions one would expect the same result for the portfolio problem. Indeed one would expect the size of the region  $\Omega_0$  about  $\xi^0$  to decrease as  $T - t$  increases so that the longer the remaining lifespan of the investor the greater his propensity to transact.

Inside the region  $\Omega_0$  the portfolio proportions  $\xi(t) = \bar{s}(t)/w(t)$  describe a diffusion process the nature of which is determined by (4) and (7). Since

<sup>19</sup> Corollary 2 should be carefully distinguished from the Tobin–Cass–Stiglitz Separation Theorem [20] which in its simplest form asserts that the composition of the portfolio of risky assets is independent of the investor's preferences, age, or financial assets. This result is the subject of a separate analysis in [24].

the covariance matrix  $\Sigma$  is positive definite and since  $\Omega_0$  is a closed, bounded region about  $\xi^0$  the process  $\xi(t)$  will pierce the boundaries of  $\Omega_0$  at random instants. Since the length of the interval  $K(\xi_j^0)$  to which the proportion  $\xi_j$  is confined increases as  $\chi^j, \chi_j$  increase, it is clear that the average frequency per unit of time with which  $\xi(t)$  pierces the boundaries of  $K(\xi_j^0)$  decreases as  $\chi^j, \chi_j$  increase. Conversely in the limit as  $\chi^j, \chi_j \rightarrow 0$  the investor trades continuously in the  $j$ th security.

COROLLARY 3. (i) *When  $\chi^j, \chi_j > 0$  the investor trades in the  $j$ th security at randomly spaced instants of time,  $j = 1, \dots, m$ .*

(ii) *The average frequency of trading in any security per unit of time decreases as the cost of transacting the security increases.<sup>20</sup>*

The portfolio policy of Proposition 1 partitions the portfolio space  $R^m$  into  $\binom{m}{k} 2^k$  distinct regions

$$\Omega_k = \{\xi \in R^m \mid \xi_j \notin K(\xi_j^0), k \text{ indices } j\} \quad k = 0, \dots, m,$$

where each  $\Omega_k$  may be called a  $k$ -transaction region since whenever  $\xi \in \Omega_k$ ,  $k$  securities are transacted, the remaining  $(m - k)$  involving no transaction.  $R^m$  is thus partitioned into  $3^m$  distinct regions.<sup>21</sup>  $\Omega_0$  is the  $m$ -dimensional rectangular solid about  $\xi^0$  with sides of length  $2\chi_i \xi_i^0, i = 1, \dots, m$  (assuming  $\chi^i = \chi_i$  to be relatively small). The regions  $\Omega_k$  surround  $\Omega_0$  and  $\Omega_k \cap \Omega_0$  are its  $(m - k)$  dimensional hyperfaces. As soon as a change in security prices causes  $\xi$  to pierce one of the hyperfaces  $\Omega_k \cap \Omega_0$ ,  $k$  securities are transacted and  $\xi$  is driven back to the hyperface. As  $\chi^i, \chi_i \rightarrow 0, i = 1, \dots, m$  the rectangular solid  $\Omega_0$  shrinks to the point  $\xi^0$  and we are back to Merton's case. Figure 1 shows the regions  $\Omega_k$  when  $m = 2, \chi^i = \chi_i = \chi, i = 1, 2$ .<sup>22</sup>

<sup>20</sup> Since the probability distribution for the frequency of trading depends crucially on the magnitude of the transaction cost rates relative to the return-covariance structure of the diffusion process (4), (7), it is clear that the empirical magnitudes of the transaction cost rates are of considerable importance if the resulting theory is to represent a substantial improvement over the earlier Merton theory. The empirical evidence available [15] suggests, as mentioned in footnote 7, that the transactions costs must be given a much broader interpretation than narrowly defined brokerage fees.

<sup>21</sup> Since there are  $2^k$  ways of buying or selling  $k$  securities and since  $k$  securities can be chosen from  $m$  in  $\binom{m}{k} = (m! / k!(m - k)!)$  ways, there are  $\binom{m}{k} 2^k$  regions  $\Omega_k$ . The Binomial Theorem then implies  $\sum_{k=0}^m \binom{m}{k} 2^k = 3^m$ . Since  $3^m > 10^{m(0.477)}$  this partition can involve an exceedingly large number of distinct regions even for a relatively small number of securities. For example  $3^{15} = 14,348,907$ .

<sup>22</sup> When  $m = 3$ , the  $(m - k)$  dimensional hyperfaces  $\Omega_k \cap \Omega_0$  are just the 8 vertices, 12 edges, and 6 faces of the rectangular solid. A transaction involving 3 securities ends at a vertex, a transaction involving 2 securities ends at an edge and a transaction involving 1 security ends at a face.

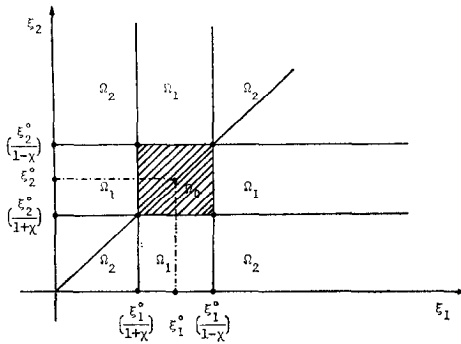


FIG. 1. Transaction regions for  $m = 2$  (small proportions).

PROPOSITION 2. *If Assumptions 1-7 are satisfied then there exists a region  $\Omega_0 \subset R^m$  such that the investor always confines his portfolio proportions to this region.*<sup>23</sup>

*Proof.* Consider the signal functions

$$\nu_j^*(\xi, \chi_{\nu}) = [\chi_{\nu_j}(\xi_j^0 - 1) - 1] \xi_j + \xi_j^0 \left( 1 + \sum_{\substack{i=1 \\ i \neq j}}^m \chi_{\nu_i} \xi_i \right) \quad j = 1, \dots, m.$$

The idea is to use these functions to define general  $k$ -transaction regions  $\Omega_k$  in which  $k$  securities are transacted. By calculating the regions  $\Omega_m$ ,  $\Omega_{m-1}$  and so on, the region  $\Omega_0$  is arrived at recursively. In the proof that follows we assume implicitly that the regions do not overlap. When regions overlap however, which arises in particular when  $\xi_j^0 < 0$  or  $\xi_j^0 > 1$  for some indices  $j$ , we proceed as in the proof of Proposition 1 and show that regions in which the  $i$ th security is both bought and sold must be regions in which the  $i$ th security is not transacted.

Consider the regions  $\Omega_m$ . Since  $\nu_j^* \geq 0$  is equivalent to  $\nu_j^* \geq 0$ , the inequalities  $\nu_j^* \geq 0 \quad j = 1, \dots, m$  define  $2^m$  regions  $\Omega_m$  in which all  $m$  securities are transacted where  $\chi_{\nu_j} = \chi^j$  if  $\nu_j^* > 0$ ,  $\chi_{\nu_j} = -\chi^j$  if  $\nu_j^* < 0$ . Next we obtain the regions  $\Omega_{m-1}$ . Suppose the first security is not transacted so that  $\nu_1^* = 0$ . Let all the remaining securities have definite signs

<sup>23</sup> The region  $\Omega_0$  can become unbounded if  $|\xi_j^0|$  is sufficiently large. In practice such cases are unlikely to arise. Suppose  $\chi^j = \chi_j = \chi$ . Then the hyperplanes  $\nu_j^* = 0$  intersect the  $\xi_j$  axis at the points  $A(\xi_j^0) = (\xi_j^0 / (1 + \chi(1 - \xi_j^0)))$  and  $B(\xi_j^0) = (\xi_j^0 / (1 - \chi(1 - \xi_j^0)))$  and these points become unbounded as  $\xi_j^0 \rightarrow 1 + (1/\chi)$  and  $\xi_j^0 \rightarrow 1 - (1/\chi)$  respectively.

for  $v_2^*, \dots, v_m^*$ , say  $v_j^* > 0, j = 2, \dots, m$ . Consider the region defined by  $v_j^* > 0, j = 2, \dots, m$  with  $\chi_{v_1} = \chi^1$ : intersect this with a similar region obtained by setting  $\chi_{v_1} = -\chi^1$ . Subtract out the regions where  $v_1^* > 0, v_j^* > 0, j = 2, \dots, m$  and  $v_1^* < 0, v_j^* > 0, j = 2, \dots, m$  (namely the two  $\Omega_m$  regions) and we are left with the region in which the first security is not transacted but the remaining securities are transacted in a definite way. Since there are  $2^{m-1}$  ways of buying and selling the remaining  $(m - 1)$  securities there are  $2^{m-1}$  such regions in which the first security is not transacted and since any of the  $(m - 1)$  securities can be chosen in place of the first security there are  $m2^{m-1}$  regions  $\Omega_{m-1}$  involving transactions in  $(m - 1)$  securities. The recursive procedure should now be evident. Proceeding in this way we obtain all the different transactions regions in  $R^m$

$$\Omega_m, \Omega_{m-1}, \dots, \Omega_k, \dots, \Omega_0.$$

By construction  $\Omega_0$  is then the region to which the investor confines his portfolio. ■

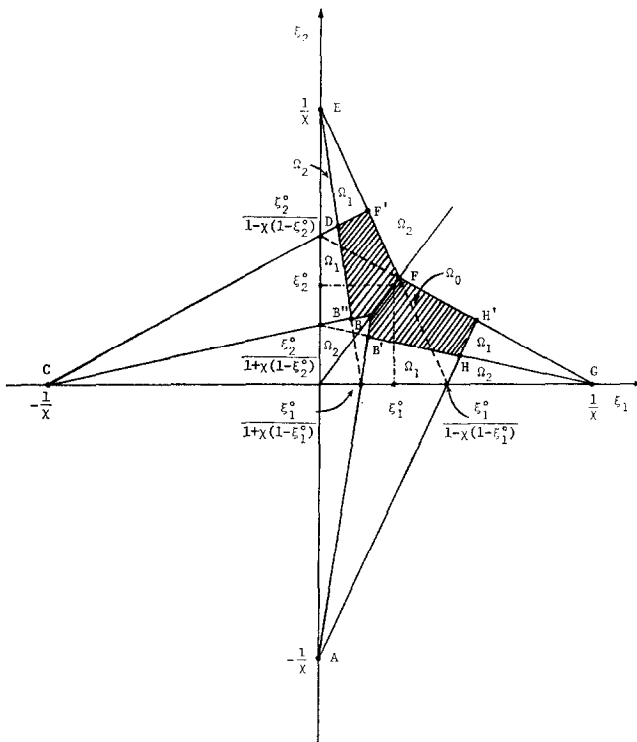


FIG. 2. Transaction regions for  $m = 2$  (general case).



Figure 2 shows the 9 regions  $\Omega_2, \Omega_1, \Omega_0$  when  $m = 2, \chi^i = \chi_i = \chi, \xi_i^0 > 0, i = 1, 2$ .<sup>24</sup>

Theorem 1, (9), and (15) imply that the investor's consumption policy becomes, as  $\epsilon \rightarrow 0^+$ ,

$$c^*(t) = \hat{c}(t) + \frac{D}{1 - e^{-D(T-t)}} \left( s_0 + \sum_{i=1}^m (1 + \chi_{v_i}) s_i + Y(t) - \hat{C}(t) \right)$$

where  $D = \frac{1}{1 - \eta} (\rho - \eta(r + q)).$  (20)

This leads at once to

**PROPOSITION 3.** *If Assumptions 1-7 are satisfied the investor's consumption policy depends (i) upon his current portfolio policy, (ii) upon his wealth and the length of his remaining lifespan.*<sup>25</sup>

When  $\chi_v = 0$ , (20) coincides with Merton's consumption policy [2, 3]; in this case as Samuelson observed [23], the consumption policy and the portfolio policy are independent financial decisions. When transactions costs are present, however, (20) implies that consumption varies depending on the region of the portfolio space in which  $\xi$  lies. The factor  $\sum_{i=1}^m \chi_{v_i} s_i$  adjusts his effective wealth in such a way that whenever the investor is purchasing (selling) a security, a factor is added to (subtracted from) his wealth, this nominal increase (reduction) in his wealth leading to an increase (reduction) in his consumption. The increased (decreased) consumption must however be drawn out of (kept in) cash which leads to a real reduction (increase) in his wealth thereby helping to increase (decrease) the proportion of the security in his portfolio. (ii) is immediate — though the fact that the consumption policy depends upon his remaining lifespan marks an important qualitative difference between the portfolio and the consumption policies.

<sup>24</sup> Using the recursive procedure of the proof we obtain the following transaction regions. Below *ABC* buy 1, buy 2: to left of *CDE* buy 1, sell 2: to right of *EFG* sell 1, sell 2: below *GHA* sell 1, buy 2. These are the 4  $\Omega_2$  regions. In *B'CD* buy 1: in *DEF'* sell 2: in *H'GH* sell 1: in *HAB'* buy 2. These are the 4  $\Omega_1$  regions. In *BB'DF'FH'HB'* do not transact. This is the region  $\Omega_0$ .

<sup>25</sup> Condition (16) ensures  $c^* - \hat{c} \geq 0$  as required by (10) provided

$$s_0 + \sum_{i=1}^m (1 + \chi_{v_i}) s_i + Y - \hat{C} \geq 0.$$

The economic interpretation of (16) is familiar. For when  $\eta < 0, u \rightarrow -\infty$  as  $c - \hat{c} \rightarrow 0^+$  while when  $0 < \eta < 1, u \rightarrow 0$  as  $c - \hat{c} \rightarrow 0^+$ . Thus when  $0 < \eta < 1$  the pure rate of time preference  $\rho > 0$  must be sufficiently large to ensure that consumption always exceeds the minimum level  $\hat{c}$ . We may also note that if (16) is satisfied (15) and (20) are well defined as  $T \rightarrow \infty$ , so that the portfolio problem is well defined for the infinite horizon case. As  $T \rightarrow \infty, c^*$  converges to a time-independent consumption policy.

## 4. CONCLUSION

This paper has shown a number of fundamental qualitative changes that arise in the portfolio behavior of an investor when trading opportunities on the capital market are no longer available costlessly. The most basic change is that the investor substantially modifies his concept of an optimal portfolio which now consists of a whole region in the portfolio space. A direct consequence of this is that the investor only seeks to make use of trading opportunities at randomly spaced instants of time. Both of these properties are likely to hold more generally for the class of concave utility and transaction cost functions. The wider economic significance of trading costs must now be sought in their impact on the capital market as a whole. As one result in this direction trading costs can be shown to be an important factor explaining the existence of financial intermediaries such as mutual funds, as is shown in [24]. The methods developed in this paper may also prove useful in determining the impact of trading costs on capital market equilibrium.

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