CHAPTER 3

The nonneutrality of money in a production economy with nominal assets

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1 Introduction

Recent work on the theory of incomplete markets (see Magill and Shafer, 1991) can be viewed as an attempt to broaden the scope and applicability of standard general equilibrium theory by introducing a structure of markets, contracts, and constraints on agent participation which conforms more closely with that observed in the real world. The key idea is that once we recognize that we live in a world in which time and uncertainty enter in an essential way, we must acknowledge that the system of markets is incomplete.

Drèze (1974) was one of the first to extend the traditional Arrow–Debreu theory to the framework of incomplete markets, following the earlier lead of Diamond (1967). He recognized the importance of differences in agents' rates of substitution for the single most significant type of firm ownership structure – the shareholder-owned firm (Drèze, 1987, ch. 15, 16). He emphasized the inevitably controversial role of a firm that seeks to make decisions in the typically conflicting interests of its three sets of clienteles – its shareholders, its employees, and the consumers in the marketplace (Drèze, 1989). These contributions have greatly increased our understanding of the central role played by shareholder-owned corporations in a modern economy.

Although the subsequent literature on incomplete markets has emphasized the relationship between the *real* and *financial* markets, there has

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not been sufficient recognition of the far-reaching changes that arise once *money* is explicitly introduced. In a modern economy, money is primarily used as a medium of exchange: its presence permits the introduction of *nominal contracts,* which are promises to deliver amounts of money in future states. In an economy with money, price levels are influenced by monetary policy: If agents exchange nominal assets then their opportunity sets are altered by changes in price levels. Thus, in an economy with money and nominal contracts, monetary changes have real effects.

The problem of introducing money into a model with incomplete markets has been studied for the case of an *exchange* economy by Magill and Quinzii (1991). Our object here is to extend the analysis to the case of a *production* economy. After laying out the assumptions of a *smooth* production economy (Section 2), we introduce the system of spot and financial markets (equity and bonds) and show how money circulates through these markets, performing its role as a medium of exchange (Section 3). Since we allow firms as well as consumers to trade on the financial markets, some clarification is required to obtain a well-defined objective function for each firm: the present value of profit for each firm is obtained by using the average present value vector of the firm's *original* shareholders. We stress that at the present time, no definitive criterion for the firm has been discovered for this class of equilibrium models. There are a number of plausible criteria, and this one is chosen among these because it simplifies the qualitative analysis of the effects of changes in monetary policy.

After introducing the concept of a *monetary equilibrium* (Section 4), we explain a simplification relative to the earlier exchange analysis of Magill and Quinzii (1991): We restrict the qualitative analysis to those equilibria in which money is used solely as a medium of exchange, the *positive nominal interest-rate equilibria.* For those parameter values for which agents are induced to use money as a store of value in competition with the other financial assets, the induced changes in the velocity of circulation of money greatly complicate the analysis in Section 6. We thus confine our analysis to parameter values leading to the positive interest-rate case. We analyze monetary equilibria by reducing them to an equivalent *reduced-form* concept of equilibrium (Section 5). This enables us to establish generic existence of monetary equilibria by a straightforward application of the powerful fixed-point theorem of Husseini, Lasry, and Magill (1990) (see also Hirsch, Magill, and Mas-Colell, 1990 for a geometric proof of the theorem).

Section 6 contains the basic qualitative analysis of the effects of changes in monetary policy. We distinguish between *anticipated* and *unanticipated* monetary changes. We say that a monetary change has *real effects* if it alters a given equilibrium allocation, that is, if it alters either the consumption plan of at least one agent or the production plan of at least one firm or both; we say that it has *production effects* if it alters the production plan of at least one firm. We show that if the asset markets are complete, then anticipated monetary changes have no real effects (Theorem 2). However, if the asset markets are incomplete and if the number of agents (I) exceeds the number of nominal assets ($K \geq 1$), then generically there is a subspace Λ such that all nontrivial anticipated marginal monetary changes chosen from this subspace have real effects. Since the dimension of Λ is at least $S-(J+K)$ (where S is the number of states at date 1 and J is the number of firms), there are always some local monetary changes that have real effects. To be sure that there are production effects, we need an additional assumption that ensures that the initial ownership of firms is not too diffused among consumers (Theorem 3). Unanticipated monetary changes always have real effects; under the assumption of non-diffused initial ownership, there are production effects provided the induced income effects do not cancel (Theorem 4).

This chapter should be viewed as a preliminary general equilibrium study of the way the presence of nominal assets and contracts enables monetary policy to have real effects by altering price levels and, hence, the purchasing power of nominal asset returns. The analysis can thus be viewed as a generalization of the *real balance effect,* which has played such a central role in macroeconomics (see, for example, Tobin, 1980, ch. 1). The analysis of this chapter has been confined to the case of nominal assets in zero net supply: The results thus depend crucially on the general equilibrium analysis of income effects, an analysis that is typically ignored in the macroeconomic literature.

2 The production economy

We consider the simplest production economy with time and uncertainty. The economy consists of a finite number of consumers ($i=1,\dots,I$) and firms ($j=1,\dots,J$) and a finite number of goods ($h=1,\dots,H$). There are two time periods ($t=0,1$) and one of S states of nature ($s=1,\dots,S$) occurs at date 1. For convenience, we call date $t=0$, state $s=0$, so that in total there are $S+1$ states.

Since there are H goods available in each state ($s=0,1,\dots,S$) the *commodity space* is $\mathbb{R}^n$, with $n=H(S+1)$. Each consumer i ($i=1,\dots,I$) has an *initial endowment* of the H goods in each state $w^i=(w_0^i, w_1^i,\dots,w_S^i)$. The preference ordering of agent i is represented by a *utility function:*

$$u^i: \mathbb{R}_+^n \to \mathbb{R}, \quad i=1,\dots,I$$

defined over *consumption bundles* $x^i=(x_0^i, x_1^i,\dots,x_S^i)$ lying in the nonnegative orthant $\mathbb{R}_+^n$, which satisfies the following:

Assumption A (Preferences). (1) *$u^i: \mathbb{R}^n_+ \to \mathbb{R}$ is continuous on $\mathbb{R}^n_+$ and $\mathcal{C}^2$ on $\mathbb{R}^n_{++}$.* (2) *If $U^i(\xi) = \{x \in \mathbb{R}^n_+ \mid u^i(x) \geq u^i(\xi)\}$, then $\overline{U^i(\xi)} \subset \mathbb{R}^n_{++}$, $\forall \xi \in \mathbb{R}^n_{++}$.* (3) *For each $x \in \mathbb{R}^n_{++}$, $Du^i(x) \in \mathbb{R}^n_{++}$ and $h^T D^2 u^i(x) h < 0$ for all $h \neq 0$ such that $Du^i(x)h = 0$.*

Production in the economy is carried out by the J firms ($j = 1, \ldots, J$), each characterized by a production set $Y^j \subset \mathbb{R}^n$. The managers of firm j choose a production plan $y^j \in Y^j$, where $y^j = (y^j_0, y^j_1, \ldots, y^j_S)$, and $y^j_s = (y^j_{s1}, \ldots, y^j_{sH})$ with $y_{sh} < 0$ (> 0), indicating that good h is used in state s as an input (is produced in state s as an output). In order to obtain a smooth supply function for firm j, we assume that the *boundary* ∂Y^j of Y^j is a smooth submanifold of a subspace $E^j \subset \mathbb{R}^n$. The production sets are assumed to satisfy the following conditions.

Assumption B (Production sets). (1) *$Y^j \subset \mathbb{R}^n$ is closed, convex and $\mathbb{R}^n_- \subset Y^j$.* (2) *$(\sum_{i=1}^I w^i + \sum_{j=1}^J Y^j) \cap \mathbb{R}^n_+$ is compact.* (3) *Y^j is a full dimensional submanifold of $E^j \subset \mathbb{R}^n$, whose boundary ∂Y^j is a $\mathcal{C}^2$ manifold with strictly positive Gaussian curvature at each point.*

Since we use genericity arguments, we need a way of *parametrizing* the decision of agents. We parametrize the decisions of consumers by the vector of initial endowments $\omega = (w^1, \ldots, w^I) \in \mathbb{R}^{nI}_+$ of the I consumers. To parametrize the production activity of firms, we assume that the production of each firm consists of two components, the *endogenously* chosen $y^j \in Y^j$ and an exogenously given vector of outputs $\eta^j \in \mathbb{R}^n_+$, so that the total production of firm j is $y^j + \eta^j$. We call η^j the *initial endowment* vector of firm j and let $\eta = (\eta^1, \ldots, \eta^J)$. To obtain genericity results, we parametrize the decisions of consumers and producers by the initial endowment vector

$$(\omega, \eta) \in \Omega \subset \mathbb{R}^{nI}_+ \times \mathbb{R}^{nJ}_+.$$

If we choose utility functions $u = (u^1, \ldots, u^I)$, production sets $Y = (Y^1, \ldots, Y^J)$, and an initial endowment vector $(\omega, \eta) \in \Omega$, then we obtain an economy $\mathcal{E}(u, Y; \omega, \eta)$. The assumptions made on the set Ω depend on whether the economy is being viewed as an *exchange* economy - in which case, we can set $\Omega = \mathbb{R}^{nI}_{++} \times \{0\}$ - or as a *production* economy - in which case, the goods with which consumers are initially endowed are typically different from those produced by firms. Thus, we made the following assumption.

Assumption C (Initial endowments). (a) *(ω, η) lie in an open subset $\Omega \subset \mathbb{R}^{nI}_{++}$ or* (b) *$\sum_{i=1}^I w^i + \sum_{j=1}^J \eta^j \in \mathbb{R}^n_{++}$ and $w^i_{s1} > 0$, $s = 1, \ldots, S$ for $i = 1, \ldots, I$.*

3 Markets, transactions, and money

Our object is to study the allocation of resources obtained when we adjoin to the economy $\mathcal{E}(u, Y; \omega, \eta)$ a market structure consisting of *n spot markets* and *J+K asset markets.* It is convenient to begin by describing the financial assets: These are basically of two kinds, the *J securities* of the firms and *K nominal* assets (bonds). At date 0, the ownership of firm j is distributed among a set of *initial shareholders,* with z^{ij}_0 denoting the proportion of firm j owned by consumer i, where $z^{ij}_0 \geq 0$ for all i, j and $\sum_{i=1}^I z^{ij}_0 = 1$. Agent i receives the proportion z^{ij}_0 of the dividends D^j_0 of firm j at date 0, where D^j_0 is measured in francs (the unit of account of the medium of exchange). The securities of firm j are then traded on a *stock market.* The market price of firm j is q_j (francs) and full ownership of firm j conveys the right to the dividend stream $D^j_1 = (D^j_1, \ldots, D^j_S)$ at date 1. We assume that both consumers and firms can trade on the equity market for firm j ($j = 1, \ldots, J$). Let z^{ij} denote the proportion of firm j purchased (sold if $z^{ij} < 0$) by consumer i and let $\xi^{j'j}$ denote the proportion purchased by firm j'. Clearly, market clearing will require $\sum_{i=1}^I z^{ij} + \sum_{j'=1}^J \xi^{j'j} = 1$, $j = 1, \ldots, J$.

At date 0, consumer i inherits an ownership of z^{ik}_0 units of nominal asset k: To simplify the later analysis, we assume that firms initial holdings of bonds are zero ($\xi^{jk}_0 = 0$, $j = 1, \ldots, J$). The bond holdings are in zero net supply ($\sum_{i=1}^I z^{ik}_0 = 0$). Asset k gives consumer i the right to receive $N^k_0 z^{ik}_0$ francs at date 0. These one-period contracts are then traded on a *bond market.* For the price q_k, an agent receives the right to the income stream $N^k_1 = (N^k_1, \ldots, N^k_S)$ (francs) at date 1. We let $z^{ik}(\xi^{jk})$ denote the number of units bought (sold if $z^{ik} < 0$ ($\xi^{jk} < 0$)) by consumer i (firm j). Each nominal asset is in *zero net supply,* so that market clearing requires $\sum_{i=1}^I z^{ik} + \sum_{j=1}^J \xi^{jk} = 0$, $k = J+1, \ldots, J+K$.

In addition to these financial assets, there is another financial instrument, *fiat money,* which serves as the medium of exchange and defines the unit of account (*francs*) for all transactions on markets. While in a *barter* economy, goods can be exchanged for goods directly, in a *monetary* economy, a good is always exchanged for money and this money is then used to purchase other commodities. We use the procedure introduced in Magill and Quinzii (1991) to describe the way in which money circulates in the economy. The transactions activities of each period ($s = 0, 1, \ldots, S$) are decomposed into transactions in three separate subperiods (s_1, s_2, s_3). At date 0, in the first subperiod 0_1, consumers and firms sell their initial endowments

$$(\omega_0, \eta_0) = (w^1_0, \ldots, w^I_0, \eta^1_0, \ldots, \eta^J_0) \tag{1}$$

to a central exchange. They thus receive the amounts of fiat money

$$(m_0^1, \ldots, m_0^I, \tilde{m}_0^1, \ldots, \tilde{m}_0^J),$$

where $m_0^i = p_0 w_0^i$, and $\tilde{m}_0^j = p_0 \eta_0^j$, $p_0 = (p_{01}, \ldots, p_{0H})$, denoting the vector of *spot prices* at date 0. In the second subperiod 0_2, armed with the money balances acquired in subperiod 0_1, agents proceed to the financial markets. Consumers and firms receive (make) dividend payments implied by their *initial portfolios*

$$(z_0, \xi_0) = (z_0^1, \ldots, z_0^I, \xi_0^1, \ldots, \xi_0^J). \tag{2}$$

To simplify the analysis, we assume $\xi_0 = 0$ so that firms have no initial holdings of financial assets. Consumers and firms then purchase *new portfolios* in the stock and bond markets

$$(z, \xi) = (z^1, \ldots, z^I, \xi^1, \ldots, \xi^J), \tag{2'}$$

where z^i and ξ^j lie in $\mathbb{R}^{J+K}$. All transactions on the financial markets are made with fiat money. The full amount of the money balances acquired in the subperiods 0_1 and 0_2 will be used for purchasing commodities in subperiod 0_3, *provided agents do not have an incentive to use money as a store of value*[1] from date 0 to date 1. Although this circumstance is of considerable interest, in this chapter, we concentrate on the simpler case where money is used purely as a medium of exchange. A sufficient condition for this is that money be dominated by some financial asset as a method for transferring purchasing power between date 0 and date 1. This can be ensured in two steps as follows. The first step consists in assuming that all agents (consumers and firms) have access to riskless borrowing and lending. For convenience, we introduce the full set of conditions imposed on the bond matrix N_1 as follows.

Assumption D (Bond matrix). *The bond matrix N_1 satisfies* (i)

$$\operatorname{rank} N_1 = K,$$

(ii) *the first column of N_1 is the riskless bond*

$$N_1^1 = (N_1^1, \ldots, N_S^1) = (1, \ldots, 1).$$

The price q_1 of the riskless bond is related to the *riskless nominal rate of interest* r_1 by the equation $q_1 = 1/(1+r_1)$. *If $r_1 > 0$, no agent will use money as a store of value between dates 0 and 1.* The second step consists in restricting attention to those parameter values of the economy

[1] Clearly, money is always used as a store of value between the subperiods 0_1 and 0_3 – but this is qualitatively different from its use in storing value between dates 0 and 1, where it competes in the portfolio role with other financial assets: This case has been studied in some detail for an exchange economy in Magill and Quinzii (1991).

(explained in what follows) for which each monetary equilibrium has a positive nominal rate of interest, $r_1 > 0$.

Under these assumptions, all the money balances acquired in the subperiods 0_1 and 0_2 are used by consumers and firms to purchase commodities

$$(x_0, y_0) = (x_0^1, \ldots, x_0^I, y_0^1, \ldots, y_0^J) \tag{3}$$

from the central exchange and the total amount of money

$$M_0 = \sum_{i=1}^{I} m_0^i + \sum_{j=1}^{J} \tilde{m}_0^j \tag{4}$$

injected into the economy by the central exchange in subperiod 0_1 is returned to it in subperiod 0_3.

Transactions in each state s at date 1 are similarly divided into transactions in three separate subperiods (s_1, s_2, s_3). In the first subperiod s_1, consumers and firms sell the commodities

$$(w_s, y_s + \eta_s) = (w_s^1, \ldots, w_s^I, y_s^1 + \eta_s^1, \ldots, y_s^J + \eta_s^J) \tag{5}$$

to the central exchange. They thus receive the amounts of fiat money

$$(m_s^1, \ldots, m_s^I, \tilde{m}_s^1, \ldots, \tilde{m}_s^J),$$

where $m_s^i = p_s w_s^i$, and $\tilde{m}_s^j = p_s(y_s^j + \eta_s^j)$, $p_s = (p_{s1}, \ldots, p_{sH})$, denoting the vector of *spot prices* in state s. In the second subperiod s_2, consumers and firms receive (make) the dividend payments implied by their contractual commitments (z, ξ) in (2′) at date 0. In the final subperiod s_3, consumers use the money balances at their disposal to purchase the goods

$$x_s = (x_s^1, \ldots, x_s^I) \tag{6}$$

from the central exchange. Since no money is transferred from date 0, the total amount of money returned to the central exchange in subperiod s_3 coincides with the amount of money

$$M_s = \sum_{i=1}^{I} m_s^i + \sum_{j=1}^{J} \tilde{m}_s^j \tag{7}$$

that it injects into the economy in subperiod s_1. Figure 1 gives a schematic breakdown of the transactions and money flows involved in (1)-(7).

We assume that the vector of money supplies (the *monetary policy*)

$$M = (M_0, M_1, \ldots, M_S) \in \mathcal{M} = \mathbb{R}_{++}^{S+1}$$

(where $\mathcal{M}$ is the *monetary policy space*) is exogenously determined by the government. Just as the decisions of consumers and firms are parametrized by the initial endowment vector $(\omega, \eta) \in \Omega$, so the action of the government can be viewed as being parametrized by the monetary policy M. Thus, the triple

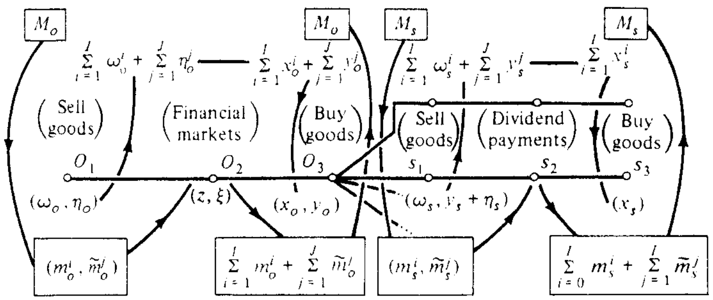

Figure 1. Transactions and money flows.

$$(\omega, \eta, M) \in \Omega \times \mathfrak{M}$$

constitutes the basic set of parameters for the economy.

4 Monetary equilibrium

To make precise our concept of a monetary equilibrium, we need to define the opportunity sets, objectives, and resulting actions of consumers and firms. Consider, first, the consumers. Let $z_0^i = (z_0'^i, z_0''^i) \in \mathbb{R}^J \times \mathbb{R}^K$ and $z^i = (z'^i, z''^i) \in \mathbb{R}^J \times \mathbb{R}^K$ denote the *initial* (inherited) and *new* (posttrade) portfolios of stocks and bonds held by consumer i and let

$$q = (q', q'') = (q_1, \dots, q_J, q_{J+1}, \dots, q_{J+K}),$$
$$(D_s, N_s) = (D_s^1, \dots, D_s^J, N_s^1, \dots, N_s^K), \quad s = 0, 1, \dots, S,$$

denote the vector of stock and bond prices and their vectors of dividend payments across the states, measured in francs. Then the previous transactions flows lead to the following budget equations for consumer i:

$$\begin{aligned} &p_0 x_0^i = p_0 w_0^i + (q' + D_0, N_0) z_0^i - q z^i, && \text{(i)} \\ &p_s x_s^i = p_s w_s^i + (D_s, N_s) z^i, \quad s = 1, \dots, S. && \text{(ii)} \end{aligned} \qquad (8)$$

For $x^i \in \mathbb{R}^{H(S+1)}$ and $p \in \mathbb{R}^{H(S+1)}$ define the *box product*

$$p \,\square\, x^i = (p_0 x_0^i, p_1 x_1^i, \dots, p_S x_S^i)$$

and let $e_0 = (1, 0, \dots, 0) \in \mathbb{R}^{S+1}$. If we let $W_0 = (q' + D_0, N_0)$,

$$W = \begin{bmatrix} -q' & -q'' \\ D_1 & N_1 \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_J & -q_{J+1} & \cdots & -q_{J+K} \\ D_1^1 & \cdots & D_1^J & N_1^1 & \cdots & N_1^K \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ D_S^1 & \cdots & D_S^J & N_S^1 & \cdots & N_S^K \end{bmatrix} \qquad (9)$$

denote the financial asset *returns matrices,* then the budget set of consumer i can be written as $(i = 1, \dots, I)$

$$B^i(p, q, D, N, \omega, \eta) = \{x^i \in \mathbb{R}_+^n \mid \exists z^i \in \mathbb{R}^{J+K} \text{ such that } p \,\square\, (x^i - w^i) = W_0 z_0^i e_0 + W z^i\}. \qquad (10)$$

Define the *subspace of income transfers* in $\mathbb{R}^{S+1}$ generated by the columns of the matrix W

$$\langle W \rangle = \{\tau \in \mathbb{R}^{S+1} \mid \exists z \in \mathbb{R}^{J+K} \text{ such that } \tau = Wz\}$$

and the *orthogonal (dual) subspace of present-value vectors* (state prices)

$$\langle W \rangle^\perp = \{\beta \in \mathbb{R}^{S+1} \mid \beta W = 0\}.$$

A variant of the Minkowski–Farkas lemma asserts that

$$\textit{either} \quad \langle W \rangle \cap (\mathbb{R}_+^{S+1} \setminus 0) \neq \emptyset \quad \textit{or} \quad \langle W \rangle^\perp \cap \mathbb{R}_{++}^{S+1} \neq \emptyset. \qquad (11)$$

It is easy to see that *consumer i's maximum problem*

$$\max_{x^i \in B^i(p, q, D, N, \omega, \eta)} u^i(x^i), \quad i = 1, \dots, I, \qquad (12)$$

has a solution if and only if $\langle W \rangle \cap (\mathbb{R}_+^{S+1} \setminus 0) = \emptyset$. This is just the *no-arbitrage* condition, which by (11) is equivalent to the existence of a vector of state prices $\beta \in \langle W \rangle^\perp \cap \mathbb{R}_{++}^{S+1}$, which can be written in the *normalized* present value form $\beta = (1, \beta_1)$ and satisfies

$$\beta W = 0. \qquad (13)$$

Let $\lambda^i = (\lambda_0^i, \lambda_1^i, \dots, \lambda_S^i) \in \mathbb{R}_{++}^{S+1}$ denote the Lagrange multipliers induced by the $S+1$ budget constraints in (12) and let

$$\pi^i = (1, \pi_1^i, \dots, \pi_S^i) = (1, \lambda_1^i/\lambda_0^i, \dots, \lambda_S^i/\lambda_0^i), \quad i = 1, \dots, I,$$

denote the *normalized* multipliers. We call π^i the *present-value vector* of consumer i since π_S^i denotes the value to consumer i at date 0 of a contract that promises to pay one franc in state s. From the first-order conditions for the choice of portfolio z^i in (12)

$$\pi^i W = 0, \quad i = 1, \dots, I. \qquad (14)$$

Thus, the first-order conditions imply that $\pi^i \in \langle W \rangle^\perp \cap \mathbb{R}_{++}^{S+1}$.

Given production and portfolio decisions

$$(y, \xi) = (y^1, \dots, y^J, \xi^1, \dots, \xi^J)$$

for the firms and market prices (p, q), the dividends of the firms are defined by the relations $(j = 1, \dots, J)$

$$\begin{aligned} &D_0^j = p_0(y_0^j + \eta_0^j) - q\xi^j, && \text{(i)} \\ &D_s^j = p_s(y_s^j + \eta_s^j) + (Ds, Ns)\xi^j, \quad s = 1, \dots, S. && \text{(ii)} \end{aligned} \qquad (15)$$

As Duffie and Shafer (1986) first observed, in order that equations (15) lead to well-defined dividends for the firms, the matrix

$$I-\xi'=\begin{bmatrix} 1-\xi^{11} & -\xi^{21} & \cdots & -\xi^{J1} \\ -\xi^{12} & 1-\xi^{22} & \cdots & -\xi^{J2} \\ \vdots & \vdots & \ddots & \vdots \\ -\xi^{1J} & -\xi^{2J} & \cdots & 1-\xi^{J2} \end{bmatrix} \tag{16}$$

must be invertible, ξ' being the $J\times J$ matrix of *interfirm shareholdings.* When this condition is satisfied, the $S\times J$ matrix of date 1 dividends can be written as

$$D_1=[p_1\square(y_1+\eta_1)+N_1\xi''][1-\xi']^{-1}, \tag{17}$$

where

$$\xi''=\begin{bmatrix} \xi^{1,J+1} & \cdots & \xi^{J,J+1} \\ \vdots & \ddots & \vdots \\ \xi^{1,J+K} & \cdots & \xi^{J,J+K} \end{bmatrix}$$

is the $K\times J$ matrix of *firm bondholdings.* To obtain an objective function for each firm, it suffices to introduce a vector of state prices

$$\beta^j=(1,\beta_1^j,\ldots,\beta_S^j)\in\mathbb{R}_{++}^{S+1},\quad j=1,\ldots,J, \tag{18}$$

which translates the stream of dividends into a present value at date 0. Recall the asset-returns matrix $W=W(q,D_1,N_1)$ defined in (9). Using (15) and (18) leads to the following expression for the present value of firm j's dividends

$$\beta^jD^j=\beta^j(p\square(y^j+\eta^j))+\beta^jW(q,D_1,N_1)\xi^j,\quad j=1,\ldots,J.$$

Thus, firm j's profit-maximizing problem

$$\max_{(y^j,\xi^j)\in Y^j\times\mathbb{R}^{J+K}}\beta^j\cdot D^j,\quad j=1,\ldots,J,$$

has a solution only if β^j *satisfies the orthogonality condition* $\beta^jW=0$, *which is equivalent to*

$$\beta^j\in\langle W\rangle^{\perp}\cap\mathbb{R}_{++}^{S+1},\quad j=1,\ldots,J,$$

so that

$$\beta^j\cdot D^j=\beta^j\cdot(p\square(y^j+\eta^j)). \tag{19}$$

If the firm's maximum problem is to have a solution, then the present value of its profit can only depend on its production decision (y^j) *and must not be influenced by its choice of financial policy* (ξ^j). This irrelevance of ξ^j for the firms profit is the first part of the Modigliani–Miller theorem.

The problem of selecting a discount vector β^j for firm j thus reduces to selecting a present-value vector which in some sense represents the collective interests of the firm's owners – namely, its *shareholders.* It was Drèze (1974) who first proposed a criterion of this kind with

$$\beta^j=\sum_{i=1}^{I}z^{ij}\pi^i\quad(\text{for } z^{ij}\geq 0),\quad j=1,\ldots,J,$$

being the average present-value vector of the *new* shareholders $(z^{ij}\geq 0)$ of firm j. Although this criterion has substantial *normative* appeal (see Geanakoplos et al., 1990), adopting it would make our subsequent comparative static analysis very complex. We could employ the elegant refinement of this criterion subsequently introduced by Drèze (1985) by which in essence only a subgroup of the largest shareholders has a dominant say in the firms decisions – a criterion with great *descriptive* appeal – but this criterion also makes comparative static analysis difficult. We are thus led to adopt the other natural candidate criterion introduced by Grossman and Hart (1979) by which

$$\beta^j=\sum_{i=1}^{I}z_0^{ij}\pi^i,\quad j=1,\ldots,J,$$

is the average present-value vector of the firm's *original* shareholders (z_0^{ij}). To prove that monetary changes have production effects, we will also require that the original ownership of the firms is not too diffused among the set of consumers. Even though the technical difficulties of working with the Drèze criterion are substantially greater, it is our belief that the qualitative results that we obtain do not depend on the particular criterion that we have adopted.

We are now in a position to define the concept of a monetary equilibrium which forms the basis for the analysis that follows.

Definition 1. *A* monetary equilibrium *for the economy* $\mathcal{E}(u,Y,N,\omega,\eta,M)$ *is a pair of actions and prices* $((\bar{x},\bar{z}),(\bar{y},\bar{\xi});(\bar{p},\bar{q},\bar{\pi}))$ *such that*

(i) $(\bar{x}^i,\bar{z}^i,\bar{\pi}^i)$, $i=1,\ldots,I$, *satisfy*

$$\bar{x}^i=\underset{x^i\in B^i(\bar{p},\bar{q},\bar{D},N,\omega,\eta)}{\arg\max}\;u^i(x^i),\qquad \bar{p}\square(\bar{x}^i-w^i)=W_0z_0^ie_0+\bar{W}\bar{z}^i.$$

$\bar{\pi}^i$ *is the present-value vector of agent i and* $(\bar{W},\bar{D})$ *are defined by* (9), (15) (i), *and* (17).

(ii) $(\bar{y}^j,\bar{\xi}^j)$, $j=1,\ldots,J$, *satisfy*

$$\bar{y}^j=\underset{y^j\in Y^j}{\arg\max}\;\bar{\beta}^j\cdot(\bar{p}\square y^j),\qquad \bar{\beta}^j=\sum_{i=1}^{I}z_0^{ij}\bar{\pi}^i$$

and the matrix $(I-\bar{\xi}')$ *in* (16) *is invertible.*

(iii) $\sum_{i=1}^{I} (\bar{x}^i - w^i) = \sum_{j=1}^{J} \bar{y}^j + \eta^j.$

(iv) $\sum_{i=1}^{I} \bar{z}'^i + \sum_{j=1}^{J} \xi'^j = e, \qquad e = (1, \dots, 1) \in \mathbb{R}^J,$

$\sum_{i=1}^{I} \bar{z}''^i + \sum_{j=1}^{J} \xi''^j = 0.$

(v) $\bar{p}_0 \left(\sum_{i=1}^{I} w_0^i + \sum_{j=1}^{J} \eta_0^j \right) = M_0,$

$\bar{p}_s \left(\sum_{i=1}^{I} w_s^i + \sum_{j=1}^{J} (y_s^j + \eta_s^j) \right) = M_s, \quad s = 1, \dots, S.$

We say that the equilibrium has a positive nominal interest rate *if* $q_1 < 1$. *If* $\text{rank}[D_1, N_1] = \rho$, *we say that the equilibrium has* rank ρ.

Remark 1. A positive nominal interest-rate equilibrium will not exist for all parameter values $(\omega, \eta, M) \in \Omega \times \mathfrak{M}$. For a given production economy $\mathcal{E}(u, Y; \omega, \eta)$, if the money supplies at date 1, M_s $(s = 1, \dots, S)$, are sufficiently small relative to M_0, then the marginal utility of a franc at date 1 $(\sum_{s=1}^{S} \lambda_s^i)$ is greater than the marginal utility of a franc at date 0 (λ_0^i)

$$\sum_{s=1}^{S} \lambda_s^i > \lambda_0^i \Leftrightarrow \sum_{s=1}^{S} \pi_s^i > 1$$

and consumers will respond by carrying forward part of their date 0 money balances for use at date 1. But then the quantity of money exchanged for goods is not the same at subperiods 0_1 and 0_3 although the total quantity of goods exchanged remains unchanged. Thus, the prices at which agents sell their initial endowment to the central exchange at 0_1 must differ from the prices to which they buy their consumption bundles at 0_3. The budget constraint is then different from the one given in (i) of Definition 1 and the previous concept of a monetary equilibrium does not apply.

Since the money supplies M that lead to positive interest-rate equilibria depend upon the data of the economy $\mathcal{E}(u, Y; \omega, \eta)$ in a complex way, it is difficult to analyze monetary equilibria if we restrict attention to positive interest-rate equilibria from the start. We are thus led to extend the concept of a monetary equilibrium to include *negative* interest-rate equilibria $(q_1 > 1)$. We will show that with this extended concept, a monetary equilibrium exists for a generic subset of $\Omega \times \mathfrak{M}$. Furthermore, such equilibria are locally finite and smooth functions of the parameters (ω, η, M). Once these results have been obtained, it is straightforward to find the positive interest-rate equilibria and to carry out the basic qualitative analysis of the effects of changes in monetary policy.

Remark 2. It is clear that if agents can carry money from one period to the next, the interest rate cannot become negative. However, the negative nominal interest-rate equilibria can be given an economic interpretation if we assume that *separate currencies* are issued for use at dates 0 and 1 and that the date 0 currency is not legal tender for transactions at date 1. Although there are circumstances under which such an interpretation may be of interest, it does not deal with the important issue of extending the concept of equilibrium so as to allow agents to hoard part of their money balances for use at date 1. Such an extension has been given by Magill and Quinzii (1991) for the case of an exchange economy: A similar analysis for the case of a production economy becomes complex – especially if we are to obtain interesting comparative statics results. We have, therefore, chosen to concentrate on the more tractable positive nominal interest-rate equilibria.

5 Analysis via reduced form and pseudoequilibrium

The first step is to establish that monetary equilibria exist. This, however, is not a trivial problem since the budget sets, (10), do not vary continuously with the spot prices. We will show that the existence proof of Husseini, Lasry, and Magill (HLM) (1990) for an *exchange economy* with incomplete financial assets, consisting solely of real assets (equity and futures contracts), can be extended to a *production economy* in which there are both real and nominal assets in addition to money. To do this, we will apply the fixed-point theorem of HLM.

To carry out the existence proof and also for the subsequent comparative static analysis, it is convenient to introduce the concept of a *reduced-form equilibrium* and the associated concept of a *pseudoequilibrium*. These are essentially two types of *constrained* Arrow-Debreu equilibria in which asset trades and asset prices are eliminated by using the no-arbitrage condition (13). The $J + K$ asset prices q are replaced by a vector of state prices $\beta = (1, \beta_1) \in \mathbb{R}_{++}^{S+1}$ in such a way that the map $q \leftrightarrow \beta$ is one to one. A pseudoequilibrium has the additional property that the date 1 subspace of income transfers achievable by trading in the assets is replaced by a surrogate $J + K$ dimensional subspace of income transfers.

5.1 *Derivation of reduced-form equilibrium*

This concept is derived as follows. Substituting (13) written as

$$q' = \beta_1 D_1, \qquad q'' = \beta_1 N_1 \tag{20}$$

into the date 0 budget equation (8) (i) gives

$$p_0(x_0^i - w_0^i) = \beta D z_0'^i + N_0 z_0''^i - \beta_1[D_1, N_1]z^i$$
$$= \beta\cdot[p\,\Box\,(y+\eta)]z_0'^i + N_0 z_0''^i - \beta_1(p_1\,\Box\,(x_1^i - w_1^i))$$

in view of (19) and (8) (ii). Let

$$\gamma_0^i(p, \beta, y, \eta) = \beta\cdot(p\,\Box\,(y+\eta))z_0'^i + N_0 z_0''^i$$

denote the *initial financial wealth* of consumer i, then the date 0 budget equation becomes

$$\beta\cdot(p\,\Box\,(x^i - w^i)) = \gamma_0^i(p, \beta, y, \eta). \tag{21}$$

Substituting D_1 in (17) into the date 1 equation 8 (ii) gives

$$p_1\,\Box\,(x_1^i - w_1^i) = [p_1\,\Box\,(y_1+\eta_1), N_1]\gamma^i, \tag{22}$$

where $\gamma^i = (\gamma'^i, \gamma''^i) = ([1-\xi']^{-1}z'^i, \xi''[1-\xi']^{-1}z'^i + z''^i)$.

Thus, each consumer has access to any date 1 *income transfers lying in the subspace of* $\mathbb{R}^S$

$$\langle p_1\,\Box\,(y_1+\eta_1), N_1\rangle$$

regardless of the financial policies $\xi = (\xi', \xi'') \in \mathbb{R}^J \times \mathbb{R}^K$ *chosen by the firms.* In other words, since consumers and firms have access to the same financial markets, consumers can undo any financial policy chosen by the firms. The assertion that the budget sets are uninfluenced by firms choices of financial policies constitutes the second part of the Modigliani–Miller theorem (see DeMarzo, 1987).

Let $V = [D_1, N_1]$. If $\dim\langle V\rangle < S$, we say that the (asset) markets are *incomplete.* It is clear that the dimension of the set of no-arbitrage vectors satisfies

$$\dim\{\beta \in \mathbb{R}_{++}^{S+1} \mid \beta_1 V = q\} = S - \dim\langle V\rangle.$$

Despite this multiplicity of no-arbitrage vectors, we can make the transformation $q \mapsto \beta = (1, \beta_1)$ one-to-one by selecting a *particular* no-arbitrage vector – any other choice of no-arbitrage β will then generate the same budget sets for the agents. In particular, by (14) the present-value vector $\pi^i = (1, \pi_1^i)$ of any consumer is a no-arbitrage vector. Thus, we may select $\beta = \pi^1$. This choice of β is equivalent to eliminating the date 1 constraints (22) for consumer 1. Thus, if we introduce the date 0 *present-value prices*

$$P = \beta\,\Box\,p, \tag{23}$$

then the budget set of consumer 1 becomes

$$\mathbb{B}^1(P, y; \omega, \eta) = \{x^1 \in \mathbb{R}_+^n \mid P(x^1 - w^1) = \gamma_0^1(P, y, \eta)\}. \tag{24}$$

If we define the diagonal matrix

$$[\beta_1] = \begin{bmatrix} \beta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_S \end{bmatrix}$$

and the ρ-dimensional subspace of $\mathbb{R}^S$ $(K \le \rho \le J+K)$

$$L = \langle P_1\,\Box\,(y_1+\eta_1), [\beta_1]N_1\rangle, \tag{25}$$

then the budget sets of consumers $i = 2, \ldots, I$ can be written as

$$\mathbb{B}^i(P, y, L; \omega, \eta) = \left\{x^i \in \mathbb{R}_+^n \,\middle|\, \begin{matrix} P(x^i - w^i) = \gamma_0^i(P, y, \eta) \\ P_1\,\Box\,(x_1^i - w_1^i) \in L \end{matrix}\right\}. \tag{26}$$

Let $G^\rho(\mathbb{R}^S)$ denote the *Grassmanian manifold* of *ρ-dimensional subspaces of* $\mathbb{R}^S$. Any $L \in G^\rho(\mathbb{R}^S)$ can be represented by a system of equations. More precisely for any $L \in G^\rho(\mathbb{R}^S)$, there is a permutation $\sigma \in \Sigma_{1,\ldots,S}$ (the set of permutations of $1, \ldots, S$) and a $\rho \times (S-\rho)$ matrix $A_\sigma(L)$ such that

$$L = \{v \in \mathbb{R}^S \mid [I \mid A_\sigma(L)]E_\sigma v = 0\},$$

where E_σ is the permutation matrix induced by σ. Although there are several permutations $\sigma \in \Sigma_{1,\ldots,S}$ that can be chosen, for any *given* permutation σ, the matrix $A_\sigma(L)$ is unique. (For a discussion of the problem of representing $G^\rho(\mathbb{R}^S)$, see, for example, Duffie and Shafer, 1985.) Thus, if we define

$$[P_1] = \begin{bmatrix} P_1^T & 0 & \cdots & 0 \\ 0 & P_2^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_S^T \end{bmatrix}, \qquad Q_\sigma(L) = [I \mid A_\sigma(L)]E_\sigma$$

$$\Phi_\sigma(P, L) = \left[\begin{array}{c|ccc} P_0^T & P_1^T & \cdots & P_S^T \\ \hline 0 & & Q_\sigma(L)[P_1] & \end{array}\right]. \tag{27}$$

Then the budget set (26) can be written as

$$\mathbb{B}^i(P, L, y, \omega, \eta) = \left\{x^i \in \mathbb{R}_+^n \,\middle|\, \Phi_\sigma(P, L)(x^i - w^i) = \begin{bmatrix} \gamma_0^i(P, y, \eta) \\ 0 \end{bmatrix}\right\}, \tag{28}$$

where L is given by (25). The change in the representation of the budget set from (10) to (28) leads to a change in the associated Lagrange multipliers from $\pi^i \in \mathbb{R}_{++}^{S+1}$ to $\mu_\sigma^i = (\mu_{\sigma 0}^i, \ldots, \mu_{\sigma, S-\rho}^i) \in \mathbb{R}^{S-\rho+1}$. To express the objective functions of the firms, we must be able to recover the π^i vectors in the new variables. This can be done by noting that the first-order conditions under the two representations lead to the same consumption bundle x^i so that

$$\lambda_0^i \pi^i \square p = D_{x^i} u^i = \mu_\sigma^i \Phi_\sigma(P, L).$$

Since $\lambda_0^i = \mu_{\sigma 0}^i$, if we define the *normalized* multiplier

$$\nu_\sigma^i = (1, \nu_{\sigma 1}^i, \ldots, \nu_{\sigma, S-\rho}^i) = \left(1, \frac{\mu_{\sigma 1}^i}{\mu_{\sigma 0}^i}, \ldots, \frac{\mu_{\sigma, S-\rho}^i}{\mu_{\sigma 0}^i}\right),$$

then $\pi^i \square p = \nu_\sigma^i(P, y, L)\Phi_\sigma(P, L)$. Thus, even though the terms ν_σ^i and Φ_σ separately depend on the permutation σ (i.e., the representation of L) chosen, the product $\nu_\sigma^i \Phi_\sigma$ is independent of σ and depends only on L. Thus, we can write

$$\pi^i \square p = (\nu^i \Phi)(P, y, L).$$

For the purpose of writing the objective of the firms, agent 1 can be treated symmetrically with the other agents with the convention that $\mu^1 = (\mu_0^1, 0, \ldots, 0)$ and $\nu^1 = (1, 0, \ldots, 0)$. Let $\nu = (\nu^1, \ldots, \nu^I)$. Then the previous transformations have reduced the study of a monetary equilibrium to the study of the following analytically more tractable concept of equilibrium.

Definition 2. *A* reduced-form (RF) equilibrium *of rank ρ ($K \le \rho \le J+K$) for the economy $\mathcal{E}$ is a pair of actions and prices*

$$((\bar{x}, \bar{y}), (\bar{P}, \bar{L}, \bar{\beta}, \bar{\nu})) \in \mathbb{R}_+^{nI} \times \mathbb{R}^{nJ} \times \mathbb{R}_{++}^n \times G^\rho(\mathbb{R}^S) \times \mathbb{R}_{++}^{S+1} \times \mathbb{R}^{(S-\rho+1)I},$$

such that

(i) $(\bar{x}^i, \bar{\nu}^i)$, $i = 1, \ldots, I$, *satisfy*

$$\bar{x}^i = \underset{x_i \in \mathbb{B}^i(\bar{P}, \bar{y}, \bar{L}; \omega, \eta)}{\arg\max} u^i(x^i), \quad i = 1, \ldots, I,$$

with multipliers $\bar{\nu}^i$, $i = 1, \ldots, I$,

where $\mathbb{B}^1$ *is defined by* (24) *and* $\mathbb{B}^i$ *is defined by* (28), $i = 2, \ldots, I$.

(ii) $\bar{y}^j$, $j = 1, \ldots, J$, *satisfy*

$$\bar{y}^j = \underset{y^j \in Y^j}{\arg\max} \sum_{i=1}^I z_0^{ij}(\bar{\nu}^i \Phi)(\bar{P}, \bar{y}, \bar{L}) \cdot y^j, \quad j = 1, \ldots, J.$$

(iii) $$\sum_{i=1}^I (\bar{x}^i - w^i) = \sum_{j=1}^J (\bar{y}^j + \eta^j).$$

(iv) $\langle \bar{P}_1 \square (\bar{y}_1 + \eta_1), [\bar{\beta}_1] N_1 \rangle = \bar{L}$.

(v) $$\bar{P}_0\left(\sum_{i=1}^I w_0^i + \sum_{j=1}^J \eta_0^j\right) = M_0,$$

$$\bar{P}_s\left(\sum_{i=1}^I w_s^i + \sum_{j=1}^J (\bar{y}_s^j + \eta_s^j)\right) = \bar{\beta}_s M_s, \quad s = 1, \ldots, S.$$

We say that a reduced-form equilibrium has a positive nominal interest rate *if $\sum_{s=1}^S \bar{\beta}_s < 1$, where $\bar{\beta} = (1, \bar{\beta}_1)$.*

This analysis has established part (i) of the following proposition; part (ii) is left to the reader.

Proposition 1. (i) *If $((\bar{x}, \bar{z}), (\bar{y}, \bar{\xi}); (\bar{p}, \bar{q}, \bar{\pi}))$ is a (positive interest rate) monetary equilibrium of rank ρ then there exist $(\bar{\beta}, \bar{L}, \bar{\nu})$ such that $((\bar{x}, \bar{y}), (\bar{\beta} \square \bar{p}, \bar{L}, \bar{\beta}, \bar{\nu}))$ is a (positive interest rate) reduced-form equilibrium of rank ρ.* (ii) *If $((\bar{x}, \bar{y}), (\bar{P}, \bar{L}, \bar{\beta}, \bar{\nu}))$ is a (positive interest rate) reduced-form equilibrium of rank ρ then there exist $((\bar{z}, \bar{\xi}), (\bar{p}, \bar{q}, \bar{\pi}))$ with $\bar{\beta} \square \bar{p} = \bar{P}$, $\bar{q} = \bar{\beta}_1[\bar{p}_1 \square (\bar{y}_1 + \eta_1), N_1]$ such that $((\bar{x}, \bar{z}), (\bar{y}, \bar{\xi}); (\bar{p}, \bar{q}, \bar{\pi}))$ is a (positive interest rate) monetary equilibrium of rank ρ.*

5.2 *Existence of equilibrium*

To establish the existence of a reduced-form equilibrium of rank $\rho = J+K$, we need to relax the requirement of equality in Definition 2 (iv): We want the dimension of the subspaces L to remain unchanged so that L can be parametrized by $G^\rho(\mathbb{R}^S)$, while allowing for the fact that the dimension of $\langle P_1 \square (y_1 + \eta_1), [\beta_1] N_1 \rangle$ will change as the prices P and production plans y change. This leads to the following concept.

Definition 3. *A* pseudoequilibrium *for the economy $\mathcal{E}$ is a reduced-form equilibrium of rank $J+K$ in which Definition 2* (iv) *is replaced by*

(iv)′ $\langle \bar{P}_1 \square (\bar{y}_1 + \eta_1), [\bar{\beta}_1] N_1 \rangle \subset \bar{L}$.

We need one more restriction on the model in order to be sure of having an equilibrium. If the initial level of indebtedness of agent i is excessive relative to his initial resources then he may not be able to afford any positive consumption bundle: In this case, the budget set $\mathbb{B}^i$ is empty and his demand function is not well-defined. Let

$$\hat{\gamma}_0^i(\omega, \eta, M) = \min\left\{P_0 w_0^i \;\middle|\; P_0 \in \mathbb{R}_+^H, P_0\left(\sum_{i=1}^I w_0^i + \sum_{j=1}^J \eta_0^j\right) = M_0\right\},$$

$i = 1, \ldots, I$.

Assumption E (Initial indebtedness). $N_0 z_0^{\prime\prime i} > -\hat{\gamma}_0^i(\omega, \eta, M)$, $i = 1, \ldots, I$.

A set $A \subset \Omega$ is said to be *generic* if its complement is a closed set of measure zero in Ω.

Theorem 1 (Existence and smoothness). *Under Assumptions A to E, there exists a generic set $\Gamma \subset \Omega \times \mathfrak{M}$ such that for all $(\omega, \eta, M) \in \Gamma$ (i) there exists a reduced-form equilibrium of rank $J+K$; (ii) there is at most a finite number of reduced-form equilibria of rank $J+K$, each of which is locally a smooth function of the parameters (ω, η, M).*

Proof: We first establish the following lemma.

Lemma 1. *Under the hypotheses of Theorem 1, there exists a pseudo-equilibrium for all $(\omega, \eta, M) \in \Omega \times \mathfrak{M}$.*

Proof: The idea is to construct appropriate maps so that the fixed-point theorem of HLM (1990) can be applied. The maps that handle the *production* side of the economy are borrowed from the existence proof for an economy with increasing returns by Beato and Mas-Colell (1985).

Step 1 (Truncated sets). Let $X^i = \mathbb{R}^n_+$ denote the consumption set of agent i $(i = 1, \ldots, I)$. Assumption B implies that the set of *feasible* allocations

$$\mathfrak{F} = \left\{(x, y) \in \prod_{i=1}^{I} X^i \times \prod_{j=1}^{J} Y^j \;\middle|\; \sum_{i=1}^{I} (x^i - w^i) \leq \sum_{j=1}^{J} (y^j + \eta^j)\right\}$$

is compact. Let $\hat{X}^i$ denote the projection of $\mathfrak{F}$ on X^i and let $\hat{Y}^j$ denote the projection of $\mathfrak{F}$ on Y^j. For each firm, we modify the production set $Y^j \to \tilde{Y}^j$ in such a way that $\tilde{Y}^j$ coincides with Y^j on $\hat{Y}^j$ and production plans in $\tilde{Y}^j$ involving large amounts of inputs are inefficient (no additional output with additional inputs). To simplify the analysis, we consider first the case where the subspace E^j in Assumption B(3) satisfies $E^j = \mathbb{R}^n$. $\tilde{Y}^j$ is then constructed as follows. Pick $\hat{a}^j \in \mathbb{R}^n_{++}$ such that $\hat{Y}^j \subset \mathbb{R}^n_+ - \hat{a}^j$. Then choose $a^j \gg \hat{a}^j$ and construct $\tilde{Y}^j$ (see Figure 2) such that $\partial\tilde{Y}^j$ is smooth and

$$y^j \in \tilde{Y}^j \Leftrightarrow y^j \in Y^j \quad \text{for all} \quad y^j \in \mathbb{R}^n_+ - \hat{a}^j,$$
$$y^j \in \tilde{Y}^j \Leftrightarrow y^j \in (Y^j \cap (\mathbb{R}^n_+ - a^j)) - \mathbb{R}^n_+ \quad \text{for all} \quad y^j \in (\mathbb{R}^n_+ - a^j)^c.$$

Let $\tilde{X}^i$ be a compact convex subset of $\mathbb{R}^n_+$ such that $\tilde{X}^i \subset \hat{X}^i$, $i = 1, \ldots, I$, and let

$$\tilde{\mathfrak{F}} = \left\{(x, y) \in \prod_{i=1}^{I} \tilde{X}^i \times \prod_{j=1}^{J} \tilde{Y}^j \;\middle|\; \sum_{i=1}^{I} (x^i - w^i) \leq \sum_{j=1}^{J} (y^j + \eta^j)\right\};$$

then $\tilde{\mathfrak{F}} = \mathfrak{F}$. Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ denote the economies with consumption and production sets $(X^1, \ldots, X^I, Y^1, \ldots, Y^J)$ and $(\tilde{X}^1, \ldots, \tilde{X}^I, \tilde{Y}^1, \ldots, \tilde{Y}^J)$, re-

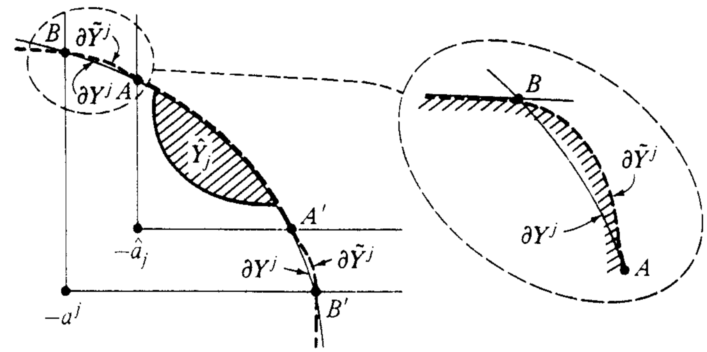

Figure 2. The smooth truncation $\tilde{Y}^j$ of Y^j.

spectively. Then, since agents maximize over convex sets the equilibria of $\mathcal{E}$ and $\tilde{\mathcal{E}}$ coincide.

Step 2 (Maps). Let $\Delta = \{v \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} v_j = 1\}$ denote the $n-1$ dimensional simplex. We will construct a pair of continuous functions (ϕ, ψ)

$$\phi : \Delta^{J+1} \times G^{J+K}(\mathbb{R}^S) \to \Delta^{J+1},$$
$$\psi : \Delta^{J+1} \times G^{J+K}(\mathbb{R}^S) \to \mathbb{R}^{(J+K)S}, \tag{29}$$

and apply the fixed-point theorem of HLM (1990, Theorem A) by which, if $(P, \hat{y}, L)$ is a typical element of $\Delta \times \Delta^J \times G^{J+K}(\mathbb{R}^S)$, then there exists $(P^*, \hat{y}^*, L^*) \in \Delta^{J+1} \times G^{J+K}(\mathbb{R}^S)$ such that

$$\phi(P^*, \hat{y}^*, L^*) = (P^*, \hat{y}^*), \quad \langle\psi(P^*, \hat{y}^*, L^*)\rangle \subset L^*. \tag{30}$$

By the assumption of free disposal, the boundary $\partial\tilde{Y}^j \cap (\mathbb{R}^n_+ - a^j)$ of $\tilde{Y}^j \cap (\mathbb{R}^n_+ - a^j)$ is homeomorphic to the simplex Δ under the homeomorphism

$$\alpha_j : \partial\tilde{Y}^j \cap (\mathbb{R}^n_+ - a^j) \to \Delta, \quad \alpha_j = \frac{y^j + a^j}{\sum_{s,h}(y^j_{sh} + a^j_{sh})}, \quad j = 1, \ldots, J,$$

and the *unit normal* map (i.e., $\sum_{s,h} n^j_{sh} = 1$)

$$n^j : \partial\tilde{Y}^j \cap (\mathbb{R}^n_+ - a^j) \to \Delta, \quad j = 1, \ldots, J,$$

is well-defined. In the analysis that follows, let $y^j = \alpha_j^{-1}(\hat{y}^j)$. If we let

$$\beta_0(P) = \frac{P_0(\sum_{i=1}^{I} w^i_0 + \sum_{j=1}^{J} \eta^j_0)}{M_0}$$

and define the modified date 0 revenue functions

$$\tilde{\gamma}_0^i(P,\hat{y}) = \sum_{j=1}^{J} z_0^{ij}\max(0, P(y^j+\eta^j)) + \beta_0 N_0 z_0''^{i}, \quad i=1,\dots,I, \tag{31}$$

and the associated demand functions

$$\begin{aligned} f^1(P,\hat{y},L) &= \arg\max\{u^1(x^1), x^1 \in \tilde{X}^1 \mid P(x^1-w^1) = \tilde{\gamma}_0^1(P,\hat{y})\}, \\ f^i(P,\hat{y},L) & \\ &= \arg\max\left\{u^1(x^i), x^i \in \tilde{X}^i \;\middle|\; \begin{matrix} P(x^i-w^i) = \tilde{\gamma}_0^i(P,\hat{y}) \\ P_1 \square (x_1^i - w_1^i) \in L \end{matrix}\right\}, \quad i=2,\dots,I, \end{aligned} \tag{32}$$

then the aggregate excess demand function

$$Z(P,\hat{y},L) = \sum_{i=1}^{I}(f^i(P,\hat{y},L)-w^i) - \sum_{j=1}^{J}(y^j+\eta^j)$$

satisfies Walras Law $PZ(P,\hat{y},L)=0$ whenever $P(y^j+\eta^j) \geq 0$, $j=1,\dots,J$. Furthermore, the truncation $\tilde{X}^1$ can be made sufficiently large so that $P_{sh}=0$ implies $Z_{sh}>0$ for all $s=0,\dots,S$, $h=1,\dots,H$. Let $\beta(P,\hat{y}) = (\beta_0(P), \beta_1(P,\hat{y}),\dots,\beta_S(P,\hat{y}))$, where

$$\beta_s(P,\hat{y}) = \frac{P_s(\sum_{i=1}^I w_s^i + \sum_{j=1}^J (y_s^j+\eta_s^j))}{M_s}, \quad s=1,\dots,S, \tag{33}$$

and let

$$b^j(P,\hat{y},L) = \frac{\sum_{i=1}^I z_0^{ij}(\nu^i\Phi)(P,y,L)}{\sum_{i=1}^I z_0^{ij}\sum_{s,h}(\nu^i\Phi)_{sh}(P,y,L)}, \quad j=1,\dots,J,$$

denote the vector of present-value prices of firm j, normalized to lie in the simplex, $\Phi(P,L)$ being given by (27) (for an appropriate choice of σ) and $\nu^i = \nu_\sigma^i(P,\hat{y},L)$ being the normalized multiplier induced by consumer i's maximum problem (32). Then the maps $(\phi,\psi) = (\phi^0,\phi^1,\dots,\phi^J,\psi)$ in (29) are defined as follows, for $s=0,1,\dots,S$, $h=1,\dots,H$:

$$\phi_{sh}^0(P,\hat{y},L) = \frac{(P_{sh}+\max(0, Z_{sh}(P,\hat{y},L)))}{1+\sum_{s,h}\max(0, Z_{sh}(P,\hat{y},L))},$$

$$\phi_{sh}^j(P,\hat{y},L) = \frac{(\hat{y}_{sh}^j + \max(0, b_{sh}^j(P,\hat{y},L) - n_{sh}^j(y^j))}{1+\sum_{s,h}\max(0, b_{sh}^j(P,\hat{y},L) - n_{sh}^j(y^j))},$$

$$\psi(P,\hat{y},L) = [P_1 \square (y_1+\eta_1), [\beta_1(P,\hat{y})]N_1].$$

Step 3 (The fixed point is an equilibrium). Let $(P^*,\hat{y}^*,L^*)$ denote the fixed point defined by (30). We will show that $((\bar{x},\bar{y}),(\bar{P},\bar{L},\bar{\beta},\bar{\nu}))$ defined by

$$\bar{x} = (f^1(P^*,L^*,\hat{y}^*),\dots,f^I(P^*,L^*,\hat{y}^*)), \quad \bar{y} = \alpha^{-1}(\hat{y}^*),$$

$$\bar{P} = \frac{P^*}{\beta_0(P^*)}, \quad \bar{L} = L^*, \quad \bar{\beta} = \frac{\beta(P^*,\hat{y}^*)}{\beta_0}, \quad \bar{\nu} = \nu^*,$$

is a pseudoequilibrium. $\phi^{j*} = \hat{y}^{j*} \Rightarrow b^{j*} = n^{j*} \Rightarrow$ firm j maximizes profit under the prices $\sum_{i=1}^I z_0^{ij}(\nu^{i*}\Phi^*) = \beta_0^* \sum_{i=1}^I z_0^{ij}(\bar{\nu}^i\bar{\Phi})$. Since $\eta^j \geq 0 \Rightarrow \sum_{i=1}^I z_0^{ij}(\bar{\nu}^{i*}\Phi^*)(y^{j*}+\eta^j) \geq 0$. $\langle\psi^*\rangle \subset L^* \Rightarrow Q_\sigma(L^*)[P_1](y_1^*+\eta_1) = 0 \Rightarrow 0 \leq \sum z_0^{ij}(\nu^{i*}\Phi^*)(y^{j*}+\eta^j) = P^*(y^{j*}+\eta^j)$. $P^* \gg 0$ since $P_{sh}^* = 0 \Rightarrow Z_{sh}^* > 0$. $\phi_{sh}^{0*} = P_{sh}^* \Rightarrow \max(0, Z_{sh}^*) = 0 \Rightarrow Z_{sh}^* \leq 0$. Since $\sum_{s,h} P_{sh}^* Z_{sh}^* = 0 \Rightarrow Z_{sh}^* = 0$, $\forall s,h$. Finally, equations (33) and $\bar{P} = P^*/\beta_0(P^*)$ ensure that the monetary equations in Definition 2(v) hold and that the budget sets in (32) coincide with those in (24) and (28). □

The proof of Theorem 1 is completed by establishing the following lemma.

Lemma 2. *Under the hypotheses of Theorem 1:* (i) *there exists a generic set* $\Gamma' \subset \Omega \times \mathfrak{M}$ *such that for all* $(\omega,\eta,M) \in \Gamma'$, *there are at most a finite number of pseudoequilibria and each pseudoequilibrium is locally a smooth function of the parameters* (ω,η,M); (ii) *there exists a generic set* $\Gamma \subset \Gamma'$ *such that for all* $(\omega,\eta,M) \in \Gamma$ *every pseudoequilibrium is a reduced-form equilibrium of rank* $J+K$.

Proof: The ideas behind the proofs of (i) and (ii) are simple. To show (i), we show that a pseudoequilibrium can be written as a solution of a system of equations in which the number of equations equals the number of unknowns. To prove (ii), we show that generically there are no pseudoequilibria of rank $\rho < J+K$. When the condition rank$[P_1 \square (y_1+\eta_1), [\beta_1]N_1] < J+K$ is expressed as a system of equations and added to the equations of a pseudoequilibrium, we obtain a system with *more* equations than unknowns, which generically has no solution. To prove (i) and (ii) *without perturbing the nominal returns matrix* N_1, we must be careful to express precisely that we work with the subset of $G^{J+K}(\mathbb{R}^S)$ *consisting of subspaces L that contain* $\langle\beta_1 N_1\rangle$ (see Step 1 of the proof that is completed in the Appendix).

6 Effects of changes in monetary policy

In this section, we study the way changes in monetary policy affect the equilibrium allocation $(\bar{x},\bar{y})$. In macroeconomics, a distinction is made between *anticipated* and *unanticipated* changes in monetary policy. If agents inherit no initial holdings of bonds ($z_0''=0$), then they are free to adapt their portfolios to any change in M. In this case, we say that the agents can *anticipate* monetary changes. If the agents inherit initial holdings of bonds ($z_0'' \neq 0$), then these initial portfolios cannot be adapted to changes in the date 0 monetary policy. In this case, we say that date 0 monetary changes are *unanticipated.* We want to analyze the effect of anticipated and unanticipated monetary changes on positive interest-rate equilibria.

6.1 *Positive interest-rate parameters*

In the previous section, we have shown that there is a generic set $\Gamma \subset \Omega \times \mathfrak{M}$ such that monetary equilibria of rank $J+K$ exist for all parameter values $(\omega, \eta, M) \in \Gamma$ and all such equilibria are locally smooth functions of the parameters. *Let $\hat{\Gamma} \subset \Gamma$ denote the subset of parameters for which all equilibria have a positive nominal interest rate.* Note that if (P, y, L, β_1) is a reduced-form equilibrium price for the parameter value (ω, η, M), then $(P, y, L, \beta_1/\alpha)$ is a reduced-form equilibrium price for the parameter value (ω, η, M') with $M' = (M_0, \alpha M_1)$ for $\alpha \neq 0$: If the money supply is doubled in each state s $(s \geq 1)$, then the value of a contract promising one franc at date 1 is halved. Thus, a sufficient (proportional) increase in the money supply at date 1 will always force the nominal interest rate to be positive. Consider the correspondence $\gamma: \Gamma \to \Gamma$ defined by

$$\gamma(\omega, \eta, M) = \{(\omega, \eta, M_0, \alpha M_1) \mid \alpha > \alpha^*(\omega, \eta, M)\},$$

where $\alpha^*(\omega, \eta, M) = \max\{\sum_{s=1}^{S} \beta_s \mid (P, y, L, \beta)$ is an RF equilibrium for $(\omega, \eta, M)\}$. Then $\hat{\Gamma} = \gamma(\Gamma)$.

6.2 *Anticipated monetary changes*

Consider a regular parameter value $(\bar{\omega}, \bar{\eta}, \bar{M}) \in \hat{\Gamma}$ and an associated equilibrium "price" $(\bar{P}, \bar{y}, \bar{L}, \bar{\beta})$. Since at an equilibrium of rank $J+K$,

$$\bar{L} = \langle \bar{P}_1 \square (\bar{y}_1 + \eta_1), [\bar{\beta}_1] N_1 \rangle,$$

the variables $(\bar{P}, \bar{y}, \bar{\beta})$ determine completely the equilibrium. For equilibria with parameter values in $\hat{\Gamma}$, we can thus view

$$\mathcal{P} = \mathbb{R}^{n}_{++} \times \mathbb{R}^{nJ} \times \mathbb{R}^{S}_{++}$$

as the space of "prices." By Theorem 1(ii), we know that there exists a neighborhood $\mathfrak{N}_{(\bar{\omega}, \bar{\eta}, \bar{M})} \subset \hat{\Gamma}$ and a function $g: \mathfrak{N}_{(\bar{\omega}, \bar{\eta}, \bar{M})} \to \mathcal{P}$ that gives the equilibrium prices (P, y, β) for each $(\omega, \eta, M) \in \mathfrak{N}_{(\bar{\omega}, \bar{\eta}, \bar{M})}$. Since the comparative static analysis is confined to local changes in M, we consider the neighborhood

$$\mathfrak{N}_{\bar{M}} = \{M \in \mathfrak{M} \mid (\bar{\omega}, \bar{\eta}, M) \in \mathfrak{N}_{(\bar{\omega}, \bar{\eta}, \bar{M})}\}.$$

Let $g(\bar{\omega}, \bar{\eta}, M) = (P(M), y(M), \beta(M))$ and let $f(g(\bar{\omega}, \bar{\eta}, M), \bar{\omega}, \bar{\eta}) = x(M)$, where $f = (f^1, \ldots, f^I)$ is the vector of demand functions $f^i: \mathcal{P} \times \Omega \to \mathbb{R}^n$ in Definition 2(i).

Theorem 2 (Neutrality of money). *Let Assumptions A to E hold, let $z_0'' = 0$, and let $(\bar{\omega}, \bar{\eta}, \bar{M}) \in \hat{\Gamma}$.*

(i) *For all $M = (\alpha \bar{M}_0, \alpha' \bar{M}_1) \in \mathfrak{N}_{\bar{M}}$ with $\alpha > 0$, $\alpha' > 0$,*

$$(x(M), y(M)) = (x(\bar{M}), y(\bar{M})). \tag{34}$$

(ii) *If $J+K = S$, then (34) holds for all $M \in \mathfrak{N}_{\bar{M}}$.*

Proof: (i) It is easy to check in Definition 2(i) to (v) that

$$(P(M), y(M), \beta(M)) = (\alpha P(\bar{M}), y(\bar{M}), (\alpha/\alpha')\beta(\bar{M}))$$

and $x(M) = x(\bar{M})$. (ii) follows because each agent's budget set (26) reduces to the Arrow–Debreu budget set with shareholdings $(z_0'^i)$, and each firm's objective function reduces to Py^j since $\nu \Phi = P$. □

Remark 3. In Theorem 2(i), when $\alpha = \alpha' \neq 1$, spot prices p change by the factor α, but the nominal interest rate r_1 is unchanged. When $\alpha > 1$, $\alpha' = 1$, date 0 spot prices increase and the nominal interest rate falls; when $\alpha = 1$, $\alpha' > 1$, date 1 spot prices increase and the nominal interest rate increases. These are nominal changes that do not affect consumption or production decisions.

Theorem 2 points out that there are certain monetary changes that have no real effects. To take the analysis further, we restrict ourselves to the study of marginal changes at the equilibrium under consideration. By Theorem 1, the equilibrium allocation map

$$(x, y): \mathfrak{N}_{\bar{M}} \to \mathbb{R}^{nI} \times \mathbb{R}^{nJ}$$

is differentiable at $\bar{M}$ and we let

$$(D_{\bar{M}} x, D_{\bar{M}} y): \mathbb{R}^{S+1} \to \mathbb{R}^{nI} \times \mathbb{R}^{nJ}$$

denote the derivative map. We say that a monetary change $dM \in \mathbb{R}^{S-1}$ has *real effects* if $(dx, dy) = (D_{\bar{M}} x, D_{\bar{M}} y) dM \neq 0$; we say that it has *production effects* if $dy = (D_{\bar{M}} y) dM \neq 0$. To describe precisely the marginal monetary changes that have real effects (which amounts to describing $\ker D_{\bar{M}}(x, y)$) requires a more precise knowledge of the structure of the bond matrix N_1. *Without adding any further conditions on N_1, we can, however, give a lower bound on the number of linearly independent directions in which monetary changes dM have real effects.* As we will see in the case where the riskless bond is the only nominal asset $(K = 1)$, this bound is exact. We will need an assumption regarding the distribution of ownership of firms amongst consumers to show that monetary changes have production effects. To this end, let $\mathcal{I}' = \{i \mid z_0^{ij} > 0$ for some $j \in \{1, \ldots, J\}\}$ and let $I' = \# \mathcal{I}'$ (the number of agents in $\mathcal{I}'$).

Theorem 3 (Real effects of anticipated monetary changes). *Let Assumptions A to E hold and let* $z_0'' = 0$.

(i) *If* (a) $J+K<S$, (b) $K<I$, *then there exists a generic set* $\hat{\Gamma}^* \subset \hat{\Gamma}$ *such that if* $(\bar{\omega}, \bar{\eta}, \bar{M}) \in \hat{\Gamma}^*$, *there is a subspace* $\Lambda \subset \mathbb{R}^{S-1}$ *with* $\dim \Lambda \geq S-J-K$ *for which*

$$(dx, dy) = (D_{\bar{M}}x, D_{\bar{M}}y)dM \neq 0, \quad \forall dM \in \Lambda \backslash \{0\}.$$

(ii) *If* (a) $J+K<S$, (b) $K<I' \leq J$, (c) $(z_0^{i}, i \in \mathcal{I}')$ *are linearly independent,* (d) $E^j = \mathbb{R}^n$, $j=1,\ldots,J$, *then* (i) *holds with* $dy \neq 0$.

Proof: (i) Let $\bar{L} = \langle \bar{P}_1 \square (\bar{y}_1 + \bar{\eta}_1), [\bar{\beta}_1]N_1 \rangle$ and pick $d\tilde{\beta}_1 \in \bar{L}^\perp$. Consider the change dM such that $dM_0 = 0$, $dM_s/\bar{M}_s = -d\tilde{\beta}_s/\bar{\beta}_s$, $s=1,\ldots,S$. We show that $(dx, dy) = (D_{\bar{M}}x, D_{\bar{M}}y)dM \neq 0$. Suppose not. Then differentiating the first-order condition for agent 1 gives $dx^{1T}D^2u^1 = d\lambda_0^1\bar{P} + \bar{\lambda}_0^1 dP$, and since $dx^1 = 0$, $dP = -(d\lambda_0^1/\bar{\lambda}_0^1)\bar{P}$. From the date 0 monetary equation, $d(P_0(\sum_{i=1}^I \bar{w}_0^i + \sum_{j=1}^J \bar{\eta}_0^j)) = dM_0 = 0 \Rightarrow dP = 0$. From the date s monetary equation, since $dy = 0$ and $dP_s = 0$, $0 = d(\beta_s M_s) = \bar{\beta}_s dM_s + d\beta_s \bar{M}_s$; thus $d\beta_s = d\tilde{\beta}_s$, $s=1,\ldots,S$. Since $d\beta_1 \in \bar{L}^\perp$, $[\bar{\beta}_1 + d\beta_1]N_1^1 = \bar{\beta}_1 + d\beta_1 \notin \bar{L}$; thus

$$\langle \bar{P}_1 \square (\bar{y}_1 + \eta_1), [\bar{\beta}_1 + d\beta_1]N_1 \rangle \neq \bar{L}. \tag{35}$$

Since $dy_1 = 0$, $dP_1 = 0$, and $dx_1 = 0$,

$$\bar{P}_1 \square (\bar{x}_1^i - \bar{w}_1^i) \in \langle \bar{P}_1 \square (\bar{y}_1 + \bar{\eta}_1), [\bar{\beta}_1 + d\beta_1]N_1 \rangle, \quad i = 2, \ldots, J+1. \tag{36}$$

Lemma 3, which follows, implies that (35) contradicts (36). This argument holds for all $dM \in \Lambda'$ defined by

$$\Lambda' = \left\{ dM \,\middle|\, dM_0 = 0,\ dM_1 = -\left[\frac{\bar{M}_1}{\bar{\beta}_1}\right] d\beta_1,\ d\beta_1 \in \bar{L}^\perp \right\}, \tag{37}$$

where

$$\left[\frac{\bar{M}_1}{\bar{\beta}_1}\right] = \begin{bmatrix} \bar{M}_1/\bar{\beta}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{M}_S/\bar{\beta}_S \end{bmatrix}$$

so that $\Lambda' \cap \ker D_{\bar{M}}(x, y) = \{0\}$. Thus, there exists a subspace Λ'' such that

$$\ker D_{\bar{M}}(x, y) \oplus \Lambda' \oplus \Lambda'' = \mathbb{R}^{S+1}.$$

Since $\dim \bar{L}^\perp = S - (J+K)$, it follows that $\Lambda = \Lambda' \oplus \Lambda''$ satisfies $\dim \Lambda \geq S-(J+K)$ and the proof of (i) is complete.

Lemma 3. *If* $I > K$, *then there exists a generic set* $\hat{\Gamma}^* \subset \hat{\Gamma}$ *such that if* $((\bar{x}, \bar{y}), (\bar{P}, \bar{y}, \bar{\beta}))$ *is a reduced-form equilibrium for* $(\bar{\omega}, \bar{\eta}, \bar{M}) \in \hat{\Gamma}^*$, *then*

$$\bar{L} = \langle \bar{P}_1 \square (\bar{x}_1^{i_1} - \bar{w}_1^{i_1}), \ldots, \bar{P}_1 \square (\bar{x}_1^{i_K} - \bar{w}_1^{i_K}), \bar{P}_1 \square (\bar{y}_1^1 + \bar{\eta}_1^1), \ldots, \bar{P}_1 \square (y_1^J + \bar{\eta}_1^J) \rangle$$

for every K-element subset $\{i_1, \ldots, i_K\}$ *of* $\{1, \ldots, I\}$.

Proof: See the Appendix.

(ii) Consider $dM \in \Lambda'$ defined by (37) and suppose $dy = 0$. Let $n^j : \partial Y^j \to \Delta$ denote the unit normal map for the boundary ∂Y^j of firm j's production set. By the first-order conditions for profit maximization, there exists $\alpha_j \neq 0$ such that $\alpha_j n^j(\bar{y}^j) = \sum_{i=1}^I z_0^{ij}(\bar{\nu}^i \bar{\Phi})$. Differentiating and using $dy^j = 0$ gives $d\alpha_j n^j(\bar{y}^j) = \sum_{i=1}^I z_0^{ij}(d\nu^i \bar{\Phi} + \bar{\nu}^i d\Phi)$ and since $\bar{\nu}^i = (1, \bar{\nu}_1^i, \ldots, \bar{\nu}_{S-(J+K)}^i)$ and $\sum_{i=1}^I z_0^{ij} = 1$, the date 0 component implies $dP_0 = d\alpha_j n^j(\bar{y}^j) = (d\alpha_j/\alpha_j)\bar{P}_0$. Since $\bar{P}_0(\sum_{i=1}^I w_0^i + \sum_{j=1}^J \eta_0^j) = \bar{M}_0$ and $dM_0 = 0$, $dP_0 = 0 \Rightarrow d\alpha_j = 0$, $j = 1, \ldots, J$, so that $\sum_{i=1}^I z_0^{ij}(d\nu^i \bar{\Phi} + \bar{\nu}^i d\Phi) = 0$, $j = 1, \ldots, J$. Assumption (c) in (ii) then implies

$$d\nu^i \bar{\Phi} + \bar{\nu}^i d\Phi = 0, \quad \forall i \in \mathcal{I}'. \tag{38}$$

Without loss of generality, we may assume agent 1 satisfies $z_0'^1 \neq 0$ $(1 \in \mathcal{I}')$. Since $\bar{\nu}^1 = (1, 0, \ldots, 0)$, (38) implies $dP = 0$. Differentiating the first-order conditions for agent i $D_{\bar{x}^i}u^i = \bar{\mu}^i \bar{\Phi}$ (where $D_{\bar{x}^i}u^i$ is viewed as a row vector) gives

$$dx^{iT} D_{\bar{x}^i}^2 u^i = d\mu^i \bar{\Phi} + \bar{\mu}^i d\Phi, \quad \forall i \in \mathcal{I}'.$$

Since $\nu^i = \mu^i/\mu_0^i$, $d\mu^i = d\mu_0^i \nu^i + \mu_0^i d\nu^i$. By (38),

$$dx^{iT} D_{\bar{x}^i}^2 u^i = d\mu_0^i \bar{\nu}^i \bar{\Phi}, \quad \forall i \in \mathcal{I}'.$$

Differentiating the budget constraint

$$\Phi(\bar{x}^i - \bar{w}^i) = \begin{bmatrix} \bar{\gamma}_0^i \\ 0 \end{bmatrix}, \quad \forall i \in \mathcal{I}', \tag{39}$$

gives $\bar{\Phi} dx^i = -d\Phi(\bar{x}^i - \bar{w}^i)$, since $dP_0 = 0$, $dy = 0 \Rightarrow d\gamma_0^i = 0$. Thus, by (38), (39) and $d\nu_0^i = 0$,

$$\nu^i \bar{\Phi} dx^i = -\bar{\nu}^i d\Phi(\bar{x}^i - \bar{w}^i) = d\nu^i \bar{\Phi}(\bar{x}^i - \bar{w}^i) = 0, \quad \forall i \in \mathcal{I}'.$$

Thus, $\forall i \in \mathcal{I}'$, $\bar{\nu}^i \bar{\Phi} dx^i = 0 \Rightarrow D_{\bar{x}^i}u^i dx^i = 0$, and $dx^{iT} D_{\bar{x}^i}^2 u^i dx^i = 0$. From Assumption A(3), we conclude $dx^i = 0$, $i \in \mathcal{I}'$. Since $I' > K$, applying the argument in (i) previously completes the proof. □

Remark 4. If the riskless bond is the only nominal asset $(K=1)$, then it is clear from the proof of (i) that

$$\ker(D_{\bar{M}}(x, y)) = \tilde{\Lambda} = \left\{ dM \,\middle|\, dM_1 = -\left[\frac{\bar{M}_1}{\bar{\beta}_1}\right] d\beta_1,\ d\beta_1 \in \bar{L} \right\}$$

so that there are exactly $J+K+1$ directions of change dM that do not affect the equilibrium allocation. If there are other bonds ($K \geq 2$), then $\ker(D_{\bar{M}}(x,y))$ can be a strict subspace of $\tilde{\Lambda}$, since $d\beta_1 \in \bar{L}$ does not imply $[d\beta_1]N^k \in \bar{L}$ for $k \geq 2$. If we are prepared to view the bond returns $N^2, \ldots, N^K$ as being "drawn at random," then a result of Geanakoplos and Mas-Colell (1989, Theorem 2) implies that *generically in* $(\omega, \eta, N^2, \ldots, N^K)$,

$$\ker(D_{\bar{M}}(x,y)) = \{dM \mid dM_1 = \alpha \bar{M}_1, \alpha \in \mathbb{R}\}$$

so that all marginal changes differing from those given in Theorem 2(i) *have real effects.* It can be argued, however, that using genericity with respect to $N^2, \ldots, N^K$ is a somewhat implausible economic assumption since it implies in particular that each bond has a different return in each state.

6.3 *Unanticipated monetary changes*

When agents have initial holdings of nominal assets, the conditions under which monetary policy is neutral can be stated as follows.

Theorem 2' (Neutrality of money). *Let Assumptions A to E hold and let* $(\bar{\omega}, \bar{\eta}, \bar{M}) \in \hat{\Gamma}$.

(i) *For all* $M = (\bar{M}_0, \alpha' \bar{M}_1) \in \mathfrak{N}_{\bar{M}}$ *with* $\alpha' > 0$,

$$(x(M), y(M)) = (x(\bar{M}), y(\bar{M})). \tag{40}$$

(ii) *If* $J+K=S$, *then* (40) *holds for all* $M \in \mathfrak{N}_{\bar{M}}$ *with* $M_0 = \bar{M}_0$.

Thus, date 1 monetary changes that were neutral in previous Section 6.2 remain neutral. Date 0 monetary changes are not, however, neutral, as we will now show. Since the effect of a change in M_0 depends fundamentally on *income effects,* we need the extension of the *Slutsky analysis* of consumer demand to the case of incomplete markets. This can be obtained as follows.

Slutsky analysis. Consider the first-order conditions for an agent with a budget set given by (28)

$$D_{x^i} u^i = \mu^i \Phi, \qquad \Phi(x^i - w^i) = \begin{bmatrix} \gamma_0^i \\ 0 \end{bmatrix}.$$

Differentiating gives ($i = , \ldots, I$)

$$J^i \begin{bmatrix} dx^i \\ d\mu^{iT} \end{bmatrix} = \begin{bmatrix} d\Phi^T \mu^{iT} \\ d\Phi(x^i - w^i) - \begin{bmatrix} d\gamma_0^i \\ 0 \end{bmatrix} \end{bmatrix} \quad \text{where } J^i = \begin{bmatrix} D^2_{x^i} u^i & -\Phi^T \\ -\Phi & 0 \end{bmatrix}. \tag{41}$$

It can be shown that J^i is invertible. Writing the inverse in block form similar to that used for J^i gives

$$[J^i]^{-1} = \begin{bmatrix} A_1^i & A_2^i \\ A_2^{iT} & A_3^i \end{bmatrix},$$

where A_1^i is $n \times n$ and A_2^i is $n \times (S+1)$. The following properties analogous to those in standard consumer demand theory are readily derived (see Balasko and Cass, 1991):

(i) *The Slutsky matrix* A_1^i *is symmetric,* $\operatorname{rank} A_1^i = n - (S+1-(J+K))$ *and*

$$\ker A_1^i = \langle \Phi^T \rangle.$$

(ii) *The matrix of income effects satisfies* $-\Phi A_2^i = I_{S+1-(J+K)}$.

Since $[J^i]^{-1}$ exists, solving from (41) gives

$$dx^i = A_1^i d\Phi^T \mu^{iT} + A_2^i d\Phi(x^i - w^i) - A_2^{i0} d\gamma_0^i, \tag{42}$$

where A_2^{i0}, the first column of A_2^i, gives the effect on consumption of a change $d\gamma_0^i$ in the present value of consumer i's wealth.

Theorem 4 (Real effects of unanticipated monetary changes). *Let Assumptions A to E hold and let* $(\bar{\omega}, \bar{\eta}, \bar{M}) \in \hat{\Gamma}$.

(i) *If* $z_0'' \neq 0$, *then*

$$(dx, dy) = (D_{\bar{M}} x, D_{\bar{M}} y) dM \neq 0, \quad \forall dM = (dM_0, 0) \neq 0.$$

(ii) *If* (a) $\sum_{i=1}^{I} (N_0 z_0''^i) A_2^{i0} \neq 0$, (b) $K < I' \leq J$, (c) $(z_0'^i, i \in \mathcal{I}')$ *are linearly independent,* (d) $E^j = \mathbb{R}^n$, $j = 1, \ldots, J$, *then* (i) *holds with* $dy \neq 0$.

Proof: (i) As in the proof of Theorem 3(i), $dx^1 = 0$ implies dP is proportional to $\bar{P}$. Differentiating the date 0 monetary equation gives $dP = (dM_0/\bar{M}_0)\bar{P}$ and differentiating the date s monetary equation gives $d\beta_s = (dM_0/\bar{M}_0)\bar{\beta}_s$, $s = 1, \ldots, S$. Thus, the subspace $\bar{L}$ and, hence, the matrix $\bar{Q}$ in (27) does not change $\Rightarrow d\Phi = (dM_0/\bar{M}_0)\bar{\Phi}$. Since

$$d\gamma_0^i = \frac{dM_0}{\bar{M}_0} (\bar{P} \square (\bar{y} + \bar{\eta})) z_0'^i$$

and

$$(d\Phi(\bar{x}^i - \bar{w}^i))_0 = \frac{dM_0}{\bar{M}_0} (\bar{P} \square (\bar{y} + \bar{\eta})) z_0'^i + N_0 z_0''^i,$$

and since by property (i) of $[J^i]^{-1}$, $\ker A_1^i = \langle \Phi^T \rangle$, (40) implies

$$dx^i = \frac{dM_0}{\bar{M}_0}(N_0 z_0''^i) A_2^{i0}, \quad i = 1, \dots, I. \tag{43}$$

Since $A_2^{i0} \neq 0$ by property (ii) of $[J^i]^{-1}$, $dx = 0$ contradicts $z_0'' \neq 0$.

(ii) Suppose $dy = 0$. By the first-order conditions for profit maximization, there exists $\alpha_j \neq 0$ such that $\alpha_j n^j(\bar{y}^j) = \sum_{i=1}^{I} z_0^{ij}(\bar{\nu}^i \bar{\Phi})$. Differentiating gives

$$\sum_{i=1}^{I} z_0^{ij}(d\nu^i \bar{\Phi} + \bar{\nu}^i d\Phi) = d\alpha_j n^j(\bar{y}^j) = \frac{d\alpha_j}{\alpha_j} \sum_{i=1}^{I} z_0^{ij}(\bar{\nu}^i \bar{\Phi}).$$

The date 0 component implies $dP_0 = (d\alpha_j/\alpha_j)\bar{P}_0$ and the date 0 monetary equation then implies $d\alpha_j/\alpha_j = dM_0/\bar{M}_0$ so that

$$\sum_{i=1}^{I} z_0^{ij}\left(\bar{\nu}^i d\Phi + \left(d\nu^i - \frac{dM_0}{\bar{M}_0}\bar{\nu}^i\right)\bar{\Phi}\right) = 0, \quad j = 1, \dots, J.$$

It follows from (c) that $\bar{\nu}^i d\Phi + (d\nu^i - (dM_0/\bar{M}_0)\bar{\nu}^i)\bar{\Phi} = 0$, $\forall i \in \mathcal{I}'$. Since there is no loss of generality in assuming that agent 1 lies in $\mathcal{I}' \Rightarrow dP = (dM_0/\bar{M}_0)\bar{P}$. But then $dy = 0$ implies $0 = \sum_{j=1}^{J} dy^j = \sum_{i=1}^{I} dx^i$, which in view of (43) and the hypothesis in (ii)(a),

$$\sum_{i=1}^{I} dx^i = \frac{dM_0}{\bar{M}_0} \sum_{i=1}^{I} (N_0 z_0''^i) A_2^{i0} \neq 0,$$

leads to a contradiction. □

Remark 5. In view of Theorem 2′(i), the monetary change $dM = (dM_0, 0)$ is equivalent to the monetary change $dM = (dM_0, (dM_0/\bar{M}_0)\bar{M}_1)$: *a local proportional change in all the money supplies has the same real effects as a local change in the date 0 money supply alone.*

Remark 6. There is a well-known case where condition (ii)(a) cannot hold. When the consumer side of the economy can be represented by a single consumer, that is, when all agents have identical homothetic preferences (an example frequently used in macroeconomics), then $A_2^{i0} = A_2^0$, $i = 1, \dots, I$, so that $\sum_{i=1}^{I} z_0''^i = 0$ implies $\sum_{i=1}^{I}(N_0 z_0''^i)A_2^0 = 0$. This example is not robust, however, to a slight perturbation in the utility functions of the agents. More generally, if we add a suitable finite dimensional perturbation $\delta = (\delta^1, \dots, \delta^I)$ of the utility functions $u = (u^1, \dots, u^I)$ to the basic parameters (ω, η, M), then it can be shown (along the lines of Geanakoplos and Polemarchakis, 1986) that generically in $(\omega, \eta, \delta, M)$, condition (ii)(a) holds. *Thus, generically, an unanticipated monetary change has production effects.*

Appendix

Proof of Lemma 2: We break the proofs of (i) and (ii) into a sequence of steps.

Step 1. To simplify notation, we drop the subscript and let $\beta_1 = \beta$ and $N_1 = N$. We show that *for fixed* (β, N), *the subset of the Grassmanian* $G^{J+K}(\mathbb{R}^S)$ *defined by*

$$G_{\beta N}^{J+K}(\mathbb{R}^S) = \{L \in G^{J+K}(\mathbb{R}^S) \mid L \supset \langle [\beta] N \rangle\}$$

is diffeomorphic to $G^J(\mathbb{R}^{S-K})$.

Let $\{e^1, \dots, e^S\}$ denote the standard basis of $\mathbb{R}^S$, and let $\{N^1, \dots, N^K, N^{K+1}, \dots, N^S\}$ be another basis for $\mathbb{R}^S$, where $N^1, \dots, N^K$ are the K columns of N and $N^{K+1}, \dots, N^S$ are $S-K$ vectors of $\mathbb{R}^S$ that span $\langle N \rangle^\perp$. Let $N^\perp = [N^{K+1} \dots N^S]$ denote the associated $S \times (S-K)$ matrix. It is clear that for any $\beta \in \mathbb{R}_{++}^S$, the columns of the matrix $[[\beta]N, [\beta]^{-1}N^\perp]$ form a basis for $\mathbb{R}^S$, since $\langle [\beta]^{-1} N^\perp \rangle = \langle [\beta] N \rangle^\perp$. For each $L \in G_{\beta N}^{J+K}(\mathbb{R}^S)$, there exists a unique J-dimensional subspace $\tilde{l}$ in $\langle [\beta] N \rangle^\perp$ such that

$$L = \langle [\beta] N \rangle \oplus \tilde{l}.$$

Let $\theta: \mathbb{R}^{S-K} \to \langle [\beta]^{-1} N^\perp \rangle$ denote the isomorphism that sends the standard basis of $\mathbb{R}^{S-K}$ onto the basis $\{[\beta]^{-1}N^{K+1}, \dots, [\beta]^{-1}N^S\}$; then θ induces a diffeomorphism $\tau: G^J(\mathbb{R}^{S-K}) \to G^J(\langle [\beta]^{-1} N^\perp \rangle)$. Thus, there exists $l \in G^J(\mathbb{R}^{S-K})$ such that $\tau(l) = \tilde{l}$. By the standard representation of $G^J(\mathbb{R}^{S-K})$, there exists a permutation $\sigma' \in \Sigma_{K+1, \dots, S}$ (the set of permutations of $K+1, \dots, S$) and a unique matrix $A' = A_{\sigma'}(l) \in \mathbb{R}^{(S-K-J)J}$ such that

$$\tilde{l} = \{v \in \langle [\beta]^{-1} N^\perp \rangle \mid [I \mid A'] E_{\sigma'}[v] = 0\},$$

where I is the $(S-K-J) \times (S-K-J)$ identity matrix, $E_{\sigma'}$ is the permutation matrix for σ', and the vectors v are written $v = \sum_{i=1}^{S-K} v_{K+i}[\beta]^{-1}N^{K+i}$ with $[v] = (v_{K+1}, \dots, v_S)$. Whenever $\{M_1, \dots, M_{S-K-J}\}$ is a basis for the subspace of $\langle [\beta] N \rangle^\perp$, orthogonal to $\tilde{l}$, then $\{M_1, \dots, M_{S-K-J}\}$ is a basis for $L^\perp$ in $\mathbb{R}^S$ (see Figure 3).

Thus, if we write

$$\begin{array}{c} \\ M_1 \\ \vdots \\ M_{S-K-J} \end{array} \begin{array}{c} \begin{array}{ccccccc} [\beta]N^1 & \cdots & [\beta]N^K & [\beta]^{-1}N^{K+1} & \cdots & [\beta]^{-1}N^S \end{array} \\ \left[\begin{array}{ccc|c} 0 & \cdots & 0 & \\ \vdots & \ddots & \vdots & [I \mid A'_{\sigma'}]E_{\sigma'} \\ 0 & \cdots & 0 & \end{array}\right] \end{array},$$

then L can be represented as

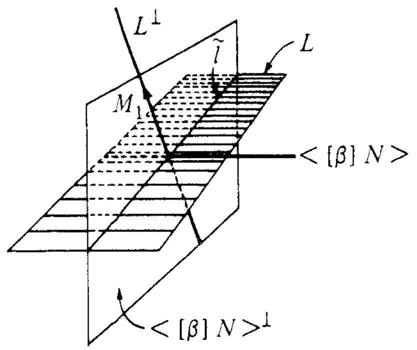

Figure 3. Geometry of subspaces ($S=3$, $K=J=1$).

$$L=\{v\in \mathbb{R}^S \mid [0 \mid [I \mid A'_{\sigma'}]E_{\sigma'}][v]_{\beta N}=0\}, \tag{A.1}$$

where $[v]_{\beta N}$ denotes the coordinates of v in the basis $\{[\beta]N^1,\dots,[\beta]N^K, [\beta]^{-1}N^{K+1},\dots,[\beta]^{-1}N^S\}$. Thus, if we let

$$B(\beta,N)=[[\beta]N,[\beta]^{-1}N^{\perp}]^{-1} \tag{A.2}$$

denote the matrix for the change of basis, then L can be written as

$$L=\{v\in\mathbb{R}^S \mid Q_{\sigma'}v=0\} \tag{A.3}$$

where

$$Q_{\sigma'}=[0 \mid [I \mid A'_{\sigma'}]E_{\sigma'}]B(\beta,N) \tag{A.4}$$

Step 2. Let $f^i(P,y,L;\omega,\eta)$, $i=1,\dots,I$, denote the consumer *demand* functions induced by Definition 2(i) and let $g^j(P,y,L;\omega,\eta)$, $j=1,\dots,J$, denote the producer *supply* functions induced by Definition 2(ii). To express all pseudoequilibria as solutions of a system of equations, we consider for each permutation $\sigma'\in\Sigma_{K+1,\dots,S}$ a system of equations

$$F_{\sigma'}(P,y,A',\beta_1;\omega,\eta,M)=0,$$

where the unknowns are the "prices" (P,y,A',β_1) that lie in the *price space*

$$\mathcal{P}=\mathbb{R}^n_{++}\times\mathbb{R}^{nJ}\times\mathbb{R}^{J(S-K-J)}\times\mathbb{R}^S_{++}$$

and the parameters (ω,η,M) lie in the parameter $\Omega\times\mathfrak{M}$. The functions

$$F_{\sigma'}:\mathcal{P}\times\Omega\times\mathfrak{M}\to\mathbb{R}^{n-1}\times\mathbb{R}^{nJ}\times\mathbb{R}^{S+1}\times\mathbb{R}^{J(S-K-J)}$$

are defined by

$$\begin{aligned}
&\sum_{i=1}^{I}(\hat{f}^i(P,y,L(\beta_1,A');\omega,\eta)-\hat{w}^i)-\sum_{j=1}^{J}(\hat{y}^j+\hat{\eta}^j)=0, && \text{(i)}\\
&g^j(P,y,L(\beta_1,A');\omega,\eta)-y^j=0,\quad j=1,\dots,J, && \text{(ii)}\\
&P_0\Big(\sum_{i=1}^{I}w_0^i+\sum_{j=1}^{J}\eta_0^j\Big)-M_0=0, && \text{(iii)}\qquad \text{(A.5)}\\
&P_S\Big(\sum_{i=1}^{I}w_s^i+\sum_{j=1}^{J}(y_s^j+\eta_s^j)\Big)-\beta_sM_s=0,\quad s=1,\dots,S, && \text{(iv)}\\
&Q_{\sigma'}(\beta_1,A')P_1\,\square\,(y_1^j+\eta_1^j)=0,\quad j=1,\dots,J, && \text{(v)}
\end{aligned}$$

where $L(\beta_1,A')$ and $Q_{\sigma'}(\beta_1,A')$ are defined by (A.3) and (A.4) and $\hat{f}^i$ is the truncated demand function obtained from f^i by omitting the demand for good 1 in state 0. The proof that

$$\operatorname{rank}(D_{\omega,\eta,M}F_{\sigma'})=n-1+nJ+S+1+J(S-K-J)$$

is straightforward and left to the reader. A standard argument based on Sard's Theorem proves that there exists a generic set $\Gamma'\subset\Omega\times\mathfrak{M}$ such that for each $(\omega,\eta,M)\in\Gamma'$, there is at most a finite number of pseudoequilibria, each of which is locally a smooth function of the parameters. This completes the proof of (i).

Step 3. We show that for a generic set $\Gamma''\subset\Gamma'$, all pseudoequilibria satisfy $\operatorname{rank}[P_1\,\square\,(y_1+\eta_1)]=J$. If the rank is less than J, then there exist $(\alpha_1,\dots,\alpha_J)\in\mathbb{S}^{J-1}$ (the $J-1$ dimensional sphere) such that

$$\begin{aligned}
\sum_{j=1}^{J}\alpha_jP_1\,\square\,(y_1^j+\eta_1^j)=0 &\Leftrightarrow \sum_{j=1}^{J}\alpha_jB(\beta,N)P_1\,\square\,(y_1^j+\eta_1^j)=0\\
&\Leftrightarrow \sum_{j=1}^{J}\alpha_j[P_1\,\square\,(y_1^j+\eta_1^j)]_{\beta N}=0.
\end{aligned}$$

Let $\sigma'=$ identity, so that $E_{\sigma'}=I$. For a vector $v\in\mathbb{R}^S$, make the decomposition $[v]_{\beta N}=[v]_a,[v]_b,[v]_c$ (viewed as a column vector) so as to be conformable with the matrix $[0 \mid [I \mid A']]$ in (A.1). If a pseudoequilibrium is such that $\operatorname{rank}[P_1\,\square\,(y_1^j+\eta_1^j)]<J$, then the system of equations

$$F_{\sigma'}(P,y,A',\beta_1;\omega,\eta,M)=0, \tag{A.5}$$

$$\sum_{j=1}^{J}\alpha_j[P_1\,\square\,(y_1^j+\eta_1^j)]_c=0 \tag{A.6}$$

in the unknowns (P,y,A',β_1,α) must have a solution. Since equations (A.6) introduce more equations (J) than new unknowns ($J-1$), the overall system, equations (A.5) and (A.6), generically has no solution if each equation in (A.6) can be locally controlled without affecting the others. To control equation k in (A.6), we pick a vector $v \in \mathbb{R}^S$ such that $[v]_a = 0$, $[v]_c = (0, \dots, 1, \dots, 0)$ (1 in the kth component) and $[v]_b$ such that $[0 \mid [I \mid A']][v]_{\beta N} = 0$. For some j, $\alpha_j \neq 0$. Pick $d\eta_1^j$ such that $B(P_1 \square d\eta_1^j) = [v]_{\beta N}$ and $d\eta_0^j$ such that $P \cdot d\eta^j = 0$. If we let $dw^1 = -d\eta^j$, then all the equations except for the kth equation in (A.6) are unchanged. When σ' is not the identity, the decomposition of $[v]_{\beta N}$ is made conformably with σ'. By running over all the permutations $\sigma' \in \Sigma_{K+1,\dots,S}$ and applying the standard transversality argument in each case, we obtain the desired set Γ''.

Step 4. We show that for a generic set $\Gamma \subset \Gamma''$, all pseudoequilibria satisfy $\langle P_1 \square (y_1 + \eta_1) \rangle \cap \langle \beta_1 N_1 \rangle = \{0\} \Leftrightarrow P_1 \square (y_1^j + \eta_1^j) \notin \langle \beta_1 N_1 \rangle$, $j = 1, \dots, J$. If at a pseudoequilibrium, $P_1 \square (y_1^j + \eta_1^j) \in \langle \beta_1 N_1 \rangle$ for some j, then the system of equations

$$F_{\sigma'}(P, y, A', \beta_1; \omega, \eta, M) = 0, \tag{A.5}$$

$$[P_1 \square (y_1^j + \eta_1^j)]_b = 0, \tag{A.7}$$

$$[P_1 \square (y_1^j + \eta_1^j)]_c = 0 \tag{A.8}$$

has a solution for some $\sigma' \in \Sigma_{K+1,\dots,S}$. Since (A.7) and (A.8) introduce new equations but no new unknowns, the system, equations (A.5), (A.7), and (A.8), generically has no solution if one of the equations in (A.7) or (A.8) can be locally controlled without perturbing (A.5). This is straightforward to show. A standard transversality argument then leads to the desired generic set Γ. □

Proof of Lemma 3: Since $\bar{P}_1 \square (\bar{x}_1^i - \bar{w}_1^i) \in \bar{L}$, there exist $\bar{z}^i = (\bar{z}'^i, \bar{z}''^i)$ such that for $i = 1, \dots, I$,

$$\bar{P}_1 \square (\bar{x}_1^i - \bar{w}_1^i) = [\bar{P}_1 \square (\bar{y}_1 + \bar{\eta}_1)] \bar{z}'^i + [[\bar{\beta}_1] N_1] \bar{z}''^i. \tag{A.9}$$

Let $(i_1, \dots, i_K)$ denote a K-element subset of $\{1, \dots, I\}$, let $[\bar{z}']_K$ and $[\bar{z}'']_K$ denote the induced $J \times K$ and $K \times K$ matrices, respectively, of portfolio holdings, and let $[\bar{P}_1 \square (\bar{x}_1 - \bar{w}_1)]_K$ denote the $S \times K$ matrix of excess expenditures for these K agents; then (A.9) implies

$$[\bar{P}_1 \square (\bar{x}_1 - \bar{w}_1)]_K - [\bar{P}_1 \square (\bar{y}_1 + \bar{\eta}_1)][\bar{z}']_K = [[\beta_1] N_1][\bar{z}'']_K.$$

It remains to prove the existence of a generic set $\hat{\Gamma}^* \subset \hat{\Gamma}$ such that $[\bar{z}'']_K$ is invertible for every K-element subset of $\{1, \dots, I\}$. Suppose that for some K-element subset $\operatorname{rank}[\bar{z}'']_K < K$, then there exists $\alpha \in \mathbb{S}^{K-1}$ such that the system of equations

$$F_{\sigma'}(P, y, A', \beta_1; \omega, \eta, M) = 0, \tag{A.5}$$

$$\sum_{i=i_1}^{i_K} \alpha_i z''^i = 0 \tag{A.10}$$

in the unknowns $(P, y, A', \beta_1, \alpha)$ must have a solution. Again, since equations (A.10) introduce more equations (K) than unknowns $(K-1)$, the overall system, equations (A.5) and (A.10), generically has no solution if each of the equations in (A.10) can be locally controlled without affecting the others. To control equation k in (A.10), let i_1 be such that $\alpha_{i_1} \neq 0$. To induce a one-unit change in agent i_1's demand for nominal asset k, $dz''^{i_1} = (0, \dots, 1, \dots, 0)$, consider a change $dw_s^{i_1}$ such that $\bar{P}_s dw_s^{i_1} = -\bar{\beta}_s N_s dz''^{i_1}$, $s = 1, \dots, S$, $\bar{P}_0 dw_0^{i_1} + \sum_{s=1}^{S} \bar{P}_s dw_s^{i_1} = 0$. Choose a new agent $i_{K+1} \notin \{i_1, \dots, i_K\}$ and set $dw^{i_{K+1}} = -dw^{i_1}$ to restore the equation of equilibrium (A.5). □

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