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Some New Results on the Local Stability of the Process of Capital Accumulation*

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1. SHORT- AND LONG-RUN DYNAMICS

In the "Mécanique Analytique" (1788) Lagrange proved the following theorem. *If the potential function of a conservative dynamical system attains a minimum (maximum) at a position of equilibrium then the motion in a neighborhood of this equilibrium point is stable (unstable).*[1] In 1885 Poincaré showed that if the potential function is made a function not only of the state of the system but also of an *exogenous parameter* and if the equilibria induced by the potential function are considered as functions of the parameter we obtain an *equilibrium surface* in the parameter–state space for which the stable and unstable branches are separated by *bifurcation equilibria* [50, pp. 43–55]. If we combine these two ideas for a class of dynamical systems that arises in economic theory we obtain the beginings of a rich and interesting theory of economic dynamics.

Such a theory has two parts, a *short-run dynamics* and a *long-run dynamics.* For a fixed value of the exogenous parameter the short-run dynamics classifies the equilibria associated with this parameter value into stable and unstable equilibria, and shows the local nature of the motion in the neighborhood of each equilibrium point when viewed in the state space. Such an analysis carried out for all feasible values of the parameter leads to a classification of the equilibrium manifold into *stable* and *unstable submanifolds.* A system with only stable equilibria will in general have associated with it a *continuous* long-run equilibrium manifold. When the exogenous parameter, taken as fixed in the short-run dynamics, is allowed to vary in a slow and systematic way, the system will trace out a trajectory along the equilibrium

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[1] See [34, pp. 69–76]. The proof of instability was completed by Liapunov [35, pp. 62, 377–386]. It was Lagrange's theorem and the proof of it by Dirichlet [34, pp. 457–459] that suggested to Liapunov his general approach to the problem of stability [35, pp. 56–64].

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manifold. When the manifold is smooth we are in essence carrying out the continuous-time analog of the classical problem of *comparing equilibria* [4, Chap. 10; 11; 59, Chap. 3].

If a *unique equilibrium* for a given value of the parameter is normally stable, *multiple equilibria* are normally associated with the presence of both stable and unstable equilibria. While in a static theory uniqueness seems to be essential to obtain a meaningful theory [4, Chap. 9] in the dynamical theory the presence of multiple equilibria is likely in some cases to lead to a much richer theory.

The reason is as follows. Since the short-run dynamics ensures that in the long run the system moves along the stable submanifolds, we are led to remove the unstable submanifolds. Thus a system with unstable equilibria will normally have associated with it a *discontinuous* long-run equilibrium manifold. *When the parameter changes and leads the system along the equilibrium manifold to a bifurcation equilibrium the system essentially jumps across the unstable submanifold to the next stable submanifold, thereby inducing a major change in the long-run dynamical behavior of the system.* In the Thom–Zeeman terminology [66, 68] *crossing a bifurcation equilibrium produces a catastrophe.* In Section 5 we will show that such phenomena may arise quite naturally in economic dynamics.

The equilibria we will consider in this paper are the simplest type of dynamical equilibria—the *point equilibria.* Equilibria consisting of *periodic orbits* such as those that appear in the paper of Ryder and Heal [56] will call for a more sophisticated analysis.

Section 2 outlines the basic class of dynamical models, while Section 3 presents some preliminary results for the classification of equilibria. Except when $n = 1$ the results are far from complete and involve exclusively *sufficient conditions for characterizing stable equilibria.* These stability conditions are given a geometric and economic interpretation in Section 4. Some preliminary results on long-run dynamics are presented in Section 5, while Section 6 illustrates the theory with some examples.

2. Competitive Paths

Consider the following *family of extremum problems*[2]

$$\sup_v \int_0^T \mathscr{U}(t, k, v; \rho)\, dt, \qquad \dot{k} = \phi(k, v; \rho), \qquad k \geqslant 0, \qquad k(0) = k_0\,, \quad (\mathscr{T}_\rho)$$

where $\rho = (\rho_1, \ldots, \rho_s) \in R^s$ is a vector of *exogenous parameters* (taken as

[2] The analysis of a family of dynamical systems depending on a parameter has a long and established tradition in celestial mechanics where the parameters are given by the *masses* (see [51, p. 325]).

fixed in Sections 2 through 4), $k = (k_1, \ldots, k_n) \in R^n$ denotes the *state* of the system (typically the vector of *capital stocks*), $v = (v_1, \ldots, v_m) \in R^m$ is piecewise continuous and denotes the vector of *controls* (typically *consumption* and *investment*), $\mathscr{U} \in R^1$, $\mathscr{U} \in \mathscr{C}^3$ and denotes the *objective* (*utility*) *function*, $\phi = (\phi^1, \ldots, \phi^n) \in R^n$, $\phi^i \in \mathscr{C}^3$, $i = 1, \ldots, n$ and characterize the *technology*. In the present paper we will consider the special case where $\rho = (\delta, \beta)$ with

$$\mathscr{U}(t, k, v; \rho) = e^{-\delta t} u(k, v; \beta), \qquad \phi(k, v; \rho) = f(k, v; \beta), \qquad -\infty < \delta < \infty. \quad (1)$$

To simplify the notation we will normally write $u(k, v)$ and $f(k, v)$ omitting the parameter β in (1).

DEFINITION. $(\mathscr{T}_\rho)$ and (1) are called the *control problem*. In the special case where $\dot{k} = f(k, v) = v$ we write $u(k, v) = L(k, \dot{k})$ and refer to $(\mathscr{T}_\rho)$ and (1) as the *variational problem*.

ASSUMPTION (Strict Concavity). $u(k, v)$ is strictly concave and $f^1(k, v), \ldots, f^n(k, v)$ are concave functions on $R^{n+} \times R^n$, where R^{n+} denotes the nonnegative orthant.

DEFINITION. A path (k, v) is *feasible* if $k(t) \geqslant 0$, $k(0) = k_0$, and $\dot{k} = f(k, v)$ for all $t \in [0, T]$.

We are interested in the class of problems for which $T \to \infty$. To this end we introduce the following criterion due to Gale [25].

DEFINITION. A path $(\bar{k}, \bar{v})$ *catches up to* the path (k, v) if for any $\epsilon > 0$ there exists T_ϵ such that

$$\int_0^T e^{-\delta\tau}(u(\bar{k}, \bar{v}) - u(k, v))\, d\tau \geqslant -\epsilon \qquad \text{for all} \quad T \geqslant T_\epsilon .$$

This is equivalent to the condition

$$\varliminf_{T\to\infty} \int_0^T e^{-\delta\tau}(u(\bar{k}, \bar{v}) - u(k, v))\, d\tau \geqslant 0.$$

DEFINITION. A feasible path $(\bar{k}, \bar{v})$ is *optimal* if it catches up to all other feasible paths.

The following definition is basic to all of the analysis that follows.

DEFINITION. A feasible path $(\bar{k}, \bar{v})$ is *competitive* if there exists an absolutely continuous path of *prices* $(\dot{\bar{p}} - \delta\bar{p}, \bar{p})$ such that

$$u(\bar{k}, \bar{v}) + \bar{p} f(\bar{k}, \bar{v}) + (\dot{\bar{p}} - \delta\bar{p})\, \bar{k} \geqslant u(k, v) + \bar{p} f(k, v) + (\dot{\bar{p}} - \delta\bar{p})\, k \quad (2)$$

for all $(k, v) \in R^{n+} \times R^n$, $t \in [0, \infty)$.

Remark. A competitive path is a path that has associated with it a dual path of prices under which it maximizes profit at each instant, for $\dot{\bar{p}} - \delta\bar{p}$ denotes the vector of *unit rental costs* and $(1, \bar{p})$ denotes the vector of *unit output prices* so that $u + \bar{p}\dot{k} + (\dot{\bar{p}} - \delta\bar{p})\, k$ is the profit which is maximized at each instant along a competitive path.

DEFINITION. $\hat{G}(k, p; v) = u(k, v) + pf(k, v)$ is called the *pre-Hamiltonian.* In view of the assumption of strict concavity we have the following.

LEMMA 1. *A path (k, v, p) is competitive if and only if*

$$\hat{G}_k(k, p; v) + \dot{p} - \delta p = 0, \qquad \hat{G}_v(k, p; v) = 0, \qquad \dot{k} = \hat{G}_p(k, p; v), \qquad t \in [0, \infty), \tag{3}$$

where $\hat{G}_k$, $\hat{G}_v$ and $\hat{G}_p$ denote the gradients of $\hat{G}$ with respect to k, v, and p, respectively.

Remark. In the variational case (3) reduces to the *Euler–Lagrange equation*

$$L_{\dot{k}\dot{k}}\ddot{k} + L_{\dot{k}k}\dot{k} - (L_k + \delta L_{\dot{k}}) = 0. \tag{4}$$

DEFINITION. $G(k, p) = \max_v \hat{G}(k, p; v)$ is called the *Hamiltonian.*

Remark. In terms of the Hamiltonian, the equations (3) reduce to the *canonical equations*

$$\dot{p} = -G_k(k, p) + \delta p, \qquad \dot{k} = G_p(k, p). \tag{5}$$

LEMMA 2. *Under the assumption of strict concavity a competitive path $(\bar{k}, \bar{v}, \bar{p})$ which satisfies the transversality conditions*

$$\left.\begin{aligned} \overline{\lim_{t\to\infty}}\, e^{-\delta t}\bar{p}(t)\, \bar{k}(t) &= \overline{K} < \infty, \\ \underline{\lim_{t\to\infty}}\, e^{-\delta t}\bar{p}(t)\, k(t) &= \underline{K} > -\infty, \end{aligned}\right\} \quad \text{for constants } \overline{K}, \underline{K} \tag{6}$$

for any feasible path $k(t)$, is optimal.

Proof. We will present a continuous-time version of McKenzie's proof [41, p. 274]. Introduce the *profit–loss (value–loss) function*

$$\begin{aligned} \mathscr{L}(k, v) = {} & u(\bar{k}, \bar{v}) + \bar{p}f(\bar{k}, \bar{v}) + (\dot{\bar{p}} - \delta\bar{p})\, \bar{k} \\ & - [u(k, v) + \bar{p}f(k, v) + (\dot{\bar{p}} - \delta\bar{p})\, k] \end{aligned} \tag{7}$$

which evaluates the *profit–loss* of any feasible path (k, v) under the competitive prices $(\dot{\bar{p}} - \delta\bar{p}, \bar{p})$ of the path $(\bar{k}, \bar{v}, \bar{p})$. Premultiplying by $e^{-\delta t}$, integrating, and using the fact that $\dot{k} = f(k, v)$ gives

$$\int_0^T e^{-\delta\tau}(u(k, v) - u(\bar{k}, \bar{v}))\, d\tau$$
$$= \bar{p}(0)(k(0) - \bar{k}(0)) + e^{-\delta T}\bar{p}(T)(\bar{k}(T) - k(T)) - \int_0^T e^{-\delta\tau}\mathscr{L}(k, v)\, d\tau. \quad (8)$$

Equation (6) and the assumption of feasibility $k(0) = \bar{k}(0) = k_0$ imply

$$\varlimsup_{T\to\infty} \int_0^T e^{-\delta\tau}(u(k, v) - u(\bar{k}, \bar{v}))\, d\tau \leqslant \bar{K} - \underline{K} - \lim_{T\to\infty} \int_0^T e^{-\delta\tau}\mathscr{L}(k, v)\, d\tau.$$

Without loss of generality we may suppose there is a constant ℓ such that

$$\ell < \varlimsup_{T\to\infty} \int_0^T e^{-\delta\tau}(u(k, v) - u(\bar{k}, \bar{v}))\, d\tau$$

so that

$$\lim_{T\to\infty} \int_0^T e^{-\delta\tau}\mathscr{L}(k, v)\, d\tau < \bar{K} - \underline{K} - \ell$$

which implies $e^{-\delta t}\mathscr{L}(k, v) \to 0$ as $t \to \infty$. The assumption of strict concavity implies $\mathscr{L}(k, v) \geqslant 0$ for all $t \geqslant 0$ and $e^{-\delta t}\bar{p}(t)(\bar{k}(t) - k(t)) \to 0$ as $t \to \infty$, from which the result follows. △

ASSUMPTION (Utility Saturation). For $\delta < 0$, $u(k, v) < u(\bar{k}, \bar{v}) = 0$ for all $(k, v) \neq (\bar{k}, \bar{v})$ for some $(\bar{k}, \bar{v}) \in \mathring{R}^{n+} \times R^n$ and $f(\bar{k}, \bar{v}) = 0$, where $\mathring{R}^{n+}$ denotes the interior of the nonnegative orthant.

Remark. This is the familiar Ramsey assumption [53]. For a careful analysis of the problems that can arise when $\delta < 0$, see Koopmans' original paper [32].

DEFINITION. A competitive path (k^*, v^*, p^*) which is *stationary* ($\dot{k}^* = \dot{p}^* = 0$ for all $t \in [0, \infty)$) is called an *optimal stationary state* (OSS).

ASSUMPTION (OSS). $(k^*, p^*) \in \mathring{R}^{n+} \times \mathring{R}^{n+}$ for $\delta \geqslant 0$, $k^* \in \mathring{R}^{n+}$ and $p^* = 0$ for $\delta < 0$.

The following definition is central to our analysis.

DEFINITION. Let $\rho = (\delta, \beta)$ denote the vector of exogenous parameters. The manifolds

$$\mathscr{E} = \left\{(\rho, k^*, p^*) \,\middle|\, \begin{matrix} \delta p^* - G_k(k^*, p^*; \beta) = 0 \\ G_p(k^*, p^*; \beta) = 0 \end{matrix}\right\} \subset R^s \times \mathring{R}^{n+} \times \mathring{R}^{n+},$$
$$\mathscr{E}_k = \{(\rho, k^*) \mid L_k(k^*, 0; \beta) + \delta L_{\dot{k}}(k^*, 0; \beta) = 0\} \subset R^s \times \mathring{R}^{n+}, \quad (9)$$

are called the *equilibrium manifolds* for the family of control and variational problems $(\mathscr{T}_\rho)$ and (1).

ASSUMPTION. $\mathscr{E} \neq \varnothing (\mathscr{E}_k \neq \varnothing)$.

Remark. See Peleg and Ryder [49] and Cass and Shell [21].

As is usual in the analysis of problems of stability, it is useful to make the equilibrium point (OSS) the *origin* for the analysis.

DEFINITION. The *local coordinates* around an OSS $(k^*, v^*, p^*) = (k^*(\rho), v^*(\rho), p^*(\rho))$ are defined by the transformation

$$(\chi, \gamma, \eta) = (k - k^*, v - v^*, p^* - p). \tag{10}$$

DEFINITION. The profit–loss function induced by the competitive prices of an OSS (k^*, v^*, p^*) is defined by

$$\begin{aligned}\mathscr{L}^0(\chi, \gamma) = \mathscr{L}(k^* + \chi, v^* + \gamma) &= \hat{G}(k^*, p^*; v^*) \\ &\quad - \hat{G}(k^* + \chi, p^*; v^* + \gamma) + \delta p^* \chi.\end{aligned} \tag{11}$$

DEFINITION. An OSS is *accessible* if there exists a local control $\hat{\gamma}$ and an associated feasible local path $(\hat{\chi}, \hat{\gamma})$ such that $\int_0^\infty e^{-\delta\tau} \mathscr{L}^0(\hat{\chi}, \hat{\gamma})\, d\tau < \infty$.

LEMMA 3 (Brock). *If an OSS (k^*, v^*, p^*) is accessible and if a feasible local path $(\bar{\chi}, \bar{\gamma})$ minimizes profit–loss $\int_0^\infty e^{-\delta\tau} \mathscr{L}^0(\chi, \gamma)\, d\tau$ then the path $(k^* + \bar{\chi}, v^* + \bar{\gamma})$ is optimal.*

Proof. We give a continuous-time version of Brock's proof [10, p. 279] omitting details. Let $(\bar{\chi}, \bar{\gamma})$ be a feasible local path that minimizes profit–loss and let (χ, γ) denote a second feasible local path, then using (8)

$$\begin{aligned}&\int_0^T e^{-\delta\tau}(u(k^* + \bar{\chi}, v^* + \bar{\gamma}) - u(k^* + \chi, v^* + \gamma))\, d\tau \\ &\quad = e^{-\delta T} p^*(\chi(T) - \bar{\chi}(T)) + \int_0^T e^{-\delta\tau} \mathscr{L}^0(\chi, \gamma)\, d\tau - \int_0^T e^{-\delta\tau} \mathscr{L}^0(\bar{\chi}, \bar{\gamma})\, d\tau.\end{aligned}$$

Since the OSS is accessible $\int_0^\infty e^{-\delta\tau} \mathscr{L}^0(\bar{\chi}, \bar{\gamma})\, d\tau < \infty$, and hence by the strict concavity of $\mathscr{L}^0(\chi, \gamma)$, $e^{-\delta T} p^* \bar{\chi}(T) \to 0$ as $T \to \infty$. If $\int_0^\infty e^{-\delta\tau} \mathscr{L}^0(\chi, \gamma)\, d\tau = \infty$, $(\bar{\chi}, \bar{\gamma})$ is clearly a better path. But if $\int_0^\infty e^{-\delta\tau} \mathscr{L}^0(\chi, \gamma)\, d\tau < \infty$, $e^{-\delta T} p^* \chi(T) \to 0$ as $T \to \infty$ and the fact that $(\bar{\chi}, \bar{\gamma})$ minimizes profit–loss gives the result. △

COROLLARY. *Under the assumptions of the lemma if $\delta \leqslant 0$, $(\bar{\chi}, \bar{\gamma}) \to (0, 0)$ as $t \to \infty$.*

Remark. When $\delta \leqslant 0$ once strict concavity is assumed convergence depends only upon accessibility. *It is not evident however, that an OSS is always accessible.* In the next section we give local conditions at an OSS that ensure accessibility.

For a market description of the process of capital accumulation the following definition and theorem are of central importance.

DEFINITION. A time-invariant linear system $\left[\begin{smallmatrix}\dot{x}\\ \dot{y}\end{smallmatrix}\right] = A\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]$, A a $2n \times 2n$ constant matrix, has the *saddle-point property* if n eigenvalues have negative real parts and the remaining n eigenvalues have positive real parts.

A proof of the following theorem may be found in [30, pp. 57–60, 242–244].

THEOREM 1 (Existence of Stable Manifold).

(i) Linear manifold ($\mathcal{N}$). *Let* $\left[\begin{smallmatrix}\dot{x}\\ \dot{y}\end{smallmatrix}\right] = A\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]$ *be a 2n-dimensional time-invariant linear system with the saddle-point property, then there exists a real n-dimensional linear stable manifold* $\mathcal{N} \in R^n \times R^n$ *such that if* $(x(0), y(0)) \in \mathcal{N}$ *then* $(x(t), y(t)) \in \mathcal{N}$ *for* $t \in [0, \infty)$ *and* $(x(t), y(t)) \to (0, 0)$ *as* $t \to \infty$.

(ii) Nonlinear manifold ($\mathcal{N}^*$). *Let* $\left[\begin{smallmatrix}\dot{x}\\ \dot{y}\end{smallmatrix}\right] = A\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] + h(x, y)$, *where* $h(x, y) \in \mathscr{C}^1$, $h(0, 0) = 0$, $h_x(0, 0) = 0$, *if the associated linear system* $\left[\begin{smallmatrix}\dot{x}\\ \dot{y}\end{smallmatrix}\right] = A\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]$ *has the saddle-point property, then there exists a real n-dimensional* $\mathscr{C}^1$ *stable manifold* $\mathcal{N}^* \in R^n \times R^n$ *containing the origin and an* $\epsilon > 0$ *such that if* $\| x(0), y(0)\| < \epsilon$ *and if* $(x(0), y(0)) \in \mathcal{N}^*$ *then* $(x(t), y(t)) \in \mathcal{N}^*$ *for* $t \in [0, \infty)$ *and* $(x(t), y(t)) \to (0, 0)$, *as* $t \to \infty$, *where* $\| \;\|$ *is the standard Euclidean metric. Furthermore* $\mathcal{N}^*$ *is tangent to* $\mathcal{N}$ *at the origin.*

3. SHORT-RUN DYNAMICS

We address ourselves to the following problem.

PROBLEM (Characterizing Stable Submanifolds of $\mathscr{E}$ and $\mathscr{E}_k$).

I. Control problem [(k, p) space]. Find sufficient conditions on the utility function $u(k, v)$ and the technology $f(k, v)$ such that if $(\bar{k}(t), \bar{p}(t))$ is an optimal competitive path then there exist a real n-dimensional $\mathscr{C}^1$ manifold $\mathscr{M}^* \in R^n \times R^{n^+}$ containing $(k^*(\rho), p^*(\rho))$ [where $(\rho, k^*, p^*) \in \mathscr{E}$] and an $\epsilon > 0$ such that if $\|(\bar{k}(0), \bar{p}(0)) - (k^*(\rho), p^*(\rho))\| < \epsilon$ and $(\bar{k}(0), \bar{p}(0)) \in \mathscr{M}^*$ then $(\bar{k}(t), \bar{p}(t)) \in \mathscr{M}^*$ for $t \in [0, \infty)$ and $(\bar{k}(t), \bar{p}(t)) \to (k^*(\rho), p^*(\rho))$ as $t \to \infty$. Furthermore if $(\bar{k}(t), \bar{p}(t)) \in \mathscr{M}^*$ then $(\bar{k}(t) - k^*)(\bar{p}(t) - p^*) < 0$ for $\bar{k}(t) \neq k^*$.

II. Variational problem [k space]. Find sufficient conditions on the utility function $L(k, \dot{k})$ such that if $\bar{k}(t)$ is an optimal solution of the variational problem then there exists an $\epsilon > 0$ such that if $\| \bar{k}(0) - k^*(\rho)\| < \epsilon$ [where $(\rho, k^*) \in \mathscr{E}_k$] then $\bar{k}(t) \to k^*(\rho)$ as $t \to \infty$.

Remark. (I) is the *conditional asymptotic stability* of a saddle-point. (II) is standard *asymptotic stability.*

Let $(k^*, v^*, p^*) = (k^*(\rho), v^*(\rho), p^*(\rho))$ be an OSS. We are interested in the qualitative stability properties of the paths generated by $(\mathscr{T}_\rho)$ and (1) in the case where $T \to \infty$. Brock's Lemma assures us that paths which are solutions to the problem

$$\inf_\gamma \int_0^\infty e^{-\delta\tau} \mathscr{L}^0(\chi, \gamma)\, d\tau, \qquad \dot{\chi} = f(k^* + \chi, v^* + \gamma), \qquad k^* + \chi \geqslant 0, \qquad k^* + \chi(0) = k_0 \tag{$\mathscr{L}_\rho{}^0$}$$

are solutions to $(\mathscr{T}_\rho)$ and (1) as $T \to \infty$, *where the dependence of $\mathscr{L}^0(\chi, \gamma)$ and $f(k^* + \chi, v^* + \gamma)$ on β has been omitted to simplify the notation.* In view of the assumption of strict concavity we will find that the *stability properties* of the paths generated by $(\mathscr{L}_\rho{}^0)$ are the *same* as the stability properties of the paths generated by the problem

$$\inf_\gamma \int_0^\infty e^{-\delta\tau} L^0(\chi, \gamma)\, d\tau, \qquad \dot{\chi} = F\chi + G\gamma, \qquad k^* + \chi \geqslant 0, \qquad k^* + \chi(0) = k_0\,, \tag{$L_\rho{}^0$}$$

where

$$L^0(\chi, \gamma) = \frac{1}{2} \begin{bmatrix} \chi \\ \gamma \end{bmatrix}' \begin{bmatrix} A & N \\ N' & B \end{bmatrix} \begin{bmatrix} \chi \\ \gamma \end{bmatrix},$$

$$\begin{bmatrix} A & N \\ N' & B \end{bmatrix} = \begin{bmatrix} \mathscr{L}^0_{\chi\chi}(0, 0) & \mathscr{L}^0_{\chi\gamma}(0, 0) \\ \mathscr{L}^0_{\gamma\chi}(0, 0) & \mathscr{L}^0_{\gamma\gamma}(0, 0) \end{bmatrix},$$

and

$$F = f_k(k^*, v^*), \qquad G = f_v(k^*, v^*). \tag{12}$$

Note that by (11), $\mathscr{L}^0(0, 0) = 0$, $\mathscr{L}_\chi{}^0(0, 0) = 0$, $\mathscr{L}_\gamma{}^0(0, 0) = 0$ and from (11) and the definition of the pre-Hamiltonian

$$\begin{bmatrix} A & N \\ N' & B \end{bmatrix} = - \left[\begin{bmatrix} u_{kk}(k^*, v^*) & u_{kv}(k^*, v^*) \\ u_{vk}(k^*, v^*) & u_{vv}(k^*, v^*) \end{bmatrix} + \sum_{i=1}^n p_i{}^* \begin{bmatrix} f^i_{kk}(k^*, v^*) & f^i_{kv}(k^*, v^*) \\ f^i_{vk}(k^*, v^*) & f^i_{vv}(k^*, v^*) \end{bmatrix} \right] \tag{13}$$

which in the *variational case* reduces to the simpler matrix

$$\begin{bmatrix} A & N \\ N' & B \end{bmatrix} = - \begin{bmatrix} L_{kk}(k^*, 0) & L_{k\dot{k}}(k^*, 0) \\ L_{\dot{k}k}(k^*, 0) & L_{\dot{k}\dot{k}}(k^*, 0) \end{bmatrix}. \tag{14}$$

All the stability conditions that we shall give will reduce to conditions on the matrices (12)–(14).

Remark. $(L_\rho{}^0)$ is the *accessory variational problem* for $(\mathscr{L}_\rho{}^0)$, familiar in the analysis of conjugate points, and generates the *linearized equations* for $(\mathscr{L}_\rho{}^0)$ at the OSS.

Approach to stability analysis. The *Hamilton–Jacobi theory* is concerned with the relationship between the *trajectories* of the system (the solutions of the *canonical equations*) and certain *transversal surfaces* (the surfaces $W(\chi) =$ constant, where W is a solution of the *Hamilton–Jacobi equation*).[3] *But how does this help in a stability analysis?*

Consider the value–loss function induced by the problem $(L_\rho{}^0)$[4]

$$W(\chi) = \inf_{\gamma} \int_0^\infty e^{-\delta\tau} L^0(\chi, \gamma)\, d\tau. \tag{15}$$

Since $L^0(\chi, \gamma)$ is a positive convex function which attains a unique minimum at the OSS $(L^0(0, 0) = 0)$, *$W(\chi)$ is a positive convex function which attains a unique minimum at the OSS* $(W(0) = 0)$. Thus the surfaces $W(\chi) =$ constant will form concentric contours around the OSS (see [61, p. 545]). *Will the attempt to minimize value–loss, from Brock's Lemma, lead the trajectories to cut across the surfaces $W(\chi) =$ constant, towards the OSS?* Not necessarily. But if we ask that this be so, we are led directly to a natural stability condition.

THEOREM 2 (Hamilton–Jacobi equation). *If γ minimizes $\int_0^T e^{-\delta\tau} L^0(\chi, \gamma)\, d\tau$ subject to $\dot{\chi} = F\chi + G\gamma$ then*

(i) *the value-loss function* $W(\chi, t) = \inf_\gamma \int_t^T e^{-\delta(\tau - t)} L^0(\chi, \gamma)\, d\tau$ *is a solution of the Hamilton–Jacobi equation*

$$W_t - \delta W + G^0(\chi, W_\chi) = 0 \tag{16}$$

with transversality condition $e^{-\delta T} W(\chi, T) = 0$;

(ii) *the control γ minimizes*

$$\hat{G}^0(\chi, \eta; \gamma) = L^0(\chi, \gamma) + \eta'(F\chi + G\gamma),$$

$$G^0(\chi, \eta) = \min_\gamma \hat{G}^0(\chi, \eta; \gamma)$$

[3] For a full development of these ideas see my monograph "On a General Economic Theory of Motion" [45, Chap. V].

[4] As Lionel McKenzie has pointed out to me, this approach to the problem of stability may be viewed as a natural continuous-time generalization of the *value–loss method* originally introduced by Radner [52] and subsequently extensively developed by McKenzie [38–42]. In the analysis that follows if we replace $L^0(\chi, \gamma)$ by $\mathscr{L}^0(\chi, \gamma)$ then we obtain *global* stability theorems, provided the value–loss function which replaces (15) is a $\mathscr{C}^2$ function. *The ideas that follow are thus not restricted to a local analysis.*

Proof. [23, p. 84].

The necessary conditions for γ to minimize $\hat{G}^0(\chi, \eta; \gamma)$ are[5]

$$\hat{G}_\gamma{}^0 = \chi' N + \gamma' B + \eta' G = 0, \qquad z' \hat{G}^0_{\gamma\gamma} z = z' B z \geqslant 0 \quad \text{for all} \quad z \neq 0, \quad z \in R^m \tag{17}$$

If we assume $|B| \neq 0$ *then* (17) *implies B is positive definite.* Thus

$$\gamma = -B^{-1}(G'\eta + N'\chi)$$

so that the Hamiltonian becomes

$$G^0(\chi, \eta) = \frac{1}{2} \begin{bmatrix} \chi \\ \eta \end{bmatrix}' \begin{bmatrix} A - NB^{-1}N' & F' - NB^{-1}G' \\ F - GB^{-1}N' & -GB^{-1}G' \end{bmatrix} \begin{bmatrix} \chi \\ \eta \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \chi \\ \eta \end{bmatrix}' \begin{bmatrix} G^0_{xx} & G^0_{x\eta} \\ G^0_{\eta x} & G^0_{\eta\eta} \end{bmatrix} \begin{bmatrix} \chi \\ \eta \end{bmatrix}.$$

The solution of (16) is given by

$$W(\chi, t) = \tfrac{1}{2}\chi' Q(t)\, \chi, \tag{18}$$

where $Q(t)$ is the solution of the *matrix–Riccati equation*

$$\dot{Q} + QG^0_{\eta x} + (G^0_{x\eta} - \delta I)\, Q + QG^0_{\eta\eta} Q + G^0_{xx} = 0, \qquad Q(T) = 0. \tag{19}$$

Equation (18) implies $\gamma = -B^{-1}(G'Q(t) + N')\,\chi$.

LEMMA 4. *Let $Q(t, T)$ be the solution of* (19). *If $L^0(\chi, \gamma)$ is positive definite and if for each $\chi(0) \in R^n$, $\|\chi(0)\| < \infty$ there exists a control $\gamma^0(t)$ with associated trajectory $\chi^0(t)$ for the system $\dot{\chi} = F\chi + G\gamma$, and constants $a > 0$, $b > 0$, $\alpha < \delta$ such that*

$$\|\gamma^0(t)\| \leqslant a e^{(\alpha/2)t}, \qquad \|\chi^0(t)\| \leqslant b e^{(\alpha/2)t} \qquad \text{for all } t \geqslant 0 \tag{20}$$

then $\lim_{T\to\infty} Q(t, T) = Q^*$ *where Q^* is positive definite.*

Proof. Let $W(\chi, t, T; \gamma) = \int_t^T e^{-\delta(\tau - t)} L^0(\chi, \gamma)\, d\tau$. Let $\lambda^* > 0$ denote the maximum eigenvalue of the quadratic form $L^0(\chi, \gamma)$, then

$$\chi' Q(t, T)\, \chi \leqslant W(\chi^0, t, T; \gamma^0) \leqslant \frac{\lambda^*(a^2 + b^2)}{\delta - \alpha}\, e^{\alpha t}(1 - e^{-(\delta-\alpha)(T-t)})$$

Since $\delta - \alpha > 0$, $\lim_{T\to\infty} \chi' Q(t, T)\, \chi < \infty$. Since $L^0(\chi, \gamma)$ is positive definite, $\chi' Q(t, T)\, \chi$ is positive definite for all $t < T$ and $\chi' Q(t, T + h)\, \chi \geqslant \chi' Q(t, T)\, \chi$ for $h > 0$ from which the result follows. △

[5] When matrices enter into the equations we use primes to denote the *transpose*.

COROLLARY. *Under the assumptions of Lemma* 4

$$\gamma(t) \to \gamma^* = -B^{-1}(G'Q^* + N')\chi = P^*\chi \qquad as \quad T \to \infty. \tag{21}$$

Remark. The fact that finite horizon control policies converge to the infinite horizon policy is a result of great economic importance. For if under γ^* the process $\dot{\chi} = F\chi + G\gamma^*$ converges to the OSS then the corollary implies that *if the horizon is sufficiently far in the future, $\gamma(t)$ can be set equal to γ^* for most of the planning horizon, except, that is, for a transient terminal interval during which $Q(t)$ goes from Q^* to its terminal value $Q(T) = 0$.* The corollary thus leads to the standard *finite horizon Turnpike Theorems* of Radner, Samuelson, and McKenzie [38–42, 52, 57, 58].

Accessibility. We will now consider two types of local accessibility some variant of which plays a basic role in both the Radner–Samuelson–McKenzie finite horizon Turnpike Theorems and the Furuya–Inada–Gale *asymptotic Turnpike Theorems* [24, 25]. Some variant of the first definition is necessary for the finite horizon Turnpike results, while some variant of the second is necessary to obtain asymptotic results.

DEFINITION. If for each $\chi(0) \in R^n$, $\|\chi(0)\| < \infty$ there exists a control $\gamma^0(t)$ and associated trajectory $\chi^0(t)$ for the system $\dot{\chi} = F\chi + G\gamma$ for which

(1) there is a time T^0 such that $\|\chi^0(t)\| \to 0$ as $t \to T^0$, then the OSS will be called *locally finitely accessible*:[6] If, on the other hand,

(2) there are constants $a > 0$, $b > 0$, $\alpha < 0$ such that

$$\|\gamma^0(t)\| \leqslant ae^{\alpha t}, \qquad \|\chi^0(t)\| \leqslant be^{\alpha t} \qquad \text{for all } t \geqslant 0 \tag{22}$$

then the OSS will be called *locally asymptotically accessible of degree* α. It will sometimes be convenient to say more briefly that (F, G) is locally finitely accessible under (1) and locally asymptotically accessible of degree α under (2).

In the following proposition (i) and (ii) are well known [26, 31]. (iii) strengthens the result of Gal'perin and Krasovskii [26] and provides a convenient test for local asymptotic accessibility of degree α when the simplest test (i) fails.

[6] The assumption of *local finite accessibility* is closely related to the concept of *primitivity* which implies that a positive amount of every good can be produced within a finite number of periods (discrete case) given any nonzero, nonnegative initial endowment. See [24, Assumption 6, p. 99] and [52, Remark 3, p. 104]. Local finite accessibility was first introduced and characterized in the important paper of Kalman [31, p. 107]. He called this property *complete controllability*. For our purposes in view of the new definition (2), the term *local finite accessibility* seems preferable.

PROPOSITION 1. (i) *The collection of initial states* $\chi(0)$ *from which the OSS is locally finitely accessible is the subspace of* R^n *spanned by the* $r \leqslant n$ *linearly independent columns of the matrix*

$$S = [G, FG, ..., F^{n-1}G]. \tag{23}$$

The OSS is locally finitely accessible if and only if $r = n$.

(ii) *Let* $(s_1, ..., s_r)$ *be r linearly independent columns of S and let* $(v_{r+1}, ..., v_n)$ *be* $(n - r)$ *vectors which span the remainder of* R^n. *Let* $y = U^{-1}\chi$, *where* $U = [s_1, ..., s_r, v_{r+1}, ..., v_n]$, *then* $\dot{y} = \bar{F}y + \bar{G}\gamma$ *where*

$$\bar{F} = U^{-1}FU = \begin{bmatrix} \bar{F}_{11} & \bar{F}_{12} \\ 0 & \bar{F}_{22} \end{bmatrix}, \qquad \bar{G} = U^{-1}G = \begin{bmatrix} \bar{G}_1 \\ 0 \end{bmatrix}.$$

$\bar{F}_{11}$ *is* $r \times r$, $(\bar{F}_{11}, \bar{G}_1)$ *is locally finitely accessible and the eigenvalues of* $\bar{F}_{22}$ *are independent of the choice of* $(v_{r+1}, ..., v_n)$.[7]

(iii) *If* $r < n$ *and if the eigenvalues of* $\bar{F}_{22}$ *satisfy* $\text{Re}(\lambda_i) < \alpha < 0$, $i = 1, ..., n - r$ *then the OSS is locally asymptotically accessible of degree* α.

Proof of (iii). Let $y = U^{-1}\chi = (z, w), z \in R^r, w \in R^{n-r}$, then $\dot{y} = \bar{F}y + \bar{G}\gamma$ becomes

$$\dot{z} = \bar{F}_{11}z + \bar{F}_{12}w + \bar{G}_1\gamma, \tag{24}$$

$$\dot{w} = \bar{F}_{22}w. \tag{25}$$

Consider the problem of finding γ associated with the system $\dot{z} = \bar{F}_{11}z + \bar{G}_1\gamma$ which minimizes $\int_0^\infty e^{-2\alpha\tau}(z'Cz + \gamma'D\gamma)\,d\tau$ for some C, D positive definite. Let $\gamma^* = P^*z$ denote the associated control given by (21). Since $(\bar{F}_{11}, \bar{G}_1)$ is locally finitely accessible γ^* exists and by Theorem 3 there exist constants $a_1 > 0$, $b > 0$ such that the solution $\tilde{z}(t)$ of the homogeneous system

$$\dot{z} = \bar{F}_{11}z + \bar{G}_1\gamma^* = (\bar{F}_{11} + \bar{G}_1P^*)\,z = Vz$$

satisfies $\| \tilde{z}(t)\| \leqslant a_1e^{\alpha t}$, $\| \gamma^*(t)\| \leqslant be^{\alpha t}$ for all $t \geqslant 0$.

No generality is lost in the argument that follows if we assume $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j = 1, ..., n - r$ so that (25) implies $w(t) = \sum_{i=1}^{n-r} v_ie^{\lambda_i t}$, $v_i \in R^{n-r}$. Since $\text{Re}(\lambda_i) < \alpha$, $\| w(t)\| \leqslant a_2e^{\alpha t}$ for some $a_2 > 0$. Let $\bar{z}$ denote a particular solution of

$$\dot{z} = \bar{F}_{11}z + \bar{G}_1\gamma^* + \bar{F}_{12}w = Vz + \bar{F}_{12}\left(\sum_{i=1}^{n-r} v_ie^{\lambda_i t}\right),$$

then $\| \bar{z}(t)\| < a_3e^{\alpha t}$ for some $a_3 > 0$. Thus the general solution $z = \tilde{z} + \bar{z}$

[7] Recall that $|F - \lambda I| = |U^{-1}FU - \lambda I| = |\bar{F}_{11} - \lambda I|\,|\bar{F}_{22} - \lambda I|$ implies that the eigenvalues of $(\bar{F}_{11}, \bar{F}_{22})$ are the same as the eigenvalues of F.

of (24) under γ^* satisfies $\| z(t)\| \leqslant a_4 e^{\alpha t}$ for some $a_4 > 0$. Thus for some $a > 0$

$$\| \chi(t)\| = \| Uy(t)\| \leqslant \| U \|(\| z(t)\| + \| w(t)\|) \leqslant ae^{\alpha t} \qquad \text{for all } t \geqslant 0.$$

Since (22) is satisfied under γ^*, the proof is complete. △

In the proof of (iii) we also established the following

COROLLARY. *If (F, G) is locally finitely accessible then (F, G) is locally asymptotically accessible of any degree α, $-\infty < \alpha < 0$.*

Stability. We are now in a position to analyze the stability properties of the trajectories associated with the problem $(L_\rho{}^0)$. Using (21), the equation $\dot{\chi} = F\chi + G\gamma^*$ when written in terms of the Hamiltonian $G^0(\chi, \eta)$ becomes

$$\dot{\chi} = (G^0_{\eta x} + G^0_{\eta\eta} Q^*)\, \chi. \tag{26}$$

Since $\eta = Q^*\chi$ the system may also be characterized by the *dual price equation*[8]

$$\dot{\eta} = -(G^0_{x\eta} - \delta I + G^0_{xx} Q^{*-1})\, \eta. \tag{27}$$

DEFINITION.

$$K^\delta = \begin{bmatrix} -G^0_{xx} & (\delta/2)\, I \\ (\delta/2)\, I & G^0_{\eta\eta} \end{bmatrix} \qquad \text{for} \quad \delta \in [0, \infty)$$

will be called the *curvature matrix* generated by the Hamiltonian $G^0(\chi, \eta)$.

DEFINITION.

$$R = G^{0\prime}_{\eta x}(-G^0_{\eta\eta})^{-1} + (-G^0_{\eta\eta})^{-1}\, G^0_{\eta x}\,,$$

$$M^\delta = [\delta I - G^0_{x\eta}]'\, (G^0_{xx})^{-1} + (G^0_{xx})^{-1}\, [\delta I - G^0_{x\eta}] \qquad \text{for} \quad \delta \in [0, \infty)$$

will be called the *Liapunov matrices* generated by the Hamiltonian $G^0(\chi, \eta)$.

THEOREM 3 (Asymptotic Stability).[9] (i) *If $\delta \leqslant 0$, $L^0(\chi, \gamma)$ is positive*

[8] Equations (26) and (27) are the *linearized equations* for the two *reduced form* equations in χ and η, respectively, that are obtained from the canonical equations (5) associated with the problem $(\mathscr{L}_\rho{}^0)$ when we make the substitutions $\eta = W_\chi(\chi)$ and $\chi = W_\eta{}^*(\eta)$, respectively, $W(\chi)$ and $W^*(\eta)$ being the value–loss and the dual value–loss function (the Fenchel conjugate of $W(\chi)$), respectively. From this observation the reader can see how to generalize the results below to the *global* case.

[9] (i) is a generalization of the theorem of Kalman [31]. A global version of Kalman's theorem was established by Samuelson [60] and Rockafellar [54]. The K^δ condition was first established in [43]. Similar results in the global case were subsequently obtained by Cass and Shell [21], Brock and Scheinkman [14], and Rockafellar [55]. The R condition was first established in [15] and is here given an alternative proof. The M^δ condition is proved here for the first time.

definite and the OSS is locally asymptotically accessible of degree $\alpha/2$, $\alpha < \delta$, *or if*

(ii) $\delta > 0$ *and* (a) K^δ *is negative definite or* (b) K^0 *is negative definite and R is nonpositive definite or* (c) K^0 *is negative definite and* M^δ *is nonpositive definite, then the trajectories of* (26) *and* (27) *are asymptotically stable. Furthermore if* (i) *or* (ii) *are satisfied then there exist constants* $a > 0$, $b > 0$, $c > 0$, $\theta < 0$, $\theta < \delta$ *such that*

$$\|\chi(t)\| \leqslant ae^{(\theta/2)t}, \qquad \|\gamma(t)\| \leqslant be^{(\theta/2)t}, \qquad \|\eta(t)\| \leqslant ce^{(\theta/2)t}, \qquad t \in [0, \infty). \quad (28)$$

Proof. The proof is in two steps. We first show that (i) and (ii) imply that the assumptions of Lemma 4 are satisfied, we then exhibit an appropriate Liapunov function for each case. Consider therefore the quadratic form $z'\begin{bmatrix} A & N \\ N' & B \end{bmatrix} z$, where $z = (\chi, \gamma)$. Let $z = Uz^0 = \begin{bmatrix} I & 0 \\ -B^{-1}N' & I \end{bmatrix} z^0$, then $U'\begin{bmatrix} A & N \\ N' & B \end{bmatrix} U = \begin{bmatrix} A-NB^{-1}N' & 0 \\ 0 & B \end{bmatrix}$. Since U is nonsingular, $\begin{bmatrix} A & N \\ N' & B \end{bmatrix}$ is positive definite if and only if $\begin{bmatrix} A-NB^{-1}N' & 0 \\ 0 & B \end{bmatrix}$ is positive definite. K^δ, $\delta \geqslant 0$ negative definite implies $A - NB^{-1}N'$ and $GB^{-1}G'$ positive definite. The latter implies rank $(G) = n$ and B positive definite. Thus K^δ, $\delta \geqslant 0$ negative definite implies that $L^0(\chi, \gamma)$ is positive definite. Since K^δ, $\delta \geqslant 0$ negative definite implies rank $(G) = n$, by (23) (F, G) is locally finitely accessible and hence by the corollary to Proposition 1 is locally asymptotically accessible of any degree $\alpha < 0$. Thus if (i) or (ii) hold then the assumptions of Lemma 4 are satisfied and $\lim_{T\to\infty} Q(t, T) = Q^*$, Q^* positive definite.

To prove (i) and (ii)(a) we consider the value–loss function (15)

$$W(\chi) = \tfrac{1}{2}\chi' Q^* \chi,$$

$W(0) = 0$, $W(\chi) > 0$, $\chi \neq 0$, $W(\chi) \in \mathscr{C}^2$. As the system moves along the trajectories (26)

$$\dot{W}(\chi) = (W_x)'\,\dot{\chi} = \tfrac{1}{2}(\chi' Q^* + \chi' Q^{*\prime})(G^0_{\eta x} + G^0_{\eta\eta} Q^*)\,\chi$$

$$= \tfrac{1}{2}\chi'(Q^* G^0_{\eta x} + G^0_{x\eta} Q^* + 2Q^* G^0_{\eta\eta} Q^*)\,\chi.$$

Since Q^* is the solution of (19) with $\dot{Q}^* = 0$,

$$Q^* G^0_{\eta x} + G^0_{x\eta} Q^* = -Q^* G^0_{\eta\eta} Q^* - G^0_{xx} + \delta Q^*$$

so that

$$\dot{W}(\chi) = \tfrac{1}{2}\chi'(Q^* G^0_{\eta\eta} Q^* - G^0_{xx} + \delta Q^*)\,\chi. \quad (29)$$

Suppose $\delta \leqslant 0$. $L^0(\chi, \gamma)$ positive definite implies $A - NB^{-1}N'$ and B positive definite. Thus $G^0_{\eta\eta} = -GB^{-1}G'$ and hence $Q^* G^0_{\eta\eta} Q^*$ is nonpositive definite and $-G^0_{xx} = -(A - NB^{-1}N')$ is negative definite. Since $\delta \leqslant 0$ and since

Q^* is positive definite, δQ^* is nonpositive definite. Thus $\dot{W}(\chi) < 0$, $\chi \neq 0$. Suppose $\delta > 0$. Since $\eta = Q^*\chi$, $\dot{W}(\chi)$ in (29) can be written as

$$\dot{W}(\chi) = \frac{1}{2}\begin{bmatrix}\chi\\ \eta\end{bmatrix}'\begin{bmatrix}-G^0_{xx} & (\delta/2)\,I\\ (\delta/2)\,I & G^0_{nn}\end{bmatrix}\begin{bmatrix}\chi\\ \eta\end{bmatrix}, \tag{30}$$

so that K^δ negative definite implies $\dot{W}(\chi) < 0$, $\chi \neq 0$. Thus if (i) or (ii)(a) are satisfied by the Liapunov stability theorem [30, p. 39] the trajectories of (26) are asymptotically stable. Since $\eta = Q^*\chi$ a similar result for the trajectories of (27) is immediate.

If (29)(i) and (30(ii)(a) are satisfied, there exists $\theta < 0$ such that

$$\dot{W}/W \leqslant \theta, \qquad \theta < \delta.$$

Let $0 < \lambda^0 \leqslant \lambda^*$ denote the minimum and maximum eigenvalues of Q^*, then

$$\lambda^0\chi'(t)\,\chi(t) \leqslant \chi'(t)\,Q^*\chi(t) = W(t) \leqslant W(0)\,e^{\theta t} \leqslant \lambda^*\chi'(0)\,\chi(0)\,e^{\theta t}$$

so that $\| \chi(t)\| \leqslant \| \chi(0)\|\,(\lambda^*/\lambda^0)^{1/2}\,e^{(\theta/2)t}$, $\| \gamma(t)\| = \| P^*\chi(t)\| \leqslant \| P^* \|\,\| \chi(t)\|$, $\| \eta(t)\| = \| Q^*\chi(t)\| \leqslant \| Q^* \|\,\| \chi(t)\|$ and (28) follows for (i) and (ii)(a) with $a = \| \chi(0)\|\,(\lambda^*/\lambda^0)^{1/2}$, $b = \| P^* \|\,a$, $c = \| Q^* \|\,a$.

To prove (ii)(b) we consider the Liapunov function $V(\chi) = \chi'(-G^0_{nn})^{-1}\,\chi$ and note that $\dot{V}(\chi) = \chi'(-2Q^* + R)\,\chi$ so that R nonpositive definite implies $\dot{V}(\chi) < 0$, $\chi \neq 0$, since Q^* is positive definite. Similarly, to prove (ii)(c) we consider the Liapunov function $U(\eta) = \eta'(G^0_{xx})^{-1}\,\eta$ and note that along the trajectories (27), $\dot{U}(\eta) = \eta'(-2Q^{*-1} + M^\delta)\,\eta$ so that M^δ nonpositive definite implies $\dot{U}(\eta) < 0$, $\eta \neq 0$. For an appropriate $\theta < 0$ in each case $\dot{V}/V \leqslant \theta$, $\dot{U}/U \leqslant \theta$. The rest is immediate. △

Geometric interpretation. The K^δ conditions implies $(W_x)'\,\dot{\chi} < 0$ for $\chi \neq 0$. *Whenever the system is not at the OSS, the angle between the tangent vector $\dot{\chi}$ of the trajectory and the gradient W_x of the surface of constant value-loss is an obtuse angle. Thus the trajectories of* (26) *cut across the surfaces $W(\chi) = $ constant, towards the OSS.* A similar interpretation follows for the R and M^δ conditions (see Fig. 1).

THEOREM 4 (Stable Submanifolds of $\mathscr{E}$ and $\mathscr{E}_k$). *Let $(\rho, k^*, p^*) \in \mathscr{E}$ and $(\rho, k^*) \in \mathscr{E}_k$. Under the conditions of Theorem* 3 *there exists an ϵ-neighborhood*

(1) *of (k^*, p^*) such that $(\mathscr{T}_\rho)$ and* (1) *has a unique optimal solution which is stable in the sense of Problem* I *for $\delta \geqslant 0$,*

(2) *of k^* such that $(\mathscr{T}_\rho)$ and* (1) *has a unique optimal solution which is stable in the sense of Problem* II.

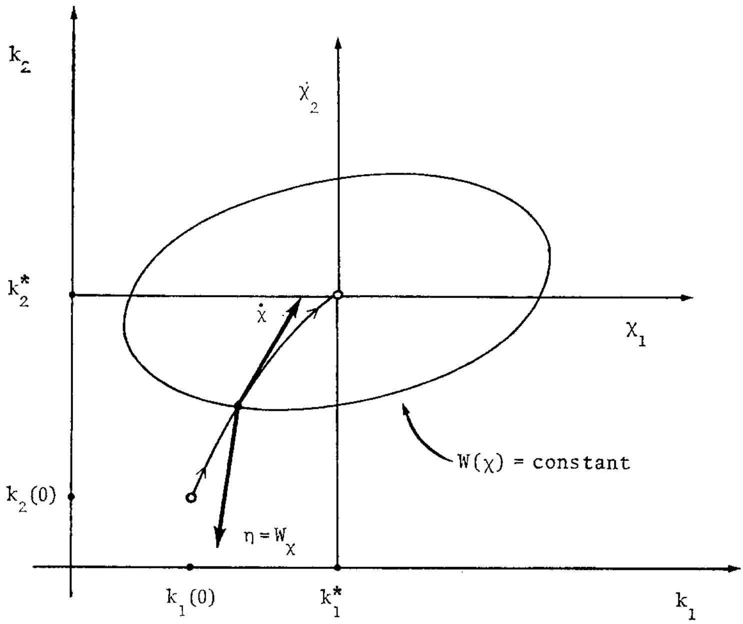

FIG. 1. Under the K^δ condition the tangent vector $\dot{\chi}$ leads the trajectory across the surfaces $W(\chi)$ = constant, towards the OSS.

Proof. Since $\eta = Q^*\chi$ (26) and (27) are equivalent to the canonical equations

$$\begin{bmatrix}\dot{\chi}\\ \dot{\eta}\end{bmatrix} = \begin{bmatrix} G^0_{\eta x} & G^0_{\eta\eta} \\ -G^0_{xx} & -(G^0_{\eta x} - \delta I)'\end{bmatrix}\begin{bmatrix}\chi\\ \eta\end{bmatrix}. \tag{31}$$

The asymptotic stability of Theorem 3 implies that the system has n eigenvalues with negative real parts. By the theorem of Kurz [33] the eigenvalues of (31) are symmetric about $(\delta/2)$ in the complex plane. Since $\delta \geqslant 0$, (31) has the saddle-point property. By Theorem 1 there exists a linear stable manifold $\mathscr{N}$. It is clear that $\mathscr{N} = \{(\bar{\chi}, \bar{\eta}) \mid \bar{\eta} = Q^*\bar{\chi}\}$. Equations (31) are the linearized canonical equations of the problem $(\mathscr{L}_\rho^0)$. Hence by Theorem 1(ii) there exists a nonlinear stable manifold $\mathscr{N}^*$ which is tangent to $\mathscr{N}$ at the OSS. Thus there exists an $\epsilon > 0$ such that $\bar{\eta}'\bar{\chi} > 0$ whenever $(\bar{\chi}, \bar{\eta}) \in \mathscr{N}^*$, $\|(\bar{\chi}, \bar{\eta})\| < \epsilon$, $\bar{\chi} \neq 0$, since Q^* is positive definite. The canonical equations (5) characterizing competitive prices $(\bar{k}(t), \bar{p}(t))$ are equivalent to the canonical equations of $(\mathscr{L}_\rho^0)$ under the transformation (10) [55, p. 80] and hence lead under (10) to a manifold $\mathscr{M}^*$ generated by $\mathscr{N}^*$ with the properties of Problem I. Since (6) is satisfied along the manifold $\mathscr{M}^*$ the competitive paths along $\mathscr{M}^*$ are optimal. To establish (2) we need only note that (28) ensures that (6) is satisfied when $\delta < 0$ for some $\epsilon > 0$. △

4. Interpretation of Stability Conditions

Geometry of Hamiltonian. It was the original paper of Shell and Stiglitz [63, 64] that first brought to light the rich economic interpretation that underlies the *saddle-point property* for a competitive market system. Translated into the present framework the stable manifold leading to the OSS carries the one and only infinite sequence of market clearings that is optimal over all future time. Cass and Shell [21] have shown that under certain conditions a Hamiltonian and equations akin to (5) may provide a useful framework for analyzing competitive dynamical systems. Although the Hamiltonian is not the economic entity that is of real interest in a theory of resource allocation, since it represents *revenue* rather than *profit*, its *curvature properties* may provide a useful analytical tool for analyzing stability, since the Hamiltonian is the function by means of which the simultaneous evolution of prices and quantities is described. We will consider therefore the curvature properties of the Hamiltonian implied by the basic condition in Theorem 3, the K^δ condition.

Definition. Let $f(x) \in \mathscr{C}^2$ be defined on a convex domain $D \subset R^n$, let $\alpha \geqslant 0$, $\alpha \in R$, and let B be an $n \times n$ positive definite matrix. $f(x)$ will be called *α-convex in the metric of B* if $w' f_{xx}(x)\, w \geqslant \alpha w' B w$ for all $x \in D$, $w \in R^n$. If $B = I$, $f(x)$ will be called *α-convex*. $f(x)$ is *α-concave* if $-f(x)$ is α-convex.

Definition. Let A, B be $n \times n$ matrices, B positive definite, symmetric. ν_i is an *eigenvalue of A in the metric of B* if there exists $x^i \neq 0$ such that $Ax^i = \nu^i B x^i$. It is well known that *if in addition A is positive definite, symmetric, then A has n real, positive eigenvalues in the metric of B* [27, pp. 310–319]. Let $\nu_1, \ldots, \nu_n$ denote the real, positive eigenvalues of G^0_{xx} in the metric of $(-G^0_{\eta\eta})^{-1}$ and let

$$\nu^0(G^0_{xx}\,;(-G^0_{\eta\eta})^{-1}) = \min_{1 \leqslant i \leqslant n} \{\nu_i\}.$$

Proposition 2. *The following statements are equivalent*:

(i) *K^δ is negative definite,*

(ii) $\nu^0(G^0_{xx}\,;(-G^0_{\eta\eta})^{-1}) > (\delta/2)^2$,

(iii) *$G^0(\chi, \eta)$ is θ-convex in χ in the metric of $(-G^0_{\eta\eta})^{-1}$ for some* $\theta > (\delta/2)^2$.

Proof. The transformation

$$\begin{bmatrix}\chi\\ \eta\end{bmatrix} = \begin{bmatrix} I & 0\\ (\delta/2)(-G^0_{\eta\eta})^{-1} & I\end{bmatrix}\begin{bmatrix}\bar\chi\\ \bar\eta\end{bmatrix} \quad \text{implies} \quad \begin{bmatrix}\chi\\ \eta\end{bmatrix}' \begin{bmatrix} -G^0_{xx} & (\delta/2)\, I\\ (\delta/2)\, I & G^0_{\eta\eta}\end{bmatrix}\begin{bmatrix}\chi\\ \eta\end{bmatrix} < 0$$

for all $(\chi, \eta) \neq 0$ is equivalent to the two conditions

$$\begin{aligned} \chi'[G^0_{xx} - (\delta/2)^2 (-G^0_{\eta\eta})^{-1}]\, \chi > 0 \qquad & \text{for all} \quad \chi \neq 0, \\ \bar{\eta}' G^0_{\eta\eta} \bar{\eta} < 0 \qquad & \text{for all} \quad \bar{\eta} \neq 0 \end{aligned} \tag{32}$$

Note that since $G^0(\chi, \eta)$ is quadratic in χ and η, G^0_{xx} and $G^0_{\eta\eta}$ are independent of (χ, η).

(i) ⇒ (ii). Since (32) implies $\chi'(-G^0_{\eta\eta})^{-1} \chi > 0$, $\chi \neq 0$, (32) also implies $[\chi' G^0_{xx} \chi / \chi'(-G^0_{\eta\eta})^{-1} \chi] > (\delta/2)^2$ for all $\chi \neq 0$ so that $\min_{\chi \neq 0} [\chi' G^0_{xx} \chi / \chi'(-G^0_{\eta\eta})^{-1} \chi] > (\delta/2)^2$. But

$$\nu^0(G^0_{xx}\,;\, (-G^0_{\eta\eta})^{-1}) = \min_{\chi \neq 0} \frac{\chi' G^0_{xx} \chi}{\chi'(-G^0_{\eta\eta})^{-1}\, \chi}$$

[22, pp. 37–39].

(ii) ⇒ (iii). Since $\nu^0 = \min_{\chi \neq 0} [\chi' G^0_{xx} \chi / \chi'(-G^0_{\eta\eta})^{-1}\, \chi]$, $[\chi' G^0_{xx} \chi / \chi'(-G^0_{\eta\eta})^{-1} \chi] \geqslant \nu^0$ for all $\chi \neq 0$ so that $\chi' G^0_{xx} \chi \geqslant \theta \chi'(-G^0_{\eta\eta})^{-1} \chi$ for all $\chi \in R^n$ with $\theta = \nu^0 > (\delta/2)^2$.

(iii) ⇒ (i). If $G^0(\chi, \eta)$ is θ-convex in χ in the metric of $(-G^0_{\eta\eta})^{-1}$ with $\theta > (\delta/2)^2$, then $w' G^0_{xx} w \geqslant \theta w'(-G^0_{\eta\eta})^{-1}\, w > (\delta/2)^2\, w'(-G^0_{\eta\eta})^{-1}\, w$ for all $w \neq 0$ and $v' G^0_{\eta\eta} v < 0$ for all $v \neq 0$, which implies (32). △

The following proposition and corollary relate Rockafellar's [55] α convexity, β-concavity for $G^0(\chi, \eta)$, to the more general condition of Proposition 2(iii).

PROPOSITION 3. *If $G^0(\chi, \eta)$ is α-convex in χ and β-concave in η with $\beta > 0$ then $G^0(\chi, \eta)$ is $\alpha\beta$-convex in χ in the metric of $(-G^0_{\eta\eta})^{-1}$.*

Proof. $w' G^0_{xx} w \geqslant \alpha w' w$ for all $w \in R^n$ implies $(w' G^0_{xx} w / w' w) \geqslant \alpha$ for all $w \neq 0$ so that $\mu^0(G^0_{xx}) = \min_{w \neq 0} (w' G^0_{xx} w / w' w) \geqslant \alpha$. Similarly $\lambda^0(-G^0_{\eta\eta}) = \min_{w \neq 0} [w'(-G^0_{\eta\eta})\, w / w' w] \geqslant \beta$. Thus

$$\alpha\beta \leqslant \mu^0(G^0_{xx})\, \lambda^0(-G^0_{\eta\eta}) = \frac{\mu^0(G^0_{xx})}{\lambda^*(-G^0_{\eta\eta})^{-1}} = \frac{\min_{v \neq 0} [v' G^0_{xx} v / v' v]}{\max_{y \neq 0} [y'(-G^0_{\eta\eta})^{-1}\, y / y' y]}$$

$$\leqslant \min_{v \neq 0} \frac{v' G^0_{xx} v}{v'(-G^0_{\eta\eta})^{-1}\, v} \leqslant \frac{v' G^0_{xx} v}{v'(-G^0_{\eta\eta})^{-1}\, v}$$

for all $v \neq 0$. Since $\lambda^0(-G^0_{\eta\eta}) \geqslant \beta > 0$, $v'(-G^0_{\eta\eta})^{-1}\, v > 0$ for $v \neq 0$ so that $v' G^0_{xx} v \geqslant \alpha\beta v'(-G^0_{\eta\eta})^{-1}\, v$ for all $v \in R^n$. △

COROLLARY. (i) *If $\mu^0(G^0_{xx})\, \lambda^0(-G^0_{\eta\eta}) = [\mu^0(G^0_{xx}) / \lambda^*(-G^0_{\eta\eta})^{-1}] > (\delta/2)^2$ then $\nu^0(G^0_{xx}\,;\, (-G^0_{\eta\eta})^{-1}) > (\delta/2)^2$.*

(ii) *If $G^0(\chi, \eta)$ is α-convex in χ and β-concave in η with $\alpha\beta > (\delta/2)^2$ then K^δ is negative definite.*

Benefit–cost calculation. The transformation $\begin{bmatrix} x \\ \gamma \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^{-1}N' & I \end{bmatrix} \begin{bmatrix} \bar{\chi} \\ \bar{\gamma} \end{bmatrix}$ reduces $(L_\rho{}^0)$ to the *equivalent problem*

$$\begin{gathered} \inf_{\gamma} \tfrac{1}{2} \int_0^\infty e^{-\delta\tau}[\chi'(A - NB^{-1}N')\chi + \bar{\gamma}'B\bar{\gamma}]\, d\tau, \\ \dot{\chi} = (F - GB^{-1}N')\chi + G\bar{\gamma}, \quad k^* + \chi \geqslant 0, \quad k^* + \chi(0) = k_0 . \end{gathered} \tag{33}$$

DEFINITION. $A - NB^{-1}N'$, $F - GB^{-1}N'$, and $(GB^{-1}G')^{-1}$ will be called the *effective state–loss matrix*, the *underlying system matrix*, and the *effective control–cost matrix*, respectively.

Remark. $\chi'(A - NB^{-1}N')\chi$ measures the value–loss induced by a given deviation χ of the state from the OSS when the interaction between χ and the control γ, measured by N, is taken into account: It is a measure of the *benefit* generated by the OSS. $F - GB^{-1}N'$ determines the nature of the motion of the underlying uncontrolled system ($\bar{\gamma} = 0$). $(GB^{-1}G')^{-1}$ measures control costs when allowance is made for the effectiveness of control, as measured by G: It is a measure of the *cost* of reaching the OSS. *These three matrices are precisely the information revealed by the curvature properties of the Hamiltonian, since* $G^0_{xx} = A - NB^{-1}N'$, $G^0_{\eta x} = F - GB^{-1}N'$, $(-G^0_{\eta\eta})^{-1} = (GB^{-1}G')^{-1}$.

Proposition 2(ii) leads to the *benefit–cost ratio*

$$\nu^0 = \min_{\chi \neq 0} [\chi' G^0_{xx}\chi / \chi'(-G^0_{\eta\eta})^{-1}\chi] = \min_{\chi'(-G^0_{\eta\eta})^{-1}\chi = 1} \chi' G^0_{xx}\chi.$$

Adjusting the dimensions of ν^0 so that it coincides with that of the interest rate δ, leads us to the condition

$$2(\nu^0)^{1/2} > \delta. \tag{34}$$

An OSS for which the benefit–cost ratio $2(\nu^0)^{1/2}$ exceeds the interest rate δ is locally asymptotically stable, independent of the underlying system matrix $G^0_{\eta x}$ for the economy.

Remark. The strength of the K^δ condition lies in the fact that it is invariant with respect to $G^0_{\eta x}$. It requires in return, however, a minimum benefit–cost ratio relative to the interest rate, for the OSS. If more is known about the underlying system matrix $G^0_{\eta x}$ then the R condition or the M^δ condition may be appropriate. *The condition R nonpositive definite implies by the Liapunov theorem*[10] *that the underlying system $\dot{\chi} = (G^0_{\eta x})\chi$ must be stable. The eigen-*

[10] *If there exists a positive definite matrix P such that $F'P + PF$ is nonpositive then the solutions of $\dot{x} = Fx$ are stable.* To prove the result use the Liapunov function $V(x) = x'Px$ and the basic Liapunov stability theorem [30, p. 39].

values of $(G^0_{\eta x})$ *must thus satisfy* $\text{Re}(\lambda_i) \leqslant 0$, $i = 1,..., n$ *so that* $G^0_{\eta x}$ *is a quasi-stable matrix.*[11] *The condition* M^δ *nonpositive definite implies that the underlying system for the dual* $\dot{\eta} = (\delta I - G^0_{x\eta})\,\eta$ *must be stable. The eigenvalues of* $G^0_{\eta x} = G^{0\prime}_{x\eta}$ *must thus satisfy* $\text{Re}(\lambda_i) \geqslant \delta > 0$, $i = 1,..., n$ *so that* $G^0_{\eta x}$ *is unstable of degree* δ.

Remark. I originally conjectured that if $G^0_{\eta x}$ is a stable matrix the OSS would be locally asymptotically stable. Brock [9, Appendix] subsequently showed that this is not the case. A closely related result however, in the variational case, is the following.

PROPOSITION 4. *If* $L_{kk}(k^*, 0)$, $L_{\dot{k}\dot{k}}(k^*, 0)$ *are negative definite and if* $L_{k\dot{k}}(k^*, 0) + L_{\dot{k}k}(k^*, 0)$ *is nonpositive definite then the OSS is locally asymptotically stable.*

Proof. Set $F = O$, $G = I$ in Theorem 3(ii)(a). △

DEFINITION. The utility function $L(k, \dot{k})$ is *additively separable* if $L(k, \dot{k}) = u(k) + w(\dot{k})$.

COROLLARY. *If* $L_{kk}(k^*, 0)$, $L_{\dot{k}\dot{k}}(k^*, 0)$ *are negative definite and if the utility function is additively separable then the OSS is locally asymptotically stable.*

Remark. This result is closely related to the theorem of Lagrange mentioned at the start of the paper. In classical physics the Lagrangean $L(k, \dot{k}) = T(\dot{k}) - V(k)$ is additively separable, $T(\dot{k})$ being *kinetic energy* (a positive definite function) and $V(k)$ being the *potential function.*

Remark. Thus we see that in the variational case if the *gyroscopic terms* $L_{k\dot{k}}(k^*, 0)$ are either absent or such that $L_{k\dot{k}}(k^*, 0) + L_{\dot{k}k}(k^*, 0)$ is quasi-stable then instability does not arise.

Remark. The idea that stability should depend upon the *curvature properties* of the utility function at the OSS is an old one: it is implicit in Lagrange's theorem and is explicitly used by Liapunov to study gradient processes [35, p. 61]. More complex gradient processes were extensively analyzed in the classical investigations of Arrow and Hurwicz [3, 5, 6]. Their Theorem 2, [6, p. 125] is closely related to the K^δ condition, although it should be recalled that gradient processes are a much simpler class of processes than those that arise from the control problem.

Mirage variables. An interesting economic interpretation may be given of those cases where a sufficient increase in the rate of interest δ leads to instability.

[11] Let $\lambda_1,..., \lambda_n$ denote the eigenvalues of an $n \times n$ matrix A. We say A is (i) *stable* if $\text{Re}(\lambda_i) < 0$, (ii) *quasi-stable* if $\text{Re}(\lambda_i) \leqslant 0$, (iii) *unstable* if $\text{Re}(\lambda_i) > 0$, for $i = 1,..., n$.

DEFINITION. The variables $(\hat{\chi}(t), \hat{\gamma}(t), \hat{\eta}(t)) = e^{-(\delta/2)t}(\chi(t), \bar{\gamma}(t), \eta(t))$ will be called the *mirage variables*.

Remark. The variables $(\hat{\chi}(t), \hat{\gamma}(t), \hat{\eta}(t))$ are taken to be the Planners' (society's) distorted or miraged view of the *actual future variables* $(\chi(t), \bar{\gamma}(t), \eta(t))$. If the Planners view the future path of capital accumulation in terms of the mirage variables then *the future path always appears to lie closer to the OSS than is actually the case.* If, furthermore, the Planners do not discount future losses, then the effect is the same as if they correctly assessed the actual future path but applied a systematic discounting to future losses.

In the mirage variables (33) reduces to the *equivalent problem*

$$\begin{aligned} &\inf_{\gamma} \tfrac{1}{2} \int_0^{\infty} [\hat{\chi}'(A - NB^{-1}N')\,\hat{\chi} + \hat{\gamma}'B\hat{\gamma}]\, d\tau, \\ &\dot{\hat{\chi}} = (F - GB^{-1}N' - (\delta/2)\, I)\,\hat{\chi} + G\hat{\gamma}, \\ &k^* + \hat{\chi} \geqslant 0, \qquad k^* + \hat{\chi}(0) = k_0, \end{aligned} \tag{35}$$

where $-(\delta/2)\, I\hat{\chi}$ is the factor by which the actual growth rate is distorted in the mirage variables. The equations (26) and (27) now become

$$\dot{\hat{\chi}} = (G^0_{\eta\chi} - (\delta/2)\, I + G^0_{\eta\eta}Q^*)\,\hat{\chi}, \tag{36}$$

$$\dot{\hat{\eta}} = -(G^0_{\chi\eta} - (\delta/2)\, I + G^0_{\chi\chi}Q^{*-1})\,\hat{\eta}. \tag{37}$$

Since the K^δ condition applied to the original system (33) implies $A - NB^{-1}N'$ and B are positive definite and rank $(G) = n$, Proposition 1(i), its corollary and Theorem 3(i) imply that (36) and (37) are asymptotically stable. *In the mirage variables the optimal path of capital accumulation and the associated competitive prices converge for all values of* δ, *in some* ϵ*-neighborhood of the OSS.*

The Planners conscientiously outline the future path of capital accumulation following closely the advice of Ramsey, assigning equal importance to present and future losses [53]. *Viewing the economy in terms of the mirage variables the Planners are convinced the economy will converge to the OSS. And yet if the degree of distortion of the actual path passes above a critical level, the actual path which the economy will follow will not converge to the OSS.* This is the great social fraud thrown upon the society by the mirage variables. The deception they face is not unlike that of the traveler in the desert who is convinced he sees an oasis ahead. But in reality it does not exist. It is only a mirage.

Remark. The mirage variables are of considerable interest in connection with the earlier analysis of Kurz [33]. For if the capital stocks and prices are analyzed simultaneously (36) and (37) reduce to

$$\begin{bmatrix} \dot{\hat{\chi}} \\ \dot{\hat{\eta}} \end{bmatrix} = \begin{bmatrix} (G^0_{\eta x} - (\delta/2)\, I) & G^0_{\eta\eta} \\ -G^0_{xx} & -(G^0_{\eta x} - (\delta/2)\, I)' \end{bmatrix} \begin{bmatrix} \hat{\chi} \\ \hat{\eta} \end{bmatrix}. \tag{38}$$

The canonical equations in the mirage variables generate a saddle-point in the 2n-dimensional space $(\hat{\chi}, \hat{\eta})$ *for all values of* δ. This result follows at once from the theorem of Poincaré[12], by which if μ_i is an eigenvalue of the matrix in (38), then $-\mu_i$ is also an eigenvalue, furthermore $\mathrm{Re}(\mu_i) > 0$ by the K^δ condition so that periodic trajectories are eliminated. If we transform back to the *actual capital stocks and prices* we obtain the standard canonical equations (31). Assuming μ_i distinct for simplicity, the local optimal path becomes

$$\hat{\chi}(t) = \sum_{i=1}^{n} a_i e^{-\mu_i t}, \qquad \chi(t) = \sum_{i=1}^{n} a_i e^{[(\delta/2)-\mu_i]t}, \qquad a_i \in \mathbb{C}^n, \quad i = 1,\dots, n,$$

so that $\chi(t)$ is asymptotically stable if and only if $\min_{1\leqslant i\leqslant n} \mathrm{Re}(\mu_i) > \delta/2$.

5. Long-Run Dynamics

Bifurcation. Consider the equilibrium manifold $\mathscr{E}_k$, defined by (9) for the variational problem in the case where $\rho = \delta$ $(s = 1)$:

$$\mathscr{E}_k = \{(\delta, k^*) \mid L_k(k^*, 0) + \delta L_{\dot{k}}(k^*, 0) = 0, \ \delta \geqslant 0, \ k^* > 0\}.$$

Definition. $k^*(\delta_0) \in \mathscr{E}_k$ is a *bifurcation equilibrium* and δ_0 is a *bifurcation point* for δ if there exist two solutions $k^*(\delta)$, $\bar{k}^*(\delta) \in \mathscr{E}_k$ such that $\| k^*(\delta) - \bar{k}^*(\delta)\| \to 0$ as $\delta \to \delta_0$.

The following result [50, p. 43] is a direct consequence of the implicit function theorem.

Lemma 5. *If* $k^*(\delta_0)$ *is a bifurcation equilibrium then*

$$\Delta = | L_{kk}(k^*, 0) + \delta_0 L_{\dot{k}k}(k^*, 0)| = 0.$$

[12] The theorem on the *characteristic exponents* of periodic orbits and stationary solutions of the variational Hamiltonian equations is due independently to Poincaré [51, p. 340] and Liapunov [35, pp. 75, 204]. The property follows from Liouville's theorem by which the vector field in the phase space behaves like an incompressible fluid, the phase volume remaining constant during the motion.

Remark. Linearizing the Euler–Lagrange equations about an OSS gives

$$L^*_{\dot{k}\dot{k}}\ddot{\chi} + (L^*_{\dot{k}k} - L^*_{k\dot{k}} - \delta L^*_{\dot{k}\dot{k}})\,\dot{\chi} - (L^*_{kk} + \delta L^*_{k\dot{k}})\,\chi = 0 \tag{39}$$

so that the characteristic polynomial reduces to

$$D(\lambda) = |\, L^*_{\dot{k}\dot{k}}\lambda^2 + (L^*_{\dot{k}k} - L^*_{k\dot{k}} - \delta L^*_{\dot{k}\dot{k}})\,\lambda - (L^*_{kk} + \delta L^*_{k\dot{k}})| = 0,$$

where $D(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_{2n} - \lambda)$, so that at a bifurcation equilibrium

$$\Delta = (-1)^n\, D(0) = (-1)^n\, \lambda_1 \cdots \lambda_{2n} = 0. \tag{40}$$

If all the eigenvalues are real as δ passes through a bifurcation point δ_0, at least one eigenvalue passes through the origin. In the standard case Δ changes sign and a single eigenvalue simultaneously changes its sign leading to a change in the stability properties of the equilibrium manifold.[13]

LEMMA 6 (Equilibrium Potential). *If $L \in \mathscr{C}^2$ and if there exists a constant $M > 0$ such that $L_{k\dot{k}}(k, 0)$ is symmetric for all $k \in K = \{k \mid 0 < k_j < M, j = 1,..., n\}$ then there exists a function $\phi(k, \delta)$ such that*

$$\phi_k(k, \delta) = L_k(k, 0) + \delta L_{\dot{k}}(k, 0), \qquad k \in K. \tag{41}$$

Proof. Since $L_{k\dot{k}}(k, 0)$ is symmetric for all $k \in K$, the result follows from the standard theorem for the existence of a potential function [1, pp. 293–297]. △

Remark. The symmetry condition on $L_{k\dot{k}}$ in Lemma 6 is also *necessary* for the existence of $\phi(k, \delta)$.

DEFINITION. Let $\Delta(\mu) = |\, \phi_{kk}(k^*, \delta) - \mu I\,| = 0$, then $\mu_1 ,..., \mu_n$ will be called the *curvature coefficients* of the utility function ϕ.

Remark. Since $\phi_{kk}(k^*, \delta)$ is a symmetric matrix $y'\phi_{kk}(k^*, \delta)\, y = \sum_{i=1}^n \mu_i z_i^2$ under an orthogonal transformation $y = Uz$. If Δ changes sign as δ passes through a bifurcation point δ_0, one of the curvature coefficients μ_i changes sign. *Thus in the case where a utility function ϕ satisfying* (41)

[13] The bifurcation equilibrium defined here is the one originally introduced by Poincaré [50, p. 50]. It is associated with one *real* eigenvalue passing through the origin. The *Hopf bifurcation*, by which a stationary solution branches into a periodic orbit, is associated with a *complex conjugate* pair of eigenvalues crossing the imaginary axis and is not considered here. See Arnol'd [2, pp. 96–98]. *Thom's transversality theorem implies that the families of vector fields with bifurcation at a single zero real eigenvalue or a conjugate pair of purely imaginary eigenvalues form an open dense set in the space of one-parameter families of vector fields* (see Arnol'd [2, p. 94]).

exists, a change in stability is associated with a change in the curvature properties of ϕ.[14]

Consider the case $n = 1$. By Lemma 6 *a utility function* ϕ *satisfying* (41) *exists*. The roots of $D(\lambda)$ are given by

$$\lambda_1, \lambda_2 = \frac{\delta \pm (\delta^2 - 4\bar{D})^{1/2}}{2}, \quad \bar{D} = -\frac{(L_{kk}^* + \delta L_{k\dot{k}}^*)}{L_{\dot{k}\dot{k}}^*} = \lambda_1\lambda_2, \quad \delta = \lambda_1 + \lambda_2.$$

Thus we have a direct analog of the theorem of Lagrange. *When* $n = 1$, *if the utility function* ϕ *attains a maximum,* $\phi_{kk}^* < 0$, *then the OSS is locally asymptotically stable, while if* ϕ *attains a minimum,* $\phi_{kk}^* > 0$, *then the OSS is unstable.*

Let $r(k^*) = -[L_k(k^*, 0)/L_{\dot{k}}(k^*, 0)]$ then $\mathscr{E}_k = \{(\delta, k^*) \mid \delta = r(k^*), \delta \geqslant 0, k^* > 0\}$ which corresponds to the statement that at an OSS the own rate of return $r(k^*)$ is equal to the rate of interest δ. Two typical curves $r(k^*)$ are shown in Figs. 2a and b.[15] Since $r'(k^*) = (L_{kk}^* + \delta L_{k\dot{k}}^*)/-L_{\dot{k}}^*$ and since $p^* = -L_{\dot{k}}^* > 0$ for $\delta \geqslant 0$ by Assumption (OSS), $\operatorname{sgn} r'(k^*) = \operatorname{sgn} \phi_{kk}^*$. Thus the manifold $\mathscr{E}_k$ is stable (unstable) according as $r'(k^*) < 0$ (>0).

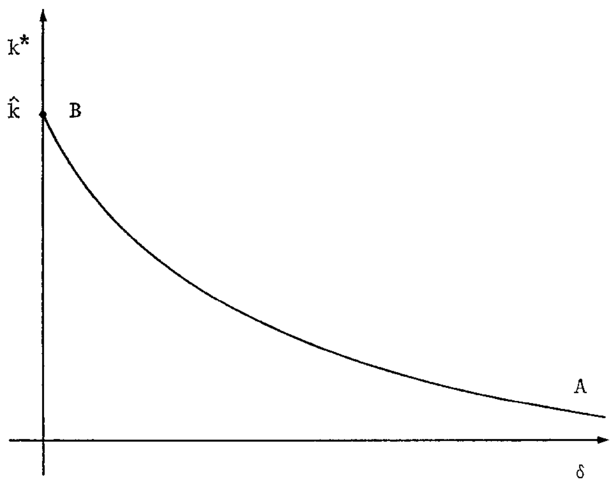

FIG. 2a. In Example 1 the equilibrium manifold *AB* is a stable manifold.

[14] Paul Samuelson has pointed out that the property associated with (40) by which the *dynamical system* changes its behavior whenever the *statical part* of the system changes its behavior, in the case where the roots are real, is closely related to the ideas that underlie his Correspondence Principle [59, Chap. IX].

[15] The cases shown in Figs. 2a and b arise in Examples 1 and 4, respectively, in Section 6. The former is the standard Cass model [19], the latter the one-sector model with joint-production of Liviatan and Samuelson [36].

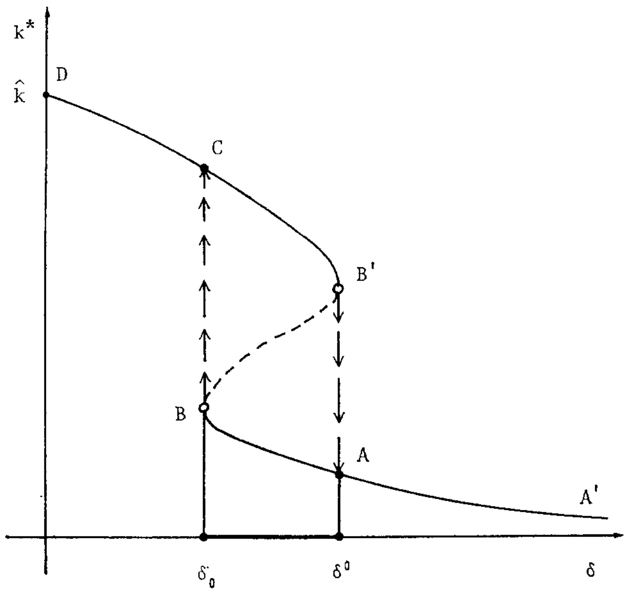

FIG. 2b. In Example 4 the equilibrium manifold has two stable submanifolds ($A'B$, $B'D$) and one unstable submanifold BB', separated by bifurcation equilibria at B and B'.

In Fig. 2a the manifold AB is stable since $r'(k^*) < 0$, $k^* > 0$, while in Fig. 2b $A'B$ and $B'D$ are stable submanifolds, while BB' is an unstable submanifold, the stable and unstable submanifolds being separated by bifurcation equilibria at B and B'. At B and B' the curvature coefficient ϕ_{kk}^* changes sign, signaling a simultaneous change in the curvature of ϕ and the stability properties of the equilibrium manifold.

Long-run dynamics. Consider the economy depicted in Fig. 2a. As the interest rate δ varies the economy moves smoothly along the equilibrium manifold AB. This is the classical case of *comparing equilibria.* Consider Fig. 2b. Suppose the system starts at A' with a high interest rate δ and a low OSS capital–labor ratio. Suppose that in a secular way the interest rate δ falls, approaching δ_0, then the economy moves smoothly up the submanifold $A'B$ toward B. *Suppose δ continues its slow monotone decrease passing below δ_0, then as δ passes through the bifurcation point δ_0, the OSS "jumps" from B to C.* In the Thom–Zeeman [66, 68] terminology, passing through the bifurcation point δ_0 produces a *catastrophe.* At a bifurcation point a small change in the interest rate can cause a major change in the OSS. In a similar way suppose the economy finds itself at D. If the interest rate rises towards δ^0, then the economy moves smoothly down the stable submanifold DB' towards B'. *If at δ^0 the interest rate continues to rise, then as δ passes through* *the bifurcation point δ^0, the economy, always seeking to stay on the relevant stable submanifold of $\mathscr{E}_k$, "jumps" from the OSS at B' to the OSS at A.*[16]

Remark. In Fig. 2b, $\mathscr{B} = [\delta_0, \delta^0]$, the collection of δ for which $k^*(\delta)$ is not unique, is called the *bifurcation set*. Whenever $\delta \notin \mathscr{B}$, ϕ has a unique maximum, corresponding to points on $A'A$ and CD. At δ^0 and δ_0, in fact at the points A and C, ϕ develops in addition to the maximum, an inflexion point at B' and B, respectively. Whenever $\delta \in \mathring{\mathscr{B}}$, ϕ has two maxima separated by a minimum, corresponding to points on AB, $B'C$, and BB'. *The change in ϕ as δ moves through the bifurcation set $\mathscr{B}$ explains the catastrophes at B and B'. As δ decreases (increases through $\mathscr{B}$, the lower (upper) maximum B (B') is eliminated, causing k^* to shift from the lower (upper) maximum near B (B'), to the upper (lower) maximum near C (A), respectively.*[17]

Paradoxical behavior. The presence of catastrophes at B and B' helps to throw some light on the familiar *paradox* of neoclassical capital theory: *Given two economies with identical technology and preferences, how can the economy with a lower rate of interest have a lower capital–labor ratio*? When the presence of joint-production leads to a return curve $r(k^*)$ of the form shown in Fig. 2b each economy essentially has two long-run modes of behavior: one along the stable submanifold $A'B$ and the second along the stable submanifold $B'D$.[18]

Comparing equilibria. As Brock [11] has suggested, the sufficient conditions of Cass and Shell [21], Brock and Scheinkman [14, 15], and Theorem 3 are likely, in the spirit of Samuelson's Correspondence Principle [59, Chap. IX], to lead to interesting theorems in comparing equilibria. Burmeister and Turnovsky [18] have shown that the following concept leads to many interesting results in capital theory.

DEFINITION. The economy exhibits *capital deepening response* at an OSS $(\delta, k^*, p^*) \in \mathscr{E}$ if $R^* = p^*(dk^*/d\delta) \leqslant 0$.

In Proposition 5 (i) is due to Brock and Burmeister [13], (ii) is new.

PROPOSITION 5. *If $\delta \geqslant 0$ and if at an OSS $(\delta, k^*, p^*) \in \mathscr{E}$ or $(\delta, k^*) \in \mathscr{E}_k$*

(i) *K^δ is nonpositive definite*, or

[16] The following terminology seems preferable. At *regular points* of the equilibrium manifold, changes in the interest rate cause *weak perturbations* in the OSS. At *bifurcation points*, changes in the interest rate cause *strong perturbations* in the OSS.

[17] The process of unfolding equilibrium potential functions as the parameter changes is the basic idea that led Thom and Zeeman to develop catastrophe theory.

[18] Smale [65] presents an interesting analysis of the catastrophe points of an equilibrium manifold closely related to $\mathscr{E}$, arising in Walrasian general equilibrium, with a vector of endowments among agents taken as the parameter ρ.

(ii) *in the variational problem, R is nonpositive definite, then the economy exhibits capital deepening response at the OSS.*

Proof. (i) Differentiating (5) at an OSS gives

$$\begin{bmatrix} \delta I - G_{kp}^* & -G_{kk}^* \\ G_{pp}^* & G_{pk}^* \end{bmatrix} \begin{bmatrix} dp^*/d\delta \\ dk^*/d\delta \end{bmatrix} = \begin{bmatrix} -p^* \\ 0 \end{bmatrix};$$

premultiplying by $(dk^*/d\delta, dp^*/d\delta)$ and canceling terms gives the result.

(ii) Differentiating (4) at an OSS yields at once by the R condition

$$R^* = p^{*\prime}(dk^*/d\delta) = p^{*\prime}(L_{kk}^* + \delta L_{\dot{k}k}^*)^{-1} p^* \leqslant 0. \qquad \triangle$$

6. Examples

Example 1. (One-Sector Model of Cass, Koopmans, Samuelson, and Ramsey [19, 32, 53, 57]).

Let k and c denote capital and consumption per worker and let δ, ν, and μ denote the pure rate of time preference, the rate of population growth, and the rate of depreciation of capital, then the state and control are k and c and

$$u(k, v) = u(c), \qquad f(k, v) = f(k) - (\nu + \mu) k - c,$$

where $u(c)$ and $f(k)$ denote the utility and production function $u' > 0$, $u'' < 0$, $c \geqslant 0$, $f(0) = 0$, $f' > 0$, $f'' < 0$, $k \geqslant 0$, $f'(k) \to \infty$ as $k \to 0$, $f'(k) \to 0$ as $k \to \infty$. The OSS is defined by $c^* = f(k^*) - (\nu + \mu) k^*$, $f'(k^*) = \nu + \mu + \delta$, $p^* = u'(c^*)$. Let $\gamma = \alpha = c - c^*$, then

$$L^0(\chi, \gamma) = -\tfrac{1}{2}[f''^*\chi^2 + (u''^*/u'^*)\,\alpha^2], \qquad F = f'^* - (\nu + \mu), \qquad G = -1.$$

The conditions of Theorem 3(ii)(c) *are satisfied since* $-G_{xx}^0 = f''^*$, $G_{\eta\eta}^0 = (u'^*/u''^*)$, $|K^0| = -G_{xx}^0 G_{\eta\eta}^0 > 0$ *imply* K^0 *negative definite, while* $G_{\eta x}^0 = \delta$ *implies* $M^\delta = 0$. The local asymptotic stability of the OSS also follows directly from the fact that (26) reduces to

$$\dot{\chi} = \tfrac{1}{2}(\delta - (\delta^2 + 4f''^*(u'^*/u''^*))^{1/2})\,\chi.$$

Example 2. (n-Sector Model of Magill, Samuelson, and Solow [44, 61]).[19] Let $k = (k_1, \dots, k_n)$, $c = (c_1, \dots, c_n)$, and $z = (z_1, \dots, z_n)$ denote per-worker vectors of capital stocks, consumption, and gross investment. For any vector

[19] The reader is referred to [44] for a further analysis of this model.

$x \in R^n$, let $\hat{x} = (x_2, \ldots, x_n)$, then the vector of controls is the consumption–investment policy $v = (c, \hat{z})$ and

$$u(k, v) = u(c), \qquad f(k, v) = z - \lambda k,$$

where

$$\lambda = \begin{bmatrix} \nu + \mu_1 & & 0 \\ & \ddots & \\ 0 & & \nu + \mu_n \end{bmatrix} \qquad \text{and} \qquad z_1 = F(\hat{c} + \hat{z}; k) - c_1,$$

$F(\hat{c} + \hat{z}; k)$ denoting the society's production frontier and $\mu_1, \ldots, \mu_n$ denoting the depreciation rates of the n capital goods. The matrices in (12) and (13) become[20]

$$F = \begin{bmatrix} F_{1'}^* - (\nu + \mu_1) & F_{2'}^* & \cdots & F_{n'}^* \\ 0 & -(\nu + \mu_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -(\nu + \mu_n) \end{bmatrix},$$

$$G = \begin{bmatrix} -1 & F_2^* & \cdots & F_n^* & F_2^* & \cdots & F_n^* \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{42}$$

$$A = -F_{kk}^*, \qquad N = -[0 \quad F_{k\hat{c}}^* \quad F_{k\hat{c}}^*],$$

$$B = -\left[\begin{array}{c|c} \left(\dfrac{u_{cc}^*}{u_1^*}\right) + \begin{bmatrix} 0 & 0 \\ 0 & F_{\hat{c}\hat{c}}^* \end{bmatrix} & \begin{matrix} 0 \\ F_{\hat{c}\hat{c}}^* \end{matrix} \\ \hline \begin{matrix} 0 & \quad F_{\hat{c}\hat{c}}^* \end{matrix} & F_{\hat{c}\hat{c}}^* \end{array}\right]. \tag{43}$$

Let $\gamma = (\alpha, \hat{\xi}) = (c - c^*, \hat{z} - \hat{z}^*)$, then $L^0(\chi, \gamma)$ reduces to

$$L^0(\chi, \gamma) = -\tfrac{1}{2}[\chi' F_{kk}^* \chi + \alpha'(u_{cc}^*/u_1^*)\,\alpha + (\hat{\alpha} + \hat{\xi})' F_{\hat{c}\hat{c}}^*(\hat{\alpha} + \hat{\xi}) + 2(\hat{\alpha} + \hat{\xi})' F_{\hat{c}k}^* \chi]. \tag{44}$$

While the solution of the Riccati equation (19) can no longer be obtained in closed form,[21] the K^δ condition is readily established.[22] Note first that (42)

[20] Let $F_{c_i} = F_i$, $F_{k_i} = F_{i'}$.

[21] The solution of the Riccati equation is readily obtained for numerical examples, however, even in cases with many capital goods. *It is this property that makes the approach of this paper most valuable for econometric analysis.*

[22] The R and M^δ conditions are not applicable for this model.

implies rank $(G) = n$. *To obtain* G^0_{xx} *and* $(-G^0_{\eta\eta})^{-1}$ *we need to determine the inverse of the matrix B defined in* (43). If we make the relatively mild economic assumption that *the utility function* $u(c_1, \ldots, c_n)$ *is additively separable in the first good so that* $u_{1j} = u_{j1} = 0, j = 2, \ldots, n$ then we find that

$$B = -\left[\begin{array}{c|c|c} \left(\dfrac{u^*_{11}}{u^*_1}\right) & 0 & 0 \\ \hline 0 & \left(\dfrac{u^*_{\hat{c}\hat{c}}}{u^*_1}\right) + F^*_{\hat{c}\hat{c}} & F^*_{\hat{c}\hat{c}} \\ \hline 0 & F^*_{\hat{c}\hat{c}} & F^*_{\hat{c}\hat{c}} \end{array}\right],$$

$$B^{-1} = \left[\begin{array}{c|c|c} \left(\dfrac{u^*_1}{u^*_{11}}\right) & 0 & 0 \\ \hline 0 & \left(\dfrac{u^*_{\hat{c}\hat{c}}}{u^*_1}\right)^{-1} & -\left(\dfrac{u^*_{\hat{c}\hat{c}}}{u^*_1}\right)^{-1} \\ \hline 0 & -\left(\dfrac{u^*_{\hat{c}\hat{c}}}{u^*_1}\right)^{-1} & \left(\dfrac{u^*_{\hat{c}\hat{c}}}{u^*_1}\right)^{-1} + (F^*_{\hat{c}\hat{c}})^{-1} \end{array}\right].$$

Substituting G, A, and N defined by (42) and (43) we find

$$G^0_{xx} = A - NB^{-1}N'$$

$$= -F^*_{kk} + \left[\begin{array}{c|c} (F^*_{k_1\hat{c}})'\,(F^*_{\hat{c}\hat{c}})^{-1}\,F^*_{k_1\hat{c}} & (F^*_{k_1\hat{c}})'\,(F^*_{\hat{c}\hat{c}})^{-1}\,(F^*_{k\hat{c}}) \\ \hline (F^*_{k\hat{c}})(F^*_{\hat{c}\hat{c}})^{-1}\,F^*_{k_1\hat{c}} & (F^*_{k\hat{c}})(F^*_{\hat{c}\hat{c}})^{-1}(F^*_{k\hat{c}}) \end{array}\right], \qquad (45)$$

$$(-G^0_{\eta\eta})^{-1} = (GB^{-1}G')^{-1}$$

$$= -\left[\begin{array}{c|c} \left(\dfrac{u^*_1}{u^*_{11}}\right) + (F^*_{\hat{c}})'\,(F^*_{\hat{c}\hat{c}})^{-1}\,F^*_{\hat{c}} & (F^*_{\hat{c}})'\,(F^*_{\hat{c}\hat{c}})^{-1} \\ \hline (F^*_{\hat{c}\hat{c}})^{-1}\,F^*_{\hat{c}} & \left(\dfrac{u^*_{\hat{c}\hat{c}}}{u^*_1}\right)^{-1} + (F^*_{\hat{c}\hat{c}})^{-1} \end{array}\right]^{-1}. \qquad (46)$$

To understand the economic meaning of the benefit–cost ratio ν^0, consider first the simplest case of *fixed proportions in the output space so that* $F_{\hat{c}\hat{c}} \to \infty$ implying $(F_{\hat{c}\hat{c}})^{-1} \to 0$. In this case

$$G^0_{xx} = -F^*_{kk}, \qquad (-G^0_{\eta\eta})^{-1} = -\begin{bmatrix} (u^*_{11}/u^*_1) & 0 \\ 0 & (u^*_{\hat{c}\hat{c}}/u^*_1) \end{bmatrix} = (-\bar{u}^*_{cc}/u^*_1)$$

so that

$$\nu^0 = \min_{\chi \neq 0} \frac{\chi'(-F^*_{kk})\,\chi}{\chi'(-\bar{u}^*_{cc}/u^*_1)\,\chi} = \min_{\chi'(-\bar{u}^*_{cc}/u^*_1)\chi=1} \chi'(-F^*_{kk})\,\chi.$$

It is evident from (44) that when $F_{\hat{c}\hat{c}} \to \infty$ no attempt will be made to alter the output proportions in which goods are produced, so that $\hat{\alpha} + \hat{\xi} = 0$. This eliminates the last two terms. Local consumption $\alpha = c - c^*$ (and hence investment) is thus the only control used. All that remains is to balance the benefits of investment, the first term in (44), against the costs, the second term in (44), both relative to the interest rate δ.

The nature of the preferences with respect to the consumption of each good as embodied in the matrix $(-\bar{u}^*_{\hat{c}\hat{c}}/u^*_1)$ determines the unit control–cost surface. *The curvature of the utility function $u(c)$ thus induces the metric in the space. The return from capital $\chi'(-F^*_{kk})\,\chi$ is then measured relative to this preference induced metric.* The crucial point is that return and cost are considered simultaneously, the return for each *profile* of capital goods χ being measured against the cost involved *for that profile.*[23] Capital good endowments χ for which the production return is high and the consumption cost low generate a high benefit–cost ratio: Endowments χ for which the return is small and the consumption cost large, generate a low benefit–cost ratio. It is the worst example of the latter that ν^0 hunts out. *The resulting minimum benefit–cost ratio $2(\nu^0)^{1/2}$ is then matched against the interest rate δ to determine whether it is worthwhile to proceed to the OSS.*

If we allow variable proportions in the output space but assume *additive separability in the production function so that $F(\hat{c} + \hat{v};\, k) = g(\hat{c} + \hat{v}) + h(k)$, then $F_{\hat{c}k} = 0$.* In this case $G^0_{xx} = -F^*_{kk}$ as before, but $(-G^0_{\eta\eta})^{-1}$ is given by (46). The unit control–cost surface must now also take into consideration *the cost of altering the output proportions in which goods are produced.* In the general case where $F_{\hat{c}k} \neq 0$, G^0_{xx} is given by (45) and the basic return from capital $\chi'(-F^*_{kk})\,\chi$ needs to be adjusted to take into account *the change in the return induced by variations in the output proportions in which goods are produced.* I shall leave it to the reader to work the details for particular preferences and technology, for example $u(c_1, \ldots, c_n) = \sum_{j=1}^{n} b_j \ln c_j$, $b_j > 0, j = 1, \ldots, n$ and $F(\hat{c};\, k) = \hat{c}'K\hat{c} + \sum_{j=1}^{n} a_j \ln k_j$, K negative definite, $a_j > 0, j = 1, \ldots, n$.

EXAMPLE 3. (Adjustment–Cost Model of Lucas, Mortensen, and Treadway [37, 48, 67]).

The instantaneous flow of profit of a firm is given by

$$L(k, \dot{k}) = h(k, \dot{k}) - qk - p\dot{k},$$

[23] From a strictly economic point of view Rockafellar's *stability condition* of α-convexity in χ and β-concavity in η for $G^0(\chi, \eta)$ with $\alpha\beta > (\delta/2)^2$, is thus somewhat misleading since return and cost are made into quite *separate* considerations.

where the output price (embodied in h) and the rental and purchase prices (q, p) are taken as constants determined on competitive markets, independent of the actions of the firm. If $\delta > 0$ denotes the interest rate then the firm maximizes its present value

$$\int_0^\infty e^{-\delta\tau}(h(k, \dot{k}) - qk - p\dot{k})\, d\tau.$$

The vector of exogenous parameters $\rho = (\delta, q, p)$ is taken as fixed in the short-run and an OSS is defined by

$$h_k(k^*, 0) + \delta h_{\dot{k}}(k^*, 0) - (q + \delta p) = 0.$$

By Proposition 4 *if* $h_{kk}(k^*, 0)$, $h_{\dot{k}\dot{k}}(k^*, 0)$ *are negative definite and if* $h_{k\dot{k}}(k^*, 0) + h_{\dot{k}k}(k^*, 0)$ *is nonpositive definite then the OSS is locally asymptotically stable.* The result of Lucas [37] where $h(k, \dot{k})$ is additively separable follows as a special case. Brock [11] presents an analysis of the problem of *comparing equilibria* when the vector of exogenous market parameters $\rho = (\delta, q, p)$ changes.

EXAMPLE 4. (One-Sector Model with Joint-Production of Liviatan and Samuelson [36]).

This model provides an example for the *long-run dynamics* developed in conjunction with Figs. 2a and b in Section 5. Let c, k, δ, and $u(c)$ be the same as in Example 1. The production function is now given by $c = T(\dot{k}, k)$, where $T_{\dot{k}} < 0$, $T_k \gtreqless 0$ for $k \lesseqgtr \hat{k} > 0$, $T_{kk} < 0$, $T_{\dot{k}\dot{k}} < 0$, $T_{kk}T_{\dot{k}\dot{k}} - (T_{\dot{k}k})^2 > 0$,

$$L(k, \dot{k}) = u(T(\dot{k}, k)) \qquad \text{and} \qquad r(k^*) = -T_k(0, k^*)/T_{\dot{k}}(0, k^*),$$

so that $r'(k^*) = (T^*_{kk} + T^*_{\dot{k}k})/-T^*_{\dot{k}}$ and $r'(k^*) \gtreqless 0$ when $T^*_{kk} + T^*_{\dot{k}k} \gtreqless 0$. Liviatan and Samuelson provide a simple geometrical construction leading to Fig. 2b. Since in Example 1, $r(k^*) = f'(k^*) - (\nu + \mu)$, $r'(k^*) = f''(k^*) < 0$, Fig. 2a illustrates the return curve and hence the long-run dynamics, for Example 1.

The last example gives a *complete analysis* of the control problem for the case $n = 1$. This provides a useful framework for analyzing the sufficient conditions of Theorems 3 and 4.

EXAMPLE 5. (General Case $n = 1$).

Let $F = f$, $G = g$, $A = a$, $B = b$, $N = n$. We assume $a > 0$, $b > 0$, $ab - n^2 > 0$, $g \neq 0$. The Hamiltonian becomes

$$G^0(\chi, \eta) = \tfrac{1}{2}\{[a - (n^2/b)]\,\chi^2 + 2[f - (ng/b)]\,\chi\eta - (g^2/b)\,\eta^2\}.$$

The Riccati equation (19) reduces to

$$\dot{Q} + (2G^0_{\eta x} - \delta)\, Q + G^0_{\eta\eta} Q^2 + G^0_{xx} = 0, \qquad Q(T) = 0,$$

the solution of which is

$$Q(t) = -(1/2G^0_{\eta\eta})\{(2G^0_{\eta x} - \delta) - \Delta^{1/2} \tanh(\tfrac{1}{2}\Delta^{1/2}(t + h - T))\},$$

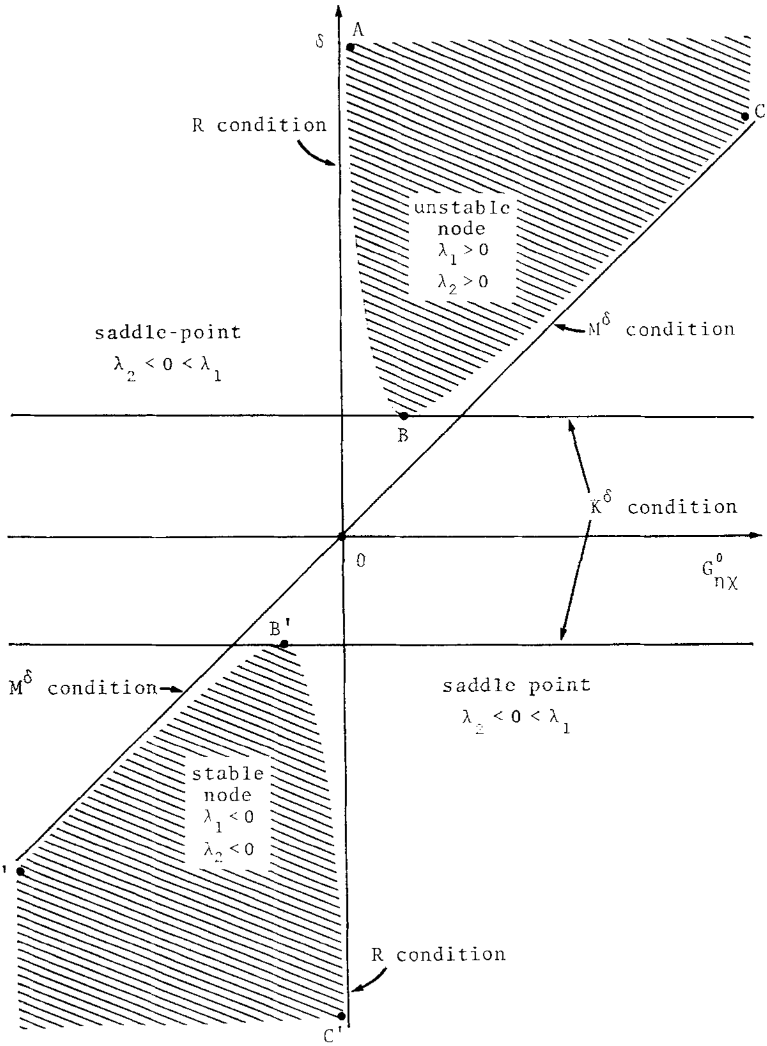

FIG. 3. When $n = 1$, the precise *stability* and *saddle-point* regions can be compared with the stability and saddle-point regions given by Theorems 3 and 4.

$$\Delta = (2G^0_{\eta x} - \delta)^2 - 4G^0_{\eta\eta}G^0_{xx}, \quad h = -(2/\Delta^{1/2}) \tanh^{-1} [(\delta - 2G^0_{\eta x})/\Delta^{1/2}],$$

$$\lim_{T\to\infty} Q(t, T) = Q^* = -(1/2G^0_{\eta\eta})\{(2G^0_{\eta x} - \delta) + [(2G^0_{\eta x} - \delta)^2 - 4G^0_{\eta\eta}G^0_{xx}]^{1/2}\} > 0,$$

so that (26) reduces to

$$\dot{\chi} = \tfrac{1}{2}(\delta - [(2G^0_{\eta x} - \delta)^2 - 4G^0_{\eta\eta}G^0_{xx}]^{1/2})\,\chi. \tag{47}$$

Since $-G^0_{\eta\eta}G^0_{xx} > 0$, *the region in the space* $(G^0_{\eta x}, \delta)$ *for which* (47) *is asymptotically stable is given by*

$$\Sigma(-G^0_{xx}G^0_{\eta\eta}) = \left\{(G^0_{\eta x}, \delta) \,\middle|\, \begin{matrix} -\infty < \delta < \infty, & G^0_{\eta x} \leqslant 0 \\ \delta < G^0_{\eta x} - (G^0_{xx}G^0_{\eta\eta}/G^0_{\eta x}), & G^0_{\eta x} > 0 \end{matrix}\right\}.$$

$\Sigma(-G^0_{xx}G^0_{\eta\eta})$ is the region not on, or above, the curve ABC in Fig. 3. Let

$$\delta^* = \sup_{\tilde{\delta}} \{\tilde{\delta} \mid (G^0_{\eta x}, \delta) \in \Sigma(-G^0_{xx}G^0_{\eta\eta}) \text{ whenever } \delta \leqslant \tilde{\delta}, \text{ for all } G^0_{\eta x}\}.$$

It is easy to see that $\delta^* = 2(-G^0_{xx}G^0_{\eta\eta})^{1/2}$. However, from the K^δ condition $\nu^0(G^0_{xx}; (-G^0_{\eta\eta})^{-1}) = -G^0_{xx}G^0_{\eta\eta}$ so that $\delta^* = 2(\nu^0)^{1/2}$ implying that $2(\nu^0)^{1/2}$ *is the supremum of the interest rates* δ *for which asymptotic stability of the process* (47) *is assured for all* $G^0_{\eta x}$. If $G^0_{\eta x} = (G^0_{\eta x})^* = (-G^0_{xx}G^0_{\eta\eta})^{1/2} > 0$ implying $[f - (ng/b)] = \{[a - (n^2/b)](g^2/b)\}^{1/2}$, as soon as $\delta \geqslant \delta^* = 2\{[a - (n^2/b)](g^2/b)\}^{1/2}$ the process (47) ceases to be asymptotically stable.

The four parts of Theorem 3 account for a major portion of the region of stability when $n = 1$. In terms of Fig. 3, (i) gives $\delta \leqslant 0$, (ii)(a) gives $0 < \delta < \delta^*$, (ii)(b) gives $G^0_{\eta x} \leqslant 0$, (ii)(c) gives $\delta \leqslant G^0_{\eta x}$ for $G^0_{\eta x} > 0$. *The regions of stability generated by* (ii)(b) *and* (ii)(c) *are asymptotically tangent to the region of stability* $\Sigma(-G^0_{xx}G^0_{\eta\eta})$. Since the region of stability is a function of the benefit–cost ratio $\nu^0 = -G^0_{xx}G^0_{\eta\eta}$, *if the benefit–cost ratio increases the region of stability is increased.*

The eigenvalues of the canonical equations (31) are given by

$$\lambda_1, \lambda_2 = (\delta/2) \pm \mu, \qquad \mu = \{[G^0_{\eta x} - (\delta/2)]^2 - G^0_{xx}G^0_{\eta\eta}\}^{1/2},$$

where $\pm\mu$ are the eigenvalues of the mirage system (38). *Thus the region in the space* $(G^0_{\eta x}, \delta)$ *for which* (31) *has the saddle-point property is given by*

$$\mathscr{S}(-G^0_{xx}G^0_{\eta\eta}) = \{(G^0_{\eta x}, \delta) \mid \delta \lesseqgtr G^0_{\eta x} - (G^0_{xx}G^0_{\eta\eta}/G^0_{\eta x}) \text{ for } G^0_{\eta x} \gtreqless 0\}.$$

$\mathscr{S}(-G^0_{xx}G^0_{\eta\eta})$ is the region below ABC and above $A'B'C'$ in Fig. 3.

When $\delta < 0$, if we replace the nonpositive definite condition on R and M^δ in Theorem 3, by the condition that *R and M^δ be nonnegative definite,* *then* (ii)(a), (b), *and* (c) *give sufficient conditions for the canonical equations* (31) *to have the saddle-point property when* $\delta < 0$.[24] When extended in this way to the case $\delta < 0$, (ii)(a), (b), and (c) account for a major portion of the region $\mathscr{S}(-G^0_{xx}G^0_{\pi\pi})$, in which the canonical equations have the saddle-point property.[25]

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[24] The proof of this result is similar to that used in establishing Theorems 3 and 4, the *forward* value–loss function (15) being replaced by the *backward* value–loss function $W^-(\chi) = \inf_\gamma \int_{-\infty}^{0} e^{-\delta\tau} L^0(\chi, \gamma) d\tau$.

[25] The presence of the region below $A'B'C'$ does *not* contradict the stability result for $\delta < 0$. The manifold leading to the OSS is, in that case, two dimensional. The appropriate *one-dimensional submanifold* leading to the OSS is obtained by an argument similar to that given by Liviatan and Samuelson [36] for the unstable case, the region above ABC.

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