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Notes, Comments, and Letters to the Editor

A Local Analysis of *N*-Sector Capital Accumulation under Uncertainty*

1. Introduction

In a recent paper [1] I presented some conditions under which the optimal trajectory arising in a class of dynamic economic models converges to a stationary state. The analysis was confined to the deterministic case. In this note I shall indicate how some of these results may be generalized to the stochastic case. I will also show how these results may be applied to obtain some new insights into the nature of business cycles by introducing the remarkable ideas of Slutsky [2] concerning the summation of random causes as the source of cyclic processes.

2. Convergence to Stationary Process

Consider the extremum problem

$$\sup_{v} E_{k(0)} \int_0^\infty e^{-\delta t} u(k(t), v(t))\, dt, \qquad \delta > 0, \tag{1}$$

$$dk = f(k, v)\, dt + \sigma(k)\, dz, \qquad k(0) = k_0, \tag{2}$$

where $u \in R^1$, $f = (f^1,..., f^n) \in R^n$,

$$\sigma = \begin{bmatrix} \sigma^{11} & \cdots & \sigma^{1m} \\ \vdots & & \vdots \\ \sigma^{n1} & \cdots & \sigma^{nm} \end{bmatrix} \in R^{nm},$$

$k \in R^n$, $v \in R^q$, and $z(t) \in R^m$ is a Brownian motion process.

Assumption (Concavity). $u(k, v)$ is strictly concave and $f^1(k, v),..., f^n(k, v)$ are concave functions in (k, v).

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Consider the pre-Hamiltonian [1]

$$\hat{G}(k, p; v) = u(k, v) + pf(k, v).$$

In the deterministic case, where $\sigma(k) = 0$, the solution of (1), (2) is reduced to an analysis of the equations

$$\hat{G}_k(k, p; v) + \dot{p} - \delta p = 0, \qquad \hat{G}_p(k, p; v) - \dot{k} = 0, \qquad \hat{G}_v(k, p; v) = 0. \quad (3)$$

I assume that the equations (3) have a stationary solution (k^*, v^*, p^*).[1] As in [1] let

$$(\chi, \gamma, \eta) = (k - k^*, v - v^*, p^* - p);$$

then (2) becomes

$$d\chi = f(k^* + \chi, v^* + \gamma)\, dt + \sigma(k^* + \chi)\, dz. \quad (4)$$

In order to apply the method of Fleming [3] by which the maximization of (1) subject to (2) is reduced to a simpler quadratic problem in (χ, γ) around the stationary solution (k^*, v^*, p^*), we need to make two additional assumptions.[2] The first is that the process starts sufficiently close to k^* so that $x_0 = k_0 - k^*$ is close to zero. The second is that each element of the matrix $\sigma(k^* + \chi)$ in (4) is sufficiently small and does not change appreciably in a small neighborhood of k^*. In this case we may use the linearizations

$$f(k^* + \chi, v^* + \gamma) = f(k^*, v^*) + f_k(k^*, v^*)\, \chi + f_v(k^*, v^*)\, \gamma, \quad (5)$$

$$\sigma(k^* + \chi) = \sigma(k^*) + \sigma_k(k^*)\, \chi. \quad (6)$$

In (5), $f(k^*, v^*) = 0$. In (6), $\sigma(k^*) \neq 0$ and by assumption $\sigma_k(k^*)\, \chi$ is insignificant relative to $\sigma(k^*)$ so that (4) becomes

$$d\chi = [f_k(k^*, v^*)\, \chi + f_v(k^*, v^*)\, \gamma]\, dt + \sigma(k^*)\, dz. \quad (7)$$

By the procedure outlined in [1, 3], (1) is replaced by the problem

$$\inf_{\gamma} E_{x(0)} \int_0^\infty e^{-\delta t} L^0(\chi, \gamma)\, dt, \quad (8)$$

[1] See [1, Sect. 2], where a competitive path (k^*, v^*, p^*), a solution of (3), which is stationary is called an *optimal stationary state* (OSS).

[2] Fleming also requires that in the limiting deterministic case, $\sigma(k) = 0$, the extremals in a neighborhood of k^* be *unique* given the initial conditions, and contain no *conjugate points*. As is well known, the concavity assumption on $u(k, v)$ and $f(k, v)$ ensures that these two properties hold.

where

$$L^0(\chi, \gamma) = \frac{1}{2}\begin{bmatrix}\chi\\ \gamma\end{bmatrix}'\begin{bmatrix}A & N\\ N' & B\end{bmatrix}\begin{bmatrix}\chi\\ \gamma\end{bmatrix},$$

$$\begin{bmatrix}A & N\\ N' & B\end{bmatrix} = -\left\{\begin{bmatrix}u_{kk}(k^*, v^*) & u_{kv}(k^*, v^*)\\ u_{vk}(k^*, v^*) & u_{vv}(k^*, v^*)\end{bmatrix}\right.$$

$$\left. + \sum_{i=1}^{n} p_i{}^* \begin{bmatrix}f^i_{kk}(k^*, v^*) & f^i_{kv}(k^*, v^*)\\ f^i_{vk}(k^*, v^*) & f^i_{vv}(k^*, v^*)\end{bmatrix}\right\}. \tag{9}$$

We also let

$$F = f_k(k^*, v^*), \qquad G = f_v(k^*, v^*). \tag{10}$$

THEOREM 1. *If* γ *minimizes* $E_{x(0)} \int_0^T e^{-\delta\tau} L^0(\chi, \gamma)\, d\tau$ *subject to* $d\chi = [F\chi + G\gamma]\, dt + \sigma(k^*)\, dz$ *then*

(i) *the value-loss function* $W(\chi, t) = \inf_\gamma E_{x(t)} \int_t^T e^{-\delta(\tau - t)} L^0(\chi, \gamma)\, d\tau$ *is a solution of the generalized Hamilton–Jacobi equation*

$$W_t + G^0(\chi, W_x) - \delta W + \tfrac{1}{2}\operatorname{tr}(\sigma(k^*)\, \sigma(k^*)'\, W_{xx}) = 0 \tag{11}$$

with boundary condition $W(\chi, T) = 0$;

(ii) *the control* γ *minimizes* $\hat{G}^0(\chi, \eta; \gamma) = L^0(\chi, \gamma) + \eta'(F\chi + G\gamma)$, *where* $G^0(\chi, \eta) = \min_\gamma \hat{G}^0(\chi, \eta; \gamma)$.

This theorem is the stochastic counterpart of [1, Theorem 2], the proof of which may be found in [4, p. 569]. The minimizing γ in (ii) satisfies

$$\hat{G}^0_\gamma = \chi' N + \gamma' B + \eta' G = 0 \qquad \text{or} \qquad \gamma = -B^{-1}(G'\eta + N'\chi), \tag{12}$$

assuming B is positive definite, so that B^{-1} is well defined. Equation (12) implies that the local Hamiltonian $G^0(\chi, \eta)$ becomes

$$G^0(\chi, \eta) = \frac{1}{2}\begin{bmatrix}\chi\\ \eta\end{bmatrix}'\begin{bmatrix}A - NB^{-1}N' & F' - NB^{-1}G'\\ F - GB^{-1}N' & -GB^{-1}G'\end{bmatrix}\begin{bmatrix}\chi\\ \eta\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}\chi\\ \eta\end{bmatrix}'\begin{bmatrix}G^0_{xx} & G^0_{x\eta}\\ G^0_{\eta x} & G^0_{\eta\eta}\end{bmatrix}\begin{bmatrix}\chi\\ \eta\end{bmatrix}.$$

Since $G^0(\chi, \eta)$ is quadratic, the solution of (11) is given by

$$W(\chi, t) = \tfrac{1}{2}(\chi' Q(t)\, \chi + r(t)), \tag{13}$$

where $Q(t)$ is the solution of the matrix Riccati equation

$$\dot{Q} + QG^0_{\eta x} + (G^0_{x\eta} - \delta I)\, Q + QG^0_{\eta\eta}Q + G^0_{xx} = 0, \qquad Q(T) = 0 \tag{14}$$

and $r(t) = \int_t^T e^{-\delta(\tau - t)} \operatorname{tr}(\sigma(k^*)\, \sigma(k^*)'\, Q(\tau))\, d\tau$ is the solution of

$$\dot{r} - \delta r + \operatorname{tr}(\sigma(k^*)\, \sigma(k^*)'\, Q) = 0, \qquad r(T) = 0.$$

Equations (12) and (13) imply

$$\gamma(t) = -B^{-1}(G'Q(t) + N')\,\chi. \tag{15}$$

LEMMA. *Let $Q(t, T)$ be the solution of* (14). *If the rank of the matrix G in* (10) *is n, then* $\lim_{T\to\infty} Q(t, T) = Q^*$ *where Q^* is positive definite.*

Since the solution to (14) is unaffected by $\sigma(k^*)$, we may assume $\sigma(k^*) = 0$. The proof of the Lemma then follows from [1, Lemma 4]. Thus as in [1]

$$\gamma(t) \to \gamma^* = -B^{-1}(G'Q^* + N')\,\chi \qquad \text{as } T \to \infty. \tag{16}$$

We are now in a position to examine the stability properties of the trajectories that result from the problem (8). Using (16), the process

$$d\chi = [F\chi + G\gamma^*]\, dt + \sigma(k^*)\, dz$$

may be written in terms of the local Hamiltonian $G^0(\chi, \eta)$ as

$$d\chi = (G^0_{\eta x} + G^0_{\eta\eta} Q^*)\, \chi\, dt + \sigma(k^*)\, dz. \tag{17}$$

If we let $Y = G^0_{\eta x} + G^0_{\eta\eta} Q^*$ then the solution of (17) *is given by*

$$\chi(t) = e^{Yt}\chi(0) + \int_0^t e^{Y(t-\tau)}\sigma(k^*)\, dz(\tau) \tag{18}$$

which is the normally distributed process with mean and covariance matrix

$$\bar{\chi}(t) = e^{Yt}\chi(0), \qquad R_x(t) = \int_0^t e^{Y(t-\tau)}\sigma(k^*)\, \sigma(k^*)'\, e^{Y'(t-\tau)}\, d\tau.$$

If all the eigenvalues of Y have negative real parts then[3]

$$\|\, \bar{\chi}(t)\| \to 0, \qquad \|\, R_x(t) - R_x{}^*\, \| \to 0 \qquad \text{as} \quad t \to \infty,$$

$$R_x^* = \int_0^\infty e^{Y\theta}\sigma(k^*)\, \sigma(k^*)'\, e^{Y'\theta}\, d\theta. \tag{19}$$

If we let $\psi(\chi, t; \chi(0))$ and $\phi(\eta, t; \eta(0))$, where $\eta = Q^*\chi$, denote the conditional density functions for the process (17) and the associated process for $\eta(t)$, then we obtain the following generalization of [1, Theorem 3].

[3] For $A \in R^{nm}$, $\|\, A\, \|^2 = \operatorname{tr}(AA')$.

THEOREM 2 (Asymptotic Convergence to Stationary Process). *Let*

$$K^\delta = \begin{bmatrix} -G^0_{\chi\chi} & (\delta/2)\, I \\ (\delta/2)\, I & G^0_{\eta\eta} \end{bmatrix} \qquad \textit{for} \quad \delta \in [0, \infty).$$

If

(a) K^δ *is negative definite or*

(b) K^0 *is negative definite and*

$$R = G^{0\prime}_{\eta\chi}(-G^0_{\eta\eta})^{-1} + (-G^0_{\eta\eta})^{-1}\, G^0_{\eta\chi}$$

is non-positive definite or

(c) K^0 *is negative definite and*

$$M^\delta = [\delta I - G^0_{\chi\eta}]'\, (G^0_{\chi\chi})^{-1} + (G^0_{\chi\chi})^{-1}\, [\delta I - G^0_{\chi\eta}]$$

is non-positive definite, then for all $(\chi, \chi(0))$ *and* $(\eta, \eta(0))$

$$|\, \psi(\chi, t; \chi(0)) - \psi^*(\chi)| \to 0 \quad \text{and} \quad |\, \phi(\eta, t; \eta(0)) - \phi^*(\eta)| \to 0 \quad \text{as } t \to \infty,$$

where $\psi^*(\chi)$ *and* $\phi^*(\eta)$ *are the normal density functions with mean and covariance matrices* $(0, R_\chi^*)$ *and* $(0, Q^* R_\chi^* Q^*)$, *where* R_χ^* *is given by* (19).

EXAMPLE 1. Let $K = (K_1, ..., K_n)$, $C = (C_1, ..., C_n)$, and $(M_1, ..., M_n)$ denote vectors of capital stocks, consumption, and gross investment (see [1, Example 2]). We assume that the rate of growth of the labor force L and the effective rate of capital accumulation of each capital good K_j are subject to random disturbances

$$dL = \nu L\, dt + \sigma_0 L dz_0\,, \qquad \nu \geqslant 0, \qquad \sigma_0 > 0,$$
$$dK_j = (M_j - \mu_j K_j)\, dt + \sigma_j K_j dz_j\,, \quad \mu_j \geqslant 0, \quad \sigma_j > 0, \quad j = 1, ..., n.$$

Let $k = (K_1/L, ..., K_n/L)$, $c = (C_1/L, ..., C_n/L)$, $m = (M_1/L, ..., M_n/L)$ and for any $x \in R^n$ let $\hat{x} = (x_2, ..., x_n)$, then the vector of controls is the consumption–investment policy $v = (c, \hat{m})$. Applying *Ito's Lemma* to k gives

$$f(k, v) = m - (\lambda - \sigma_0^2 I)\, k, \qquad \sigma(k) = \begin{bmatrix} -\sigma_0 k_1 & \sigma_1 k_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_0 k_n & 0 & \cdots & \sigma_n k_n \end{bmatrix},$$

where

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \qquad \lambda = \begin{bmatrix} \nu + \mu_1 & & 0 \\ & \ddots & \\ 0 & & \nu + \mu_n \end{bmatrix},$$

$$m_1 = F(\hat{c} + \hat{m}; k) - c_1\,,$$

$F(\hat{c} + \hat{m}; k)$ denoting the society's production frontier. In this case

$$u(k, v) = u(c).$$

The K^δ condition may now be evaluated precisely as in [1].

EXAMPLE 2. *Quadratic criterion* (1) *and linear dynamics* (2)

$$u(k, v) = -\frac{1}{2}\begin{bmatrix} k \\ v \end{bmatrix}' \begin{bmatrix} A & N \\ N' & B \end{bmatrix}\begin{bmatrix} k \\ v \end{bmatrix},$$

$$f(k, v) = Fk + Gv, \qquad \sigma(k) = \sigma,$$

where $\{A, N, B; F, G, \sigma\}$ are matrices with real constant coefficients. In this case the above procedure does not require the assumption that each element of σ be sufficiently small; *the procedure is precise for all real matrices* σ. The K^δ condition reduces to the condition that

$$\begin{bmatrix} A - NB^{-1}N' & (\delta/2)\, I \\ (\delta/2)\, I & GB^{-1}G' \end{bmatrix}$$

be positive definite. This requires rank $(G) = n$ and $q \geqslant n$.

3. CAPITAL ACCUMULATION AS A CYCLICAL PROCESS

In his classic investigation [2] Slutsky showed that *a weighted sum of independent identically distributed random variables with zero mean and finite variance leads to approximately regular cyclical motion.* When any one of the conditions (a), (b), or (c) of Theorem 2 is satisfied, this result is directly relevant to the process of capital accumulation (17). Each of these conditions implies that the eigenvalues of Y have negative real parts, so that in (18)

$$\| e^{Yt}\chi(0)\| \to 0 \qquad \text{as } t \to \infty,$$

and $\chi(t)$ *becomes a weighted sum of the independent identically distributed random disturbances* $dz(\tau)$.

A precise characterization of the periodicity of the stationary process generated by (18) is given by its *Fourier–Kolmogorov spectral decomposition* [5, Chap. 2]. In the simplest case $n = 1$, if $s(\omega)$ denotes the *normalized spectral density* of the stationary stochastic process, where ω denotes the angular frequency, then as is well known [5, p. 69]

$$s(\omega) = -Y/\pi(Y^2 + \omega^2), \tag{20}$$

so that the spectral density is completely determined by the *local investment rate* $-Y$. Equation (20) leads directly to the following observation. Suppose that the local investment rates for two economies satisfy

$$|Y_0| < |Y_1|.$$

If $s_0(\omega)$ and $s_1(\omega)$ denote the spectral densities of their stationary processes (assuming one of the conditions (a), (b), or (c) to be satisfied) then

$$\begin{aligned} s_1(\omega) &< s_0(\omega), \qquad \omega \in [0, \omega^*), \\ s_1(\omega) &> s_0(\omega), \qquad \omega \in (\omega^*, \infty), \end{aligned} \tag{21}$$

where

$$\omega^* = (Y_0 Y_1)^{1/2}.$$

Thus in the Fourier–Kolmogorov decomposition, the small frequencies (long periods) play a less important part, while the high frequencies (short periods) become more important in the second economy as compared with the first (see Fig. 1). *The greater the local rate of investment $-Y$, the shorter the typical length of the periods in the cyclical process of capital accumulation.*

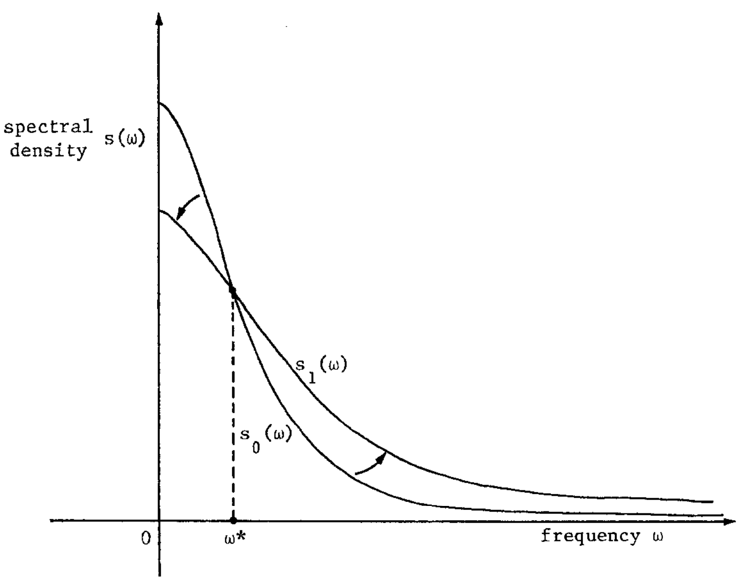

FIG. 1. Increasing the investment rate $-Y$ shifts the spectral density about the frequency ω^*.

EXAMPLE. Consider Example 1 with $n = 1$,[4]

$$Y = \tfrac{1}{2}(\delta - (\delta^2 + 4F''(k^*)(u'(c^*)/u''(c^*)))^{1/2}),$$
$$c^* = F(k^*) - \lambda k^*, \qquad F'(k^*) = \lambda + \delta. \tag{22}$$

Equations (20) *and* (22) *show precisely how the cyclical nature of the stationary process of capital accumulation is determined by the preferences and the technology.*

The spectral shift property (21) may be generalized to the case $n > 1$ in the following way. If $\mathscr{A}(\tau) = \{a_{ij}(\tau)\}$ and $\mathscr{S}(\omega) = \{s_{ij}(\omega)\}$ denote the *auto-covariance* and *spectral matrices* of the stationary process generated by (18)

$$\mathscr{S}(\omega) = (1/2\pi) \int_{-\infty}^{\infty} \mathscr{A}(\tau)\, e^{-i\omega\tau}\, d\tau,$$

then the *normalized spectral densities* for each component of the process are defined by

$$s_j(\omega) = s_{jj}(\omega)/a_{jj}(0), \qquad j = 1,\dots, n.$$

The reader may then establish the following result, by examining the expressions for $s_j(\omega)$, $j = 1,\dots, n$. *Let the eigenvalues of Y be distinct. If* $s_j(\omega)$ *and* $s_j^{\alpha}(\omega)$, $j = 1,\dots, n$ *denote the normalized spectral densities of the stationary solutions of the processes* $d\chi = Y\chi\, dt + \sigma(k^*)\, dz$ *and* $d\chi = \alpha Y\chi\, dt + \sigma(k^*)\, dz$, *respectively*, $\alpha > 1$, *then there exist frequencies* ω_0^*, ω_1^*, $0 < \omega_0^* < \omega_1^*$ *such that for* $j = 1,\dots, n$

$$s_j^{\alpha}(\omega) < s_j(\omega), \qquad \omega \in [0, \omega_0^*],$$
$$s_j^{\alpha}(\omega) > s_j(\omega), \qquad \omega \in [\omega_1^*, \infty),$$

where

$$\omega_0^* = (\alpha - 1)^{1/2}\, \lambda^0, \qquad \lambda^0 = \min_{1 \leqslant i \leqslant n} \{\lambda_i\},$$
$$\omega_1^* = (\alpha/(\alpha - 1)^{1/2})\, \lambda^*, \qquad \lambda^* = \max_{1 \leqslant i \leqslant n} \{\lambda_i\}$$

if the eigenvalues of Y are real.

The further analysis of the cyclic properties of the process of capital accumulation under uncertainty is a topic of investigation in itself which I shall not enter into here. It is, however, an important consequence of the introduction of uncertainty that *the short-run study of the business cycle*

[4] If $F''(k^*) < 0$, $u'(c^*) > 0$, $u''(c^*) < 0$ then (c) in Theorem 2 is satisfied.

becomes imbedded in a natural way into the long-run process of capital accumulation. When uncertainty is present, business cycles must be expected even in a centrally controlled economy where investment is optimally determined by planners.

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