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The Preferability of Investment Through a Mutual Fund*

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1. INTRODUCTION

One of the principal results of the theory of investor portfolio selection is the Tobin–Cass–Stiglitz Mutual Fund Theorem [1, 2]. The simplest version of the theorem asserts that in an economy with a riskless asset (money) and m risky assets, a mutual fund can be formed such that every individual is *indifferent* between investing in the mutual fund or directly purchasing the individual assets.

In a recent important contribution Merton [3] has shown that the theorem can be extended to the continuous time framework when the m risky assets are joint lognormally distributed. The theorem, however, points to an important defect of the associated capital market theory; for in such a framework financial intermediaries such as mutual funds have no real reason to exist: every investor can achieve on his own the services offered by the mutual fund.

The validity of this result depends on three basic assumptions: (i) the absence of transactions costs, (ii) the perfect divisibility of each security, so that any proportion of a security can be transacted,[1] and (iii) the availability of perfect, costless information. When any of these assumptions is dropped it would seem that an individual might *prefer* investment through a mutual fund.

The object of this paper is to show that when the first of these basic assumptions is dropped so that transactions costs are introduced explicitly a mutual fund can be formed such that individual investors prefer investment through the mutual fund to individual investment on the capital market. A preliminary step is thus made towards a formal theory

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[1] Klein [4] has suggested that (ii) follows from (i) since in a world of zero transactions costs a corporation maximizes its value when it issues perfectly divisible securities.

of financial intermediaries within the standard theory of the capital market.

The paper draws on the analysis in [5] in which it was shown how an individual investing on his own in the capital market adjusts his portfolio behavior in the presence of proportional transactions costs. In Section 2 the assumptions concerning the capital market and the individual investor are briefly summarized, while Section 3 constructs the basic Mutual Fund and establishes the preferability of investment through the Mutual Fund.

2. Assumptions Concerning the Capital Market and the Investor

In [5] seven important assumptions were made concerning the capital market and the individual investor. The reader is referred to [5] for an exact statement of the assumptions which may be summarized as follows. The capital market consists of continuous competitive markets for m risky securities each of which is perfectly divisible. The prices of the securities are lognormally distributed with instantaneous mean and covariance matrix (α, Σ) where $\alpha = (\alpha_1, \ldots, \alpha_m)$ and

$$\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & & \vdots \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{bmatrix}$$

is positive definite and all information regarding the securities is perfect, continuously available and costless. Every investor can borrow or lend an unlimited amount at a constant interest rate $r > 0$ and expects a known contractual income stream $y(t)$ over his known lifespan $[0, T]$. If v_i denotes the value of the ith security purchased ($v_i > 0$) or sold ($v_i < 0$) per unit of time then the transaction cost function $T(v_1, \ldots, v_m)$ indicating the cost of buying or selling any combination of the m securities is given by

$$T(v_1, \ldots, v_m) = \sum_{i=1}^{m} \chi_{v_i} v_i \qquad \text{where} \quad \chi_{v_i} = \begin{cases} \chi^i & v_i > 0, \\ -\chi_i & v_i < 0, \end{cases} \tag{1}$$

and where $0 < \chi^i < 1$, $0 < \chi_i < 1$, $i = 1, \ldots, m$ so that transactions costs are proportional to the value of each transaction. It was shown that if $s_i(t)$ denotes the value of the investor's holdings of the ith security at time t then

$$ds_i(t) = [\alpha_i s_i(t) + v_i(t)]\, dt + s_i(t)\, dz_i(t) \qquad i = 1, \ldots, m, \tag{2}$$

where $dz(t) = (dz_1(t), \ldots, dz_m(t))$ is the *formal increment of a Brownian motion process*

$$E(dz) = 0, \qquad E(dz\, dz') = \Sigma\, dt,$$

while the investor's stock of bank deposits (cash) $s_0(t)$ satisfies

$$ds_0(t) = \left[rs_0(t) + y(t) - c(t) - \sum_{i=1}^{m} (1 + \chi_{v_i})\, v_i(t)\right] dt, \tag{3}$$

$c(t)$ denoting the flow of consumption expenditure at time t.

Assumptions concerning the investor's preferences were made so that his objective was to choose a transaction–consumption policy $(v, c) = (v_1, \dots, v_m, c)$ which would maximize

$$E_{(s,0)}^{(v,c)} \int_0^T u(c, \tau)\, d\tau, \tag{4}$$

where $E_{(s,0)}^{(c,v)}$ denotes the *conditional expectation* given the transaction–consumption policy (v, c) over the time interval $[0, T]$ and given that his initial stock of securities is $s = (s_0, \dots, s_m)$ at $t = 0$, subject to (2) and (3). The utility function was furthermore assumed to belong to the following family characterizing an investor with *decreasing absolute risk aversion*

$$\begin{aligned} u(c, \tau) &= e^{-\rho\tau} \frac{(1-\eta)}{\eta} \left(\frac{\beta c}{1-\eta} + \gamma(\tau)\right)^{\eta} \\ &= e^{-\rho\tau}(1-\eta)^{1-\eta} \frac{\beta^{\eta}}{\eta} (c - \hat{c}(\tau))^{\eta}, \qquad c \geqslant \hat{c}(\tau), \\ \hat{c}(\tau) &= -\gamma(\tau) \frac{(1-\eta)}{\beta}, \qquad -\infty < \eta < 1, \qquad \beta > 0, \\ & \qquad\qquad -\infty < \gamma(\tau) < \infty, \qquad \rho \geqslant 0. \end{aligned} \tag{5}$$

Introducing the *effective wealth* of the investor

$$w(t) = \sum_{i=0}^{m} s_i(t) + Y(t) - \hat{C}(t)$$

where

$$Y(t) = \int_t^T y(\tau)\, e^{-r(\tau - t)}\, d\tau, \qquad \hat{C}(t) = \int_t^T \hat{c}(\tau)\, e^{-r(\tau-t)}\, d\tau$$

and the portfolio proportions $\xi_i = s_i/w$, $i = 0, \dots, m$, $\xi = (\xi_1, \dots, \xi_m)$, $\xi_y = (Y - \hat{C})/w$ so that $\sum_{i=0}^{m} \xi_i + \xi_y = 1$ it was shown that the half-spaces defined by

$$\nu_j^*(\xi, \chi_v) = [\chi_{v_j}(\xi_j^0 - 1) - 1]\, \xi_j + \xi_j^0 \left(1 + \sum_{\substack{i=1 \\ i \neq j}}^{m} \chi_{v_i} \xi_i\right) \gtreqqless 0$$

$$j = 1, \dots, m \tag{6}$$

lead recursively to a *zero-transaction region* Ω_0 about the optimal portfolio proportions in the absence of transactions costs

$$\xi^0 = \Sigma^{-1}[(\alpha - rn)/(1 - \eta)], \qquad n = (1,..., 1) \tag{7}$$

with the property that whenever $\xi \in \Omega_0$ it is optimal not to transact but as soon as $\xi \notin \Omega_0$ it is optimal to transact so as to return ξ to the boundary of Ω_0. *An investor pursuing an optimal policy of individual investment on the capital market thus obtains his best results when he confines his portfolio to the region* Ω_0. When $m = 2$ the region Ω_0 was shown to be the shaded region in Fig. 1.

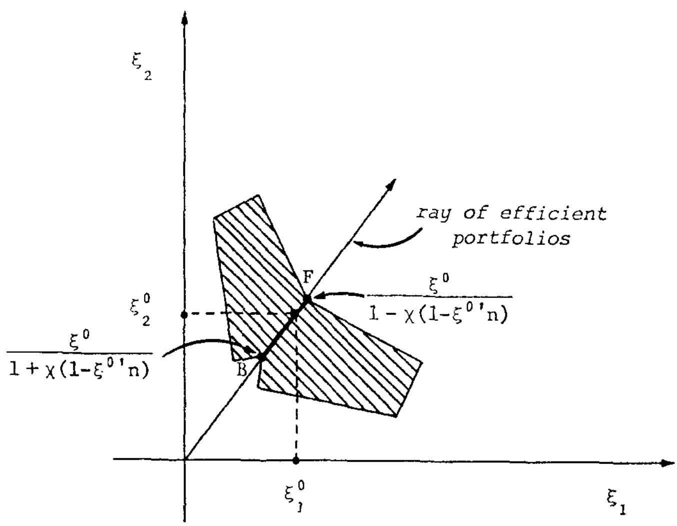

FIG. 1. The zero-transaction regions Ω_0 and Ω_0^F.

3. A Modified Mutual Fund Theorem

Consider the following idealized Mutual Fund. Let $F_1,..., F_m$ denote the value of its holdings of each of the m risky securities and let $F = \sum_{i=1}^m F_i$, $\lambda_j = F_j/F$ $j = 1,..., m$, $\lambda = (\lambda_1,..., \lambda_m)$. The portfolio proportions are chosen so as to satisfy $\lambda^* = \mu\Sigma^{-1}(\alpha - rn)$, $\mu > 0$ $\lambda^{*\prime}n = 1$ so that provided $(\alpha - rn)' \Sigma^{-1}n \neq 0$

$$\lambda^* = \Sigma^{-1}(\alpha - rn)/(\alpha - rn)' \Sigma^{-1}n. \tag{8}$$

Unlike individuals this centrally administered Mutual Fund is not involved in transactions costs when altering its portfolio.[2] This implies

$$dF(t) = \alpha'\lambda^* F(t)\,dt + F(t)\,\lambda^{*\prime}\,dz(t). \tag{9}$$

Let $N(t)$ denote the number of its shares outstanding, each share being perfectly divisible and let $P(t)$ denote the price per share. $F(t)$ changes continuously according to (9) except at certain instants when an investor either deposits or withdraws funds: then $F(t)$ increases (decreases) discontinuously but in such a way that $P(t) = F(t)/N(t)$ is unchanged. Thus $N(t)$ is unchanged except at discrete points when it alters in such a way that $F(t)/N(t)$ is unchanged. With this rule for issuing shares, *Ito's Lemma* [6, pp. 386–391] immediately implies

$$dP(t) = \alpha'\lambda^* P(t)\,dt + P(t)\,\lambda^{*\prime}\,dz(t). \tag{10}$$

Consider an investor faced with the opportunity of investment through this Mutual Fund. Let $X(t)$ denote the number of shares and $S(t) = X(t)\,P(t)$ the value of his Mutual Fund holdings. Then Ito's Lemma and (10) imply

$$dS(t) = (\bar{\alpha}S(t) + v(t))\,dt + S(t)\,d\bar{z}(t), \tag{11}$$

where $\bar{\alpha} = \alpha'\lambda^*$, $d\bar{z}(t) = \lambda^{*\prime}\,dz(t)$, and $v(t) = (dX(t)/dt)\,P(t)$ denotes the transaction rate. If $s_0(t)$ denotes the investor's stock of cash and if χ^F denotes the transaction cost rate for the Mutual Fund's shares, $0 \leqslant \chi^F < 1$ then (3) becomes

$$ds_0(t) = [rs_0(t) + y(t) - c(t) - (1 + \chi_v{}^F)\,v(t)]\,dt \tag{12}$$

where

$$\chi_v{}^F = \begin{cases} \chi^F & \text{if } v > 0, \\ -\chi^F & \text{if } v < 0. \end{cases}$$

Thus when investing through the Mutual Fund the individual's investment problem reduces to choosing (v, c), *so as to maximize* (4) *with* $s = (s_0, S)$ *subject to* (11), (12) *and the initial condition* $(s_0(0), S(0))$.

[2] This is clearly an idealized assumption. It may be interpreted either as a purely formal assumption which simplifies the construction of the Mutual Fund or as an attempt to state in extreme form the fact that transactions cost rates for a typical mutual fund are significantly smaller than those for a typical individual investor due to economies of scale in transactions. If the latter interpretation is used, note, however, that no attempt is made to develop a formal theory explaining the behavior of a typical mutual fund.

The method of analysis developed in [5] can now be applied to this alternative investment problem. The investor's *effective wealth* becomes $w(t) = s_0(t) + S(t) + Y(t) - \hat{C}(t)$. If we let $\theta = S/w$ then (6) leads to the zero-transaction region for θ

$$\left\{\theta \in R \mid \theta = \frac{\mu\theta^0}{1 + \chi^F(1 - \theta^0)} + \frac{(1 - \mu)\,\theta^0}{1 - \chi^F(1 - \theta^0)},\ 0 \leqslant \mu \leqslant 1\right\}, \qquad (13)$$

θ^0 being given by (7) which in this case reduces to

$$\theta^0 = (\bar{\alpha} - r)/\bar{\sigma}^2(1 - \eta),$$

where $\bar{\alpha} = \alpha'\Sigma^{-1}(\alpha - rn)/(\alpha - rn)'\,\Sigma^{-1}n$ and $\bar{\sigma}^2\,dt = E(d\bar{z})^2 = ((\alpha - rn)'\,\Sigma^{-1}(\alpha - rn)/[(\alpha - rn)'\,\Sigma^{-1}n]^2)\,dt$ so that

$$\theta^0 = (\alpha - rn)'\,\Sigma^{-1}n/(1 - \eta) = \xi^{0\prime}n.$$

Since the Mutual Fund's portfolio always satisfies (8), when the individual invests a proportion θ of his effective wealth in the Mutual Fund he in effect holds a portfolio ξ with two properties; ξ always lies along the ray through ξ^0 (by (7) and (8)) and $\xi'n = \theta$. Since the hyperplanes

$$\xi'n = \frac{\theta^0}{1 + \chi^F(1 - \theta^0)} \qquad \text{and} \qquad \xi'n = \frac{\theta^0}{1 - \chi^F(1 - \theta^0)}$$

cut the ray passing through ξ^0 at the points

$$\frac{\xi^0}{1 + \chi^F(1 - \xi^{0\prime}n)} \qquad \text{and} \qquad \frac{\xi^0}{1 - \chi^F(1 - \xi^{0\prime}n)},$$

the zero transaction region (13) for θ translates into the following region in the portfolio space:

$$\Omega_0{}^F = \left\{\xi \in R^m \mid \xi = \frac{\mu\xi^0}{1 + \chi^F(1 - \xi^{0\prime}n)} + \frac{(1 - \mu)\,\xi^0}{1 - \chi^F(1 - \xi^{0\prime}n)},\ 0 \leqslant \mu \leqslant 1\right\}. \qquad (14)$$

$\Omega_0{}^F$ is thus the zero-transaction region for the individual when he invests through the Mutual Fund.

THEOREM. *If the investors and the capital market satisfy Assumptions 1–7,*[3] *if $\chi^i = \chi_i = \chi > 0$, $i = 1,\ldots, m$ and $\Sigma^{-1}(\alpha - rn)$ has more than one*

[3] The numbering refers to the statement of the Assumptions in [5]. The content of these Assumptions is summarized in Section 2 above.

nonzero component then there exists a Mutual Fund such that whenever $\chi^F \leqslant \chi$ *all investors independent of their preferences, age, income, or financial assets prefer investment through the Mutual Fund to individual investment through the capital market.*[4]

Proof. Since the investor's preferences can be represented by (4) and (5) (Assumptions 6 and 7) and since Assumptions 1–5 are also satisfied the investor's two portfolio problems, the first involving individual investment through the capital market and the second involving investment through the Mutual Fund satisfying (8) and (10) are both well defined. In particular the regions Ω_0 and $\Omega_0{}^F$ are well defined.

Let $\rho(\xi) = (\alpha - rn)'\,\xi + r(1 - \xi_y)$ and $\sigma^2(\xi) = \xi'\Sigma\xi$ denote the *instantaneous mean return* and *instantaneous variance* of the portfolio ξ. Since (5) implies that each investor is risk averse, each investor prefers a portfolio which, for given $\rho(\xi)$, has a smaller $\sigma^2(\xi)$, and for given $\sigma^2(\xi)$ has a greater $\rho(\xi)$. A portfolio which for given $\rho(\xi)$ minimizes $\sigma^2(\xi)$ or for given $\sigma^2(\xi)$ maximizes $\rho(\xi)$ is called *efficient*. It is evident that a portfolio ξ is efficient if and only if $\xi = \delta\xi^0$ for some $\delta \geqslant 0$.

Suppose $\chi^F = \chi$. Recall from [5, Eq. (17)] that when $v_j{}^* > 0, j = 1,..., m$ since $\chi^i = \chi_i = \chi$, $i = 1,..., m$ the hyperplanes defined by (6) reduce to

$$\xi_j[1 + \chi(1 - \xi_j{}^0)] - \xi_j{}^0\chi\left(\sum_{\substack{i=1\\i\neq j}}^{m} \xi_i\right) = \xi_j{}^0 \qquad j = 1,..., m$$

since $v_j{}^* > 0$ implies $\chi_{v_j} = \chi$. It is easy to see that these hyperplanes intersect at the point

$$\xi^0\Big/1 + \chi\left(1 - \sum_{i=1}^{m} \xi_i{}^0\right). \tag{15}$$

Similarly when $v_j{}^* < 0$, $j = 1,..., m$ so that $\chi_{v_j} = -\chi$, the hyperplanes defined by (6) reduce to

$$\xi_j[1 - \chi(1 - \xi_j{}^0)] + \xi_j{}^0\chi\left(\sum_{\substack{i=1\\i\neq j}}^{m} \xi_i\right) = \xi_j{}^0 \qquad j = 1,..., m.$$

[4] Although the theorem can be viewed as a preliminary step toward a simple rational explanation for the existence of mutual funds within the standard capital market theory, it should be remembered that a formal theory explaining the *behavior of mutual funds* is still lacking, so that there is no assurance as yet that there will exist rational mutual funds whose behavior approximates that of the idealized Mutual Fund.

As in the proof of [5, Propositions 1 and 2] we need to make an assumption concerning the *initial portfolio proportions*, namely that $\xi(0) \in \Omega_0$ and $\theta(0) \in \Omega_0{}^F$, since it appears that there are conditions under which it is not optimal to transact into Ω_0 or $\Omega_0{}^F$. See [5, footnote 17].

It is evident that these hyperplanes intersect at the point

$$\xi^0 \Big/ 1 - \chi\left(1 - \sum_{i=1}^{m} \xi_i^0\right), \tag{16}$$

so that (15) and (16) are the boundary points of Ω_0 which lie on the ray passing through ξ^0. But then (14) implies that Ω_0^F is exactly the set of efficient portfolios in Ω_0 (the segment BF in Fig. 1). Since $\chi^i = \chi_i = \chi > 0$ and since ξ^0 has at least two nonzero components, (6) implies that Ω_0 contains many inefficient portfolios. A process confined to Ω_0^F is thus clearly preferred to a process in Ω_0. Suppose $\chi^F < \chi$. Since Ω_0^F reduces to a smaller segment of the efficient portfolios about ξ^0 the result is immediate. $\triangle$

The economic interpretation of the theorem is straightforward. Each investor, in determining his portfolio faces two problems: the problem of the *composition* of his portfolio of risky assets and the problem of the *amount* to be invested in the risky assets. When the individual invests on his own through the capital market the presence of transactions costs makes the control of both composition and amount a costly procedure. *Since all investors would like the same mix of risky securities* (a mixture which depends only on the security price parameters (Σ, α, r)) *it is feasible to establish a single Mutual Fund which solves the composition problem for all investors costlessly*. Provided $\chi^F \leqslant \chi > 0$ the investor prefers investment through the Mutual Fund since he now only has to bear the costs of adjusting the amount invested in risky assets. Indeed if $\chi^F = 0$ the investor is able to achieve through the Mutual Fund what he could otherwise only achieve individually on the capital market if there were no transactions costs.

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