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Impatience and Accumulation

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1. INTRODUCTION

In studying the concept of impatience Koopmans [8] was led to a useful class of preference orderings which generalises the familiar class of additively separable preferences. These *aggregator* preference orderings (see Definition 2.3) exhibit a richer variety of behaviour for the various concepts of impatience that can be associated with preference orderings among commodity sequences over time and provide a formal framework for the analysis of impatience along the lines of Irving Fisher [6, Chap. IV]. Beals and Koopmans [1] studied the paths of capital accumulation that arise from such preference orderings in an economy with a single (capital) good, a simple recursive technology set and a single representative consumer. They showed that if the rate of impatience (Definition 2.5) decreases whenever consumption in the first period is increased (Assumption B5′), then the optimal path of capital accumulation has a simple asymptotic behaviour—it converges to a steady state that depends only on the rate of return (Eq. (3.1)) associated with the constant path from the given initial condition. Our objective is to provide an extension of this earlier analysis.[1]

Section 2 lays out the basic assumptions on technology and preferences. In Section 3, after establishing in a natural abstract setting the basic continuity properties of an optimal path (Theorem 3.1), we show that an alternative condition (Assumption 5) leads to the same asymptotic behaviour (Lemma 3.3–Theorem 3.8). This restriction on impatience expresses in an alternative form Fisher's idea that an increase in consumption in early periods (the first two periods) reduces impatience.

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[1] Iwai [7, Sect. VII] has attempted a classification of the different types of asymptotic behaviour that can occur using a dynamic programming framework. However he does not give the restrictions on the original technology and preferences that give rise to each separate case. The analysis is thus inconclusive. A similar comment holds for the analysis of Boyer [3]. No such criticism holds for the work of Beals and Koopmans. In this paper we do not establish whether more complex types of asymptotic behaviour can occur when Assumption B5 or B5′ is not satisfied. This remains an open problem.

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Fisher argued that the rate of impatience on a constant consumption path increases as the level of the constant consumption stream is decreased and becomes extremely large as subsistence consumption is approached. In Section 4 we show that this property has an important bearing on the nature of the long-run development of an economy. It leads to a division of countries into those that are rich and those that are poor according as their initial capital exceeds or falls short of a certain critical level of capital. Rich countries can generate enough consumption to lower the rate of impatience below the rate of return on capital hence making permanent development worthwhile. Poor countries can at most generate a small consumption stream and are locked into a high rate of impatience that exceeds the rate of return on capital, forcing them to be confined in the long-run to subsistence consumption. The analysis suggests however that by lending capital to the poor countries, rich countries can enable the latter to embark on a program of permanent development. This class of preference orderings thus captures an important aspect of the process of development which is absent from the earlier analysis based on additively separable preferences [4, 9] in which every country regardless of its initial endowment embarks on a program of permanent development.

2. Technology, Preferences and Impatience

Let $\mathscr{S}$ denote the locally convex space consisting of sequences ${}_1x=(x_1, x_2,...)$, $x_t \in R$, $t \geqslant 1$, in which the topology is induced by the family of seminorms $v_t({}_1x)=|x_t|$, $t=1,2,\ldots$. A sequence ${}_1x^n \in \mathscr{S}$ is said to converge to ${}_1x$ in the *product topology* if $v_t({}_1x^n - {}_1x) \to 0$ as $n \to \infty$, $t=1,2,\ldots$. This topology is metrisable since $v_t({}_1x^n - {}_1x) \to 0$ as $n \to \infty$, $t=1,2,\ldots$ is equivalent to $d({}_1x^n, {}_1x) \to 0$ as $n \to \infty$ with $d({}_1x, {}_1z) = \sum_{t=1}^{\infty} \lambda^t \mu_t(x_t - z_t)$, $\mu_t = \min(1, v_t)$, $0<\lambda<1$. For ${}_1x \in \mathscr{S}$ let $S_\varepsilon({}_1x) = \{{}_1z \in \mathscr{S} \mid d({}_1z, {}_1x) < \varepsilon\}$.

Let $z_t \in R^+$ denote the capital stock available at time t and let $g(z_t)$ denote the maximum output producible during period t.

Assumption A. (i) $g: R^+ \to R^+$ is concave.

(ii) $g(0)=g(\hat{z})=0$ for some $0<\hat{z}<\infty$.

(iii) $g \in C^1(0, \hat{z}]$, $0<g'(0)$, $-1 \leqslant g'(\hat{z}) < 0$, $\lim_{z\to 0} g(z)=0$.

Let $f(z)=z+g(z)$ and let $Z=[0,\hat{z}]$, $Z^\infty = Z \times Z \times \cdots$. Then $f: Z \to Z$ and the correspondence $\mathscr{F}: Z \to Z^\infty$ defined by

$$\mathscr{F}(\xi) = \{{}_1z \in \mathscr{S} \mid 0 \leqslant z_{t+1} \leqslant f(z_t),\ t \geqslant 1,\ z_1 = \xi\}$$

gives the set of *feasible capital paths* attainable from the initial capital stock ξ. Let $F\colon Z^{\infty} \to \mathscr{S}^{+}$ be defined by

$$F({}_1z) = (f(z_1) - z_2, f(z_2) - z_3, \ldots),$$

then the correspondence $\mathbf{C}\colon Z \to \mathscr{S}^{+}$ defined by

$$\mathbf{C}(\xi) = \{{}_1c \in \mathscr{S}^{+} \mid {}_1c = F({}_1z),\ {}_1z \in \mathscr{F}(\xi)\}$$

gives the set of *feasible consumption paths* attainable from ξ. We let $\mathscr{C} = \mathrm{Im}(\mathbf{C}) = \bigcup_{\xi \in Z} \mathbf{C}(\xi)$ and make the following assumptions concerning preferences.

Assumption B1. Choice among consumption paths in $\mathscr{C}$ can be represented by a preference ordering $\succsim$ which is complete, transitive and continuous in the product topology.

2.1. *Remark.* It follows from the theorem of Debreu [5, Theorem 1, p. 162] that there exists a utility function $U(\cdot)\colon \mathscr{C} \to R$, which is continuous on $\mathscr{C}$ in the product topology, such that ${}_1c \succsim {}_1c'$ if and only if $U({}_1c) \geqslant U({}_1c')$ for ${}_1c$, ${}_1c' \in \mathscr{C}$.

Assumption B1 leads to the problem of finding a consumption path ${}_1c^*$ such that $U({}_1c^*) = \sup_{{}_1c \in \mathbf{C}(\xi)} U({}_1c)$ or the equivalent problem of finding a capital path ${}_1z^*$ such that

$$U(F({}_1z^*)) = \sup_{{}_1z \in \mathscr{F}(\xi)} U(F({}_1z)). \qquad (\mathscr{P})$$

${}_1c^*({}_1z^*)$ is called an *optimal consumption* (*capital*) *path*.

Assumption B2. $\mathscr{U}_{\alpha} = \{{}_1c \in \mathscr{C} \mid U({}_1c) \geqslant \alpha\}$, $\alpha \in R$, is *strictly convex*.

Assumption B3. (i) For all ${}_1c$, ${}_1c' \in \mathscr{C}$

$$U(c_1, {}_2c) \geqslant U(c_1', {}_2c) \text{ implies } U(c_1, {}_2c') \geqslant U(c_1', {}_2c'),$$

$$U(c_1, {}_2c) \geqslant U(c_1, {}_2c') \text{ implies } U(c_1', {}_2c) \geqslant U(c_1', {}_2c');$$

(ii) for some c_1 and all ${}_2c$, ${}_2c' \in \mathscr{C}$

$$U(c_1, {}_2c) \geqslant U(c_1, {}_2c') \qquad \text{if and only if} \quad U({}_2c) \geqslant U({}_2c').$$

Part (i) is the assumption of *limited independence* and (ii) is the assumption of *stationarity*. Let $\partial U({}_1c)/\partial c_t = U_t({}_1c)$, $t \geqslant 1$, denote partial derivatives of $U(\cdot)$.

Assumption B4. $U_1(\cdot)$, $U_2(\cdot)$ exist and are continuous on $\mathscr{C}$ in the product topology and $U_1({}_1c) > 0$, $U_2({}_1c) > 0$, ${}_1c \in \mathscr{C}$.

2.2. *Remark.* Since $\mathscr{C}$ is a compact[2] subset of $\mathscr{S}$ and U is continuous on $\mathscr{C}$, there exist ${}_1\underline{c}$, ${}_1\bar{c} \in \mathscr{C}$ such that

$$0 = U({}_1\underline{c}) \leqslant U({}_1c) \leqslant U({}_1\bar{c}) = 1, \qquad {}_1c \in \mathscr{C}.$$

Let $\hat{c} = \max\{c_1 \mid (c_1, c_2, \ldots) \in \mathscr{C}\} = f(\hat{z})$.

2.3. DEFINITION. If there exists a function V: $[0, \hat{c}] \times [0, 1] \to R$ such that

$$U({}_1c) = V(c_1, U({}_2c)), \qquad {}_1c \in \mathscr{C}, \tag{2.1}$$

then U is said to represent an *aggregator preference ordering*, the function $V(\cdot, \cdot)$ being called the *aggregator*.

2.4 *Remark.* If follows from Koopmans [8, pp. 388–395] that a preference ordering satisfying Assumptions B1–B4 is an aggregator preference ordering. It is easy to check that Assumption B4 implies $V_1(c_1, U)$, $V_2(c_1, U)$ exist and are continuous and positive on $(0, \hat{c}) \times (0, 1)$. The relation $U({}_1c) = V(c_1, V(c_2, \ldots V(c_t, U({}_{t+1}c))\cdots))$ can then be used to prove that $U_t(\cdot)$ satisfies Assumption B4 for $t \geqslant 3$. Thus $U(\cdot)$ is strictly increasing on $\mathscr{C}$. The next definition follows Fisher [6, p. 62].

2.5 DEFINITION. The *rate of impatience* at ${}_1c \in \mathscr{C}$ is defined by

$$R({}_1c) = \frac{U_1({}_1c)}{U_2({}_1c)} - 1, \qquad {}_1c \in \mathscr{C}.$$

Restrictions on the rate of impatience simplify the asymptotic behaviour of an optimal capital path. The natural restrictions have been discussed extensively by Fisher [6, Chap. IV]. In general he argues that either an increase in consumption in early periods or an increase in the level of the whole consumption stream reduces impatience. The following assumption made by Beals and Koopmans[3] is a restriction of the first kind.

Assumption B5′. $R(c_1, {}_2c)$ is a strictly decreasing function of c_1 for fixed ${}_2c$, for all $(c_1, {}_2c) \in \mathscr{C}$.

The following assumption which we will use instead of Assumption B5′ is also a restriction of the first kind. Let ${}^*c = (c, c, \ldots)$ and consider the function

$$\gamma(c, {}_3c) = R(c, c, {}_3c), \qquad (c, c, {}_3c) \in C,$$

$$\rho(c) = R({}^*c), \qquad {}^*c \in \mathscr{C}.$$

[2] See proof of Theorem 3.1.

[3] Condition V′ in Beals and Koopmans [1, p. 1004].

Assumption B5(i). $\gamma(c, {}^*\bar{c}) \leqslant (\geqslant) \rho(\bar{c})$ whenever $c > (<) \bar{c}$ for all $(c, c, {}^*\bar{c})$, ${}^*\bar{c} \in \mathscr{C}$.

2.6 *Remark.* If $R_1({}^*\bar{c})$ exists, then Assumptions B2 and B5(i) imply $R_1({}^*\bar{c}) \leqslant 0$ and $R_1({}^*\bar{c}) < 0$ except in the hairline case of an inflexion point. If $R_1({}_1c)$ is continuous in a neighborhood of ${}^*\bar{c}$, then $R_1({}^*\bar{c}) < 0$ implies R is a strictly decreasing function of c_1 in a neighborhood of ${}^*\bar{c}$. This is the additional property we require.

Assumption B5(ii). For each ${}^*\bar{c} \in \mathscr{C}$ there exists $\eta > 0$ such that $R(c_1, {}_2c)$ is strictly decreasing in c_1, $\forall (c_1, {}_2c) \in S_\eta({}^*\bar{c}) \cap \mathscr{C}$.

2.7 EXAMPLES. Consider a preference ordering satisfying Assumptions B1–B4 for which $\gamma(c, {}_3c)$ is strictly decreasing in c, $\forall (c, c, {}_3c) \in \mathscr{C}$. In this case both Assumptions B5 and B5′ are satisfied.

Figure 1 shows the indifference curves in the (c_1, c_2) space for a preference ordering that satisfies Assumption B5 but does not satisfy Assumption B5′. In a neighborhood of c^a (the first two coordinates of ${}^*\bar{c}$) Assumption B5(ii) holds. $R(c^b, {}^*\bar{c}) < R(c^d, {}^*\bar{c}) < R(c^a, {}^*\bar{c})$ so that Assumption B5(i) holds, while $R(c^b, {}^*\bar{c}) < R(c^{d'}, {}^*\bar{c})$, contradicting Assumption B5′. This can only happen when $\gamma(c, {}^*\bar{c})$ is increasing on a segment of the diagonal OA in the (c_1, c_2) space.

2.8 *Remark.* Fisher's argument [6, pp. 72, 247] that an increase in the level of the whole consumption stream reduces the rate of impatience would imply that $\rho(c) < \rho(\bar{c})$ whenever $c > \bar{c}$, $\forall {}^*c$, ${}^*\bar{c} \in \mathscr{C}$. The consequence of this

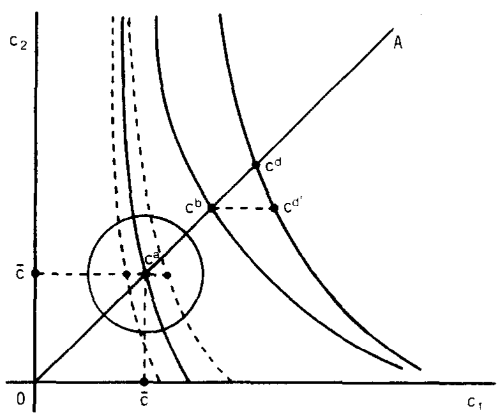

FIGURE 1

condition will be examined in Section 4. Note that $\rho(c) = 1/V_2(c, U(^*c)) - 1 > 0$, $\forall c \in (0, \hat{c})$.[4]

The final restriction on the rate of impatience, while not essential, simplifies the analysis by ensuring the interiority of an optimal path.

Assumption B6.

$$\lim_{c_1 \to 0} R({}_1c) > g'(0), \qquad c_2 > 0,\ {}_1c \in \mathscr{C},$$

$$\lim_{c_2 \to 0} R({}_1c) < g'(\hat{z}), \qquad c_1 > 0,\ {}_1c \in \mathscr{C}.$$

2.9 EXAMPLE. *Let* $U({}_1c) = \sum_{\tau=1}^{\infty} \beta^{\tau-1} u(c_\tau)$ with $\beta = 1/(1+\rho)$, $\rho > 0$, $u \in C[0, \hat{c}]$, $u'(c) > 0$, $u''(c) < 0$, $c \in [0, \hat{c}]$, $u'(c) \to \infty$ as $c \to 0$. $R({}_1c) = u'(c_1)/\beta u'(c_2) - 1$ so that $\gamma(c, {}^*\bar{c}) = \rho(\bar{c}) = \rho$. Assumptions B1–B6 are satisfied. This is the standard additively separable preference ordering.

3. ACCUMULATION

Let

$$\alpha(\xi) = \sup_{{}_1z \in \mathscr{F}(\xi)} U(F({}_1z)),$$

then the *capital* (*path*) *correspondence* $\Phi: Z \to Z^\infty$ is defined by

$$\Phi(\xi) = \{{}_1z \in \mathscr{F}(\xi) \mid U(F({}_1z)) = \alpha(\xi)\}$$

and the *consumption* (*path*) *correspondence* is given by $\Psi(\xi) = F(\Phi(\xi))$.

3.1 THEOREM. *Let Assumptions* A *and* B1–B4 *hold, then*

(i) $\Phi(\xi) \neq \varnothing$ *and is single-valued* $\forall \xi \in Z$,

(ii) Φ *is continuous on* Int Z,

(iii) *there exists a function* $\phi: Z \to Z$ *such that* $\Phi(\xi) = (\xi, \phi(\xi), \phi^2(\xi), \ldots)$, *with* ϕ *continuous on* Int Z.

Proof. (i) The continuity of f implies $\mathscr{F}$ has a closed graph. Since Z^∞ is compact by Tychonov's theorem [2, p. 79], $\mathscr{F}: Z \to Z^\infty$ is compact-valued. Since F and U are continuous, $\Phi(\xi) \neq \varnothing$, $\xi \in Z$ [2, p. 76]. Assumptions A(i) and B2 imply $\Phi(\xi)$ is single-valued.

[4] Let $h(c) = U(^*c)$, then by Remark 2.4, h is strictly increasing. By the theorem of Lebesgue (if $f: [a, b] \to R$ is increasing, then $f'(x)$ exists a.e. on $[a, b]$), $h'(c) > 0$ exists a.e. on $[0, \hat{c}]$. By (2.1), $h(c) = V(c, h(c))$ so that $0 < V_1(c, h(c)) = h'(c)(1 - V_2(c, h(c))$ a.e. on $(0, \hat{c})$ implies $\rho(c) > 0$ a.e. on $(0, \hat{c})$. By Assumption B4, ρ is continuous on $(0, c)$ so that $\rho(c) > 0$, $c \in (0, \hat{c})$.

(ii) Since $\mathscr{F}$ has a closed graph and Z^∞ is compact, $\mathscr{F}$ is upper semicontinuous [2, p. 112]. The concavity of f implies $\mathscr{F}$ has a convex graph. Since Int dom $\mathscr{F} = \text{Int } Z \neq \varnothing$ by the theorem of Ursescu,[5] $\mathscr{F}$ is lower semicontinuous on Int Z. By the maximum theorem [2, p. 116], Φ is upper semicontinuous on Int Z. Since Φ is single-valued, Φ is continuous on Int Z.

(iii) Let $\Phi(\xi) = (\xi, z_2^*, z_3^*, ...)$, then

$$V(f(\xi) - z_2^*, V(f(z_2^*) - z_3^*, U(F({}_4z^*))))$$
$$\geqslant V(f(\xi) - z_2, V(f(z_2) - z_3, U(F({}_4z)))), \qquad \forall\, {}_1z \in \mathscr{F}(\xi).$$

Let $z_2 = z_2^*$, since $V_2(\cdot, \cdot) > 0$

$$V(f(z_2^*) - z_3^*, U(F({}_4z^*))) \geqslant V(f(z_2) - z_3, U(F({}_4z))), \qquad \forall\, {}_2z \in \mathscr{F}(z_2^*)$$

so that ${}_2z^*$ is optimal from z_2^*. Thus if $z_2^* = \phi(\xi)$, then $z_3^* = \phi(z_2^*) = \phi(\phi(\xi)) = \phi^2(\xi)$. By induction $z_t^* = \phi^{t-1}(\xi)$. The continuity of ϕ follows from (ii). ▮

3.2 *Remark.* If we let $\psi_t(\xi) = f(\phi^{t-1}(\xi)) - \phi^t(\xi)$, $t \geqslant 1$ with $\phi^0(\xi) = \xi$, then the consumption correspondence is given by $\Psi(\xi) = (\psi_1(\xi), \psi_2(\xi), ...)$. Let

$$r(\xi) = g'(\xi) - \rho(g(\xi)) \tag{3.1}$$

denote the *(net) rate of return* associated with the constant path $^*\xi$. r induces a partition of the capital space Z as follows:

$$Z^+ = \{\xi \in Z \mid r(\xi) > 0\}, \qquad Z^- = \{\xi \in Z \mid r(\xi) < 0\},$$
$$Z' = \{\xi \in Z \mid \xi = 0 \text{ or } r(\xi) = 0\}.$$

We may, without loss of generality, assume that Z' contains a finite number of elements *(steady states)*[6]

$$Z' = \{\bar{z}^0, \bar{z}^1, ..., \bar{z}^n \mid 0 = \bar{z}^0 < \bar{z}^1 < \cdots < \bar{z}^n\},$$

letting

$$Z_i = (\bar{z}^i, \bar{z}^{i+1}), \qquad i = 0, ..., n-1, \qquad Z_n = (\bar{z}^n, \hat{z}].$$

[5] Let X, Y be Frechet spaces, $\mathscr{F}: X \to Y$ a correspondence whose graph is a closed convex subset of $X \times Y$. If Int dom $\mathscr{F} \neq \varnothing$, then $\mathscr{F}$ is lower semicontinuous on Int dom $\mathscr{F}$ (Ursescu [10, p. 438]). Since $\mathscr{S}$ is complete, $\mathscr{S}$ is a Frechet space (complete, metrisable locally convex space).

[6] $Z' \neq \varnothing$ since $0 \in Z'$. If $r'(\xi) \neq 0$, $\xi \in Z'$, then Z contains at most a finite number of elements.

3.3 LEMMA. *Let Assumptions* A, B1–B4 *and* B6 *hold. If* $\xi \in Z$, $\xi > 0$, *then* $\phi^t(\xi) > 0$, $\psi_t(\xi) > 0$, $t \geqslant 1$.

Proof. The result follows readily from Assumption B6; see Lemma 2 in [1]. ∎

3.4 LEMMA. *Let Assumptions* A, B1–B5(i) *and* B6 *hold. If* $\xi \in Z$, $\xi > 0$ *and if* $\Phi(\xi) = (\xi, {}^*\bar{z})$, *then* $\xi = \bar{z}$.

Proof. Lemma 3.3 and the first-order conditions for the second period of $\Phi(\xi)$ and $\Phi(\bar{z})$ imply

$$R(\Psi(\xi)) = g'(\bar{z}) = R(\Psi(\bar{z})). \tag{3.2}$$

Suppose $\xi > \bar{z}$, then $c = \psi_1(\xi) > \psi_1(\bar{z}) = \bar{c}$. Let $c' \in (\bar{c}, c)$ be such that $U(c', c', {}^*\bar{c}) = U(c, \bar{c}, {}^*\bar{c})$, then by Assumption B2, $R(c', c', {}^*\bar{c}) > R(c, \bar{c}, {}^*\bar{c})$, while Assumption B5(i) implies $R({}^*\bar{c}) \geqslant R(c', c', {}^*\bar{c})$. Thus $R({}^*\bar{c}) > R(c, \bar{c}, {}^*\bar{c})$ contradicting (3.2), so that $\xi \leqslant \bar{z}$. If $\xi < \bar{z}$, a similar contradiction forces $\xi \geqslant \bar{z}$, so that $\xi = \bar{z}$. ∎

3.5 LEMMA. *Let Assumptions* A, B1–B5(i) *and* B6 *hold. If* $\xi \in Z_i$, *then* $\phi^t(\xi) \in Z_i$, $t \geqslant 1$.

Proof. $\phi(\xi) = \bar{z}^{i+1}$ contradicts Lemma 3.4. Suppose $\phi(\xi) > \bar{z}^{i+1}$. By the continuity of $\phi(\cdot)$ there exists $\xi' \in (\bar{z}^i, \xi)$ such that $\phi(\xi') = \bar{z}^{i+1}$. But then $\Phi(\xi') = (\xi', {}^*\bar{z}^{i+1})$ is optimal, contradicting Lemma 3.4. Thus $\phi(\xi) < \bar{z}^{i+1}$. By a similar argument $\phi(\xi) > \bar{z}^i$. By induction $\phi^t(\xi) \in Z_i$, $t \geqslant 1$. ∎

3.6 LEMMA. *Let Assumptions* A *and* B1–B4 *hold. Let* $\xi \in Z^+(Z^-)$. *If* ${}_1z \in \mathscr{F}(\xi)$ *satisfies* $z_t \leqslant (\geqslant)\, \xi$, $t \geqslant 1$, *then* $U(F({}^*\xi)) \geqslant U(F({}_1z))$.

Proof. Let $\xi \in Z^+$. Pick a sequence ${}_1z^n \in \mathscr{F}(\xi)$ such that ${}_1z^n \to {}_1z$ and $z_t^n \leqslant \xi$, $1 \leqslant t \leqslant n$, $z_t^n = \xi$, $t > n$. By Lemma 5 of [1], $U(F({}^*\xi)) \geqslant U(F({}_1z^n))$. By the continuity of U and F

$$U(F({}^*\xi)) \geqslant \lim_{n\to\infty} U(F({}_1z^n)) = U(F(\lim_{n\to\infty} {}_1z^n)) = U(F({}_1z)).$$

A similar argument follows for $\xi \in Z^-$. ∎

3.7. LEMMA. *Let Assumptions* A *and* B1–B6 *hold. If* $\xi \in Z_i \subset Z^+(Z^-)$, *then there exists* $\delta > 0$ *such that if* $|\xi - \bar{z}^{i+1}(\bar{z}^i)| < \delta$, *then* $\phi^t(\xi) \to \bar{z}^{i+1}(\bar{z}^i)$.

Proof. Let $Z_i \subset Z^+$ and let $\xi' \in Z_i$, $\xi' > \xi$ satisfy $|\xi' - \bar{z}^{i+1}| < \delta$, then $\phi(\xi') > \phi(\xi)$. For suppose $\phi(\xi') = \phi(\xi)$, then $\phi^t(\xi') = \phi^t(\xi)$, $t \geqslant 1$. By Lemma 3.3

$$R(\Psi(\xi')) = g'(\phi(\xi')) = g'(\phi(\xi)) = R(\Psi(\xi)). \tag{3.3}$$

Since $\Psi(\bar{z}^{i+1}) = {}^*c$, by Assumption B5(ii) there exists $\eta > 0$ such that $R({}_1c)$ is strictly decreasing in c_1, $\forall\, {}_1c \in S_\eta({}^*c) \cap \mathscr{C}$. Let $\delta > 0$ be such that $|\xi - \bar{z}^{i+1}| < \delta$ implies $d(\Psi(\xi), \Psi(\bar{z}^{i+1})) < \eta$, then Assumption B5(ii) implies (3.3) is impossible since $\psi_1(\xi') > \psi_1(\xi)$, $\psi_t(\xi') = \psi_t(\xi)$, $t \geqslant 2$. Thus $\phi(\xi') \neq \phi(\xi)$. Suppose $\phi(\xi') < \phi(\xi)$. By Lemma 3.5, $\phi(\xi) < \bar{z}^{i+1}$. Since $\phi(\bar{z}^{i+1}) = \bar{z}^{i+1}$, by the continuity of ϕ there exists $\xi'' > \xi$ such that $\phi(\xi'') = \phi(\xi)$, contradicting $\phi(\xi'') \neq \phi(\xi)$ whenever $\xi'' > \xi$. Thus $\phi(\xi') > \phi(\xi)$. Suppose $\xi = \phi(\xi)$, then by Lemma 3.3

$$g'(\xi) = g'(\phi(\xi)) = R(\Psi(\xi)) = \rho(g(\xi)),$$

contradicting $\xi \in Z^+$. Suppose $\xi > \phi(\xi)$, then $\phi(\xi) > \phi^2(\xi)$ and $\phi^{t-1}(\xi) > \phi^t(\xi)$, $t \geqslant 1$. By Lemma 3.6, $U(F({}^*\xi)) \geqslant U(F(\Phi(\xi)))$, contradicting the nonoptimality of ${}^*\xi$. Thus we must have $\xi < \phi(\xi)$ and hence $\phi^{t-1}(\xi) < \phi^t(\xi)$, $t \geqslant 1$. Since Lemma 3.5 implies $\phi^t(\xi) < \bar{z}^{i+1}$, $t \geqslant 1$, $\lim_{t \to \infty} \phi^t(\xi) = z' \leqslant \bar{z}^{i+1}$. Suppose $z' < \bar{z}^{i+1}$. Let $\Phi(\xi) = {}_1z^*$ and consider the sequence of optimal paths $\Phi(z_n^*) = {}_nz^*$. Since $\lim_{n \to \infty} z_n^* = z'$, ${}^*z' = \lim_{n \to \infty} \Phi(z_n^*) = \Phi(\lim_{n \to \infty} z_n^*) = \Phi(z')$ by the continuity of Φ. But then $z' \in Z'$, contradicting $z' \in Z_i$. Thus $z' = \bar{z}^{i+1}$. When $\xi \in Z_i \subset Z^-$ a similar argument shows $\lim_{t \to \infty} \phi^t(\xi) = \bar{z}^i$. ∎

3.8. THEOREM. *Let Assumptions* A *and* B1–B6 *hold.*

(i) *If* $\xi \in Z_i \subset Z^+$, *then* $\phi^t(\xi) \to \bar{z}^{i+1}$, $i = 0, \ldots, n-1$;

(ii) *if* $\xi \in Z_i \subset Z^-$, *then* $\phi^t(\xi) \to \bar{z}^i$, $i = 0, \ldots, n$;

(iii) *if* $\xi = \bar{z}^i$, *then* $\phi^t(\xi) = \bar{z}^i$, $t \geqslant 1$, $i = 0, \ldots, n$.

Proof. (i) By Lemma 3.5, $\phi^t(\xi) \in (\bar{z}^i, \bar{z}^{i+1})$, $t \geqslant 1$. Suppose there exists $\tau < \infty$ such that $\phi^{\tau-1}(\xi) < \phi^\tau(\xi) = \sup_{t \geqslant 1} \phi^t(\xi)$. Let $\xi' = \phi^\tau(\xi)$. Since $\phi^t(\xi') \leqslant \xi'$, $t \geqslant 1$, by Lemma 3.6 $U(F({}^*\xi')) \geqslant U(F(\Phi(\xi')))$ so that $\Phi(\xi') = {}^*\xi'$. But then $(\phi^{\tau-1}(\xi), {}^*\xi')$ is an optimal path, contradicting Lemma 3.4. Thus

$$\phi^\tau(\xi) < \sup_{t \geqslant 1} \phi^t(\tau), \qquad \tau \geqslant 1. \tag{3.4}$$

Let $t_1 < t_2 < \cdots < t_n < \cdots$ be an increasing sequence of times such that

$$\phi^{t_n}(\xi) \to z = \sup_{t \geqslant 1} \phi^t(\xi). \tag{3.5}$$

Consider $\Phi(z)$. If there exists $\tau < \infty$ such that $\phi^\tau(z) > z$, then there exists $\varepsilon > 0$ such that $\phi^\tau(z) > z + \varepsilon$. By the convergence (3.5) and the continuity of ϕ^τ there exists t_m such that $|\phi^\tau(\phi^{t_m}(\xi)) - \phi^\tau(z)| < \varepsilon$ so that $\phi^{\tau + t_m}(\xi) = \phi^\tau(\phi^{t_m}(\xi)) > z$, contradicting (3.4). Thus $\phi^t(z) \leqslant z$, $t \geqslant 1$. By Lemma 3.6, $U(F({}^*z)) \geqslant U(F(\Phi(z)))$ so that $\Phi(z) = {}^*z$ by the uniqueness of the optimal path. Thus $z = \bar{z}^{i+1}$ and there exists t_k such that $|\phi^{t_k}(\xi) - \bar{z}^{i+1}| < \delta$. By Lemma 3.7, $\phi^t(\phi^{t_k}(\xi)) \to \bar{z}^{i+1}$ as $t \to \infty$ so that $\phi^\tau(\xi) \to \bar{z}^{i+1}$.

(ii) This follows by a similar argument.

(iii) By repeated application of the first-order conditions it is clear that $U(F(*\bar{z}^i)) \geqslant U(F({}_1z))$, $\forall\, {}_1z \in \mathscr{F}(\bar{z}^i)$ for which $z_t = \bar{z}^i$, $\forall\, t > n$ for some $n < \infty$. Any ${}_1z \in \mathscr{F}(\bar{z}^i)$ may be expressed as the limit of a sequence of such paths ${}_1z^n \to {}_1z$. But then $U(F(*\bar{z}^i)) \geqslant U(F({}_1z))$, $\forall\, {}_1z \in \mathscr{F}(\bar{z}^i)$. ∎

3.9 *Remark.* In Theorem 3.8, Assumption B5 may be replaced by Assumption B5′. This is the earlier result of Beals and Koopmans [1, Theorem 2, p. 1009].

4. Impatience and Development

Fisher has argued persuasively [6, pp. 72, 247] that the pure rate of impatience $\rho(c)$ is a strictly decreasing function of the level of the constant consumption stream $*c$ and becomes exceedingly large as $*c$ approaches subsistence consumption. Adding this condition to Assumptions B1–B6 has an important impact on the nature of the long-run development of an economy. If we assume that output is measured in such a way that zero consumption represents *subsistence* consumption, then Assumption A(ii) implies that zero capital stock enables subsistence consumption to be maintained permanently. The problem of development may now be posed as follows. What initial endowment of capital must a country possess if it is to be able to sustain permanent development beyond the subsistence level? An answer may be given in general terms as follows.

4.1 Corollary. *Let Assumptions* A *and* B1–B6 *hold, where Assumption* B5 *may be replaced by Assumption* B5′.

(i) *If* $r(0) > 0$ *and* $0 < \xi$, *then* $\lim_{t\to\infty} \phi^t(\xi) > 0$.

(ii) *If* $r(0) < 0$ *and* $\xi \in Z_0$, *then* $\lim_{t\to\infty} \phi^t(\xi) = 0$; *if* $0 < \xi \notin Z_0$, *then* $\lim_{t\to\infty} \phi^t(\xi) > 0$.

Proof. (i) By Assumption A there exists $\tilde{z}$ such that $g'(\tilde{z}) = 0$ so that $r(\tilde{z}) = -\rho(g(\tilde{z})) < 0$ by Remark 2.8. Since $r(0) > 0$ and r is continuous, there exist $0 < \bar{z}^i < \tilde{z}$ such that $r(\bar{z}^i) = 0$, $i = 1, \ldots, n$, $n \geqslant 1$. The result follows from Theorem 3.8(i).

(ii) This is immediate from Theorem 3.8(ii). ∎

A more precise answer, which is also simpler to interpret, may be given if we impose some additional simplifying restrictions on the behaviour of the return function $r(\xi)$.

4.2 COROLLARY. *Let Assumptions* A *and* B1–B6 *hold, where Assumption* B5′ *may replace Assumption* B5.

(i) *If* $r(0) > 0$ *and* $r'(\xi) < 0$, $\xi \in Z$, *then there exists a unique positive steady state* $\bar{z}$ *such that* $0 < \xi$ *implies* $\lim_{t\to\infty} \phi^t(\xi) = \bar{z}$.

(ii) *If* $r(0) < 0$ *and there exists* $\hat{\xi} \in [0, \tilde{z}]$ *such that* $r(\hat{\xi}) > 0$, $r'(\xi) > 0$, $\xi \in [0, \hat{\xi})$, $r'(\xi) < 0$, $\xi \in (\hat{\xi}, \tilde{z}]$, *then there exist two positive steady states* $\bar{z}_1 < \bar{z}_2$ *such that* $0 \leqslant \xi < \bar{z}_1$ *implies* $\lim_{t\to\infty} \phi^t(\xi) = 0$ *and* $\xi > \bar{z}_1$ *implies* $\lim_{t\to\infty} \phi^t(\xi) = \bar{z}_2$.

The earlier analysis of Cass [4] and Koopmans [9] assumed that the preference ordering is additively separable (Example 2.9). In this case the (pure) rate of impatience $\rho(c)$ is the same for all levels of the constant consumption stream $*c$. Since $r(\xi) = g'(\xi) - \rho$, $r'(\xi) = g''(\xi) < 0$, $\xi \in Z$. If $g'(0) > \rho$ (otherwise no country would ever develop beyond subsistence consumption), then Corollary 4.2(i) holds. In this case all countries with initial capital $\xi > 0$ will be led to permanent development beyond the subsistence level since $\phi^t(\xi) \to \bar{z}$ (see Fig. 2 with $\bar{z}_2 = \bar{z}$, $\bar{z}_1 = 0$).

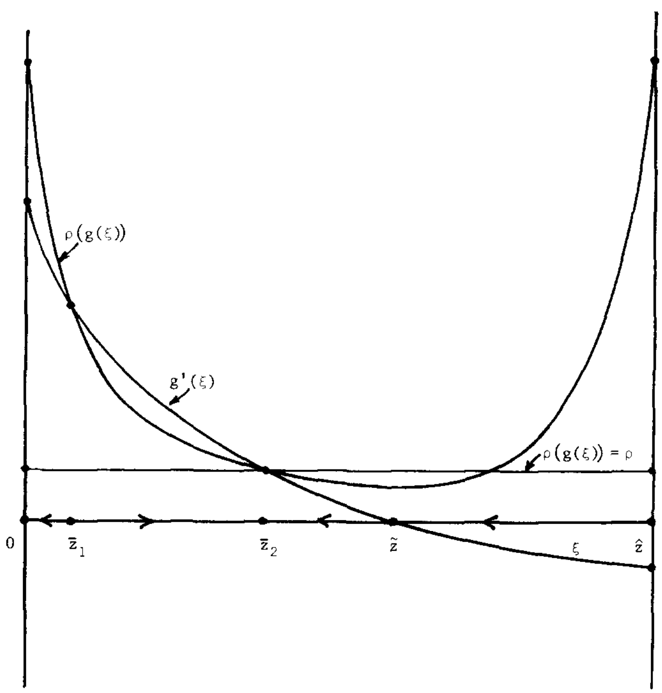

FIGURE 2

Suppose on the contrary that the preference ordering satisfies Assumptions B1–B6 (where Assumption B5′ can be substituted for Assumption B5) and that in accordance with the view of Fisher $\rho(c)$ is a strictly decreasing function of the level of the constant consumption stream $^*c$ with $g'(0) < \rho(0)$. If ρ decreases sufficiently rapidly, then Corollary 4.2(ii) holds (see Fig. 2). In this case poor countries ($\xi < z_1$) find that the rate of impatience $\rho(g(\xi))$ is too high relative to the rate of return on capital $g'(\xi)$ to warrant permanent development. Only rich countries ($\xi > z_1$) are in a position to lower the rate of impatience sufficiently to make the (net) rate of return $r(\xi)$ positive, thereby warranting permanent development. There is thus a critical level of capital z_1 that separates the poor countries from the rich. The rich countries develop, $\phi^t(\xi) \to z_2$, while the poor countries are forced to remain at the subsistence level, $\phi^t(\xi) \to 0$. The framework leads naturally to the idea of loans from rich to poor countries, for a poor country that receives a loan of at least $z_1 - \xi$ can be enabled to reduce the rate of impatience sufficiently to make permanent development worthwhile. A proper treatment of this problem would require an explicit equilibrium analysis.

References

1. R. Beals and T. C. Koopmans, Maximizing stationary utility in a constant technology, *SIAM J. Appl. Math.* **17** (1969), 1001–1015.
2. C. Berge, "Topological Spaces," Oliver & Boyd, Edinburgh, 1963.
3. M. Boyer, An optimal growth model with stationary non-additive utilities, *Canad. J. Econom.* **8** (1975), 216–237.
4. D. Cass, Optimum growth in an aggregative model of capital accumulation, *Rev. Econom. Stud.* **32** (1965), 233–240.
5. G. Debreu, Representation of a preference ordering by a numerical function, *in* "Decision Processes" (R. M. Thrall, C. H. Coombs and R. L. Davis, Eds.), pp. 159–165, Wiley, New York, 1954.
6. I. Fisher, "The Theory of Interest," Augustus M. Kelley, New York, 1965.
7. K. Iwai, Optimal economic growth and stationary ordinal utility—a Fisherian approach, *J. Econom. Theory* **5** (1972), 121–151.
8. T. C. Koopmans, Stationary ordinal utility and impatience, *Econometrica* **28** (1960), 287–309. Reprinted in "Scientific Papers of Tjalling C. Koopmans," pp. 387–428, Springer-Verlag, New York, 1970.
9. T. C. Koopmans, On the concept of optimal economic growth, *Pontif. Acad. Sci. Scr. Var.* **28** (1965), 225–287. Reprinted in "Scientific Papers of Tjalling C. Koopmans," pp. 485–547, Springer-Verlag, New York, 1970.
10. C. Ursescu, Multifunctions with convex closed graph, *Czechoslovak Math. J.* **25** (1975), 438–441.