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Futures Markets, Production and Diversification of Risk*

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This paper lays out a framework for the analysis of the risk transfer role of speculators on futures markets and the impact of their trading on the production decisions of firms. We show that when speculators diversify their portfolios over a large number of markets, the equilibrium risk premium converges to an asymptotic premium, the behaviour of which is determined by the stochastic dependence between the spot price and an index of average returns on other markets—the idiosyncratic risk arising from the variability of the spot price itself is diversified away. In the independent and negatively dependent cases this diversification of risk leads to a Pareto improving property.

1. INTRODUCTION

Futures markets may be viewed as an institutional mechanism through which speculators can perform two roles: first, that of risk transfer, and second, that of information gathering. The exchanges themselves have traditionally emphasised the importance of the former, while economists have tended to focus their attention on the latter.[1]

The exchanges point out that firms involved in the production and processing of certain commodities would, in the absence of a futures market, incur over and above the standard costs of production, the substantial risk costs arising from the wide fluctuations in the underlying commodity's price. If futures markets can through the mechanism of hedging and the trading of speculators provide such firms with price insurance at substantially smaller cost, then more output can be made available at less cost to the consumer. Under what conditions is such an argument valid?

Few attempts have been made in the economic literature to develop a

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[1] See, for example, Williams [16] and more recently Grossman [6], Danthine [5] and Bray [3].

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formal theory that explains how speculators might at reasonable cost carry such price risks and what the welfare effects might be (if any) from the introduction of a futures market. In a lucid article on the role of futures markets Keynes [8] argued that speculators spread their risks over many markets and focus their attention on the average return obtained on these markets. A similar argument was made by Knight [9, Chap. 8] in his well-known chapter entitled "Structures and Methods for Meeting Uncertainty."

This paper presents a theoretical framework in which this risk transfer role of futures markets can be analysed. We formalise the idea that speculators are specialists in the activity of risk bearing and that, unlike producers who face price risks individually or in small numbers, they face risks in large numbers and are enabled thereby to exploit the special technology for handling risks embodied in the principle of diversification and, in the extreme case, the law of large numbers. A futures market thus induces a kind of division of labor in which speculators specialise in the activity of risk bearing while firms concentrate on the activity of production.

The basic framework for the analysis is a partial equilibrium model of an industry with a finite number of firms. After introducing some preliminary concepts in Section 2, we consider in Section 3 the equilibrium that arises when the sole mechanism for selling a firm's output consists of a *spot market.* As in the paper of Danthine [5] we make the simplifying assumption that price fluctuations on the spot market arise solely from demand side fluctuations—only with this assumption have we been able to establish the qualitative properties of the spot–futures market equilibrium in Sections 4–6. With this assumption, as first shown by Danthine [5], the production and futures trading decisions of firms are separable (Lemma 2 in Section 4)—an important simplifying property which ceases to hold in general when the output of firms is random.

In Section 3, under the assumption that firms are run by risk averse managers who maximise the expected utility of their profit, we establish the existence of a spot market equilibrium and show that the associated output $\hat{x}$ is less than a certain critical output level x^* which would arise if firms could perfectly insure their price risks at an actuarily fair price.

In Section 4 we introduce a futures market and the collection of agents called speculators. The latter are viewed as agents (or firms) with a substantial pool of investable capital spread over a broad array of markets, whose concern is to maximise the expected utility of the average return obtained on these markets. Like the classical marine insurance underwriters, they balance many disparate risks against one another to obtain a fairly predictable average return. Producers, in addition to determining the output to be sold on the spot market, can now also decide the extent to which this output is to be hedged by the sale of futures contracts. We show that a joint equilibrium of the spot and futures markets exists (Proposition 4) and that in the normal case where producers are short hedgers the introduction of a futures market leads to an increase in the equilibrium output (Proposition 5) or equivalently to a decrease in the equilibrium risk premium, the equilibrium cost of risk bearing. This situation arises in particular when the average profits (A_n) earned by speculators on the other markets on which they trade are independent of or negatively dependent on the spot market equilibrium price (Proposition 6).

Section 4 concludes with an instructive example of a joint spot–futures market equilibrium which contains the example of a spot market equilibrium given in Section 3 as a special case. The von Neumann–Morgenstern utility functions of producers and speculators exhibit constant absolute risk aversion and the underlying random variables are joint normally distributed. We show that the condition which determines equilibrium output reduces to an expression for the equilibrium risk premium (Eq. (11)). This expression consists of two terms. The first term measures the premium required by producers and speculators to cover the risks arising from the variability of the spot price itself (the *idiosyncratic risk*). The second term measures the premium required by speculators to cover the other risk involved in entering the futures market, namely, that arising from the covariance of the spot price (ϕ) with the average return (A_n) on their portfolio of investments on the other $n-1$ markets on which they trade (the *covariance risk*). We show that the idiosyncratic risk premium is driven to zero as speculators diversify their portfolios over a broader array of markets ($n \to \infty$). Equilibrium output is thus asymptotically determined by the covariance risk premium.

This result is generalised to the case of an arbitrary joint distribution for the underlying random variables in Section 5. To make this generalisation we make use of an important concept of *stochastic dependence* (Definition 3) first introduced by Lehmann [10], which generalises the concept of *independence* for a pair of random variables to a concept of positive or negative dependence. In Proposition 7 we show that if the sequence of average returns A_n converges in probability to an asymptotic average return A^*, then the equilibrium risk premium converges to an asymptotic risk premium δ^* which depends solely on the risk arising from the stochastic dependence between the spot price ϕ and the asymptotic average return A^*. The idiosyncratic risk is thus asymptotically diversified away. In particular when (ϕ, A^*) are independent, δ^* is zero and the limiting output $\bar{x}$ coincides with x^*.

A close connection thus emerges between the results of Section 5 and the theory of asset pricing developed by Sharpe [15], Ross [12] and others. For when the portfolio behaviour of speculators on a broad array of markets is explicitly introduced, the risk associated with the variability of the spot price itself becomes *diversifiable risk*. The *nondiversifiable risk* arises from the stochastic dependence between ϕ and A^*. In view of the asymptotic nature of the argument, our approach is more directly related to the *arbitrage pricing theory* of Ross, where the common random factor (index) plays a role similar to that of the average return A_n in our model.

Section 6 presents a brief analysis of the welfare effect of a futures market. Proposition 8 shows that if the equilibrium cost of risk bearing is sufficiently reduced then the introduction of a futures market leads to a Pareto improvement—the welfare of producers, consumers and speculators being increased. Since this result will not in general cover the asymptotic independent case of Proposition 7 where the futures market provides complete actuarily fair price insurance, in the last remark we show how this case can be covered using Kaldor's compensation principle.

2. Preliminary Concepts

Consider an industry composed of $m > 1$ identical firms each producing y units of output of a single perishable commodity by incurring a known cost $c(y)$. Each firm makes its production decision in the spring; output appears in the fall, at which time the whole supply is offered for sale to consumers.[2] However, in the spring firms are uncertain what consumer demand will be in the fall. To formalise the idea that consumer demand is random we introduce the following standard framework for the analysis of uncertainty.

Let $(\Omega, \mathscr{F}, \mathscr{P})$ denote a probability space. Ω is the set of states of nature, $\mathscr{F}$ is a σ-field of subsets of Ω and $\mathscr{P}$ is a probability measure on $\mathscr{F}$. Let $\phi(x, \omega)$ denote the *demand price* that consumers will pay for the total amount $x \geqslant 0$ when the state of nature is $\omega \in \Omega$. The cost and demand functions $c(\cdot)$ and $\phi(\cdot)$ are assumed to have the following properties.

Assumption 1 (Cost Function). $c: R^+ \to R^+$ *satisfies* (i) $c \in \mathscr{C}^1(R^+)$, (ii) $c(0) = 0$, $0 \leqslant c'(0) \leqslant \underline{\alpha} < \infty$, $c'(\cdot)$ *is strictly increasing on* R^+, (iii) $c'(y) \to \infty$ *as* $y \to \infty$.

Assumption 2 (Demand Function). $\phi: R^+ \times \Omega \to R^+$ *satisfies* (i) $\phi(\cdot, \omega)$ *is continuous on* R^+, $\forall\, \omega \in \Omega$, $\phi(x, \cdot)$ *is measurable on* Ω, $\forall\, x \in R^+$, (ii) $\phi(\cdot, \omega)$ *is strictly decreasing on* $R^+, \forall\, \omega \in \Omega$, (iii) $\underline{\alpha} < \phi(0, \omega) < \bar{\alpha} < \infty$,

[2] Since the good is perishable the current supply is determined solely by this period's output—there is no carryover from the previous period or into a subsequent period (see Magill and Benhabib [11]).

$\forall\, \omega \in \Omega$, (iv) $\mathscr{P}(\omega \in \Omega \,|\, \phi(x, \omega) \neq \bar{\phi}_x) > 0$, $\forall\, x \in R^+$ *where* $\bar{\phi}_x = \int_\Omega \phi(x, \omega)\, d\mathscr{P}(\omega)$.

Assumption 1 asserts that there are no fixed costs, and marginal cost is positive, increasing and grows without bound as output is indefinitely increased.[3] Assumption 2 asserts that the demand price is positive, bounded above and strictly decreasing. For each output x the demand price $\phi_x = (\phi(x, \omega), \omega \in \Omega)$ is a nontrivial random variable so that the variance of the random variable ϕ_x is positive.

A particular level of output plays a special role in the analysis that follows—this output level serves to reveal some important qualitative properties of both the spot market equilibrium output (Section 3) and the spot–futures market equilibrium output (Sections 4–6). This is the output level x^* which maximises the function

$$S(x) = \int_\Omega \Phi(x, \omega)\, d\mathscr{P}(\omega) - mc\left(\frac{x}{m}\right), \qquad x \in R^+ \tag{1}$$

where $\Phi(x, \omega) = \int_0^x \phi(z, \omega)\, dz$, $x \in R^+$, $\omega \in \Omega$. That x^* exists and is unique follows at once from the fact that $S(x)$ is strictly concave and differentiable and satisfies $S'(0) > 0$, $S'(a) < 0$ where $c'(a/m) = \bar{\alpha}$. Clearly $S'(x^*) = 0$. In Sections 3–6 we shall find that the function $S'(x)$ has the important property of exhibiting the relation between the equilibrium risk premium and the equilibrium level of output in either a spot market or a spot–futures market equilibrium.

If a mechanism could be found which provides costless price insurance for the firms, so that all their output could be sold with certainty at the expected price, and if consumers were to value the benefits of trade on this market by expected consumer surplus (see Section 6), then a natural welfare property would be satisfied by x^*, for $S(x)$ is the sum of expected consumer and producer surplus (profit). We shall find that a futures market provides such a mechanism under certain idealised conditions. The first-order condition for the maximum $S'(x^*) = E\phi(x^*, \omega) - c'(x^*/m) = 0$ is related to an important empirical property of futures prices, namely, that the futures price is (approximately) an unbiased estimate of the subsequent spot price. It was an attempt to explain how this property might be satisfied that led to the analysis of Sections 4–6. For the earlier theory of Keynes [8] and the subsequent extension by Danthine [5] predict a downward bias (normal backwardation) in the futures price—the premium

[3] The differentiability assumption 1(i) plays an important role in establishing the qualitative properties of equilibrium: it is convenient but not essential for establishing existence properties. For simplicity of exposition we have decided to retain Assumption 1(i) throughout.

being the speculators' reward for risk bearing—and this has not been consistently observed on futures markets (Working [18]).

In the next section we examine the equilibrium output $\hat{x}$ that comes about in a spot market equilibrium when producers are risk averse and maximise the expected utility of profit. In equilibrium producers charge an equilibrium risk premium over and above their basic marginal cost of production—this forces the output $\hat{x}$ to lie below x^*. Note that in this framework if we reinterpret Keynes' normal backwardation as the risk premium received by firms, then Keynes' estimates of the extent of normal backwardation (much higher than 10% [8, p. 128]) may provide reasonably accurate estimates of the risk premium added by firms to their basic marginal cost of production when they themselves carry fully the risks of price fluctuations.

3. Spot Market Equilibrium

This section examines the equilibrium output produced by the m firms when the mechanism for selling their output consists solely of a spot market; firms are thus not permitted to make forward sales through a futures market. We show that a spot market equilibrium exists, that any equilibrium output $\hat{x} < x^*$ and give conditions for $\hat{x}$ to be unique. The framework is illustrated by a simple example.

3.1. *Existence of Spot Market Equilibrium*

Let $p(\omega)$ denote the spot market price for each unit of output in state $\omega \in \Omega$ then the profit of the firm is given by $\pi(y, \omega) = p(\omega) y - c(y)$.

Assumption 3 (Risk Aversion of Producers). *Each of the m producers in the industry has a common attitude toward profit risk summarised by a von Neumann–Morgenstern utility function* $u\colon R \to R$ *satisfying* (i) $u(\cdot) \in C^2(R)$, (ii) $u'(\pi) > 0$, $u''(\pi) < 0$, $\pi \in R$.

In the spring each firm chooses an output $y \in R^+$ so as to maximise the expected utility of its profit

$$U(y) = \int_\Omega u(p(\omega) y - c(y))\, d\mathscr{P}(\omega) \tag{2}$$

knowing only the distribution of spot prices $\mathscr{P}(\omega \in \Omega \mid p(\omega) \leqslant a)\ \forall\, a \in R^+$ that can arise in the fall. The demand price that consumers will pay for the total amount x in state $\omega \in \Omega$ is given by $\phi(x, \omega)$ and satisfies the conditions of Assumption 2.

The random variables that appear in this paper are elements of the following space (unless otherwise stated)—the space of real-valued (essentially) bounded measurable functions defined on Ω, denoted by $\mathscr{L}_\infty(\Omega)$. The nonnegative orthant is denoted by $\mathscr{L}_\infty^+(\Omega) = \{\xi \in \mathscr{L}_\infty(\Omega) \mid \xi(\omega) \geqslant 0$ a.s.$\}$.

DEFINITION 1. A *spot market equilibrium* is a pair $(\hat{y}, \hat{p}) \in R^+ \times \mathscr{L}_\infty^+(\Omega)$ such that

(i) the expected utility of profit $U(y)$ is maximised at $\hat{y}$ given $\hat{p}$, by each firm,

(ii) the spot market clears almost surely, $\hat{p}(\omega) = \phi(m\hat{y}, \omega)$ a.s.

This is a *rational expectations* equilibrium since firms are assumed to correctly anticipate $\hat{p}$ in the sense of knowing its probability distribution. Note that by (ii) and Assumption 2(iv), $\hat{p}$ is a nontrivial random variable.

PROPOSITION 1. *Under Assumptions* 1–3 *there exists a spot market equilibrium.*

Proof. It is easy to show that for any $p \in \mathscr{L}_\infty^+(\Omega)$ the function $U(y)\colon R^+ \to R$ defined by (2) is differentiable. Since $U(\cdot)$ is a concave function, an output $\hat{y} > 0$ satisfying $U'(\hat{y}) = 0$ maximises the expected utility of profit. It thus suffices to find a pair $(\hat{x}, \hat{p})$ such that

$$U'\left(\frac{\hat{x}}{m}\right) = 0, \qquad \hat{p}(\omega) = \phi(\hat{x}, \omega) \qquad \text{a.s.} \tag{3}$$

Let $f(x, \omega) = u'(\phi(x, \omega)(x/m) - c(x/m))(\phi(x, \omega) - c'(x/m)) \quad \forall\, (x, \omega) \in R^+ \times \Omega$ and let $F(x) = \int_\Omega f(x, \omega)\, d\mathscr{P}(\omega)$, then in view of (3) it suffices to find $\hat{x} > 0$ such that

$$F(\hat{x}) = 0. \tag{4}$$

It follows from Assumptions 1–3 that $f(0, \omega) > 0$ a.s. and $f(a, \omega) < 0$ a.s. where $a > 0$ is defined by $c'(a/m) = \bar{\alpha}$. Thus $F(0) > 0$, $F(a) < 0$. Since $f(x, \omega)$ is continuous in x a.s. $\forall\, x \in [0, a]$ and since there exists $\beta > 0$ such that $|f(x, \omega)| \leqslant \beta\ \forall\, x \in [0, a]$ a.s., for any sequence $x_n \in [0, a]$ such that $x_n \to x$, by the bounded convergence theorem [13, p. 229]

$$\lim_{n \to \infty} \int_\Omega f(x_n, \omega)\, d\mathscr{P}(\omega) = \int_\Omega \lim_{n \to \infty} f(x_n, \omega)\, d\mathscr{P}(\omega) = \int_\Omega f(x, \omega)\, d\mathscr{P}(\omega)$$

so that $F(x_n) \to F(x)$. Thus F is continuous on $[0, a]$ and there exists $x \in (0, a)$ such that $F(\hat{x}) = 0$. ∎

3.2. *Properties of Spot Market Equilibrium*

PROPOSITION 2. *If Assumptions* 1–3 *are satisfied then every spot market equilibrium output* $\hat{x} < x^*$.

Proof. Let $\Pi(x, \omega) = \phi(x, \omega)(x/m) - c(x/m)$ and let Π_x and ϕ_x denote the random variables $\Pi(x, \cdot)$ and $\phi(x, \cdot)$, $x \in R^+$. Then the condition (4) is equivalent to the condition

$$\left(\bar{\phi}_{\hat{x}} - c'\left(\frac{\hat{x}}{m}\right)\right) \int_\Omega u'(\Pi_{\hat{x}}(\omega))\, d\mathscr{P}(\omega) + \int_\Omega u'(\Pi_{\hat{x}}(\omega))(\phi_{\hat{x}}(\omega) - \bar{\phi}_{\hat{x}})\, d\mathscr{P}(\omega) = 0$$

which in view of (1) is equivalent to

$$S'(\hat{x}) = \delta(\hat{x}) \tag{5}$$

where

$$\delta(x) = -\frac{\operatorname{cov}(\phi_x, u'(\Pi_x))}{E(u'(\Pi_x))}, \qquad x \geqslant 0. \tag{6}$$

Since for $\hat{x}/m > 0$ each firm's profit is maximised if and only if $U'(\hat{x}/m) = 0$, $\hat{x}$ is a spot market equilibrium output if and only if (5) is satisfied. Since for fixed $x > 0$, $u'(\Pi_x)$ is a decreasing function of the random variable ϕ_x, by an elementary result[4] $\operatorname{cov}(\phi_x, u'(\Pi_x)) < 0$ for $x > 0$. Since $u'(\pi) > 0$, $\pi \in R$, $\delta(x) > 0$, $x > 0$. It is easy to show that $S'(\cdot)$ is continuous and strictly decreasing on R^+. Thus since $S'(x^*) = 0$ by (5) $\hat{x} < x^*$. ∎

Proposition 2 is an extension to the case of equilibrium output of the result of Baron [1, p. 467] and Sandmo [14, p. 66] which asserts (in its general form) that for a *fixed* distribution of prices, the profit maximising output of a firm is a decreasing function of the degree of its absolute risk aversion.

Let $\Delta(y)$ denote the *risk premium* that a firm is prepared to pay to avoid the risk of random profit and to obtain instead the expected value of profit with certainty, then by definition

$$u(E(\pi(y)) - \Delta(y)) = Eu(\pi(y)), \qquad y > 0$$

[4] PROPOSITION. *Let* ϕ, $g(\phi) \in \mathscr{L}_2(\Omega)$. *If* $\mathscr{P}(\omega \in \Omega \mid \phi(\omega) \neq \bar{\phi}) > 0$ *and* $g(\cdot)$ *is strictly decreasing, then* $\operatorname{cov}(\phi, g(\phi)) < 0$.

Proof. Let $A = \{w \in \Omega \mid \phi(\omega) = \bar{\phi}\}$, then $\mathscr{P}(A^c) > 0$. Since $g(\cdot)$ is strictly decreasing, $H(\omega) = (\phi(\omega) - \bar{\phi})(g(\phi(\omega)) - g(\bar{\phi})) < 0 \ \forall \ \omega \in A^c$ and $H(\omega) = 0 \ \forall \ \omega \in A$. Since $\mathscr{P}(A^c) > 0$, $0 > \int_\Omega H(\omega)\, d\mathscr{P}(\omega) = E(\phi g(\phi)) - E(\phi)\, E(g(\phi)) - E[(\phi - \bar{\phi})\, g(\bar{\phi})] = \operatorname{cov}(\phi, g(\phi))$.

where $\pi(y)$ denotes the random variable $py - c(y)$. By equating the first-order conditions for maximising the left and right sides of this expression we readily deduce that

$$\Delta'(y) = -\frac{\operatorname{cov}(p, u'(\pi(y)))}{E(u'(\pi(y)))}.$$

Thus $\delta(x)$ represents the marginal risk premium when we substitute for p the random variable ϕ_x. At the equilibrium output $\hat{x}$, $\delta(\hat{x})$ is thus the *equilibrium risk premium*, the amount charged to consumers on average in excess of the basic marginal cost of production $c'(\hat{x}/m)$ and the basic return that producers receive for incurring the risks induced by spot price variability.

Remark. The spot market equilibrium need not be unique. If the risk premium function $\delta(x)$ is decreasing for some range of output x, then there can be multiple spot market equilibria. If $\delta(x)$ is a nondecreasing function then the spot market equilibrium is unique. A related result is the following.

PROPOSITION 3. *Let Assumptions* 1–3 *hold. If* $R(\pi) = -(u''(\pi)/u'(\pi))\pi < 1$ *for* $\pi > 0$, *then the spot market equilibrium is unique.*

Proof. By a result of Cheng *et al.* [4] the firm's profit maximising output $y(p)$ is an increasing function of p, in the sense that $\tilde{p}(\omega) > p(\omega)$ a.s. implies $y(\tilde{p}) > y(p)$. Let $\hat{x}$ denote an equilibrium output and let $p_{\hat{x}}$ denote the associated price. Suppose $(x', p_{x'})$ is a second equilibrium with $x' < \hat{x}$. Then $p_{x'}(\omega) > p_{\hat{x}}(\omega)$ a.s. by Assumption 2(ii). But then $x' > \hat{x}$, contradicting $x' < \hat{x}$. Conversely if $x' > \hat{x}$.

EXAMPLE. Let $u(\pi) = -e^{-\alpha\pi}$, $\alpha > 0$, and let $\phi(x, \omega) = \mu(x) + \lambda(\omega)$ where μ is strictly decreasing, $0 \leqslant c'(0) < \mu(0) < \infty$ and λ is normally distributed with zero mean and variance $\sigma^2 > 0$. This is the simplest case of an additive random disturbance. On any interval in R^+ the probability of negative or unbounded positive prices can be made arbitrarily small[5] by suitable choice of the pair $(\mu(\cdot), \sigma)$. The risk premium function is increasing

$$\delta(x) = \alpha\left(\frac{x}{m}\right)\sigma^2, \qquad x \in R^+. \tag{7}$$

The unique equilibrium output $\hat{x} > 0$ is defined by $\mu(\hat{x}) - c'(\hat{x}/m) = \alpha(\hat{x}/m)\,\sigma^2$ and the output x^* by $\mu(x^*) - c'(x^*/m) = 0$.

[5] Let $\sigma \leqslant b\mu(x)$, $0 < b < 1$, $\forall x \in I$ (the interval under consideration). If $I = R^+$ let $0 < \underline{\mu} < \mu(x)\ \forall x \in R^+$. If $b = \frac{1}{5}$, the probability of negative prices is less than 10^{-6}.

4. Spot–Futures Market Equilibrium

The object of this and the following section is to examine the impact of the introduction of an organised futures exchange on the equilibrium output of the industry. In addition to selling their output directly on the spot market (as in Section 3), firms are given the option to trade in futures contracts. We introduce a new class of agents, called speculators, who, following the idea of Keynes and Knight, are viewed as specialists in the activity of risk bearing. Unlike the producers, whose specialty is the production of the commodity under consideration, the speculators deal extensively on this and other futures markets.

4.1. *Existence of Spot–Futures Market Equilibrium*

The producers and speculators are the sole agents that trade on the organised futures exchange and trading takes place under the following conditions. The market is open at two points in time, in the spring and in the fall. At both times the same futures contract is traded, each contract calling for the delivery of one unit of the commodity in the fall. The contracts can be purchased and sold costlessly and in perfectly divisible amounts. Whatever position a trader takes in the spring is automatically reversed in the fall. Thus each producer sells z futures contracts in the spring at the (spring) futures price q and buys back z contracts in the fall at the (fall) futures price, which by arbitrage coincides with the spot price $p(\omega)$.[6]

The profit π of a producer now comes from two sources: first, the sale of the commodity on the spot market, and second, the profit on the transaction in the futures market. Thus

$$\pi(y, z, \omega) = p(\omega)\, y - c(y) + z(q - p(\omega)), \qquad (y, z) \in R^+ \times R.$$

Similarly the profit π of a speculator comes from two sources: first, the profit on the transactions in this futures market $\pi_1(\xi, \omega) = \xi(p(\omega) - q)$, and second, the profits from transactions on $n-1$ other markets π_i, $i = 2, ..., n$, which are taken as *exogenously given*

$$\pi(\xi, \omega) = \xi(p(\omega) - q) + \pi_2(\omega) + \cdots + \pi_n(\omega), \qquad \xi \in R.$$

The exogenous profits are random variables satisfying

[6] In practice there is often a small random difference (known as *basis*) between the fall futures price and the (fall) spot price. When the analysis is extended to include this basis risk explicitly, the basic simplifying step in the analysis, Lemma 2, is no longer valid. We thus confine the analysis to the idealised case of zero basis risk.

$\pi_i \in \mathscr{L}_1(\Omega) = \{\eta \mid \int_\Omega |\eta(\omega)| \, d\mathscr{P}(\omega) < \infty\}$ and $-\infty < \gamma < \pi_i(\omega)$ a.s., $i = 1, \ldots, n$. The output and hedging decision of a producer thus reduces to

$$\sup_{(y,z) \in R^+ \times R} U(y, z) = \sup_{(y,z) \in R^+ \times R} \int_\Omega u(\pi(y, z, \omega)) \, d\mathscr{P}(\omega). \qquad (\mathscr{U})$$

Speculators are concerned with the average profit $\Pi = \pi/n$ that they earn on the n markets. In addition they are risk averse.

ASSUMPTION 4. (Risk Aversion of Speculators). *Each of the s speculators has a common attitude toward average profit risk summarised by a utility function* $w: R \to R$ *satisfying* (i) $w(\cdot) \in C^2(R)$, (ii) $w'(\Pi) > 0$, $w''(\Pi) < 0$, $\Pi = \pi/n \in R$.

With this assumption the trading decision of a speculator reduces to

$$\sup_{\xi \in R} W(\xi) = \sup_{\xi \in R} \int_\Omega w(\Pi(\xi, \omega)) \, d\mathscr{P}(\omega), \qquad \Pi = \frac{\pi}{n} \qquad (\mathscr{W})$$

where the integral is well-defined since $\Pi(\xi, \omega) \in \mathscr{L}_1(\Omega)$ and $b \leqslant \Pi(\xi, \omega)$ a.s. for some constant $b > -\infty$.

The nature of the equilibria that emerge on the joint system of spot and futures markets depends on the nature of the *stochastic dependence* between the random spot price p and the random average profit $A_n = (1/n) \sum_{i=2}^{n} \pi_i$ earned by the speculators on the $n-1$ other markets on which they trade. Without entering into the qualifying conditions that need to be made, if there is sufficient *positive dependence*[7] between the random variables p and A_n, then the equilibrium sale of futures contracts by producers may become negative ($\bar{z} \leqslant 0$): speculators will in essence be entering the futures market to shift their risks to producers. We will call such an equilibrium an *improper* equilibrium. Thus in a proper equilibrium $\bar{z} > 0$ and producers shift part of their risks to speculators. If there is sufficient *negative dependence* between p and A_n, then the equilibrium sale of futures contracts may exceed output ($\bar{z} > \bar{y}$). We will call such an equilibrium an *overhedging* equilibrium, referring to the case where $\bar{z} \leqslant \bar{y}$ as a *normal hedging* equilibrium.

A particularly simple and direct proof of the existence of a spot–futures market equilibrium can be obtained if we restrict the futures trading of producers so that $0 \leqslant z \leqslant y$. It is clear that with this restriction an equilibrium that would otherwise appear as an improper equilibrium will appear as an equilibrium with $\bar{z} = 0$, while an overhedging equilibrium will appear as an equilibrium with $\bar{z} = \bar{y}$. Since this restriction does not alter the

[7] See Definition 3 below.

qualitative properties of the equilibrium that we seek to obtain, we may without loss of generality adopt it in the analysis that follows. The producer's problem $(\mathscr{U})$ thus reduces to

$$\sup_{(y,z)\in R^{+}\times[0,y]} U(y,z). \qquad (\mathscr{U}')$$

Following the standard procedure in equilibrium theory we impose a temporary bound (Δ) on the trading of each speculator. This leads to the problem

$$\sup_{\xi\in[-\Delta,\Delta]} W(\xi). \qquad (\mathscr{W}')$$

DEFINITION 2. A *spot-futures market equilibrium* (with speculators trading on $n-1$ other markets) is a pair

$$\langle(\bar{y}_n,\bar{z}_n,\bar{\xi}_n),(\bar{p}_n,\bar{q}_n)\rangle\in R^3_+\times\mathscr{L}^+_\infty(\Omega)\times R_+ \qquad (8)$$

such that

(i) $(\bar{y}_n,\bar{z}_n)$ solves $(\mathscr{U}')$ with $(p,q)=(\bar{p}_n,\bar{q}_n)$,

(ii) $\bar{\xi}_n$ solves $(\mathscr{W}')$ with $(p,q)=(\bar{p}_n,\bar{q}_n)$,

(iii) the spot market clears almost surely, $\bar{p}_n(\omega)=\phi(m\bar{y}_n,\omega)$ a.s.,

(iv) the futures market clears in the spring and in the fall, $m\bar{z}_n=s\bar{\xi}_n$.

As in the definition of a spot market equilibrium, the expectations of agents are rational. Similarly (iii) and Assumption 2(iv) imply $\bar{p}_n$ is a nontrivial random variable. Lemmas 1–3, the proofs of which are given in the Appendix, establish the basic properties of the solutions of $(\mathscr{U}')$ and $(\mathscr{W}')$ which are needed to establish the existence of an equilibrium (Proposition 4).

LEMMA 1. *Under Assumptions* 1 *and* 3, *if* $(p,q)\in\mathscr{L}^+_\infty(\Omega)\times R_+$, *then there exists a unique solution of* $(\mathscr{U}')$.

LEMMA 2. *Let* $(y^*(p,q), z^*(p,q))$ *denote the optimal solution in Lemma* 1. *If* $z^*(p,q)>0$, *then* $y^*(p,q)=c'^{-1}(q)$.

Lemma 2 asserts that when a firm trades on the futures market, its output y^* is no longer influenced (as in the previous section) by the distribution of the spot price p, but *depends solely on the (spring) futures price* q. This result was first obtained by Danthine [5, p. 82]. Since the futures trading of producers is still influenced by their attitude toward risk, it is clear that in equilibrium the futures price and hence their production decision is indirectly influenced by their attitude toward risk. This property does, however, as we show in Section 5, when taken in conjunction with extensive diversification on the part of speculators, lead to an equilibrium which is independent of the attitude toward risk of producers.

Remark 1. In view of Lemma 2, if $(p, \bar{q}) \in \mathscr{L}_{\infty}^{+}(\Omega) \times R_{+}$ with $\bar{q} = c'(\bar{y})$ for some $\bar{y} > 0$, and if the solution $\bar{z}$ of

$$\sup_{z \in [0,y]} \int_{\Omega} u(p(\omega)\,\bar{y} - c(\bar{y}) + z(\bar{q} - p(\omega)))\,d\mathscr{P}(\omega)$$

satisfies $\bar{z} > 0$, then $(y^*(p, \bar{q}), z^*(p, \bar{q})) = (\bar{y}, \bar{z})$.

LEMMA 3. *Under Assumption* 4, *if* $(p, q) \in \mathscr{L}_{\infty}^{+}(\Omega) \times R_{+}$, *then there exists a unique solution of* $(\mathscr{W}')$.

We are now in a position to show that there is at least one well-defined equilibrium. Note that the equilibrium depends on the number of markets on which speculators trade—a property that is shown to have important consequences in Section 5.

PROPOSITION 4 (Spot–Futures Market Equilibrium). *If Assumptions* 1–4 *hold and speculators trade on* $n \geqslant 1$ *markets, then there exists a spot–futures market equilibrium with* $0 < \bar{x}_n < a$.

Proof. Let $(y^*(p, q), z^*(p, q))$ and $\xi^*(p, q)$ denote the solutions of $(\mathscr{U}')$ and $(\mathscr{W}')$ whose existence is asserted by Lemmas 1 and 3. We let $b = a/m$ be defined by $c'(b) = \bar{\alpha}$ and consider the following one-parameter family of prices

$$(p_y(\omega), q(y)) = (\phi(my, \omega), c'(y)) \qquad \text{a.s.,} \quad y \in [0, b].$$

The idea is to find an output $\bar{y} \in [0, b]$ such that

$$mz^*(p_{\bar{y}}, q(\bar{y})) - k\xi^*(p_{\bar{y}}, q(\bar{y})) = 0.$$

For fixed $y \in [0, b]$ let $z(y)$ denote the solution of

$$\sup_{z \in [0,y]} \int_{\Omega} u(p_y(\omega)\,y - c(y) + z(q(y) - p_y(\omega)))\,d\mathscr{P}(\omega). \tag{9}$$

By Remark 1 if $y > 0$ and $z(y) > 0$, then $(y^*(p_y, q(y)), z^*(p_y, q(y))) = (y, z(y))$. By the continuity of $u(\cdot)$, $\phi(\cdot, \omega)$, and $c'(\cdot)$ and by the bounded convergence theorem, the integral in (9) is a continuous function of y on $[0, b]$. By the maximum theorem [2, p. 116], $z\colon [0, b] \to R^{+}$ is a continuous function. Since by Assumptions 1 and 2

$p_0(\omega) > q(0)$ a.s. and $p_b(\omega) < q(b)$ a.s., it follows that $z(0) = 0$, $z(b) = b$. Let $\xi(y)$ denote the solution of

$$\sup_{\xi \in [-\Delta, \Delta]} \int_\Omega w\left(\frac{\xi}{n}(p_y(\omega) - q(y)) + A_n(\omega)\right) d\mathscr{P}(\omega) \tag{10}$$

where $A_n = (1/n) \sum_{i=2}^n \pi_i$. By the continuity of $w(\cdot)$, $\phi(\cdot, \omega)$, and $c'(\cdot)$ and by the dominated convergence theorem, the integral in (10) is a continuous function of y on $[0, b]$. By the maximum theorem, ξ: $[0, b] \to [-\Delta, \Delta]$ is a continuous function. It is clear that $\xi(0) = \Delta$, $\xi(b) = -\Delta$. But then

$$mz(0) - s\xi(0) = -s\Delta < 0, \qquad mz(b) - s\xi(b) = mb + s\Delta > 0.$$

By the intermediate value theorem there exists $\bar{y} \in (0, b)$ such that $mz(\bar{y}) - s\xi(\bar{y}) = 0$. Choose $\Delta \geqslant a/s$, then

$$-s\Delta < 0 \leqslant s\xi(\bar{y}) = mz(\bar{y}) \leqslant m\bar{y} < a \leqslant s\Delta$$

implies $-\Delta < \xi(\bar{y}) < \Delta$ so that $\xi(\bar{y})$ is a solution of the unconstrained problem in (10), and hence is a solution of ($\mathscr{W}$). Thus

$$\langle(\bar{y}_n, \bar{z}_n, \bar{\xi}_n), (\bar{p}_n, \bar{q}_n)\rangle = \langle(\bar{y}, z(\bar{y}), \xi(\bar{y})), (p_{\bar{y}}, q(\bar{y}))\rangle$$

satisfies (8) and (i)–(iv) in Definition 2 and hence is an equilibrium. The case where $z(\bar{y}) = 0$ is readily shown to be an equilibrium. ∎

4.2. *Properties of Spot–Futures Market Equilibrium*

A number of simple properties of an equilibrium can now be established. One would expect that if there is positive trading, since part of the price risks will be carried by speculators, output should increase relative to the spot market equilibrium level.

PROPOSITION 5. *Let the spot market equilibrium output $\hat{x}$ be unique. If $\bar{x}_n$ is the output in a proper spot–futures market equilibrium then $\hat{x} < \bar{x}_n$.*

Proof. Consider the problem of maximising

$$U(z) = \int_\Omega u\left(p_n(\omega)\frac{\bar{x}_n}{m} - c\left(\frac{\bar{x}_n}{m}\right) + z(\bar{q}_n - \bar{p}_n(\omega))\right) d\mathscr{P}(\omega)$$

subject to $0 \leqslant z \leqslant \bar{x}_n/m$. Since $\bar{z}_n > 0$, $U'(0) > 0$. Thus the function $F(\cdot)$ used to define the spot market equilibrium in (4) satisfies $F(\bar{x}_n) = -U'(0) < 0$. Since $F(\cdot)$ is continuous on $[0, a]$, $F(0) > 0$ and $\hat{x}$ is the unique solution of $F(\hat{x}) = 0$, it follows that $\hat{x} < \bar{x}_n$. ∎

We have already asserted that the stochastic dependence between the spot price ϕ and the average profits A_n earned by speculators on the $(n-1)$ other markets on which they trade has an important influence on the properties of an equilibrium. The following definition makes precise this concept, which is a natural extension of the concept of stochastic independence for a pair of random variables (see Lehmann [10]).

DEFINITION 3. A pair of random variables $\phi, \psi: \Omega \to R$ is said to be *positively (negatively) dependent* if for all $(\alpha, \beta) \in R^2$

$$\mathscr{P}(\omega \in \Omega | \phi(\omega) \leqslant \alpha, \psi(\omega) \leqslant \beta) \geqslant (\leqslant) \mathscr{P}(\omega \in \Omega | \phi(\omega) \leqslant \alpha) \mathscr{P}(\omega \in \Omega | \psi(\omega) \leqslant \beta)$$

with strict inequality for some $(\alpha, \beta) \in R^2$. We write (ϕ, ψ) are p.d. (n.d.), respectively.

Remark 2. It is readily shown that if (ϕ, ψ) are p.d. (n.d.) then $(\phi, f(\psi))$ are n.d. (p.d.) if $f(\cdot)$ is a decreasing function. Also if (ϕ, ψ) are p.d. (n.d.), then $\text{cov}(\phi, \psi) > 0$ (<0). When (ϕ, ψ) are joint normally distributed (see the example below) then (ϕ, ψ) are p.d. (n.d.) if and only if $\text{cov}(\phi, \psi) > 0$ (<0).

If (ϕ, A_n) are negatively dependent then one would expect speculators to be willing to trade on the long side of the market since they can reduce the variability of their portfolios by buying contracts on the futures market.

PROPOSITION 6. *If (ϕ, A_n) are negatively dependent or independent, then a spot–futures market equilibrium is a proper equilibrium.*

Proof. Let $r = \bar{p}_n - \bar{q}_n$ and consider the function $W(\xi)$ defined by $(\mathscr{W})$. First suppose $E(r) > 0$. Then $W'(0) = E(w'(A_n) r) \geqslant E(w'(A_n)) E(r) > 0$ since $(r, w'(A_n))$ are p.d. or independent. Since $W'(\bar{\xi}_n) = 0$ and since $w''(\cdot) < 0$ implies $W'(\cdot)$ is strictly decreasing, $\bar{\xi}_n > 0$. Now suppose $E(r) \leqslant 0$ and consider the function $U(y, z)$ defined by $(\mathscr{U})$. Then $U_z(\bar{y}_n, \bar{y}_n) = -u'(\bar{y}_n \bar{q}_n - c(\bar{y}_n)) E(r) \geqslant 0$. Since $u''(\cdot) < 0$ implies $U_z(\bar{y}_n, \cdot)$ is strictly decreasing, $\bar{z}_n \geqslant \bar{y}_n$ if it were not constrained to satisfy $\bar{z}_n \leqslant \bar{y}_n$, implying $\bar{z}_n = \bar{y}_n = \bar{x}_n / m > 0$. Thus in either case $\bar{m} z_n = s \bar{\xi}_n > 0$. ∎

It is easy to see that if (ϕ, A_n) are independent then $\bar{x}_n < x^*$ and $\bar{x}_n$ is a normal hedging equilibrium, since $\bar{z}_n > 0$ implies $\bar{q}_n < E(\bar{p}_n)$. The example introduced in Section 3.2 can be extended to the spot–futures market equilibrium of this section and can be used to illustrate further properties of an equilibrium.

EXAMPLE. Let $u(\pi) = -e^{-\alpha\pi}$, $\alpha > 0$, $w(\Pi) = -e^{-\beta\Pi}$, $\beta > 0$, $\phi(x, \omega) = \mu(x) + \lambda(\omega)$ where $(\mu(\cdot), \lambda)$ satisfy the conditions given in Section 3.2 and let $(\lambda, \pi_2, \ldots, \pi_n)$ be joint normally distributed random variables. Let

$A_n = (1/n)\sum_{i=2}^{n} \pi_i$, $V(A_n) = \sigma^2_{A_n}$, $\text{cov}(\phi, A_n) = \sigma_{\phi A_n} = \text{cov}(\lambda, A_n) = \sigma_{\lambda A_n}$, $V(\lambda) = \sigma^2$. Let $(q, E(p), \sigma_p^2)$ denote the futures price and the mean and variance of the spot price, then the output and supply of futures contracts of each producer satisfy

$$c'(y) = q, \qquad z = y + \frac{q - E(p)}{\alpha\sigma_p^2}$$

while each speculator's demand for futures contracts is given by

$$\xi = \left(\frac{n}{\beta\sigma_p^2}\right)(E(p) - q) - \frac{n\sigma_{pA_n}}{\sigma_p^2}$$

where σ_{pA_n} is the covariance between the spot price and A_n. Equilibrium on the spot market implies $E(p) = \mu(x)$, $\sigma_p = \sigma$ where $x = my$. Equilibrium on the futures market requires $mz = s\xi$—if we use the function S defined in (1) so that $S'(x) = \mu(x) - c'(x/m)$—this condition reduces to a condition defining the equilibrium output $\bar{x}_n$

$$S'(\bar{x}_n) = \underline{\delta}(x_n) + \bar{\delta} \tag{11}$$

$$\underline{\delta}(x) = \left(\frac{\alpha}{1 + \alpha sn/\beta m}\right)\left(\frac{x}{m}\right)\sigma^2, \qquad \bar{\delta} = \left(\frac{\beta}{1 + \beta m/\alpha sn}\right)\sigma_{\phi A_n}.$$

Equation (11) is the basic equilibrium condition expressing equality between the *expected rate of return* S' and the *equilibrium risk premium* $\underline{\delta} + \bar{\delta}$. *When there are no speculators* $(s = 0)$, $\underline{\delta}(x) = \delta(x)$ defined by (7) and $\bar{\delta} = 0$ so that $\bar{x} = \hat{x}$. When speculators are present the equilibrium risk premium (and hence the *bias* in the futures price) is composed of two terms. The first term $\underline{\delta}(x)$ measures the premium required jointly by producers and speculators to cover the risks arising from the variability of the spot price itself (σ^2): this may be called the *idiosyncratic risk premium.* The second term $\bar{\delta}$ measures the premium required by speculators to cover the other risk involved in introducing futures contracts into their existing portfolio A_n, namely, the one that arises from the covariance $(\sigma_{\phi A_n})$ between ϕ and A_n: this may be called the *covariance risk premium.* If speculators hold well-diversified portfolios then the first type of risk is diversified away since

$$\underline{\delta}(x) \to 0, \qquad \bar{\delta} \to \beta\sigma_{\phi A^*} \qquad \text{as } n \to \infty$$

provided $\sigma_{\phi A_n} \to \sigma_{\phi A^*}$ as $n \to \infty$: thus *the idiosyncratic risk which is the basic risk that is present in the spot market equilibrium is diversified away in a joint spot–futures market equilibrium when* $n \to \infty$. If in addition ϕ and A_n are asymptotically independent $(\sigma_{\phi A^*} = 0)$ then the covariance risk premium vanishes so that the equilibrium risk premium $\underset{\sim}{\delta}+\bar{\delta}$ is zero. In this case $\bar{x}_n \to x^*$ as $n \to \infty$ where x^* maximises (1).

A similar result holds when the number of speculators is increased, since

$$\underset{\sim}{\delta}(x) \to 0, \qquad \bar{\delta} \to \beta\sigma_{\phi A_n} \qquad \text{as } s \to \infty.$$

In this case the idiosyncratic risk becomes subdivided among a large number of speculators and once again in the joint spot–futures market equilibrium only the nature of the covariance risk matters.[8]

5. Diversification of Risk

The example of the previous section suggests a number of interesting asymptotic properties of the joint spot–futures market equilibrium which arise from asymptotic properties of the basic equilibrium risk premium expression (11). This expression has much in common with the basic risk premium expressions that are derived in the theory of capital asset pricing under uncertainty (Sharpe [15] and Ross [12]). The affinity is closest with the *arbitrage pricing theory* of Ross where the common random factor (or index) plays a role similar to that of A_n in our model. It will be recalled that in Ross' analysis as the number of assets is increased idiosyncratic risks are diversified away and only factor-dependent covariance risks remain in the resulting asymptotic pricing relations. The object of this section is to examine briefly the consequences of such a *large market* argument for the equilibrium, and in particular the eqilibrium output, of the previous section.

5.1. *Large Market Argument*

To make the large market argument work in the present context we need to assume that the random return A_n obtained by speculators settles down in a suitable way as the number of markets on which they trade is increased. We require that A_n converges in probability to a random variable A^* which thus represents the *asymptotic average profit* that they earn on other markets in the large market economy.

[8] In the present framework $\underset{\sim}{\delta}$ is always nonnegative. We have shown elsewhere that when processors are introduced who use the commodity as an input, if they also hedge on the futures market and if ϕ is interpreted as the excess demand function of agents who do not trade on the futures market, then the sign of $\underset{\sim}{\delta}$ depends on the sign of the difference between the output of producers and the input demanded by processors. Thus the *sign of the idiosyncratic risk premium depends on the basic imbalance between short and long hedgers on the futures market.* However, even when long hedgers are introduced in this way $\underset{\sim}{\delta} \to 0$ as $n \to \infty$, so that *covariance risk is the basic determinant of the risk premium in a large market economy.*

Recall that a sequence of real-valued random variables ψ_n is said to *converge in probability* to ψ^* (written $\psi_n \to_p \psi^*$) if

$$\mathscr{P}(\omega \in \Omega \mid |\psi_n(\omega) - \psi^*(\omega)| \geqslant \varepsilon) \to 0 \qquad \text{as } n \to \infty \text{ for every } \varepsilon > 0.$$

ASSUMPTION 5 (Asymptotic Average Profit). $A_n \to_p A^*$ as $n \to \infty$.

The behavior of asymptotic equilibrium output depends on the nature of the stochastic dependence between ϕ and A^*.

PROPOSITION 7. (Asymptotic Equilibrium Output). *Let Assumptions 1–5 be satisfied. If $\bar{x}_n$ denotes the output in a spot–futures market equilibrium when speculators trade on n markets, then*

$$\bar{x}_n \to \bar{x} \qquad \text{as} \quad n \to \infty$$

where (i) $\bar{x} = x^*$ *if* (ϕ, A^*) *are independent,* (ii) $x < (>)\, x^*$ *if* (ϕ, A^*) *are positively (negatively) dependent.*

Proof. By Proposition 4, for every integer $n \geqslant 1$ there exists a spot–futures market equilibrium with output satisfying $0 < \bar{x}_n < a$, $n = 1, 2, \ldots$ The sequence $\{\bar{x}_n\}_{n=1}^{\infty}$ thus has at least one point of accumulation $\bar{x}$. By picking a subsequence we may assume $\bar{x}_n \to \bar{x}$ as $n \to \infty$. By the continuity of $\phi(\cdot, \omega)$ and $c'(\cdot)$, the sequence of spot and futures prices satisfies

$$\bar{p}_n(\omega) = \phi(\bar{x}_n, \omega) \to \phi(\bar{x}, \omega) = \bar{p}(\omega), \qquad \forall \omega \in \Omega \tag{12}$$

$$\bar{q}_n = c'\left(\frac{\bar{x}_n}{m}\right) \to c'\left(\frac{\bar{x}}{m}\right) = \bar{q}. \tag{13}$$

By virtue of condition (ii) in Definition 2, each speculator's optimal decision ξ_n must satisfy the first-order condition

$$\int_{\Omega} w'\left(\frac{\xi_n}{n}(\bar{p}_n(\omega) - \bar{q}_n) + A_n(\omega)\right)(\bar{p}_n(\omega) - \bar{q}_n)\, d\mathscr{P}(\omega) = 0.$$

Using (12), (13) and following the procedure in the proof of Proposition 2, this equation can be written as follows:

$$\phi(\bar{x}_n) - c'\left(\frac{\bar{x}_n}{m}\right) = \frac{-\operatorname{cov}(\phi_{\bar{x}_n}, w')(\xi_n/n)(\phi_{\bar{x}_n} - \bar{q}_n) + A_n))}{E(w')(\xi_n/n)(\phi_{\bar{x}_n} - q_n) + A_n))} = \delta_n(\bar{x}_n, \xi_n). \tag{14}$$

Since $(\xi_n/n)(\phi_{\bar{x}_n} - \bar{q}_n) \to_p 0$, it follows from Assumption 5 that

$$\frac{\xi_n}{n}(\phi_{\bar{x}_n} - \bar{q}_n) + A_n \underset{p}{\longrightarrow} A^* \qquad \text{as } n \to \infty. \tag{15}$$

Since by Assumption 4, $w'(\cdot)$ is a continuous decreasing function on R and since $\gamma < A_n(\omega)$ a.s., (15) is readily seen to imply that

$$w'\left(\frac{\bar{\xi}_n}{n}(\phi_{\bar{x}_n} - \bar{q}_n) + A_n\right) \xrightarrow[p]{} w'(A^*) \qquad \text{as } n \to \infty. \tag{16}$$

By (14), (16) and the bounded convergence theorem

$$\lim_{n \to \infty} \delta_n(\bar{x}_n, \bar{\xi}_n) = \frac{-\operatorname{cov}(\phi_{\bar{x}}, w'(A^*))}{E(w'(A^*))} = \delta^*(\bar{x}). \tag{17}$$

Taking limits of both sides of (14), recalling that $S'(x) = \bar{\phi}(x) - c'(x/m)$ and using (17) gives the basic asymptotic equilibrium condition

$$S'(\bar{x}) = \delta^*(\bar{x}). \tag{18}$$

(i) If (ϕ, A^*) are independent random variables, then $(\phi, w'(A^*))$ are independent random variables, so that $\operatorname{cov}(\phi, w'(A^*)) = 0$ and $\delta^*(\bar{x}) = 0$. Thus $\bar{x} = x^*$.

(ii) By Remark 2 if (ϕ, A^*) are positively (negatively) dependent, since $w'(\cdot)$ is strictly decreasing $(\phi, w'(A^*))$ are negatively (positively) dependent, implying $\operatorname{cov}(\phi, w'(A^*)) < 0$ (> 0) and $\delta^*(\bar{x}) > 0$ (< 0). Since $S'(\cdot)$ is strictly decreasing on R^+, (18) implies $\bar{x} < (>) \, x^*$. ∎

5.2. *Economic Interpretation*

An intuitive economic interpretation of Proposition 7 may be given as follows. Firms are interested in using the futures market for hedging. By virtue of their production activity, firms hold long positions on the spot market and are interested in holding short positions on the futures market. By virtue of their investment activity on the $(n-1)$ other markets on which they trade, speculators start off with an initial portfolio summarised by the average return obtained on these markets A_n. Since the firms as hedgers are on the selling side of the (spring) futures market, the speculators will need to be buyers. A speculator entering the futures market as a buyer incurs two risks. The first arises from the variability of ϕ itself and may be called the *idiosyncratic risk*. The second arises from the stochastic dependence between ϕ and A_n and may be called the *stochastic dependence risk*. *Equation* (17) *asserts that the idiosyncratic risk disappears as the speculators hold progressively more widely diversified portfolios.*

The *asymptotic equilibrium risk premium* δ^* measures the risk premium that speculators require for carrying the only risk that remains, the stochastic dependence risk. If the stochastic dependence between ϕ and A^* is positive then when speculators enter the futures market as buyers and add ϕ to their portfolios, the risk of their portfolios is increased. As risk averse agents the speculators require compensation for this risk carrying activity—a positive average profit δ^* is the equilibrium reward they obtain for each unit invested. If the stochastic dependence between ϕ and A^* is negative, then the futures market in essence provides a hedging service for the speculators, since adding ϕ to their portfolios reduces the risk of their portfolios. This is a service for which speculators are prepared to pay on average—a property that reflects itself in a negative asymptotic risk premium.

Finally, and perhaps most surprisingly, when there is no stochastic dependence between ϕ and A^*—in short, when ϕ and A^* are independent—the asymptotic risk premium is zero. The idiosyncratic risk, which in the spot market equilibrium of Section 3 was carried by producers effectively charging consumers an insurance fee over and above the basic marginal cost of production, is now in the joint spot–futures market equilibrium of a large market economy diversified away through the risk-spreading portfolio activity of speculators. In the limit speculators in essence provide costless price insurance to producers.

6. Welfare Effect of Futures Market

The original motivation for introducing futures markets was the idea that these markets would provide a mechanism for reducing the cost of risk bearing—a cost otherwise incurred directly by producers over and above the basic costs of production. By reducing overall costs of production producers and consumers were expected to be better off and since speculators would only trade if such trading made them better off, the introduction of a futures market should act to the mutual benefit of producers, consumers and speculators. We shall show that such an argument is indeed valid in a *proper* equilibrium ($\bar{z}_n > 0$ and hence $\bar{x}_n > \hat{x}$) provided the equilibrium output $\bar{x}_n$ lies above a critical level $\bar{\bar{x}}$ or equivalently provided the introduction of a futures market leads to a sufficient reduction in the equilibrium risk costs (Proposition 8).

Introducing a futures market can, however, lead to an increase in the cost of risk bearing. If there is sufficient positive dependence between ϕ and A_n then speculators can induce producers to take long positions on the futures market ($\bar{z}_n < 0$ and hence $\bar{x}_n < \hat{x}$). Producers far from using the futures market as a hedging market are adding to their existing spot market risks by carrying risks for speculators. In the limiting equilibrium of Proposition 7 speculators, however, obtain no increase in welfare since $\lim_{n\to\infty} W(\bar{\xi}_n) = W(0)$. Since output is reduced consumers are worse off and in some cases producers may be worse off. Such a situation is similar to Hart's example [7] where the addition of a new market in an incomplete market framework leads to a Pareto inferior allocation. The welfare effect of a futures market thus depends on whether the cost of risk bearing is reduced or increased and this in turn depends on the nature of the stochastic dependence between the spot price and prices on other markets on which speculators trade.

To establish welfare results we need to make explicit the utility maximising behaviour of consumers. To keep the framework simple let all k consumers be identical. Each consumer faces a vector of random prices $P(\omega) \in R^r_+$ for the final goods and has a random income $M(\omega) > 0$, $\omega \in \Omega$ where $P_i \in \mathscr{L}^+_\infty(\Omega)$, $i = 1,\dots, r$ and $M \in \mathscr{L}^+_\infty(\Omega)$. Since each consumer is assumed to know the state of nature $\omega \in \Omega$ at the time the consumption decision is made, the consumer's problem reduces to the following standard maximum problem for each state of nature $\omega \in \Omega$

$$\max_{\chi \in B(P(\omega), M(\omega))} v(\chi), \qquad B(P(\omega), M(\omega)) = \{\chi \in R^r_+ \mid P(\omega)\,\chi \leqslant M(\omega)\}.$$

Let $\chi(P(\omega), M(\omega))$ denote the consumer's resulting demand in the state of nature $\omega \in \Omega$. The consumer's indirect utility function is given by

$$V(P(\omega), M(\omega)) = v(\chi(P(\omega), M(\omega))), \qquad \omega \in \Omega.$$

ASSUMPTION 6. (i) *Each consumer has a preference ordering over random price systems represented by the expected indirect utility function* $EV(P(\omega), M(\omega))$.

(ii) *The utility function* $v\colon R^r_+ \to R$ *is continuous, strictly increasing and strictly quasi-concave.*

The following proposition asserts that whenever the trading of speculators leads to a sufficient increase in the equilibrium output ($\bar{x}_n > \bar{\bar{x}}$) or equivalently to a sufficient reduction in the equilibrium risk cost ($\delta(\bar{x}_n) < S'(\bar{\bar{x}})$ where $\delta(\bar{x}_n) = E(\bar{p}_n) - \bar{q}_n$) then the introduction of a futures market leads to a Pareto improvement.

PROPOSITION 8. *Let Assumptions* 1–6 *be satisfied. There exists an output* $\bar{\bar{x}} \in (\hat{x}, b)$ *where* $c'(b/m) = \bar{\phi}(\hat{x})$ *such that whenever the equilibrium output* $\bar{x}_n$ *exceeds* $\bar{\bar{x}}$ *then the introduction of a futures market leads to a Pareto improvement.*

Proof. The welfare of speculators is increased since by the optimality and uniqueness of $\bar{\xi}_n$, $W(\bar{\xi}_n) > W(0)$. Let $P = (P_1, \tilde{P})$, $\chi = (\chi_1, \tilde{\chi})$ and let the first good be the commodity under consideration. $\phi(\cdot, \omega)$ is the inverse of the market demand function $k\chi_1(\cdot, \tilde{P}(\omega), M(\omega))$ and $P_1(\omega) = \phi(x, \omega)$ is the spot price in state $\omega \in \Omega$ when equilibrium output is x. Let $v(x) = EV(\phi(x, \omega), \tilde{P}(\omega), M(\omega))$, then since ϕ is a strictly decreasing function of x and since Assumption 6(ii) implies V is a strictly decreasing function of P_1, $v(x)$ is a strictly increasing function of x. Since $\bar{x}_n > \bar{\bar{x}} > \hat{x}$, $v(\bar{x}_n) > v(\hat{x})$ so consumers are better off. Let π_s and π_f denote the random profits earned by the representative producer in the spot market and spot–futures market equilibria, respectively, and let Π_s and Π_f denote their certainty equivalents, $u(\Pi_s) = Eu(\pi_s(\omega))$, $u(\Pi_f) = Eu(\pi_f(\omega))$. Clearly $Eu(\pi_f(\omega)) > Eu(\pi_s(\omega))$ if and only if $\Pi_f - \Pi_s > 0$ since $u(\cdot)$ is strictly increasing by Assumption 3. The producer's futures trade $(\bar{z}_n)$ can be decomposed into the sum of a hedging transaction $(\bar{x}_n/m)$ and a purely speculative trade $(\bar{\theta}_n)$

$$\bar{z}_n = \frac{\bar{x}_n}{m} + \bar{\theta}_n, \qquad s(\omega) = \bar{\theta}_n(\bar{q}_n - \bar{p}_n(\omega)), \qquad \omega \in \Omega \tag{19}$$

where $s(\omega)$ is the profit on the purely speculative component of the trade in state $\omega \in \Omega$. Let S denote the certainty equivalent of s, $u(a_n + S) = Eu(a_n + s(\omega))$ where $a_n = \bar{q}_n(\bar{x}_n/m) - c(\bar{x}_n/m)$ and let $\gamma(\pi_s)$, $\gamma(\pi_f)$ and $\gamma(s)$ denote the *risk premia* implied by Π_s, Π_f and S (thus $\Pi_s = E(\pi_s) - \gamma(\pi_s)$, etc.) Since $\gamma(\pi_f) = \gamma(s)$ it is immediate that

$$\Pi_f - \Pi_s = \psi(\bar{x}_n) + S$$

where

$$\psi(x) = c'\left(\frac{x}{m}\right)\frac{x}{m} - c\left(\frac{x}{m}\right) - \left(\bar{\phi}(\hat{x})\frac{\hat{x}}{m} - c\left(\frac{\hat{x}}{m}\right) - \gamma(\pi_s)\right).$$

Since $\bar{\theta}_n \neq (=)\, 0$ according as $x_n \neq (=)\, x^*$, it follows from the optimality and uniqueness of $\bar{\theta}_n$ that $S > (=)\, 0$ according as $\bar{x}_n (\neq) = x^*$. Since $c'(\cdot)$ is strictly increasing by Assumption 1, it follows that $\psi(\cdot)$ is strictly increasing. Thus if $\psi(\bar{\bar{x}}) = 0$ then $\Pi_f - \Pi_s > 0$, $\forall\, \bar{x}_n > \bar{\bar{x}}$. Such an $\bar{\bar{x}}$ exists and satisfies $\hat{x} < \bar{\bar{x}} < b$ where $c'(b/m) = \bar{\phi}(\hat{x})$ since $\psi(\hat{x}) = -S < 0$, $\psi(b) > 0$ and $\psi(\cdot)$ is continuous by the continuity of $c'(\cdot)$. ∎

Remark. The intuition behind the result is clear. This case will arise in particular when the return A_n earned by speculators on the other markets on which they trade is sufficiently negatively dependent on the spot price ϕ. In such a stochastic environment speculators are effectively able to use the futures market as a hedging medium, reducing the risk of their portfolios by taking long positions. For this service they are glad to pay a premium $-\delta(\bar{x}_n)$. This premium serves to increase the futures price (and hence effectively to subsidise the production of firms) to such an extent that the sure profit that producers can earn by selling all their output forward at the futures price $\bar{q}_n$ (as in Eq. (19)) exceeds the certainty equivalent of their spot market profit Π_s. This is the meaning of the equation $\psi = (\Pi_f - S) - \Pi_s > 0$. Since the producer always attaches a positive value to the random profit earned on the purely speculative trade $\bar{\theta}_n$ provided

$\bar{\theta}_n \neq 0$ $(S > (=) 0$ accordingly as $\bar{x}_n \neq (=) x^*)$ we are assured that $\Pi_f - \Pi_s > 0$. Thus producers gain and consumers gain from the increased output which is available at a reduced average price. The introduction of the futures market has thus acted to the mutual benefit of producers, consumers and speculators.

Remark. In a proper equilibrium $(\bar{x}_n > \hat{x})$ when the equilibrium output $\bar{x}_n$ lies below $\bar{\bar{x}}$ while consumers and speculators gain, producers may lose. Leaving aside the speculators, whose gains go to zero as $n \to \infty$ since $\lim_{n \to \infty} W(\xi_n) = W(0)$, we may ask under what conditions consumers can compensate producers. When the market structure changes, equilibrium output changes from $\hat{x}$ to $\bar{x}_n$ leading to a change in the spot price from $P_1(\omega) = \phi(\hat{x}, \omega)$ to $P_1'(\omega) = \phi(\bar{x}_n, \omega)$. Let $P(\omega) = (P_1(\omega), \tilde{P}(\omega))$, $P'(\omega) = (P_1'(\omega), \tilde{P}(\omega))$, then the gain to each consumer from the introduction of a futures market is given by the compensated variation in income N defined by $EV(P'(\omega), M(\omega) - N) = EV(P(\omega), M(\omega))$. The loss L incurred by each producer is defined by $Eu(\pi_f(\omega) + L) = Eu(\pi_s(\omega))$. The Kaldor criterion for a *potential* Pareto improvement requires that $\mathscr{G} = kN - mL > 0$. The fall in the spot price $P_1(\omega) \to P_1'(\omega)$ leads to a random consumer surplus gain defined by

$$\alpha(\omega) = \int_{P_1(\omega)}^{P_1'(\omega)} \chi_1(p_1, \tilde{P}(\omega), M(\omega))\, dp_1, \qquad \omega \in \Omega.$$

Let A denote the certainty equivalent of α $(EV(P'(\omega), M(\omega) - A) = EV(P'(\omega), M(\omega) - \alpha(\omega)))$ then one can show using Willig's result [17] that $N = A - \varepsilon$, where $\varepsilon \geqslant 0$ (for a normal good) is the overestimate arising from the income effect. Let $\gamma(\alpha)$ denote the consumer's *risk premium* associated with α, $A = E(\alpha) - \gamma(\alpha)$, then $N = E(\alpha) - (\varepsilon + \gamma(\alpha))$. If each producer has decreasing absolute risk aversion, then $-L \geqslant \Pi_f - \Pi_s = \psi + S$ so that $\mathscr{G} > 0$ if $kE(\alpha) + m(\psi + S) > k(\varepsilon + \gamma(\alpha))$. Furthermore it is easy to see that $kE(\alpha) + m(\psi + S) > 0$ whenever $\bar{x}_n \geqslant x^*$. Thus if the value (N) to each consumer of the introduction of a futures market lies sufficiently close to the expected consumer surplus[9] $(E(\alpha))$ and if producers have decreasing

[9] In the example of Section 4.2 where $\phi(x, \omega) = \mu(x) + \lambda(\omega)$ (this case is often useful as an approximation) the consumer surplus is nonrandom so that $\gamma(\alpha) = 0$. In this case the standard deterministic condition that the proportion of the consumer's income spent on the commodity be sufficiently small gives the result. As a second case, suppose income is nonrandom and suppose the vectors of prices can be decomposed as $P = (P^a, P^b)$ where P^a (P^b) is the vector of prices that are random (nonrandom). Let the marginal utility of income V_M be independent of P^a, then $V(P, M) = f(P) + h(P^b, M)$ and the definition of the certainty equivalent of A reduces to $h(P^b, M - A) = Eh(P^b, M - \alpha(\omega))$. If the consumer's relative risk aversion with respect to income is bounded above $-(h_{MM}/h_M)\, M \leqslant \rho$, then if each consumer is sufficiently wealthy and if his expenditure on this commodity is sufficiently small in each state of nature, then $\varepsilon + \gamma(\alpha)$ can be made arbitrarily small.

absolute risk aversion then there is a potential Pareto improvement whenever $\bar{x}_n \geqslant x^*$ or equivalently whenever $\delta(\bar{x}_n) \leqslant \delta(x^*) = 0$. This result when applied to the asymptotic output $\bar{x}$ of Proposition 7 gives conditions under which in the independent and negatively dependent cases the introduction of a futures market leads to a potential Pareto improvement.

Appendix

Proof of Lemma 1. It is straightforward to show that $U(\cdot)$ is differentiable on R^2_+. Let $\rho = \operatorname{ess\,sup} p(\omega)$ so that $p(\omega) \leqslant \rho$ a.s. By Assumption 1(iii) there exists $y_\rho > 0$ such that $c'(y_\rho) > \rho$. But then $\forall\, z \in R^+$

$$U_y(y_\rho, z) = \int_\Omega u'(\pi(\omega; y_\rho, z))(p(\omega) - c'(y_\rho))\, d\mathscr{P}(\omega)$$

$$\leqslant \int_\Omega u'(\pi(\omega; y_\rho, z))(\rho - c'(y_\rho))\, d\mathscr{P}(\omega) < 0.$$

The concavity of $U(\cdot)$ implies $U_y(\cdot, z)$ is nonincreasing. Since $z \in [0, y]$ any solution to $(\mathscr{U}')$ must lie in the region $[0, y_\rho] \times [0, y_\rho]$. The result follows from the continuity of $U(\cdot)$. Uniqueness follows from the strict concavity of $u(\cdot)$. ∎

Proof of Lemma 2. Let $y_s = y - z \geqslant 0$. Then $U(y, z) = V(y_s, z) = \int_\Omega u(qz + p(\omega)\, y_s - c(z + y_s))\, d\mathscr{P}(\omega)$. If $z^* > 0$, then the optimality of (y^*, z^*) implies

$$V_z(y_s^*, z^*) = (q - c'(z^* + y_s^*)) \int_\Omega u'(qz^* + p(\omega)\, y_s^* - c(z^* + y_s^*))\, d\mathscr{P}(\omega) = 0.$$

Since $u'(\cdot) > 0$ by Assumption 3, $q - c'(z^* + y_s^*) = 0$. ∎

Proof of Lemma 3. Since $\pi_i \in \mathscr{L}_1(\Omega)$ and $\gamma < \pi_i(\omega)$ a.s., $i = 1, ..., n$, it follows by applying the dominated convergence theorem that $W(\xi_m) \to W(\xi)$ whenever $\xi_m \to \xi$. Existence follows from the compactness of $[-\Delta, \Delta]$, and uniqueness from the strict concavity of $W(\xi)$. ∎

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