Expectations of Inflation, the Term Structure of Interest Rates and Monetary Policy

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Abstract. This paper provides a theoretical framework for studying how monetary policy can be used to control expectations of inflation. We consider a simple production economy with a cash-in-advance constraint in which monetary-fiscal policy is Ricardian. Agents' expectations are modeled as probability distributions on a finite set of possible inflation rates. The monetary authority announces a public forecast of inflation to direct agents' expectations, and a bond pricing (term structure of interest rates) policy to make the forecast credible. We study conditions under which an announced forecast is compatible with equilibrium—there must be enough weight on inflation to be compatible with a non-negative nominal interest rate. In a stationary setting we exhibit a rank condition on the payoff structure of the bonds which must be satisfied if the forecast is to be the unique probability distribution compatible with the bond pricing policy, thereby eliminating any other possible expectations of inflation for the agents. The model thus provides a formal framework for understanding the conditions under which the policy of inflation targeting can be successful.

1. Introduction

The objective of this paper is to study inflation targeting as a way of anchoring agents' expectations of inflation. As a result of the adverse experience of the 1970's with an approach based on monetary aggregates, the focus of Central Banks has shifted away from controlling monetary aggregates to using interest rate policies to control inflation, and in the last twenty years, many Central Banks, notably the Bank of England (BOE), have come to adopt the policy of *inflation targeting*.

In a stylized way, this approach to monetary policy may be characterized as follows. Price stability—the central focus of monetary policy—is expressed first by an *inflation target* $\pi^\star$, typically the annual percentage change in a consumer price index (2% in the case of BOE), and this is viewed as the average rate around which inflation should fluctuate. It is explicitly recognized that due to fundamentals or possibly due to agents' expectations regarding future inflation, the inflation rate is inevitably a random variable. Thus at any given date t the inflation rate π_t will typically differ from the target rate $\pi_t \neq \pi^\star$. The second component of a Central Bank's inflation targeting policy consists of a periodic *official announcement of its best prediction of the probability distribution of inflation for a specified horizon T into the future*, with the property that its monetary policy should lead to a sequence of probability distributions such that the mean (or for BOE, the mode) of this sequence of distributions reaches the target $\pi^\star$ in a specified horizon $\tilde{T} < T$ (where $\tilde{T} = 2$ and $T = 3$ years for BOE). This periodic announcement of its inflation forecast is an integral part of an inflation targeting policy which is typically expressed by a so-called *fan chart* showing the iso-probability contours for future inflation (see Figure 1 for the May 2009 CPI fan chart of BOE). The interest rate policy which is to achieve this policy objective is typically viewed as being the short-term (overnight) interest rate policy chosen and announced by the Central Bank at these regular intervals. In the analysis that follows we will show that our model predicts that it is not sufficient to confine monetary policy to such a short-term interest rate policy if the "announcement" of the Central Bank is to be credible.

The model we consider is a cash-in-advance economy with a simple production technology in which the short-term nominal interest rate influences output. Monetary-fiscal policy is Ricardian, i.e. monetary policy is primary and fiscal policy adapts to the resulting nominal value of the debt to ensure that the government's liabilities are always ultimately paid off (in present value terms). The novelty of the model is that agents' expectations of inflation, modeled as a sequence of probability distributions on a fixed support, are endogenous variables: the task

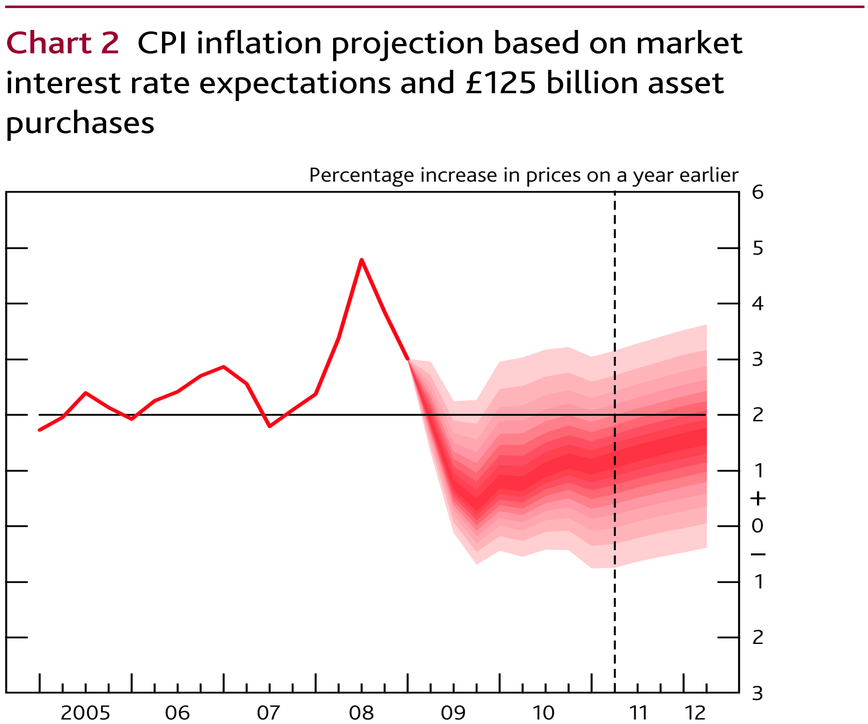

Figure 1: Fan chart issued by the Bank of England in May 2009. The fan chart shows confidence intervals around the mode of the distribution of inflation rates at each future date, the darkest band being a 10% confidence interval around the mode and each band adding 10% probability.

of the monetary authority is to induce agents to hold a sequence of probability distributions of its choice on this support, similar to the sequence of probability distributions in the fan chart that the Central Bank announces to direct agents' expectations of inflation.

The model is close to the model of Nakajima-Polemarchakis (2005)[1] who showed that if monetary policy consists solely of fixing the short-run interest rate, and fiscal policy is Ricardian, then the equilibrium is indeterminate: their objective was to study in a simplified model the indeterminacy which is present in the New-Keynesian models which follows from a monetary policy based on a short-term interest rate and a fiscal policy which is Ricardian. In such a setting, at each date, the only attribute of the probability distribution of inflation at the next period which is determined in equilibrium is the *mean* of the distribution which is tied down by the Fisher equation relating the short-term nominal interest rate to the real interest

[1] See also Bloise-Drèze-Polemarchakis (2005) for an example of indeterminate equilibrium when the government policy is specified in part in real terms, in part in nominal terms.

rate and the mean of next period's inflation. Macroeconomists use the "Taylor Principle"—increasing the interest rate more than proportionately to the deviation of the current inflation from target inflation—to focus on the unique trajectory which stays close to the steady state when the interest rate is chosen to satisfy such a Taylor rule (King (2000), Woodford (2003)). Whether or not it is appropriate to exclude all other equilibria from consideration is open to discussion (Benhabib-Schmitt-Grohe-Uribe(2002), Cochrane (2007)).

We propose an alternative approach to solving the indeterminacy of equilibrium which consists in using more instruments than just the short-term interest rate to control agents' expectations of inflation. Assuming that the support of the inflation rates that agents deem possible is discretized by a finite number of values—say S—we show that by appropriately controlling the prices of S bonds, or equivalently the yields (interest rates) of bonds of maturities 1 to S, the monetary authority can induce a unique probability distribution for inflation which is compatible with these announced prices. Thus to resolve the problem of indeterminacy we replace the interest rate rule on the short-term bond by a term-structure rule on a spectrum of bonds of S maturities.

Although the monetary authority has considerable flexibility in its choice of expectations to communicate to the private sector in the form of a forecast and to make it credible with a term-structure policy, there are limits to this flexibility. First, expectations of inflation must be compatible with non-negative nominal interest rates: this eliminates expectations which put too much weight on deflation. Under a Markov assumption, we characterize the expectations matrices which are compatible with an equilibrium with a non-negative nominal interest rate (Proposition 8). Second, in order to eliminate other possible beliefs of the agents, the expectations of inflation must be uniquely associated with a term-structure rule—a property which we call *controllability* of expectations. We derive the conditions which an expectations matrix must satisfy to be controllable (Proposition 11). The main characteristic of such matrices is that the expected future path of inflation rates must be different for different current levels of inflation. Thus a mean-reverting process to a *target inflation rate* is controllable with a term-structure rule, but not an immediate return to the target.

The plan of the paper is as follows. In Section 2 we present the model and show how the infinite horizon equilibrium can be transformed into a much simpler "reduced-form " equilibrium with a minimum number of endogenously determined variables: from these variables the complete "extensive-form" equilibrium can always be recovered, greatly enhancing the tractability of the analysis. In Section 3 we study expectations matrices that a monetary authority can control with a term-structure rule in the Markov representative agent case without real shocks to productivity.

2. Monetary Economy

Consider a monetary economy in discrete time over an infinite horizon with a finite number of agents, in which money serves not only as a unit of account but also as the medium of exchange. The objects of trade are goods and securities and the purchase (payment) for either must be made using money. There are two sets of agents: $\mathcal{H}$, the finite set of households in the private sector and a government, the monetary-fiscal authority. The monetary authority issues money and nominal (government) bonds of different maturities and seeks to control agents' expectations of inflation by announcing an inflation forecast and a bond pricing (interest rate) policy which makes the forecast credible. The fiscal authority imposes taxes on the agents commensurate with the monetary policy so that the government's debt does not grow too fast: in short the monetary-fiscal policy is *Ricardian*.

There are two sources of uncertainty in the economy: the first is nominal, the second real. The nominal uncertainty arises from the beliefs of the agents regarding the possible future course of inflation. The standard approach to modeling such beliefs in general equilibrium theory is to assume that there is an exogenously given "randomizing device" (sunspots) defined on a known state space with *fixed* probabilities, which serves to "co-ordinate" agents' expectations (Cass-Shell (1983)). The defect of this approach is that such randomizing devices are difficult (if not impossible) to identify in practice, thus severely curtailing the applicability of the approach. In this paper we suggest an alternative way of modeling "sunspot equilibria" which retains the basic idea that beliefs can be self-fulfilling and influence the equilibrium. The exogenous randomizing device is replaced by the probability distribution over future inflation adopted by the agents as their expectations, the co-ordination of beliefs being obtained through a publicly announced forecast of inflation made by the monetary authority.

Uncertainty and Event-Tree. The primitive assumption regarding agents' expectations is that there is a compact subset of the real line which serves as the support for the agents' beliefs regarding inflation at every date. To keep the analysis tractable this subset is discretized to a finite set Π, consisting of the inflation rates which agents view as possible. The real uncertainty arises from the fluctuations in the productivities of the agents in their production activities. To simultaneously model the nominal and real uncertainties, let $\mathcal{S}$ be an index set with S elements so that $\Pi = \{\pi_s, s \in \mathcal{S}\}$ denotes the possible inflation rates, and let $\mathcal{G} = \{1, \ldots, G\}$ denote the index set for the possible real shocks. Unlike the probabilities of the inflation rates which are endogenously determined, the probabilities of the real shocks are taken as exogenously given. A partial history of inflation shocks $(s_0, \ldots, s_t)$ and real shocks $(g_0, \ldots, g_t)$ up to date t leads to a typical date-event or node $\xi_t = ((s_0, g_0), \ldots, (s_t, g_t)) \in \mathcal{S}^t \times \mathcal{G}^t = \mathbb{D}_t$ where $\mathbb{D}_t$ is the set of all possible nodes at date t. The union of all the partial histories defines the event-tree

$$\mathbb{D} = \bigcup_{t=0}^{\infty} \mathbb{D}_t$$

consisting of all possible date-events ξ into the indefinite future. Any date-event ξ has a date $t(\xi)$, a unique predecessor ξ^- at date $t(\xi) - 1$ and a set of immediate successors ξ^+ at date $t(\xi) + 1$. If $\xi = ((s_0, g_0), \ldots, (s_t, g_t))$ then

$$\xi^- = ((s_0, g_0), \ldots, (s_{t-1}, g_{t-1})), \qquad \xi^+ = \{((s_0, g_0),\ \ldots, (s_t, g_t), (s, g)) \mid (s, g) \in \mathcal{S} \times \mathcal{G}\}$$

For any node $\xi \in \mathbb{D}$, let $\mathbb{D}(\xi)$ denote the sub-tree originating at ξ and let $\mathbb{D}_T(\xi)$ denote the nodes of the subtree $\mathbb{D}(\xi)$ at date T.

Agents' characteristics and actions. Each agent (household) $h \in \mathcal{H}$ has beliefs over the event-tree:[2] let B^h_ξ denote the probability that agent h assigns to passing through node ξ. Then $\sum_{\xi \in \mathbb{D}_t} B^h_\xi = 1$ and $B^h_\xi = \sum_{\xi' \in \xi^+} B^h_{\xi'}$ for all $\xi \in \mathbb{D}_t$. The basic object of interest for an agent is his consumption stream of the (single) commodity over the event-tree. However to earn the right to such a stream the agent will need to work over his lifetime, i.e. to offer labor services over the event-tree to firms who convert them into output and pay the agent for the labor services. To incorporate production into the model in the simplest way which at the same time captures the idea that production is influenced by the nominal interest rate set by the monetary authorities, we use the device introduced by Lucas-Stokey (1985) and also used by Polemarchakis-Nakajima (2005). At each date-event $\xi \in \mathbb{D}$ an agent has an endowment e^h of 'time' which can be used either for leisure (ℓ^h_ξ) or to produce labor services (L^h_ξ) with $e^h = \ell^h_\xi + L^h_\xi$, $\xi \in \mathbb{D}$. The agent sells the labor services L^h_ξ to a firm which uses them to produce $y_\xi = a^h_\xi L^h_\xi$ units of output, the productivity of the agent's labor services depending

[2]We first present the model with possibly different expectations for the agents to highlight the co-ordinating role of the forecast announced by the monetary authority.

on the real and not on the inflation shock.[3] If c^h_ξ is the agent's consumption of the good at node ξ, then the bundle $x^h_\xi = (c^h_\xi, \ell^h_\xi)$ generates the flow utility $u^h(c^h_\xi, \ell^h_\xi)$ where the functions $u^h, h \in \mathcal{H}$, satisfy the following conditions:

Assumption $\mathcal{U}$. For each $h \in \mathcal{H}$ the function $u^h : \mathbb{R}_+ \times [0, e^h] \to \mathbb{R}$ has the following properties

1. increasing and differentiably strictly concave ($u_{cc} < 0, u_{\ell\ell} < 0, u^h_{cc}u^h_{\ell\ell} - (u^h_{c\ell})^2 > 0$)
2. supermodular ($u^h_{c\ell} \geq 0$)
3. Inada conditions: $\begin{array}{l} u^h_c(c,\ell) \to \infty \ as\ c \to 0, \quad \forall\ \ell \in (0.e^h) \\ u^h_\ell(c,\ell) \to \infty \ as\ \ell \to 0, \quad \forall\ c > 0 \end{array}$
4. asymptotic satiation: $\begin{array}{l} u^h_c(c,\ell) \to 0 \ as\ c \to \infty, \quad \forall\ \ell \in (0.e^h) \\ u^h_\ell(c,\ell) \to 0 \ as\ \ell \to e^h, \quad \forall\ c > 0 \end{array}$

The consumption-leisure choice[4] $\boldsymbol{x}^h = (c^h_\xi, \ell^h_\xi)_{\xi \in \mathbb{D}}$ generates the lifetime expected utility

$$U^h(\boldsymbol{x}^h) = \sum_{\xi \in \mathbb{D}} \delta^{t(\xi)} B^h_\xi u^h(x^h_\xi), \quad 0 < \delta < 1 \tag{1}$$

In addition to the consumption-leisure decision x^h_ξ the agent needs to make a portfolio decision at each date-event which enables him to finance this consumption stream. But now an additional element in the story needs to be made clear: if money is to be modeled using Clower's idea that only money will buy goods or securities then we need to specify how the transactions take place. The simplest device is to think of the timing of the trades of securities, goods and labor as taking place in three distinct subperiods of each node. In the first subperiod securities are traded and taxes are paid to the government: in the second subperiod the available money balances are used to purchase the consumption good at its current price p_ξ: in the final subperiod firms pay agents for their labor services.

In addition to deciding how much money $\widetilde{m}^h_\xi$ to hold at each node to finance his purchase of consumption $p_\xi c^h_\xi$, the agent also decides on his holdings of the securities. These consist of zero-coupon nominal (government) bonds of maturities $\tau = 1, \ldots, T$ and a collection of private sector short-lived securities in zero net supply indexed by $j = T+1, \ldots, J$, with payoffs $V^j_{\xi'}$ (in units of money) at the immediate successors $\xi' \in \xi^+$. Let $\mathcal{J}_g$ denote the set of T government

[3] The formal assumptions will be made in section 4 where we assume a Markov structure for the shocks: if at date $t(\xi)$ the current shock is (π_s, g) then $a^h_\xi = a^h_g$.

[4] Throughout the paper we use boldface to denote a vector defined over the whole event-tree $\mathbb{D}$ (e.g. $\boldsymbol{x}^h = (x^h_\xi)_{\xi \in \mathbb{D}}$) or a vector defined over the set of agents (e.g. $\boldsymbol{x}_\xi = (x^h_\xi)_{h \in \mathcal{H}}$).

bonds, let $\mathcal{J}_p$ denote the set of private-sector securities and let $\mathcal{J} = \mathcal{J}_g \cup \mathcal{J}_p$ be the set of all securities. We assume that the combined set of securities is sufficiently rich to assure complete markets (full spanning at each node ξ of the event-tree $\mathbb{D}$). Let $q_\xi = (q_\xi^j)_{j\in\mathcal{J}}$ denote the vector of (money) prices of the securities and let $z_\xi^h = (z_\xi^{hj})_{j\in\mathcal{J}}$ denote the agent's portfolio at node ξ, the first T components consisting of the agent's holdings of the government bonds. Since a τ-period bond purchased at node ξ becomes a $\tau - 1$ period bond at each of the successors $\xi' \in \xi^+$ and since the 1-period bond at node ξ pays 1 (dollar) at each successor, the payoffs at $\xi' \in \xi^+$ of the T bonds purchased at node ξ are given by the vector $(1, q_{\xi'}^1, \ldots, q_{\xi'}^{T-1})$. Given that we focus on the bond market, we let $\hat{q}_\xi = (1, q_\xi^1, \ldots, q_\xi^{T-1}, V_\xi^j, j = T+1, \ldots, J)$ denote the payoff at node ξ of all the securities traded at node ξ^-. The $SG \times J$ matrix of payoffs of the securities traded at node ξ at the successors ξ^+ is denoted by

$$\left[\,\hat{q}_{\xi^+}\right] \equiv \left[\,\hat{q}_{\xi'}^j\right]_{\substack{j\in\mathcal{J}\\ \xi'\in\xi^+}}$$

The condition that markets are complete is equivalent to the property that $\text{rank}[\,\hat{q}_{\xi^+}] = SG$, or that $[\,\hat{q}_{\xi^+}]$ is invertible for all $\xi \in \mathbb{D}$. We consider only price processes $\boldsymbol{q}$ which do not offer arbitrage opportunities, so that each agent has a solution to the problem of choosing an optimal portfolio. For any no-arbitrage $\boldsymbol{q}$ there exists a process $\boldsymbol{P} = (P_\xi)_{\xi\in\mathbb{D}}$, where P_ξ/P_{ξ_0} is the present value at date 0 of a promise to deliver one unit of money at node ξ, such that $P_\xi q_\xi^j = \sum\limits_{\xi'\in\xi^+} P_{\xi'}\hat{q}_{\xi'}^j$. Given the assumption of complete markets, $\boldsymbol{P}$ is unique up to normalization.

Let $m_{\xi^-}^h$ denote the money balances brought into node ξ; since the agent receives the payoff $(\hat{q}_\xi, V_\xi)z_{\xi^-}^h$ on the portfolio $z_{\xi^-}^h$ purchased at the preceding node, he has the wealth $w_\xi^h = m_{\xi^-}^h + (\hat{q}_\xi, V_\xi)z_{\xi^-}^h$ available in the first subperiod of node ξ to buy a new portfolio z_ξ^h of the securities and to pay the taxes θ_ξ^h which are due. The agent lays aside enough money balances $\widetilde{m}_\xi^h \geq p_\xi c_\xi^h$ to purchase the planned consumption c_ξ^h on the goods market of the second subperiod. Thus the agent chooses $(\widetilde{m}_\xi^h, z_\xi^h)$ so that

$$\widetilde{m}_\xi^h + \theta_\xi^h + q_\xi z_\xi^h = m_{\xi^-}^h + \hat{q}_\xi z_{\xi^-}^h\ , \qquad \xi \in \mathbb{D} \tag{2}$$

$$\widetilde{m}_\xi^h \geq p_\xi c_\xi^h\ , \qquad \xi \in \mathbb{D} \tag{3}$$

Let ω_ξ denote the wage at node ξ. In the last subperiod of node ξ, the firm pays the agent $\omega_\xi a_\xi^h L_\xi^h$ for the labor services rendered at node ξ: this money and the unspent balances

$$m_\xi^h = \omega_\xi a_\xi^h L_\xi^h + (\widetilde{m}_\xi^h - p_\xi c_\xi^h)\ , \qquad \xi \in \mathbb{D} \tag{4}$$

are transferred to each of the successors $\xi' \in \xi^+$ of node ξ.

Since the agent is not willing to lend to any other agent or the government "at infinity", and since no agent is willing to lend to him "at infinity", he is obliged to confine his portfolio strategies to those for which the transversality condition

$$\lim_{T \to \infty} \sum_{\xi \in \mathbb{D}_T(\tilde{\xi})} P_\xi q_\xi z_\xi^h = 0 \ , \qquad \tilde{\xi} \in \mathbb{D} \tag{5}$$

is satisfied, where $\boldsymbol{P} = (P_\xi)_{\xi \in \mathbb{D}}$ is the valuation of income compatible with the price process $\boldsymbol{q}$ (see Magill-Quinzii (1994)).

Monetary and Fiscal Policy. Monetary policy is dominant and fiscal policy adapts itself to ensure that the government's budget is balanced. The goal of monetary policy is to control the inflation process. To this end the monetary authority announces a probabilistic forecast $\boldsymbol{B} = (B_\xi)_{\xi \in \mathbb{D}}$ for inflation/real shocks with the goal of "anchoring agents' expectations" i.e. of inducing the agents to adopt $\boldsymbol{B}^h = \boldsymbol{B}$, $h \in \mathcal{H}$ as their expectations. To make the forecast credible the monetary authority simultaneously announces a bond pricing policy $(q_\xi^\tau)_{\xi \in \mathbb{D}}$ for bonds of maturity $\tau = 1, \ldots, T$. We will discuss later the conditions under which it is reasonable to expect agents to adopt $\boldsymbol{B}$ as their expectations—namely when $\boldsymbol{B}$ is in fact credible given the bond price policy $\boldsymbol{q}$. To fix the bond prices the government must accommodate the private sector demand for money and bonds through open markets operations $(\boldsymbol{M}, \boldsymbol{Z}) = (M_\xi, Z_\xi)_{\xi \in \mathbb{D}}$ where $Z_\xi = (Z_\xi^1, \ldots, Z_\xi^T)$ is the government's issue of τ-period bond at node ξ, for $\tau = 1, \ldots, T$. Let

$$W_\xi = M_{\xi^-} + \hat{q}_\xi Z_{\xi^-}, \qquad \xi \in \mathbb{D}$$

denote the government liabilities at the beginning of node ξ, inherited from the preceding node. These liabilities need to be covered by taxes θ_ξ, and open market operations (M_ξ, Z_ξ) satisfying

$$M_\xi + \theta_\xi + q_\xi Z_\xi = M_{\xi^-} + \hat{q}_\xi Z_{\xi^-}, \qquad \xi \in \mathbb{D} \tag{6}$$

The tax-reimbursement policy of the fiscal authority is characterized by a triple $(\alpha, \beta, \gamma) \in \mathbb{R}_{++}^{\mathbb{D}} \times \mathbb{R}_{++}^{\mathbb{D} \times S \times G} \times \Delta^H$, where α determines the reimbursement policy, β determines the composition of the debt, and γ in the simplex Δ^H of $\mathbb{R}^H$ determines the share of the taxes contributed by each agent. At each node $\xi \in \mathbb{D}$ the tax θ_ξ is chosen so that in conjunction with the seignorage revenue $\left(\frac{r_\xi^1}{1+r_\xi^1}\right) M_\xi$, where r_ξ^1 is the short-term interest rate, a share α_ξ of the government's current liabilities W_ξ is paid off

$$\frac{r^1_\xi}{1+r^1_\xi}M_\xi + \theta_\xi = \alpha_\xi W_\xi, \qquad \xi \in \mathbb{D} \tag{7}$$

Assumption RC (*Ricardian condition*) There exists $\underline{\alpha}$ such that $1 > \alpha_\xi \geq \underline{\alpha} > 0, \quad \xi \in \mathbb{D}$.

This is the simplest (if not the most realistic) way of ensuring that the debt is paid off and that the transversality condition is automatically satisfied.[5] For each $\xi \in \mathbb{D}$ the vector $\beta_\xi \gg 0$ indirectly determines the maturity distribution of the bonds financing the debt by specifying the relative magnitude of the next period liabilities $W_{\xi'}$ among the successor nodes ξ' of ξ. We require that there exists a scalar $d_\xi > 0$ such that

$$(W_{\xi'})_{\xi' \in \xi^+} = d_\xi \beta_\xi \tag{8}$$

As shown in the proof of Proposition 1, without specifying β there is an indeterminacy in the equilibrium portfolios. To avoid having to specify that the vector β is in the span of the bonds' payoffs, we assume that the government can trade on the other securities if the bonds do not complete the markets to satisfy condition (35). Most of the situations considered in the paper are such that the bonds of different maturities complete the markets so that this assumption is innocuous. Thus we extend the government portfolio Z_ξ to be in $\mathbb{R}^J$. Finally the vector $\gamma \in \Delta^H$ specifies how the tax burden is shared by the agents.

$$\theta^h_\xi = \gamma^h \theta_\xi, \qquad \xi \in \mathbb{D},\ h \in \mathcal{H} \tag{9}$$

Agents should not be required to pay more taxes than they can possibly pay with their income, since this would lead to nonexistence of equilibrium: in Proposition (14) below, which establishes existence of an equilibrium, we give an assumption which ensures that each agent's after tax income is positive.

Given the government's initial liabilities (M_{-1}, Z_{-1}) and the tax reimbursement policy (α, β, γ), a monetary fiscal plan $(\boldsymbol{q}, \boldsymbol{\theta}, \boldsymbol{M}, \boldsymbol{Z}) = (q_\xi, \theta_\xi, M_\xi, Z_\xi)_{\xi \in \mathbb{D}}$ consisting of bond prices, taxes, money supply and bond issues over the event-tree is *feasible* if (6)–(9) are satisfied, where r^1_ξ is the short-term interest rate implied by the bond price q^1_ξ.

The economy is characterized by the agents' characteristics, their initial holdings of bonds and assets, and the fiscal policy of the government. Thus we let

$$\mathcal{E}(\boldsymbol{u}, \delta, \boldsymbol{e}, \boldsymbol{a}, \boldsymbol{m}_{-1}, \boldsymbol{z}_{-1}, \alpha, \beta, \gamma)$$

[5] This is essentially the class of Ricardian policies introduced by Benhabib, Schmitt-Grohe and Uribe (2001,2002), and also used in Schmitt-Grohe and Uribe (2000).

(often shortened to $\mathcal{E}$) denote an economy in which agents' preferences and endowments are given by $(\boldsymbol{u}, \delta, \boldsymbol{e}, \boldsymbol{a}) = (u^h, \delta,\ e^h, a^h)_{h\in\mathcal{H}}$, initial money holdings and bond holdings are $(\boldsymbol{m}_{-1}, \boldsymbol{z}_{-1}) = (m^h_{-1}, z^h_{-1})_{h\in\mathcal{H}}$, the initial liabilities of the government being $(M_{-1}, Z_{-1}) = \sum_{h\in\mathcal{H}}(m^h_{-1}, z^h_{-1})$. The vector (α, β, γ) characterize the government fiscal policy. We take as given the root node $\xi_0 = (s_0, g_0)$ so that the initial inflation and real shock are well defined. Without loss of generality the price $p_0 = p_{-1}(1 + \pi_{s_0})$ is taken to be equal to 1.[6]

An equilibrium consists of a monetary-fiscal policy for the government—which includes a forecast of inflation to direct agents' expectations and an associated bond pricing policy—consumption-labor choices by agents as well as their associated money and bond holdings, production plans for firms, and money prices for labor, the good and the securities across the event-tree which are mutually compatible. We focus on equilibria in which agents adopt the announced forecast as their beliefs: later we give conditions under which such an equilibrium can reasonably be expected to arise. Let $\ell^1(\mathbb{D})$ denote the space of summable sequences on the event-tree $\mathbb{D}$, $\ell^1(\mathbb{D}) = \left\{\boldsymbol{P} \in \mathbb{R}^{\mathbb{D}} \,\middle|\, \sum_{\xi\in\mathbb{D}} |P_\xi| < \infty\right\}$.

Definition 1. An *(extensive-form) equilibrium* of $\mathcal{E}$, consists of a triple

$$\left(\left((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j\in\mathcal{J}_g}), (\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})\right), \left((\bar{\boldsymbol{x}}, \widetilde{\bar{\boldsymbol{m}}}, \bar{\boldsymbol{z}}), (\bar{\boldsymbol{y}}, \bar{\boldsymbol{L}})\right), \left(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{\omega}}, (\bar{\boldsymbol{q}}^j)_{j\in\mathcal{J}_p}\right)\right)$$

such that

(i) for every node $\xi = ((s_0, g_0), \ldots, (s_t, g_t)) \in \mathbb{D}$, $\bar{p}_\xi = (1 + \pi_{s_1}) \ldots (1 + \pi_{s_t})$.

(ii) $\bar{P}_\xi \bar{q}^j_\xi = \sum_{\xi'\in\xi^+} \bar{P}_{\xi'} \tilde{\bar{q}}^j_{\xi'}$, $\forall \xi \in \mathbb{D}$, $(\bar{P}_\xi \bar{p}_\xi)_{\xi\in\mathbb{D}} \in \ell_1(\mathbb{D})$.

(iii) $\bar{q}^\tau \leq 1, \quad \tau = 1, \ldots, T$

(iv) $(\bar{\boldsymbol{x}}^h, \widetilde{\bar{\boldsymbol{m}}}^h, \bar{\boldsymbol{z}}^h)$ maximizes $\sum_{\xi\in\mathbb{D}} \delta^{t(\xi)} \bar{B}_\xi u^h(x^h_\xi)$ subject to (2)–(5) with prices $(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{\omega}}, \bar{\boldsymbol{q}})$.

(v) $(\bar{y}_\xi, \bar{L}_\xi)$ maximizes $\bar{p}_\xi y - \bar{\omega}_\xi L$, subject to $y = L$, for all $\xi \in \mathbb{D}$.

(vi) $(\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})$ satisfies (6)–(9) and RC.

(vii) $\bar{y}_\xi = \sum_{h\in\mathcal{H}} \bar{c}^h_\xi$, $\bar{L}_\xi = \sum_{h\in\mathcal{H}} a^h_\xi \bar{L}^h_\xi$, $\bar{y}_\xi = \bar{L}_\xi$, $\xi \in \mathbb{D}$.

(viii) $\sum_{h\in\mathcal{H}} \tilde{\bar{m}}^h_\xi = \bar{M}_\xi$, $\sum_{h\in\mathcal{H}} \bar{z}^h_\xi = \bar{Z}_\xi$, $\xi \in \mathbb{D}$.

[6] With an interest rate policy the price level at date 0 is not determined so we use a normalization. The paper focuses on whether or not the inflation process is determinate.

Reduced-Form Equilibrium. The sequential structure of an extensive-form equilibrium makes it a complex object to analyze directly. For analytical purposes a simpler form of equilibrium can be obtained by translating the variables to date 0, eliminating the financial variables—money and portfolios— but retaining the present value of taxes and the bond prices to capture the government fiscal and monetary policy. The resulting simplified concept of equilibrium exhibits in a clear form the duality between expectations and bond prices on one hand, and the real allocation and the present-value prices on the other. Studying the controllability of the common expectations through a bond price policy is then equivalent to studying the determinacy of this simplified concept of equilibrium for a fixed structure of bond prices over the event-tree.

In view of the assumption of complete markets the opportunity set of an agent defined by the sequence of budget constraints (2)-(4) and the transversality condition (5) can be defined equivalently by a single budget constraint in which the money and portfolio variables no longer appear. Eliminating these variables is the key to the simplifying the concept of equilibrium, but these variables can be recovered from the variables in a reduced-form equilibrium defined below. The Ricardian condition (RC) implies that when the same procedure of translating the sequence of budget constraints to date 0 is applied to the government, *the resulting present-value budget constraint is automatically satisfied, hence it does not appear in a reduced-form equilibrium.* The variables which define this simplified concept are thus the forecast/bond pricing policy of the monetary authority, the present value of taxes raised by the fiscal authority, the allocation $\bar{\boldsymbol{x}}$ in the private sector and the *nominal stochastic discount factor* $\bar{\boldsymbol{\mu}}$ which, when combined with the forecast $\bar{\boldsymbol{B}}$ defines the vector of present-value prices $\bar{P}_\xi = \bar{B}_\xi \bar{\mu}_\xi$, $\xi \in \mathbb{D}$. To simplify the analysis of equilibrium we assume that the portfolios z^h_{-1} inherited from date -1 are composed only of short-lived bonds, so that the wealth w^h_0 of each agent $h \in \mathcal{H}$ at the beginning of date 0, $w^h_0 = m^h_{-1} + z^{h1}_{-1}$, is exogenously given and does not depend on the security prices at date 0. Also to simplify the notation we do not mention $\bar{\boldsymbol{p}}$ in the reduced-form equilibrium variables, and take as given that the compatibility conditions $\bar{p}_\xi = (1+\pi_{s_1})\dots(1+\pi_{s_t})$ if $\xi = ((s_0, g_0), \dots, (s_t, g_t))$ defines $\bar{p}_\xi$ for all $\xi \in \mathbb{D}$.

Definition 2. A *reduced-form equilibrium* of the economy $\mathcal{E}$ consists of a pair

$$\Big(\big(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j\in\mathcal{J}_g}, \bar{\Theta}\big), \bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}\Big)$$

such that

(1) $\bar{x}^h \in \operatorname{argmax}\left\{ \sum_{\xi \in \mathbb{D}} \delta^{t(\xi)} \bar{B}_\xi u^h(x^h_\xi) \;\middle|\; \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{p}_\xi \Big(c^h_\xi - \frac{a^h_\xi (e^h - \ell^h_\xi)}{1 + \bar{r}^1_\xi} \Big) + \gamma^h \bar{P}_{\xi_0} \bar{\Theta} = \bar{P}_{\xi_0} w^h_0 \right\}$

(2) $\bar{P}_\xi = \bar{B}_\xi \bar{\mu}_\xi,\ \xi \in \mathbb{D}, \quad (\bar{P}_\xi \bar{p}_\xi)_{\xi \in \mathbb{D}} \in \ell_1(\mathbb{D})$

(3) $\sum_{h \in \mathcal{H}} \bar{c}^h_\xi = \sum_{h \in \mathcal{H}} \bar{a}^h_\xi (e^h - \bar{\ell}^h_\xi), \quad \xi \in \mathbb{D}$

(4) $\bar{\mu}_\xi \bar{q}^\tau_\xi = \sum_{\xi' \in \xi^+} \bar{B}_{\xi\,\xi'}\, \bar{\mu}_{\xi'}\, \bar{q}^{\tau-1}_{\xi'}, \quad \tau = 1, \ldots, T, \quad \xi \in \mathbb{D}$

(5) $\bar{q}^\tau_\xi \leq 1, \quad \tau = 1, \ldots, T, \quad \xi \in \mathbb{D}$

Once the forecast $\bar{\boldsymbol{B}}$ and the short-term interest rate process $(\bar{r}^1_\xi)_{\xi \in \mathbb{D}}$ are given, (1)-(3) define a determinate "real" equilibrium $\bar{\boldsymbol{x}}$. Given that monetary policy consists both of a forecast $\bar{\boldsymbol{B}}$ and a bond pricing policy $(\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}$, equations (4)-(5) give the constraints on the simultaneous choice of $(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g})$. The equations (4) express the duality between bond prices and expectations: for given expectations they exhibit the appropriate bond prices, and for given bond prices they exhibit the compatible expectations. These equations are central to formalizing the idea of a credible inflation targeting policy. (5) ensures that the nominal interest rates on the bonds of different maturities are non-negative. If q^τ_ξ is the price of the τ-period bond, the associated interest rate or *yield to maturity* r^τ_ξ is defined by $q^\tau_\xi = \dfrac{1}{(1 + r^\tau_\xi)^\tau}$ and $q^\tau_\xi \leq 1$ is equivalent to $r^\tau_\xi \geq 0$.

Proposition 3. (Equivalence of extensive and reduced-form equilibrium) *If financial markets are complete,*[7] $\left(\text{rank}\left[\hat{\bar{q}}_{\xi^+}\right] = SG \text{ for all } \xi \in \mathbb{D}\right)$ *then* $\left((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}, \bar{\Theta})\ \bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}\right)$ *is a reduced-form equilibrium of* $\mathcal{E}$ *if and only if there exist money holdings, portfolios, taxes and prices such that* $\left(\left((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}), (\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})\right), \left((\bar{\boldsymbol{x}}, \widetilde{\bar{\boldsymbol{m}}}, \bar{\boldsymbol{z}}), \quad (\bar{\boldsymbol{y}}, \bar{\boldsymbol{L}})\right), \left(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{\omega}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_p}\right)\right)$ *is an extensive-form equilibrium of* $\mathcal{E}$.

Proof: There are essentially two parts to the proof of Proposition 3, which is given in the Appendix. The first is to show that under the condition of complete markets the opportunity sets of the agents in the two equilibria are the same. The second is to show how the portfolio-tax policy of the government can be constructed once the consumption streams of the agents and the short-term interest rate is known. It must also be shown that the way the government

[7] If there is only one agent, or equivalently all agents are identical, the condition of complete markets is not necessary for the proposition to hold.

taxes appear in the two opportunity sets—the present value of taxes $\bar{\Theta}$ and the sequence of taxes $\bar{\boldsymbol{\theta}} = (\bar{\theta}_\xi)_{\xi\in\mathbb{D}}$—are compatible.

(4) suggests that if there are enough bond prices which are fixed by the monetary authority, then there should be a unique conditional expectation $B_{\xi\xi'}, \xi' \in \xi^+$ at each node which is compatible with the bond prices. As a result agents would have no reason to adopt beliefs different from the announced forecast $\bar{\boldsymbol{B}}$. In the sections that follow we will study the exact conditions under which equations (4) tie down agents' expectations of inflation. We begin by studying the simplest case where all agents are identical and the only source of uncertainty comes from expectations of inflation: the insights from this case carry over in an important way to the general setting.

3. Stationary Equilibrium: Identical Agents and No Real Shocks

Consider the special case of the economy in Section 2 in which there are no real shocks ($G = 1$), all agents have identical preferences and endowments, $u^h = u, a^h = 1, e^h = 1$ for all $h \in \mathcal{H}$, and the only securities are the government bonds: $\mathcal{J}_p = \emptyset$. If we write the equilibrium in per-capita terms, then the equilibrium is formally equivalent to the equilibrium of Definition 1 with $H = 1$, $\gamma^h = 1$, $\mathcal{J}_p = \emptyset$.

Proposition 4. (Equilibrium equations) *If $\mathcal{E}$ an economy with identical agents, for any bounded sequence of short-term interest rates, a reduced-form equilibrium is characterized by a pair $((\bar{\boldsymbol{B}}, \bar{\boldsymbol{q}}), \bar{\boldsymbol{c}},)$ satisfying the system of equations*

(a) $\dfrac{u_c(\bar{c}_\xi, 1-\bar{c}_\xi)}{u_\ell(\bar{c}_\xi, 1-\bar{c}_\xi)} = 1 + \bar{r}^1_\xi, \quad \xi \in \mathbb{D}$

(b1) $\bar{q}^\tau_\xi = \delta \sum_{\xi'\in\xi^+} \bar{B}_{\xi\xi'} \dfrac{u_c(\bar{c}_{\xi'}, 1-\bar{c}_{\xi'})}{u_c(\bar{c}_\xi, 1-\bar{c}_\xi)} \bar{q}^{\tau-1}_{\xi'} \dfrac{1}{1+\pi_{\xi'}}, \quad \tau = 1, \dots, T, \quad \bar{q}^0_{\xi'} = 1, \quad \xi \in \mathbb{D}$

(b2) $\bar{q}^\tau_\xi \le 1, \quad \tau = 1, \dots, T, \quad \xi \in \mathbb{D}$

Proof: see Appendix.

The striking property of Proposition 4 is that the budget constraint of the representative agent does not enter in the description of the reduced-form equilibrium: the agent's budget constraint disappears since it is the mirror image of the government budget constraint which, by the Ricardian property of its fiscal policy, is automatically satisfied and hence disappears.

It clear from (a) and (b1) that a monetary policy which only determines the price of the short-term bond (or equivalently the short-term interest rate) cannot fully determine agents' expectations of inflation. The choice of a sequence of short-term interest rates $(r^1_\xi)_{\xi\in\mathbb{D}}$ determines the real allocation $(c_\xi)_{\xi\in\mathbb{D}}$ through the equations (a). But then equations (b1) applied to the short-term bond are compatible with any belief $B_{\xi\xi'}$ satisfying

$$\frac{1}{1+r^1_\xi} = \delta \sum_{\xi'\in\xi^+} B_{\xi\xi'} \frac{u_c(c_{\xi'}, 1-c_{\xi'})}{u_c(c_\xi, 1-c_\xi)} \frac{1}{1+\pi_{\xi'}}$$

This equation only puts a restriction on a weighted average of the conditional probabilities $B_{\xi\xi'}$ and hence leaves room for expectations $\boldsymbol{B}$ different from the announced forecast $\bar{\boldsymbol{B}}$. Equation (b1) when applied to long-term bonds ($\tau \geq 2$) is normally viewed as the statement that agents' expectations of inflation determine the prices of the long-term bonds. The basic argument of this paper is to reverse this logic, and to argue that if the monetary authority can control T bond prices (yields to maturity) and if T is the branching number of the event-tree (i.e. the number of inflation rates which are thought to be possible at the successors) then it can control the expectations of the agents, forcing them to coincide with the announced forecast $\bar{\boldsymbol{B}}$.

We show below that if the matrix $\hat{\bar{Q}}_{\xi^+} = (\hat{\bar{q}})_{\xi'\in\xi^+}$ consisting of the T-vector of bond prices at each of the successors of node ξ satisfies an appropriate rank condition, then knowing $\hat{\bar{Q}}_{\xi^+}$ forces the agents' expectations of inflation to coincide with $(\bar{B}_{\xi\xi'})_{\xi'\in\xi^+}$ at node ξ. In order for the agents to anticipate next period the bond prices $\hat{\bar{Q}}_{\xi^+}$, the monetary authority must have a rule for determining the price of each long-term bond as a function of the path of inflation rates: the more complex the rule the less the chance that it will be learned or verified by the agents. Thus we focus on simple rules for which bond prices depend only on the current inflation: in this way we are led to a generalization to the term structure of interest rates of the short-term interest rate rules studied in neo-Keynesian models[8]. It is clear from Proposition 4(a) that if the security prices only depend on current inflation, then the agents' consumption, which is determined by the nominal short-term interest rate, also only depends on current inflation. Since (b1) is a system of first-order difference equations, if the bond prices and the consumption only depend on current inflation and if $\bar{\boldsymbol{B}}$ is the only belief compatible with the system of equations (b1), then it has to be Markov. We thus assume that the monetary authority announces a Markov forecast matrix $\bar{B}_{ss'}$: if $\xi = (s_0, s_1, \ldots, s)$ and $\xi' = (s_0, s_1, \ldots, s, s')$ then $\bar{B}_{\xi\xi'} = \bar{B}_{ss'}$.

[8] See Woodford (2003) for an extensive exposition of short-term interest rate rules and Cochrane (2007) for a discussion of the approach.

As a special case of Proposition 4 we characterize a Markov reduced-form equilibrium $(B, \boldsymbol{c}, \boldsymbol{q})$ as a Markov matrix $B = (B_{ss'})_{s,s' \in \mathcal{S}}$, a vector of consumption $\boldsymbol{c} = (c_s, s \in \mathcal{S})$, and bond prices $\boldsymbol{q} = (q_s, s \in \mathcal{S})$ which only depend on current inflation.[9]

Corollary 5. (Stationary equilibrium equations) *If $\mathcal{E}$ is a one-agent economy, then a Markov reduced-form equilibrium is characterized by a pair* $((\bar{\boldsymbol{B}}, \bar{\boldsymbol{q}}), \bar{\boldsymbol{c}}) = \left((\bar{B}_{ss'}, \bar{q}_s), \bar{c}_s\right)_{s,s' \in \mathcal{S}} \in \mathbb{R}_+^{SS} \times \mathbb{R}_+^S \times \mathbb{R}_+^S$ *satisfying the reduced-form equilibrium equations*

(a) $\dfrac{u_c(\bar{c}_s, 1 - \bar{c}_s)}{u_\ell(\bar{c}_s, 1 - \bar{c}_s)} = 1 + r_s^1, \quad s \in \mathcal{S}$

(b1) $\bar{q}_s^\tau = \delta \sum_{s' \in S} \dfrac{\bar{B}_{ss'}}{1 + \pi_{s'}} \dfrac{u_c(\bar{c}_{s'}, 1 - \bar{c}_{s'})}{u_c(\bar{c}_s, 1 - \bar{c}_s)} \bar{q}_{s'}^{\tau - 1}, \quad \tau = 1, \ldots, T, \;\; \bar{q}_{s'}^0 = 1, \;\; s \in S$

(b2) $\bar{q}_s^\tau \leq 1, \quad \tau = 1, \ldots, T, \quad s \in \mathcal{S}$

Expectations of inflation and monetary policy. Suppose the monetary authority wants to induce the expectations of inflation defined by a Markov matrix $\boldsymbol{B}$. We can think of this as formalizing the idea of inflation targeting. The monetary authority makes public its inflation target, but this will be credible only if it indicates how its monetary policy will lead to this target. The policy instrument at its disposal consists in fixing the prices of a family of nominal bonds of different maturities, and the monetary authority makes public the rule $(q_s^\tau, \tau, \ldots, T)_{s \in \mathcal{S}}$ by which it fixes the bond prices (or equivalently the yield to maturity of these bonds) as functions of the possible current inflation states. Since monetary policy is usually expressed in terms of interest rates rather than bond prices, we will refer to it as a *term-structure rule*, although analytically it is generally simpler to work directly with the bond prices $(q_s^\tau, \tau = 1, \ldots, T)_{s \in \mathcal{S}}$.

The analysis that follows studies the properties that a Markov matrix $\boldsymbol{B}$ must satisfy if it is to be consistent with equilibrium and be credibly sustained by a term-structure rule. The matrix must have two properties:

(1) It must be such that there exist bond prices $(q_s^\tau, \tau = 1, \ldots, T)_{s \in \mathcal{S}}$ with $q_s^\tau \leq 1$ (non-negative nominal interest rates) which satisfy the equilibrium equations (a) and (b) of Corollary 5: if this property holds we say that $\boldsymbol{B}$ is *compatible with equilibrium* or more briefly is a *compatible* expectations matrix.

[9] Although the growth of money demand $\frac{M_{\xi'}}{M_\xi} = \frac{(1+i_{s'})c_{s'}}{c_s}$ is Markov, the financial variables of the extensive-form equilibrium or their rates of growth are not necessarily Markov, especially if the government's debt reimbursement policy α_ξ is not Markov.

(2) If $\boldsymbol{B}$ satisfies (1) with bond prices $(q_s^\tau, \tau = 1, \ldots, T)_{s \in \mathcal{S}}$ then these bond prices fully determine $\boldsymbol{B}$ only if $\boldsymbol{B}$ is the *unique* solution of the equilibrium equations (b1) viewed as a system of equations in $\boldsymbol{B}$ with parameters $(\boldsymbol{c}, \boldsymbol{q})$, where $\boldsymbol{c}$ satisfies (a): if this property holds we say that $\boldsymbol{B}$ is *controllable*, or that $(\boldsymbol{B}, \boldsymbol{q})$ is *strongly credible*.

The compatibility condition (1) in essence requires that the expectations of inflation be compatible with the real interest rate. For the (simplified) Fisher relation states that

$$r_s^1 = r_s^{\text{real}} + E_s(\pi)$$

Since the real interest rate r_s^{real} can be negative, expectations of inflation $E_s(\pi)$ must be sufficiently large to ensure that the nominal interest rate r_s^1 is always non-negative: as we point out after Proposition 8 this makes it difficult to implement a target inflation rate which is negative.

If (2) does not hold then many expectations matrices are compatible with the interest rate policy and there is no guarantee that agents will choose the announced forecast $\boldsymbol{B}$ as their expectations, so that the objective of inflation targeting may not be achieved.

Compatibility of expectations with equilibrium. If the monetary authority wants to induce a matrix of beliefs $\boldsymbol{B}$ it needs to set the term-structure rule in such a way that equations (b) of Corollary 5 are satisfied. Since these equations involve both the expectations matrix $\boldsymbol{B}$ and the stochastic discount factor which, by the equations (a), is determined by the short-term interest rates $\boldsymbol{r}^1 = (r_s^1, s \in \mathcal{S})$, there is a compatibility or fixed-point problem which needs to be solved. To study this problem it is convenient to use the gross returns

$$R_s = 1 + r_s^1$$

on the short-term nominal bonds as the basic variables. We first show that for a given return R there is a unique solution to the consumption/leisure choice problem of the agent characterized by equation (a) of Corollary 5.

Lemma 6. (Equilibrium consumption) *If u satisfies Assumption $\mathcal{U}$(1)-(3) then*

(i) *for all $R > 0$, the equation*

$$\frac{u_c(c, 1-c)}{u_\ell(c, 1-c)} = R \tag{10}$$

has a unique solution $c(R)$, where $c(R)$ is a strictly decreasing function of R.

(ii) $\Phi(R) \equiv u_c(c(R), 1 - c(R))$ *is strictly increasing on* $(0, \infty)$.

(iii) $\widetilde{\Phi}(R) \equiv \dfrac{\Phi(R)}{R}$ *is strictly decreasing on* $(0, \infty)$.

Proof. (i) Let $h(c) \equiv \dfrac{u_c(c, 1-c)}{u_\ell(c, 1-c)}$. Then $h'(c) = \frac{1}{u_\ell^2}(u_{cc}u_\ell - u_{\ell c}u_c - u_{c\ell}u_\ell + u_{\ell\ell}u_c)$. Since $u_{cc} < 0, u_{\ell\ell} < 0, u_c > 0, u_\ell > 0, u_{c\ell} \geq 0$, it follows that $h'(c) < 0$ and h is decreasing. By the Inada condition $h(c) \to \infty$ as $c \to 0$ and $h(c) \to 0$ as $c \to 1$. Thus (10) has a unique solution $c(R)$. Differentiating $h(c(R)) = R$ gives $h'(c(R))c'(R) = 1$: $h' < 0$ implies $c'(R) < 0$.

(ii) $\Phi'(R) = (u_{cc} - u_{c\ell})c'(R) > 0$ by (i).

(iii) $\widetilde{\Phi}'(R) = \frac{1}{R^2}\Big((u_{cc} - u_{c\ell})c'(R)\, R - u_c\Big)$. Using $c'(R) = 1/h'(c(R))$ where $h'(c)$ has been calculated in (i), and $R = u_c/u_\ell$, we obtain $\widetilde{\Phi}'(R) = \frac{u_c^2}{D}(u_{c\ell} - u_{\ell\ell}) < 0$, where the derivatives are calculated at $c(R)$ and $D = R^2(u_{cc}u_\ell - u_{\ell c}u_c - u_{c\ell}u_\ell + u_{\ell\ell}u_c)$. Since $D < 0$ and $u_{c\ell} \geq 0$, $\widetilde{\Phi}'(R) < 0$. □

In view of Lemma 6 the FOCs (a) and the FOCs (b1) for the short-term nominal bond of Corollary 5, can be combined into the system of equilibrium equations

$$\frac{1}{R_s} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1 + \pi_{s'}} \frac{\Phi(R_{s'})}{\Phi(R_s)}, \quad s \in S \tag{11}$$

For each $s \in S$ this is the 'true' stochastic Fisher equation relating the price $q_s^1 = \frac{1}{R_s}$ of the short-term nominal bond to the price of the *real* bond

$$\frac{1}{1 + r_s^{\text{real}}} = \delta \sum_{s' \in S} B_{ss'} \frac{\Phi(R_{s'})}{\Phi(R_s)}, \quad s \in S$$

and inflation $(\pi_{s'})_{s' \in S}$ next period, when the current inflation is π_s. Since the nominal interest rate determines the real wage and hence output and consumption, it affects the real interest rate and the system of equations (11) only implicitly defines the nominal interest rates associated with an expectations matrix B. If the condition $R_s \geq 1, s \in S$ (non-negative nominal interest rates) were omitted then the equations (11) would always have a solution (this can be deduced from the fixed point argument given below). However when the condition $R_s \geq 1, s \in S$, is imposed, conditions of compatibility have to be imposed on the matrix B.

Definition 7. A Markov matrix $\boldsymbol{B}$ is said to be *compatible* with equilibrium if the system of equations (11) has a solution $\boldsymbol{R} \geq \boldsymbol{1} = (1, \ldots, 1)$.

Note that if $\bar{\boldsymbol{R}} \geq \boldsymbol{1}$ is a solution of (11) then $\bar{q}_s^1 = \frac{1}{R_s} \leq 1$, $s \in S$, and (b2) of Corollary 5 is satisfied for $\tau = 1$. (b1) of the same corollary, which can be written as

$$\bar{q}_s^\tau = \delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} \frac{\Phi(R_{s'})}{\Phi(R_s)} \bar{q}^{\tau-1}, \quad s \in S, \quad \tau = 2, \dots, T$$

then gives by successive substitution the prices of the bonds of higher maturities and the inequality $\bar{q}_s^\tau \leq 1$ is transferred to these prices. Thus all the conditions of Corollary 5 are satisfied and there exists a reduced form equilibrium.

Using the function $\widetilde{\Phi}$ defined in Lemma 6(iii), (11) can be written as

$$\widetilde{\Phi}(R_s) = \delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} \Phi(R_{s'}), \quad s \in S \tag{12}$$

Since $\widetilde{\Phi}$ is decreasing, if for each s the right side of (12) lies in the image of $\widetilde{\Phi}$ (a condition for this is given below) then $\widetilde{\Phi}$ can be inverted and (12) is equivalent to the system of equations

$$R_s = \widetilde{\Phi}^{-1}\Big(\delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} \Phi(R_{s'})\Big) \equiv \Psi_s(R_1, \dots, R_S), \quad s \in S \tag{13}$$

where Ψ_s is decreasing for each $s \in S$. Let $\boldsymbol{\Psi} = (\Psi_1, \dots, \Psi_S)$ denote the vector-valued map which associates with $\boldsymbol{R}$ the new vector of returns $\boldsymbol{\Psi}(\boldsymbol{R})$. An equilibrium $\bar{\boldsymbol{R}}$ is a fixed point of $\boldsymbol{\Psi}$: $\bar{\boldsymbol{R}} = \boldsymbol{\Psi}(\bar{\boldsymbol{R}})$.

Since a vector of nominal returns must satisfy $\boldsymbol{R} \geq \boldsymbol{1} = (1, \dots, 1)$ and since $\boldsymbol{\Psi}$ is decreasing, the minimal return vector $\boldsymbol{1}$ maps into the maximal return vector $\boldsymbol{R}^{\max} = (R_1^{\max}, \dots, R_S^{\max}) = \boldsymbol{\Psi}(\boldsymbol{1})$. Consider the rectangular subset of the non-negative orthant of $\mathbb{R}^S$

$$K = \{\boldsymbol{R} \in \mathbb{R}_+^S \mid \boldsymbol{1} \leq \boldsymbol{R} \leq \boldsymbol{R}^{\max}\}$$

If $\boldsymbol{R}^{\max} \geq \boldsymbol{1}$ then $K \neq \phi$, and if $\boldsymbol{\Psi}(\boldsymbol{R}^{\max}) = \boldsymbol{\Psi}(\boldsymbol{\Psi}(\boldsymbol{1})) \geq \boldsymbol{1}$ then $\boldsymbol{\Psi}(K) \subset K$ so that Brouwer's Theorem can be applied.

It remains to give conditions which ensure that the two properties $K \neq \phi$ and $\boldsymbol{\Psi}(K) \subset K$ are satisfied. The maximum achievable consumption c^* occurs when the nominal interest rate is zero, $c^* = c(1)$: this is also what an agent's consumption would be without a cash-in-advance constraint. $\boldsymbol{R}^{\max} = \boldsymbol{\Psi}(\boldsymbol{1})$ is equivalent to

$$\frac{1}{R_s^{\max}} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} \frac{u_c(c^*, 1-c^*)}{u_c(c(R_s^{\max}), 1-c(R_s^{\max}))}, \quad s \in S \tag{14}$$

(14) must have a solution for each s, and this solution must satisfy $R_s^{\max} \geq 1$. Condition (1) in Proposition 8 below ensures that these two properties hold: the first inequality implies that

$R_s^{\max}$ exists and the second inequality ensures that it is greater than or equal to 1. The right side of (14) gives an upper bound on the real interest rate since it assumes that the consumption in each state s' next period is maximal (at c^*) while it is minimal today (at $c(R_s^{\max})$). Thus $R_s^{\max} \geq 1$ requires that the nominal interest rate, which is essentially the real interest rate plus the expected rate of inflation, be positive when the real interest rate is at its highest possible value. This is clearly a necessary condition. The condition $E_s^B\left(\frac{\delta}{1+\pi}\right) \leq 1$, which ensures that $R_s^{\max} \geq 1$ is not a demanding condition, but it still requires that high deflation states are not given too much weight.

To find a condition which ensures that the vector $\boldsymbol{\Psi}(\boldsymbol{\Psi}(\mathbf{1})) = \boldsymbol{\Psi}(\boldsymbol{R}^{\max}) \geq \mathbf{1}$, consider the vector $\boldsymbol{R}^{\min} = (R_1^{\min}, \ldots, R_S^{\min})$ where $R_s^{\min}$ defined by

$$\frac{1}{R_s^{\min}} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} \frac{u_c(c(R_{s'}^{\max}), 1 - c(R_{s'}^{\max}))}{u_c(c^*, 1-c^*)}, \quad s \in S \tag{15}$$

The return $R_s^{\min}$ would occur if consumption today were maximal (at c^*) and consumption tomorrow were expected to be at its minimal value $c(R_{s'}^{\max})$ in each state s': this gives a lower bound on the real interest rate in each state. Condition (2) in Proposition 8 requires that the nominal interest rate is positive even when the real interest rate is at this lower bound, a more demanding requirement than condition (1).

Proposition 8. (Existence of stationary equilibrium) *If $\boldsymbol{B}$ is a $S \times S$ Markov matrix such that*

(1) $\lim_{R_s \to \infty} \frac{u_c(c(R_s), 1-c(R_s))}{R_s u_c(c^*, 1-c^*)} < E_s^B\left(\frac{\delta}{1+\pi}\right) \leq 1, \; s \in S$

(2) $R_s^{\min} \geq 1, \; s \in S$ *where $R_s^{\min}$ is defined by (15) and $c(R_s^{\max})$ is defined by (14)*

then there exists a stationary equilibrium of $\mathcal{E}$, and $\boldsymbol{B}$ is a compatible expectations matrix.

Proof: Equation (14) is equivalent to

$$\frac{\widetilde{\Phi}(R_s^{\max})}{\widetilde{\Phi}(1)} = a_s, \qquad a_s = E_s^B\left(\frac{\delta}{1+\pi}\right)$$

Since $\widetilde{\Phi}$ is decreasing, if $a_s \leq 1$ (which is the second inequality in (1)) then the solution, if it exists, will satisfy $R_s^{\max} \geq 1$. The equation will have a solution if $a_s\widetilde{\Phi}(1) > \inf_{R \geq 1} \widetilde{\Phi}(R) = \lim_{R \to \infty} \widetilde{\Phi}(R)$, which is the first inequality in (1). This proves that, when (1) is satisfied, $K \neq \phi$.

It remains to show that $\boldsymbol{\Psi}(\boldsymbol{R}^{\max}) \geq \mathbf{1}$ to ensure $\boldsymbol{\Psi}(K) \subset K$. For each state $s \in S$

$$\Psi_s(\boldsymbol{R}^{\max}) = \widetilde{\Phi}^{-1}\Big(\delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} \Phi(R^{\max}_{s'})\Big) \geq 1 \iff \delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} \Phi(R^{\max}_{s'}) \leq \widetilde{\Phi}(1) = \frac{\Phi(1)}{1}$$
$$\iff \frac{1}{R^{\min}_s} \leq 1$$

By Brouwer's Theorem $\boldsymbol{\Psi}$ has a fixed point $\bar{\boldsymbol{R}}$ in K which defines a positive short-term interest rate compatible with the expectations matrix $\boldsymbol{B}$. □

As is well known, in the deterministic version of our economy with a cash-in-advance constraint, the Friedman rule, which requires that the nominal interest rate is zero, is Pareto optimal. The associated inflation rate $\pi^* = \delta - 1$ is negative. Anticipating the analysis of controllability of Proposition 11, which requires that there be a minimum of fluctuation in inflation for expectations to be controllable, Proposition 8 shows that in a stochastic economy it is not possible to implement an approximate version of the Friedman rule with an inflation target $\pi^* = \delta - 1$ without creating much variability in inflation. For if the target inflation rate is set at π^* there will be states where the deflation π_s will be lower than π^*. An expectations matrix which mainly puts weights on the deflation rates between π_s and π^* to induce a reversion to the target rate will not satisfy (1) of Proposition 8 since $\frac{\delta}{1+\pi*} = 1$ and $\frac{\delta}{1+\pi_s} > 1$. To satisfy (1) and (2) sufficient weight must be given to inflation rates strictly greater than π^*, creating realized trajectories of inflation which alternate between inflation and deflation. Thus even if the Friedman rule is optimal in the deterministic framework, when agents have "sunspot beliefs" it may be not be optimal to set a target with negative inflation since in the domain of deflation it becomes difficult to control agents' expectations while maintaining non-negative nominal interest rates—the monetary authority is in essence caught in a "liquidity trap".

Controllability of expectations. Suppose that $\boldsymbol{B}$ is a compatible expectations matrix which the monetary authority wants to induce as the only expectations for the agents. Then the short-term interest rate must be fixed in such a way that the equilibrium equations (11) are satisfied. If $\boldsymbol{r}^1(\boldsymbol{B}) = (r^1_1(\boldsymbol{B}), \ldots, r^1_S(\boldsymbol{B}))$ is such a short-run interest rate rule announced by the Central Bank, then there are many Markov matrices $\widetilde{\boldsymbol{B}} = (\widetilde{B}_{ss'})_{ss' \in \mathcal{S}}$ which also satisfy

$$\frac{1}{1+r^1_s} = \delta \sum_{s' \in S} \widetilde{B}_{ss'} \frac{u_c(c(R_{s'}(\boldsymbol{B})), 1 - c(R_{s'}(\boldsymbol{B})))}{u_c(c(R_s(\boldsymbol{B})), 1 - c(R_s(\boldsymbol{B})))} \frac{1}{1+\pi_{s'}}, \quad s \in \mathcal{S} \tag{16}$$

since this is a system of S linear equations in the $S \times (S-1)$ unknown coefficients of the Markov matrix. If $\widetilde{\boldsymbol{B}}$ satisfies (16) then the same collection of short-run equilibrium interest rates is compatible with $\widetilde{\boldsymbol{B}}$: $\boldsymbol{r}^1(\boldsymbol{B}) = \boldsymbol{r}^1(\widetilde{\boldsymbol{B}})$. Thus $\boldsymbol{B}$ and $\widetilde{\boldsymbol{B}}$ are two different expectations of inflation compatible with the same short-term interest rate rule, so that agents can have their own views $\widetilde{\boldsymbol{B}}$ of the transition probabilities of inflation which do not need to coincide with the Central Bank's announced expectations $\boldsymbol{B}$.

If the CB controls more interest rates on bonds of longer maturities then there will be more no-arbitrage equations similar to (16) which an alternative matrix $\widetilde{\boldsymbol{B}}$ will need to satisfy to be compatible with the given interest rate rule. It may be possible, by fixing sufficiently many interest rates, to restrict the expectations to the unique matrix $\boldsymbol{B}$, but this requires that the equations of compatibility with prices of the bonds of different maturities be independent.

The interest rates on the long-term bonds which are compatible with $\boldsymbol{B}$ can be calculated recursively. First the prices $q_s^1(\boldsymbol{B}) = \frac{1}{1+r_s^1(\boldsymbol{B})} = \frac{1}{R_s(\boldsymbol{B})}$ are calculated by solving the fixed-point equations (11), which is possible if $\boldsymbol{B}$ is a compatible expectations matrix. Then the prices of the two-period bonds across the inflation states are deduced from the prices of the one-period bonds by

$$q_s^2(\boldsymbol{B}) = \delta \sum_{s' \in S} B_{ss'} \frac{u_c(c(R_{s'}(\boldsymbol{B})), 1 - c(R_{s'}(\boldsymbol{B})))}{u_c(c(R_s(\boldsymbol{B})), 1 - c(R_s(\boldsymbol{B})))} \frac{1}{1+\pi_{s'}} q_{s'}^1(\boldsymbol{B}), \quad s \in \mathcal{S}$$

Replacing $q_{s'}^1(\boldsymbol{B})$ by $q_{s'}^2(\boldsymbol{B})$ and $q_s^2(\boldsymbol{B})$ by $q_s^3(\boldsymbol{B})$ in this equation gives the price of the three-period bond in each inflation state, and so on up to maturity T chosen be the Central Bank. To write the prices of all the bonds up to maturity T in a condensed form, define the diagonal matrices

$$D_1(\boldsymbol{B}) = \begin{bmatrix} \frac{1}{\Phi(R_1(\boldsymbol{B}))} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\Phi(R_S(\boldsymbol{B}))} \end{bmatrix} \quad D_2(\boldsymbol{B}) = \begin{bmatrix} \frac{\delta\Phi(R_1(\boldsymbol{B}))}{1+\pi_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\delta\Phi(R_S(\boldsymbol{B}))}{1+\pi_S} \end{bmatrix}$$

where Φ is the function defined earlier in Lemma 6: $\Phi(R_s(\boldsymbol{B})) = u_c\Big(c(R_s(\boldsymbol{B})), 1-c(R_s(\boldsymbol{B})\Big)$, $s \in S$. Then the matrix $\Delta(\boldsymbol{B})$ defined by

$$\Delta(\boldsymbol{B}) = D_1(\boldsymbol{B})\boldsymbol{B}D_2(\boldsymbol{B}) \tag{17}$$

is the *matrix of (date t) present-value prices* between any pair of dates t and $t+1$: omitting the dependence on $\boldsymbol{B}$, the term $\Delta_{ss'}$ in row s and column s' gives the present value in inflation state s at date t of the promise to pay one unit of money at date $t+1$ in inflation state s'. Using this present-value matrix the price of the short-term bond in state s can be written as

$q_s^1 = \sum_{s' \in S} \Delta_{ss'} = \Delta_s \mathbf{1}$ where $\Delta_s = (\Delta_{s1}, \ldots, \Delta_{sS})$ is row s of Δ and $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^S$. Let $\boldsymbol{q}^1 = (q_1^1, \ldots, q_S^1)'$ denote the vector of short-term bond prices across the inflation states then

$$\boldsymbol{q}^1(\boldsymbol{B}) = \Delta(\boldsymbol{B})\mathbf{1}$$

From Corollary 5 (b1), if $\boldsymbol{q}^2 = (q_1^2, \ldots, q_S^2)'$ denotes the vector of two-period bond prices across the inflation states then

$$\boldsymbol{q}^2(\boldsymbol{B}) = \Delta(\boldsymbol{B})\boldsymbol{q}^1(\boldsymbol{B}) = \Delta^2(\boldsymbol{B})\mathbf{1}$$

and more generally the vector of prices for the τ period bond across the states is given by

$$\boldsymbol{q}^\tau(\boldsymbol{B}) = \Delta(\boldsymbol{B})\boldsymbol{q}^{\tau-1}(\boldsymbol{B}) = \Delta^\tau(\boldsymbol{B})\mathbf{1}, \qquad \tau = 1, \ldots, T \tag{18}$$

If the Central Bank fixes the prices of the long-term bonds of several maturities then it may further restrict the expectations matrices which are compatible with the interest rate rule. If the originally announced forecast $\boldsymbol{B}$ is the only matrix compatible with these bond prices then we say that the combined announcement of $\boldsymbol{B}$ and the term-structure rule is strongly credible.

Definition 9. Let $\boldsymbol{B}$ be a compatible expectations matrix and let $\boldsymbol{q}(\boldsymbol{B}) = (\boldsymbol{q}^1(\boldsymbol{B}), \ldots, \boldsymbol{q}^T(\boldsymbol{B}))$ be a compatible term-structure rule satisfying (18). The monetary policy $(\boldsymbol{B}, \boldsymbol{q}(\boldsymbol{B}))$ is said to be *strongly credible* if $\widetilde{\boldsymbol{B}} = \boldsymbol{B}$ is the only solution to the system of linear equations

$$q_s^\tau(\boldsymbol{B}) = \sum_{s' \in \mathcal{S}} \widetilde{B}_{ss'} \frac{\lambda_{ss'}(\boldsymbol{R}(\boldsymbol{B}))}{1 + \pi_{s'}} q_{s'}^{\tau-1}(\boldsymbol{B}), \qquad s \in \mathcal{S}, \quad \tau = 1, \ldots, T \tag{19}$$

where

$$\lambda_{ss'}(\boldsymbol{R}(\boldsymbol{B})) \equiv \frac{\delta\, u_c(c(R_{s'}(\boldsymbol{B})), 1 - c(R_{s'}(\boldsymbol{B})))}{u_c(c(R_s(\boldsymbol{B})), 1 - c(R_s(\boldsymbol{B})))}$$

is the real stochastic discount factor induced by the short-term interest rate rule $\boldsymbol{r}^1(\boldsymbol{B})$.

Definition 10. A Markov matrix $\boldsymbol{B}$ is said to be a *controllable* expectations matrix if there is an associated term-structure rule $\boldsymbol{q}(\boldsymbol{B})$ such that $(\boldsymbol{B}, \boldsymbol{q}(\boldsymbol{B}))$ is strongly credible.

The requirement of strong credibility is essential if the Central Bank wants to be sure to control the expectations of agents in the private sector when it announces its forecast $\boldsymbol{B}$. Supporting the forecast—as is currently done—by a compatible short-term interest rate rule only makes the forecast "weakly credible", in the sense that the agents may believe the forecast since the interest-rate rule is compatible with it, but they can also have their own expectations $\widetilde{\boldsymbol{B}}$ on the process of inflation which is compatible with $\boldsymbol{r}^1(\boldsymbol{B})$ but different from $\boldsymbol{B}$. The need to fix the prices of additional long-term bonds to support the announced forecast is made precise by the following proposition.

Proposition 11. (Controlling expectations) *Let $(\boldsymbol{B}, \boldsymbol{q}(\boldsymbol{B}))$ be a monetary policy consisting of a forecast $\boldsymbol{B}$ and a term-structure rule $\boldsymbol{q}(\boldsymbol{B}) = (\boldsymbol{q}^1(\boldsymbol{B}), \ldots, \boldsymbol{q}^T(\boldsymbol{B}))$ satisfying (18). If the payoff matrix*

$$\widehat{Q} = \left[\mathbf{1}, \Delta(\boldsymbol{B})\mathbf{1}, \ldots, (\Delta(\boldsymbol{B}))^{S-1}\mathbf{1}\right] \tag{20}$$

is invertible, then the policy $(\boldsymbol{B}, \boldsymbol{q}(\boldsymbol{B}))$ is strongly credible and $\boldsymbol{B}$ is a controllable expectations matrix.

Proof: Let $\widetilde{\boldsymbol{B}}$ be a matrix satisfying the system of equations (19). This can be written as

$$Q = \widetilde{\Delta}\,\widehat{Q}$$

where Q is the matrix of bond prices $Q = [\boldsymbol{q}^1(\boldsymbol{B}), \ldots, \boldsymbol{q}^T(\boldsymbol{B})]$ and $\widetilde{\Delta} = D_1(\boldsymbol{B})\widetilde{\boldsymbol{B}}D_2(\boldsymbol{B})$. Since $\boldsymbol{B}$ also satisfies the equations (19), $\widetilde{\Delta} = Q\widehat{Q}^{-1} = \Delta(\boldsymbol{B})$. But then $D_1(\boldsymbol{B})\widetilde{\boldsymbol{B}}D_2(\boldsymbol{B}) = D_1(\boldsymbol{B})\boldsymbol{B}D_2(\boldsymbol{B})$ implies $\widetilde{\boldsymbol{B}} = \boldsymbol{B}$. □

In order that the rank condition $\text{rank}(\widehat{Q}) = S$ can be satisfied, the monetary authority must fix the prices of S bonds,[10] and for any possible current inflation the payoff matrix of these bonds in the different inflation states next period must have full rank, i.e. markets must be dynamically complete. This is expressed as the requirement that the transforms of the sure payoff $\mathbf{1}$ under iterates of the present-value map $\Delta(\boldsymbol{B})$ are linearly independent. Note that the condition of complete markets, which usually ensures that there is optimal risk sharing, is used here for another purpose. In this representative-agent model, risk sharing is not the issue: controlling expectations is the issue, and Proposition 10 points to the fact that the more markets there are for long-term bonds, the less divergent the expectations of the market participants can be.

Propositions 8 and 11 impose conditions on a Markov matrix $\boldsymbol{B}$ for it to represent expectations which are sustainable by a term-structure rule. To better understand these restrictions

[10] In principle this condition can be weakened by requiring that the Central bank controls the prices of only $S-1$ bonds, using the equations $\widetilde{\boldsymbol{B}}\mathbf{1} = \mathbf{1}$ which must be satisfied if the matrix $\widetilde{\boldsymbol{B}}$ is to be a Markov matrix. This works only if the equations $\widetilde{\boldsymbol{B}}\mathbf{1} = \mathbf{1}$ are independent of the other bond pricing equations (19) for $\tau = 1, \ldots, S-1$. Since it is not easy to give conditions which ensure that this property of independence holds, we prefer to require that the rank condition which implies uniqueness of $\widetilde{\boldsymbol{B}}$ is obtained from the full rank condition on the payoff matrix of the bonds.

consider the simple case where the agent's utility function is quasi-linear in consumption

$$u(c, \ell) = c + v(\ell) \tag{21}$$

where v is increasing, differentiable, strictly concave and satisfies $v'(\ell) \to \infty$ when $\ell \to 0$ and $v'(\ell) \to 0$ when $\ell \to 1$. The FOC (a) in Corollary 5

$$v'(1-c) = \frac{1}{1+r_s} = \frac{1}{R_s}$$

defines the optimal consumption $\bar{c}(R_s)$ as a function of the current nominal gross return R_s of the short-term bond. Since $u_c(\bar{c}(R_s), 1-\bar{c}(R_s)) = 1$, the pricing of the bonds is risk neutral. In the notation introduced above, $\Phi(R) = 1$, $\tilde{\Phi}(R) = \frac{1}{R}$. Thus the pricing equations reduce to

$$\frac{1}{R_s} = \delta \sum_{s' \in S} \frac{B_{ss'}}{1+\pi_{s'}} = \delta E_s^B \left(\frac{1}{1+\pi} \right), \qquad s \in S$$

and $R_s^{min} = R_s^{max} = R_s$. Conditions 1 and 2 of Proposition 8 reduce to

$$\delta E_s^B \left(\frac{1}{1+\pi} \right) \leq 1, \qquad s \in S \tag{22}$$

If the smallest inflation state π_1 which agents consider possible satisfies $\frac{\delta}{1+\pi_1} \leq 1 \iff \pi_1 \geq \delta - 1$, then (22) is satisfied for any Markov matrix $\boldsymbol{B}$, so that all Markov matrices are 'compatible' expectations matrices. The condition starts to bite if $1 + \pi_1 < \delta$. Then any row which puts weight on the lowest deflation states must compensate by putting sufficient positive weight on positive (or at least less negative) inflation states to ensure that the nominal interest rate is non-negative: in short, in order that expectations of large deflation be compatible with equilibrium they must be accompanied by the expectations of regular occurrence of periods of inflation.

Consider the restrictions on $\boldsymbol{B}$ imposed by the rank condition in Proposition 11. The date t present-value matrix $\Delta(\boldsymbol{B})$ defined by (17) becomes

$$\Delta(\boldsymbol{B}) = \delta\, \boldsymbol{B} \operatorname{diag}\left(\frac{1}{1+\pi} \right) = \begin{bmatrix} \frac{\delta B_{11}}{1+\pi_1} & \cdots & \frac{\delta B_{1S}}{1+\pi_S} \\ \vdots & \vdots & \vdots \\ \frac{\delta B_{S1}}{1+\pi_1} & \cdots & \frac{\delta B_{SS}}{1+\pi_S} \end{bmatrix}$$

where $\operatorname{diag}(v)$ denotes the diagonal matrix whose diagonal elements are the coordinates of the vector v. Since $\Delta(\boldsymbol{B})(\boldsymbol{1}+\boldsymbol{\pi}) = \delta \boldsymbol{B}\, \boldsymbol{1} = \delta \boldsymbol{1}$, the vector $\boldsymbol{1}$ is in the range of $\Delta(\boldsymbol{B})$ and so are all the vectors $\Delta^n(\boldsymbol{B})\boldsymbol{1}$, for $1 \leq n \leq S-1$. Thus the rank condition can hold only if the subspace spanned by the columns of the matrix $\Delta(\boldsymbol{B})$ is of dimension S, i.e. if the matrix $\Delta(\boldsymbol{B})$ is invertible. But this is equivalent to the matrix $\boldsymbol{B}$ being invertible. Thus for example matrices which require that, whatever the initial inflation state, inflation in the next period is the desired inflation state $\pi^* = \pi_{s^*}$, of the form

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$$

do not satisfy the controllability condition of Proposition 11, and hence can not be sustained by a term-structure rule of the type we consider. Our framework cannot force an immediate return of inflation to a target inflation rate π^*.

The requirement that $\boldsymbol{B}$ has full rank also means that the rows of $\boldsymbol{B}$ must be different. Since $\Delta(\boldsymbol{B})\mathbf{1} = \left(E_s\left(\frac{1}{1+\pi}\right)\right)_{s\in S}$ the condition that $\Delta(\boldsymbol{B})\mathbf{1}$ is independent of $\mathbf{1}$ requires that the rows of the matrix $\boldsymbol{B}$ must be sufficiently different: thus the conditional expectation of inflation next period given the current inflation state must change in a systematic way as the current inflation state changes. Some permanence in the matrix $\boldsymbol{B}$—sufficient weight on diagonal and near diagonal terms—seems a reasonable way to ensure that this condition holds. The additional conditions that $\Delta^2(\boldsymbol{B})\mathbf{1}, \ldots, \Delta^{S-1}(\boldsymbol{B})$ are independent, and independent of $\mathbf{1}$ and $\Delta(\boldsymbol{B})\mathbf{1}$, reinforce this need for the rows of $\boldsymbol{B}$ to be "sufficiently different".

Example. To illustrate how a term-structure policy can be used to control agents' expectations of inflation, suppose that the interval of inflation rate is $\Pi = \{-0.02, -0.01, \ldots, 0.06\}$, by increments of 0.01 so that there are 9 inflation rates indexed by $s = 1, \ldots, 9$. Suppose the target rate is $\pi^* = 0.02$, corresponding to the index $s^* = 5$, and that the monetary authority seeks to return inflation to the target when the current rate π_s deviates π^* by choosing the mean-reverting expectations matrix B defined as follows

for $s = 1$		$B(1,1) = \frac{b}{s^*}$,	$B(1,2) = 1 - \frac{b}{s^*}$,	$B(1,\sigma) = 0$ otherwise
for $1 < s < s^*$	$B(s,s-1) = \frac{ab}{s^*-s+1}$,	$B(s,s) = \frac{(1-a)b}{s^*-s+1}$,	$B(s,s+1) = 1 - \frac{b}{s^*-s+1}$,	$B(s,\sigma) = 0$ otherwise
for $s = s^*$	$B(s^*,s^*-1) = ab$,	$B(s^*,s^*) = 1-2ab$,	$B(s^*,s^*+1) = ab$,	$B(s^*,\sigma) = 0$ otherwise
for $s^* < s < 9$	$B(s,s-1) = \frac{b}{s-s^*+1}$,	$B(s,s) = \frac{(1-a)b}{s-s^*+1}$,	$B(s,s+1) = \frac{ab}{s-s^*+1}$,	$B(s,\sigma) = 0$ otherwise
for $s = 9$	$B(s,s-1) = 1 - \frac{b}{s-s^*+1}$,	$B(s,s) = \frac{b}{s-s^*+1}$,		$B(s,\sigma) = 0$ otherwise

with two parameters $a \geq 0, b > 0$. a is the "noise" which helps with the rank of the matrix and b represents the intensity of the reversion to π^*. For every inflation rate π_s in the interior of Π ($s \neq 1, 9$), with some probability the inflation rate is unchanged, with smaller probability it moves to "the wrong side" ($s-1$ if $s < s*$ or $s+1$ if $s > s^*$), and with greater probability it moves toward the target s^*, the probability being proportional to the deviation of s to s^*. The smaller b the higher the probability of moving toward the target rate π^*. Suppose the preferences are given by $u(c, \ell) = (c^{1-\alpha} + \beta \ell^{1-\alpha})$ with $\alpha = 3, \beta = 0.4$ and a discount factor $\delta = 0.98$.

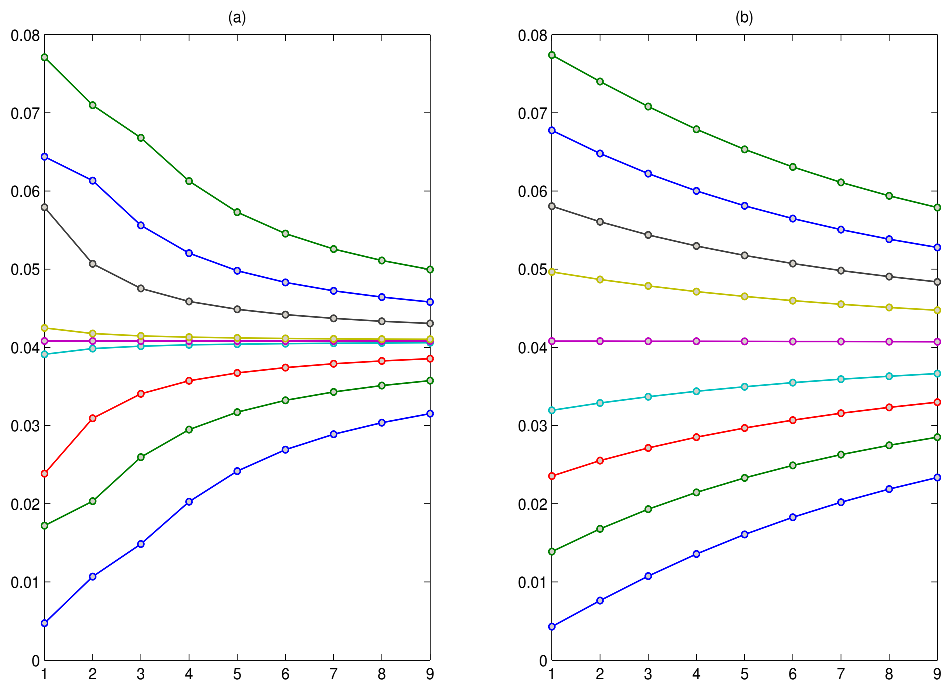

Figure 2: Term structure rules associated to the expectations matrix B in the example. In Figure(a) b=0.2; in Figure (b), b=1.5. Each curve represents the term structure $(r_s^\tau)_{\tau=1}^9$ for a given current inflation rate π_s.

Figure 2 shows the term structure for the bond prices defined in (18) for two values of the intensity parameter $b = 0.2$ (Figure (a)) and $b = 1.5$ (Figure (b)). Each curve represents the term structure $(r_s^\tau)_{\tau=1}^9$ for a given current inflation rate π_s: the top curve corresponds to $\pi_s = 6\%$ and the lowest curve to $\pi_s = -2\%$. For both values of the parameter when current inflation is at the target rate π^* the short term nominal interest rate is close to 4%, the real interest rate being close to $1/\delta - 1$ namely 2.04% and the expected inflation rate next period being 2%. In view of the symmetry of B and the approximate symmetry of interests rates r_s^1 around $r_{s^*}^1$, the long run average of the short term interest rate $\bar{r}^1$ is also approximately 4%. Since the fluctuations in consumption induced by the fluctuations in the nominal interest rates are small the risk premia are small and the yield to maturity r_s^τ on τ-period bond converges to r^∞ which also close to 4%.

In Figure 1(a) the term structure rises fast if there is deflation or low inflation ($\pi_s < \pi^*$) and decreases fast when current inflation is high ($\pi_s > \pi^*$), leading agents to anticipate a fast return to the target inflation rate of 2%. In Figure 1(b), for each current inflation rate the term structure is flatter, leading agents to anticipate more permanence and a sluggish return to the target rate.

Relation to literature. In this section we have outlined a representative-agent model of the term structure of interest rates parameterized by the expectations of the agent. Models of this kind have been studied with the objective of best replicating the data, rather than as a way of studying how the term structure can be used as an instrument of monetary policy. The qualitative and quantitative characteristics of the term structures generated by such models under different assumptions on the primitives are surveyed in Piazzesi-Schneider (2007): they show that a combination of Eipstein-Zin recursive utility for the representative agent and realistic assumptions of the inflation/consumption growth process—in particular permanence in both inflation and growth, and the fact that high inflation tends to be followed by low consumption growth—generates a term structure which is on average upward sloping. This is a stylized fact which previous representative-agent models have had difficulty replicating. In our model, as shown by the example above, if there are no real shocks and the monetary authority succeeds in keeping inflation close to the target "most of the time" then the average term structure will approximately be flat. If there are real shocks and inflation is kept close to the target then the term structure should essentially depend on the process of consumption growth.

4. Stationary Equilibrium: Heterogeneous Agents and Real Shocks

In this section we generalize the analysis of the previous section to multi-agent economies with both inflation and real shocks. Let $\eta = (s, g) \in \mathcal{S} \times \mathcal{G}$ identify the current inflation π_s and the real shock g which determines the productivities $a_g^h = a_\eta^h$ of the agents $h \in \mathcal{H}$.

The exogenous shocks are assumed to have a Markov structure described by a $G \times G$ matrix $\boldsymbol{A} = (A_{gg'})_{g,g' \in \mathcal{G}}$. We assume that the forecast of the Central Bank is given by a transition matrix $\boldsymbol{B} = (B_{\eta\eta'})_{\eta,\eta' \in \mathcal{S}\times\mathcal{G}}$ compatible with $\boldsymbol{A}$. As in the previous section we assume that the term structure rule $(r^\tau_\xi)_{\tau=1,\ldots,T}$ only depends on the current state η, so that the rule is of the form $(r^\tau_\eta)_{\tau=1,\ldots,T}$ for each $\eta \in \mathcal{S} \times \mathcal{G}$.

An equilibrium of an economy with heterogeneous agents and a cash-in-advance constraint is not Pareto optimal: there is thus no social welfare function which is maximized at an equilibrium, and in this sense no representative agent. However what is really needed to derive properties of an equilibrium with security markets is a common stochastic discount factor for pricing the securities which only depends on the aggregate state of the economy. In Proposition 12 we show that we can exploit the property that the marginal rates of substitution of the agents are equalized—they all face the same prices and the nominal interest rate distorts the real wage in the same way for all agents—to derive a *social marginal utility of consumption* at equilibrium, denoted by $\left(\Phi_\eta\right)_{\eta \in \mathcal{S}\times\mathcal{G}}$, which when discounted to date 0, leads to the *real stochastic discount factor* for pricing the securities. Lemma 13 will show that this social marginal utility of consumption is a function of the income distribution, the real shock and the nominal short-term interest rate. Once the function Φ is introduced, many of the constructions of the previous section can be extended to the multi-agent case, for Φ_η plays a role akin to the marginal utility u_c of the representative agent in the previous section. The fixed-point argument however needs to be extended to include not only the vector of returns $\boldsymbol{R}$ on the short-term bond, but also the vector of weights for the agents characterizing the distribution of income in the economy.

The first step of the analysis is given by Proposition 12 which provides the multi-agent generalization of Corollary 5 of the previous section: it characterizes a stationary reduced-form equilibrium of the economy in which the consumption and leisure $(c^h_\xi, \ell^h_\xi) = (c^h_\eta, \ell^h_\eta)$ only depend on the current state η, assuming that the agents adopt the forecast $\boldsymbol{B}$ of the Central Bank as their beliefs. The maximization of each agent in Definition 2(i) is replaced by the corresponding first-order conditions (a1, a2) and the budget equation (a4): the first-order conditions are expressed as the statement that the marginal utility of consumption of each agent is proportional to the social marginal utility of consumption Φ_η, the vector of coefficients of proportionality $\boldsymbol{\nu} = (\nu^h)_{h \in \mathcal{H}}$ in the simplex $\Delta^H \subset \mathbb{R}^H$ capturing the relative wealth of the agents. These weights are determined by the life-time budget equations of the agents which can be expressed (in (a4)) as functions of the variables $(\Phi_\eta, c_\eta, \ell_\eta, r^1_\eta)_{\eta \in \mathcal{S}\times\mathcal{G}}$ which are state, and not path, dependent. Let $R_\eta = 1 + r^1_\eta$ denote the gross return on the short-term bond in state η. Then the following equations characterize a stationary reduced-form equilibrium of the multi-agent economy.

Proposition 12. (Stationary equilibrium equations) *Under Assumption $\mathcal{U}$, a stationary reduced-form equilibrium is characterized by a pair $((\bar{\boldsymbol{B}}, \bar{\boldsymbol{q}}, \bar{\Theta}), (\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{x}}, \bar{\Phi}))$ satisfying the following system of equations*

(a1) $\bar{\nu}^h u^h_c(\bar{c}^h_\eta, \bar{\ell}^h_\eta) = \bar{\Phi}_\eta, \quad \eta \in \mathcal{S} \times \mathcal{G}, \quad h \in \mathcal{H}$

(a2) $\bar{\nu}^h u^h_\ell(\bar{c}^h_\eta, \bar{\ell}^h_\eta) = \dfrac{a^h_\eta \bar{\Phi}_\eta}{\bar{R}_\eta}, \quad \eta \in \mathcal{S} \times \mathcal{G}, \quad h \in \mathcal{H}$

(a3) $\displaystyle\sum_{h\in\mathcal{H}} \bar{c}^h_\eta = \sum_{h\in\mathcal{H}} a^h_\eta(e^h - \bar{\ell}^h_\eta), \quad \eta \in \mathcal{S} \times \mathcal{G}$

(a4) $\displaystyle\sum_{\eta\in\mathcal{S}\times\mathcal{G}} [I - \delta\bar{\boldsymbol{B}}]^{-1}_{\eta_0\,\eta} \bar{\Phi}_\eta\Big(\bar{c}^h_\eta - \frac{a^h_\eta(e^h - \bar{\ell}^h_\eta)}{\bar{R}_\eta}\Big) + \gamma^h\bar{\Theta} = \bar{\Phi}_{\eta_0} w^h_0, \quad h \in \mathcal{H}$

(b1) $\displaystyle\bar{\Phi}_\eta \bar{q}^\tau_\eta = \delta \sum_{\eta'\in\mathcal{S}\times\mathcal{G}} \frac{\bar{B}_{\eta\eta'}}{1+\pi_{\eta'}} \bar{\Phi}_{\eta'} \bar{q}^{\tau-1}_{\eta'}, \quad \eta \in \mathcal{S} \times \mathcal{G} \quad \tau = 1, \ldots, T$

(b2) $\bar{q}^\tau_\eta \leq 1 \quad \tau = 1, \ldots, T, \quad \eta \in \mathcal{S} \times \mathcal{G}$

Proof: Consider a reduced-form equilibrium $\Big(\Big(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j\in\mathcal{J}_g}, \bar{\Theta}\Big), \bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}\Big)$ as defined in Definition 2. Assuming that the present value of after tax income is $\bar{P}_{\xi_0}(w^h_0 - \gamma^h\bar{\Theta}) + \sum_\xi \bar{P}_\xi \alpha^h_\xi e^h/\bar{R}_\xi$ is positive, in view of Assumption $\mathcal{U}$, the solution to the maximum problem of agent h is interior, i.e. $c^h_\xi > 0$ and $0 < \ell^h_\xi < e^h$ for all $\xi \in \mathbb{D}$, and is characterized by the FOCs for optimal consumption/leisure and the present-value budget equation. The FOCs for the maximum problem of agent h are

$$\begin{array}{rcl} \delta^{t(\xi)} u^h_c(\bar{c}^h_\xi, \bar{\ell}^h_\xi) & = & \bar{\lambda}^h \bar{\mu}_\xi\, \bar{p}_\xi \\ \delta^{t(\xi)} u^h_\ell(\bar{c}^h_\xi, \bar{\ell}^h_\xi) & = & \bar{\lambda}^h \bar{\mu}_\xi\, \bar{p}_\xi \frac{a^h_\xi}{\bar{R}_\xi} \end{array} \tag{23}$$

for all $\xi \in \mathbb{D}$ and some $\bar{\lambda}^h > 0$. Let $\bar{\nu}^h = \frac{1}{\bar{\lambda}^h}$. Since the budget equations in (1) in Definition 2 are homogeneous in $(\bar{\mu})$, $\bar{\mu}$ can be normalized so that $\sum_{h\in\mathcal{H}} \bar{\nu}^h = 1$. This amounts to choosing $\bar{\mu}_{\xi_0}$ so that $\bar{\mu}_{\xi_0} \sum_h \frac{1}{u^h_c(\bar{c}^h_{\xi_0}, \bar{\ell}^h_{\xi_0})} = 1$.

Suppose the equilibrium is stationary, i.e. that $(\bar{c}_\xi, \bar{\ell}_\xi, \bar{R}_\xi, i_\xi, (a^h_\xi)_{h\in\mathcal{H}})$ only depend on the current state $\eta = (s, g)$ at date $t(\xi)$. In order for the FOCs (23) to be satisfied the common factor $\frac{\bar{\mu}_\xi \, \bar{p}_\xi}{\delta^{t(\xi)}}$ must only depend on the current state η at node ξ. Thus for all $\eta \in \mathcal{S} \times \mathcal{G}$ there must exist $\bar{\Phi}_\eta$, the social marginal utility of consumption in state η, such that

$$\begin{aligned} \bar{\nu}^h u_c^h(\bar{c}_\eta^h, \bar{\ell}_\eta^h) &= \bar{\Phi}_\eta \\ \bar{\nu}^h u_\ell^h(\bar{c}_\eta^h, \bar{\ell}_\eta^h) &= \bar{\Phi}_\eta \frac{a_\eta^h}{R_\eta} \end{aligned}$$

and the nominal stochastic discount factor for one unit of money at a node ξ with current state η is then equal to

$$\bar{\mu}_\xi = \frac{\delta^{t(\xi)} \bar{\Phi}_\eta}{\bar{p}_\xi}$$

Each agent's budget equation can then be written as

$$\sum_{\eta \in \mathcal{S} \times \mathcal{G}} \; \sum_{\{\xi \in \mathbb{D} \,|\, \eta(\xi) = \eta\}} \Big(\delta^{t(\xi)} B_\xi \Big) \bar{\Phi}_\eta \Big(c_\eta^h - \frac{a_\eta^h (e^h - \ell_\eta^h)}{R_\eta} \Big) - \gamma^h \bar{\Phi}_{\eta_0} \Theta = \bar{\Phi}_{\eta_0} w_0^h \tag{24}$$

For a given date t, $\sum_{\{\xi \in \mathbb{D},\, t(\xi) = t,\, \eta(\xi) = \eta\}} B_\xi$ is the probability of going from η_0 to η in t periods and is thus equal to $[\boldsymbol{B}^t]_{\eta_0 \eta}$, where $[\boldsymbol{B}^t]$ is the t^{th} power of the matrix $\boldsymbol{B}$. The sum after the second summation sign in (24) can thus be written using the $(S \times G) \times (S \times G)$ matrix

$$[I - \delta \boldsymbol{B}]^{-1} = \sum_{t=0}^{\infty} \delta^t \boldsymbol{B}^t$$

where the series converges since $\delta < 1$. This leads to the expression (a4) for the budget constraint in Proposition 12. □

Comparing the characterization of a stationary equilibrium for the multi-agent economy in Proposition 12 with that of the representative-agent economy in Corollary 5, (a1)-(a3) replace condition (a) in Corollary 5. In (a4), for symmetry, we have written the budget constraints for all the agents $h \in \mathcal{H}$. It would suffice to write the budget constraints for all but one of the agents, the Ricardian condition ensuring that it holds for the remaining agent: omitting one agent's budget constraint corresponds to the characterization of equilibrium in Corollary 5 which omits the budget constraint for the representative agent. (b1, b2) of Proposition 12 are equivalent to (b1, b2) of Corollary 5, the marginal utility of the representative agent being replaced by the social marginal utility Φ.

The next step is to characterize the expectations matrices $\boldsymbol{B}$ for which there exist a stationary reduced-form equilibrium: as in the previous section (Definition 7) we call any such matrix a *compatible expectations matrix*. To generalize Proposition 8 and obtain conditions for a matrix $\boldsymbol{B}$ to be a compatible expectations matrix we need the equivalent of Lemma 6 for the multi-agent case. We show that under Assumption $\mathcal{U}$, for a fixed vector of positive weights $\boldsymbol{\nu}$ and a fixed vector of productivities $\boldsymbol{a} = (a^h)_{h\in\mathcal{H}}$, the first-order conditions (a1)-(a2) and the market-clearing equations (a3) uniquely define the consumption and leisure of each agent and the social marginal utility of consumption.

Lemma 13. ($\boldsymbol{\nu}$-equilibrium consumption) *Let u^h satisfy Assumption $\mathcal{U}$ for all $h \in \mathcal{H}$.*

(i) *For any $(\boldsymbol{\nu}, \boldsymbol{a}, R) \in \Delta^H \times \mathbb{R}^H_{++} \times \mathbb{R}_{++}$ the equations*

$$\text{(a1)}\quad \nu^h u^h_c(c^h, \ell^h) = \Phi \quad \textit{if} \quad \nu^h > 0, \quad c^h = 0 \quad \textit{if} \quad \nu^h = 0, \qquad h \in \mathcal{H}$$

$$\text{(a2)}\quad \nu^h u^h_\ell(c^h, \ell^h) = \frac{a^h \Phi}{R} \quad \textit{if} \quad \nu^h > 0, \quad \ell^h = 0, \quad \textit{if} \quad \nu^h = 0 \qquad h \in \mathcal{H}$$

$$\text{(a3)}\quad \sum_{h\in\mathcal{H}} c^h = \sum_{h\in\mathcal{H}} a^h(e^h - \ell^h)$$

have a unique solution $\Big(c^h(\boldsymbol{\nu}, \boldsymbol{a}, R), \ell^h(\boldsymbol{\nu}, \boldsymbol{a}, R), \Phi(\boldsymbol{\nu}, \boldsymbol{a}, R)\Big)$ continuous on $\Delta^H \times \mathbb{R}^H_{++} \times \mathbb{R}_{++}$.

(ii) *$\Phi(\boldsymbol{\nu}, \boldsymbol{a}, R)$ is strictly increasing in R.*

(iii) *$\widetilde{\Phi}(\boldsymbol{\nu}, \boldsymbol{a}, R) \equiv \dfrac{\Phi(\boldsymbol{\nu}, \boldsymbol{a}, R)}{R}$ is strictly decreasing in R.*

Proof: See Appendix.

Lemma 13 is the multi-agent analogue of Lemma 6 in the previous section: it permits the equations (a1)-(a3), and (b1) of Proposition 12 for $\tau = 1$ to be combined into the system of equations

$$\frac{1}{R_\eta} = \delta \sum_{\eta' \in \mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1+\pi_{\eta'}} \frac{\Phi(\bar{\boldsymbol{\nu}}, \boldsymbol{a}_{\eta'}, R_{\eta'})}{\Phi(\bar{\boldsymbol{\nu}}, \boldsymbol{a}_\eta, R_\eta)}, \quad \eta \in \mathcal{S}\times\mathcal{G} \tag{25}$$

which has the same form as the equilibrium equations (11) for the single-agent economy. Finding a reduced-form equilibrium for the single-agent economy reduced to finding a solution $\bar{\boldsymbol{R}} \geq \mathbf{1}$ to equations (11). For the multi-agent economy, in addition to solving the system of equations (25) in the proper domain, we must find relative weights $\bar{\boldsymbol{\nu}}$ for the agents which are compatible with the distribution of wealth implied by the budget equations (a4) in Proposition 12. Thus for the multi-agent economy finding an equilibrium reduces to *finding a pair* $(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}})$ *such that the budget equations (a4) and the bond pricing equations (25) are satisfied.*

The conditions which imply the existence of a reduced-form equilibrium thus naturally reduce to two sets of conditions: the first set is analogous to conditions (1) and (2) in Proposition 8 which ensure that there is a solution $\bar{\boldsymbol{R}}$ to the short-term bond pricing equations (25) satisfying $\bar{\boldsymbol{R}} \geq \mathbf{1}$; the second ensures that the *tax burden* is shared among the agents in a way which is commensurate with their wealth, so that each agent can afford positive consumption and leisure in all states $\eta \in \mathcal{S} \times \mathcal{G}$ after paying his/her share of the present value of the taxes $\bar{\Theta}$.

To give conditions which ensure (25) has a solution in the right domain, we need to bound the possible values of $\boldsymbol{R}$. For each $\boldsymbol{\nu} \in \Delta^H$, define $R_\eta^{\max}(\boldsymbol{\nu})$ as the solution of the equation

$$\widetilde{\Phi}(\boldsymbol{\nu}, \boldsymbol{a}_\eta, R_\eta^{\max}) = \frac{\Phi(\boldsymbol{\nu}, \boldsymbol{a}_\eta, R_\eta^{\max})}{R_\eta^{\max}} = \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} B_{\eta\eta'} \frac{\Phi(\boldsymbol{\nu}, \boldsymbol{a}_{\eta'}, 1)}{1 + \pi_{\eta'}}, \quad \eta \in \mathcal{S} \times \mathcal{G} \tag{26}$$

(condition (1) in Proposition 14 below ensures that the equation has a solution) and then for each $\eta \in \mathcal{S} \times \mathcal{G}$ define

$$R_\eta^{\max} = \max_{\boldsymbol{\nu} \in \Delta^H} R_\eta^{\max}(\boldsymbol{\nu})$$

As before conditions which ensure $\boldsymbol{R}^{\max} \geq \mathbf{1}$ place restrictions on the matrix $\boldsymbol{B}$ given the inflation/technology/preference characteristics of the economy. We then define, for each $\boldsymbol{\nu} \in \Delta^H$, $R_\eta^{\min}(\boldsymbol{\nu})$ by

$$\frac{1}{R_\eta^{\min}(\boldsymbol{\nu})} = \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta\eta'}}{1 + \pi_{\eta'}} \frac{\Phi(\boldsymbol{\nu}, \boldsymbol{a}_{\eta'}, R_{\eta'}^{\max})}{\Phi(\boldsymbol{\nu}, \boldsymbol{a}_\eta, 1)} \quad \eta \in \mathcal{S} \times \mathcal{G}$$

$R_\eta^{\min}(\boldsymbol{\nu})$ gives a lower bound on the nominal interest rate since it corresponds to the lowest possible real interest rate, and hence the assumption that $R_\eta^{\min}(\boldsymbol{\nu}) \geq 1$ (condition 2 below) imposes stronger restrictions on $\boldsymbol{B}$.

To understand the tax-sharing assumption consider each agent's present value budget equation in the original form given in (1) of Definition 2. Summing the budget equations of the households implies that when the market clearing equations (3) in Definition 2 hold, then

$$\sum_{\xi \in \mathbb{D}} \bar{P}_\xi \frac{\bar{r}_\xi^1 M_\xi}{1 + \bar{r}_\xi^1} + \bar{P}_{\xi_0} \bar{\Theta} = \bar{P}_{\xi_0} \sum_{h \in \mathcal{H}} w_0^h = \bar{P}_{\xi_0} W_0 \tag{27}$$

where $M_\xi = \bar{p}_\xi \sum_{h \in \mathcal{H}} \bar{c}_\xi^h$. (27) expresses the property that the government *asymptotically withdraws its initial liabilities* $\bar{P}_{\xi_0} W_0$ (which correspond to the initial wealth of the private sector) by a combination of *seignorage* (the first term on the left side) and *direct taxes* ($\bar{P}_{\xi_0}\bar{\Theta}$). Since $\bar{r}_\xi^1 \geq 0$, it follows from (27) that

$$\bar{\Theta} \leq W_0 \tag{28}$$

We want to be sure that each agent h has a positive after-tax present-value of income

$$\gamma^h \bar{P}_{\xi_0} \bar{\Theta} < \bar{P}_{\xi_0} w_0^h + \sum_{\xi \in \mathbb{D}} \frac{\bar{P}_\xi a_\xi^h e^h}{\bar{R}_\xi}, \quad h \in \mathcal{H}$$

and, to ensure that it holds in equilibrium, we require that it holds for the 'lowest' possible values of $\frac{\bar{P}_\xi}{\bar{R}_\xi}$ and the highest possible $\bar{P}_{\xi_0}$. In the stationary case this can be expressed using the highest returns $R_\eta^{\max}$ and leads to condition 3 in the following Proposition.

Proposition 14. (Existence of stationary equilibrium) *Let $\mathcal{E}$ be an economy in which the agents' utility functions satisfy Assumption $\mathcal{U}$. If $\boldsymbol{B}$ is a Markov matrix such that*

(1) $\lim_{R\to\infty} \dfrac{\Phi(\boldsymbol{\nu}, \boldsymbol{a}_\eta, R)}{R} < \delta \displaystyle\sum_{\eta' \in \mathcal{S}\times\mathcal{G}} B_{\eta\eta'} \dfrac{\Phi(\boldsymbol{\nu}, \boldsymbol{a}_{\eta'}, 1)}{1+\pi_{\eta'}} \leq \Phi(\boldsymbol{\nu}, \boldsymbol{a}_\eta, 1), \quad \forall\, \eta \in \mathcal{S}\times\mathcal{G}, \quad \forall\, \boldsymbol{\nu} \in \Delta^H$

(2) $R_\eta^{\min}(\boldsymbol{\nu}) \geq 1, \quad \forall\, \eta \in \mathcal{S}\times\mathcal{G}, \quad \forall\, \boldsymbol{\nu} \in \Delta^H$

and if the tax burden is distributed among agents so that $\gamma = (\gamma^h)_{h\in\mathcal{H}} \in \Delta^H$ satisfies

(3) $\gamma^h W_0 < w_0^h + \displaystyle\sum_{\eta \in \mathcal{S}\times\mathcal{G}} [I - \delta \boldsymbol{B}]^{-1}_{\eta_0\eta} \dfrac{\Phi(\boldsymbol{\nu}, \boldsymbol{a}_\eta, R_\eta^{\max})\ a_\eta^h e^h}{R_\eta^{\max}\ \Phi(\boldsymbol{\nu}, \boldsymbol{a}_{\eta_0}, R_{\eta_0}^{\max})}, \quad \forall\, \boldsymbol{\nu} \in \Delta^H, \quad \forall\, h \in \mathcal{H}$

then there exists a stationary equilibrium of $\mathcal{E}$, and $\boldsymbol{B}$ is a compatible expectations matrix.

Proof: See Appendix.

For an expectations matrix $\boldsymbol{B}$ compatible with equilibrium, let $\boldsymbol{\nu}(\boldsymbol{B})$ denote the vector of relative weights of the agents in an equilibrium induced by $\boldsymbol{B}$, and let $R_\eta(\boldsymbol{B})$ be the return on the short-term bond traded in state η, for each $\eta \in \mathcal{S}\times\mathcal{G}$. $\Phi_\eta(\boldsymbol{B}) \equiv \Phi(\boldsymbol{\nu}(\boldsymbol{B}), a_\eta, R_\eta(\boldsymbol{B}))$ is then marginal social utility of an additional unit of consumption in state η, for $\eta \in \mathcal{S}\times\mathcal{G}$. Equations (b1) in Proposition 12 give by successive substitutions the prices of the bonds of maturities $\tau \geq 2$: as in the previous section, the complete system of bond prices across the different states $\eta \in \mathcal{S}\times\mathcal{G}$ can then be written in compact form by defining the matrices

$$D_1(\boldsymbol{B}) = \operatorname{diag}\left(\frac{1}{\Phi_\eta(\boldsymbol{B})}\right)_{\eta\in\mathcal{S}\times\mathcal{G}}, \quad D_2(\boldsymbol{B}) = \operatorname{diag}\left(\frac{\delta\Phi_\eta(\boldsymbol{B})}{1+\pi_\eta}\right)_{\eta\in\mathcal{S}\times\mathcal{G}}, \quad \Delta(\boldsymbol{B}) = D_1(\boldsymbol{B})\boldsymbol{B}D_2(\boldsymbol{B})$$

where, for a vector $u \in \mathbb{R}^{SG}$, diag(u) denotes the $SG \times SG$ diagonal matrix whose n^{th} entry is the n^{th} component of u. The prices of the bonds compatible with B are then given by (18) of the previous section. The definition of a *strongly credible* monetary policy $(\boldsymbol{B}, \boldsymbol{q}(\boldsymbol{B}))$, and of a *controllable* matrix $\boldsymbol{B}$ (Definitions 9 and 10) are the same as in the previous section with s replaced by η: if $\boldsymbol{B}$ is the only matrix compatible with the bond pricing (or term structure) rule $\boldsymbol{q}(\boldsymbol{B})$, then the forecast $\boldsymbol{B}$ is made strongly credible by the bond pricing rule $\boldsymbol{q}(\boldsymbol{B})$, and the agents will adopt $\boldsymbol{B}$ as their expectations matrix. Proposition 10 immediately generalizes by replacing S, the number of independent bonds whose price should be fixed, by SG.

Proposition 15 (Controlling expectations) *Let $(\boldsymbol{B}, \boldsymbol{q}(\boldsymbol{B}))$ be a monetary policy consisting of a forecast $\boldsymbol{B}$ and a term-structure rule $\boldsymbol{q}(\boldsymbol{B}) = (\boldsymbol{q}^1(\boldsymbol{B}), \ldots, \boldsymbol{q}^T(\boldsymbol{B}))$ satisfying (18). If the payoff matrix*

$$\widehat{Q} = \left[\mathbf{1}, \Delta(\boldsymbol{B})\mathbf{1}, \ldots, (\Delta(\boldsymbol{B}))^{SG-1}\mathbf{1}\right] \tag{29}$$

is invertible, then the policy $(\boldsymbol{B}, \boldsymbol{q}(\boldsymbol{B}))$ is strongly credible and $\boldsymbol{B}$ is a controllable expectations matrix.

Thus the main difference between a heterogeneous-agent and a representative-agent economy is that the social marginal utility of consumption, which replaces the marginal utility of the representative agent, depends on the distribution of income among the agents. Thus the equilibrium vector of returns $\bar{\boldsymbol{R}}$ on the short-term bond must be determined simultaneously with the equilibrium vector of weights $\bar{\boldsymbol{\nu}}$ for the agents by solving the fixed-point problem (46) in the appendix. The pair $(\boldsymbol{\nu}, \boldsymbol{R})$ induces a vector of social marginal utilities of consumption $\left(\Phi_\eta\right)_{\eta \in \mathcal{S} \times \mathcal{G}}$ which, from the bond pricing equations, lead to the term-structure rule associated with the expectations matrix $\boldsymbol{B}$. As in the previous section, a rank condition is then required to ensure the uniqueness of the compatible expectations matrix.

5. Conclusion

We have proposed a framework for studying the policy of inflation targeting. The point of departure is that in a model in which monetary-fiscal policy is Ricardian if monetary policy consists solely of controlling the short-term nominal interest rate—and if we do not invoke a local steady state analysis based on the Taylor rule—the agents' expectations of inflation are indeterminate. We are used to the idea that prices of long-term nominal bonds depend on agents' expectations of inflation: the basic idea of the paper is to reverse the argument and suggest that controlling the prices of long-term bonds can be a natural instrument for controlling agents' expectations of inflation. We have proposed a framework in which the

Central Bank seeks to anchor agents' expectations of inflation by making a public forecast of inflation—represented in the stationary case by a Markov transition matrix—and choosing a compatible term structure of interest rates to make the forecast credible.

In practice controlling more than the short-term interest rate would require either co-ordination between the Central Bank and the fiscal authority to choose the maturity of the bonds used to finance the government debt, or open market operations by the Central Bank on long-term government bonds. Once the Central Bank has acquired credibility, since the term structure it wants to control is that which is compatible with the expectations it wants to promote, it will not have to move "against the market"—thus implementing the policy does not seem call on the use of extensive funds.

It is well-known that the problem of indeterminacy of expectations of inflation present in a flexible- price model is also present in models—such as the New-Keynesian models—with staggered price setting by firms. As Nakajima-Polemarchakis (2005) have made clear by converting models with different price setting assumptions to a finite horizon framework, the indeterminacy comes from the Ricardian policy which removes a market-clearing equation on each trajectory. For this reason we have chosen the simplest flexible-price framework in which the choice of the short-term nominal interest rate affects real output to expose our proposal for anchoring agents' expectations of inflation: it would certainly be of interest to see how this highly stylized monetary module could be incorporated into a richer model of the real side of the economy.

Appendix

Proof of Proposition 3: Step 1. Show that the extensive and reduced-form budget sets are the same. The budget set $B^h(\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta})$ of agent h in an extensive-form equilibrium is given by

$$\mathbb{B}^h(\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta}) = \left\{ x^h \in (\ell^+_\infty(\mathbb{D}))^2 \;\middle|\; \begin{array}{l} \exists\ \boldsymbol{z}^h \in \mathbb{R}^{\mathbb{D}J} \text{ such that } \forall\ \xi \in \mathbb{D} \\ p_\xi c^h_\xi + \gamma^h \theta_\xi + q_\xi z^h_\xi = p_{\xi^-} L^h_{\xi^-} + \hat{q}_\xi z^h_{\xi^-} \\ \lim\limits_{T\to\infty} \sum\limits_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} q_{\xi'} z^h_{\xi'} = 0 \end{array} \right\}$$

where $\ell^+_\infty(\mathbb{D})$ is the space of non-negative bounded sequences on $\mathbb{D}$, and where $p_{\xi_0^-} L^h_{\xi_0^-} + \hat{q}_{\xi_0} z^h_{\xi_0^-} = w^h_0$ denotes the agent's initial wealth at date 0. While the extensive-form budget set is defined by an infinite sequence of budget constraints (one at each node of the event-tree) the reduced-form budget set of agent h, denoted by $B^h(\boldsymbol{p},\boldsymbol{P},\boldsymbol{r}^1,\Theta)$ is defined by a single present-value budget equation

$$B^h(\boldsymbol{p}, \boldsymbol{P}, \boldsymbol{r}^1, \Theta) = \left\{ x^h \in (\ell_\infty^+(\mathbb{D}))^2 \;\middle|\; \sum_{\xi \in \mathbb{D}} P_\xi p_\xi \Big(c_\xi^h - \frac{L_\xi^h}{1 + r_\xi^1} \Big) + \gamma^h P_{\xi_0} \Theta = P_{\xi_0} w_0^h \right\}$$

where $\boldsymbol{r}^1 = (\boldsymbol{r}_\xi^1)_{\xi \in \mathbb{D}}$ is the sequence of short-term interest rates over the event-tree $\mathbb{D}$. Let $\ell_1(\mathbb{D}) = \{\boldsymbol{\pi} \in \mathbb{R}^{\mathbb{D}} | \sum_{\xi \in \mathbb{D}} |P_\xi| < \infty\}$ denote the space of summable sequences on $\mathbb{D}$. We want to show that if

(i) $\operatorname{rank} [\, \hat{q}_{\xi^+} \,] = SG, \;\; \forall \, \xi \in \mathbb{D}$ (complete markets)

(ii) $P_\xi q_\xi^j = \sum_{\xi' \in \xi^+} P_{\xi'} \hat{q}_{\xi'}^j, j \in \mathcal{J}, \;\forall \, \xi \in \mathbb{D}$ (no-arbitrage security prices)

(iii) $(P_\xi p_\xi)_{\xi \in \mathbb{D}} \in \ell_1(\mathbb{D})$ (summable prices)

(iv) $\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} \to 0$ as $T \to \infty$, $\;\Theta = \sum_{\xi \in \mathbb{D}} P_\xi \theta_\xi$, (summable taxes)

then $\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}) = B^h(\boldsymbol{p}, \boldsymbol{P}, \boldsymbol{r}^1, \Theta)$ for all $h \in \mathcal{H}$.

($\Longrightarrow$) We show $\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}) \subset B^h(\boldsymbol{p}, \boldsymbol{P}, \boldsymbol{r}^1, \Theta)$. Pick $x^h \in \mathbb{B}^h$. Multiplying the budget equation at node ξ by P_ξ for all $\xi \in \mathbb{D}$, summing over the event-tree up to date T and using the no-arbitrage condition (ii) gives (remember that $\mathbb{D}^T$ denotes the set of nodes up to date T and $\mathbb{D}_T$ the set of nodes at date T)

$$\sum_{\xi \in \mathbb{D}^T} P_\xi p_\xi c_\xi^h + \gamma^h \sum_{\xi \in \mathbb{D}^T} P_\xi \theta_\xi + \sum_{\xi \in \mathbb{D}_T} P_\xi q_\xi z_\xi^h = \sum_{\xi \in \mathbb{D}^{T-1}} \Big(\sum_{\xi' \in \xi^+} P_{\xi'} \Big) p_\xi L_\xi^h + p_{\xi_0} w_0^h$$

Since no arbitrage applied to the short-term bond implies that, for all $\xi \in \mathbb{D}$, $\frac{P_\xi}{1+r_\xi^1} = \sum_{\xi' \in \xi^+} P_{\xi'}$, this equation can be written as

$$\sum_{\xi \in \mathbb{D}^{T-1}} P_\xi p_\xi \Big(c_\xi^h - \frac{L_\xi^h}{1 + r_\xi^1} \Big) + \gamma^h \sum_{\xi \in \mathbb{D}^T} P_\xi \theta_\xi + \sum_{\xi \in \mathbb{D}_T} P_\xi p_\xi c_\xi^h + \sum_{\xi \in \mathbb{D}_T} P_\xi q_\xi z_\xi^h = p_{\xi_0} w_0^h$$

Since $(P_\xi p_\xi)_{\xi \in \mathbb{D}} \in \ell_1(\mathbb{D})$ and $c^h \in (\ell_\infty^+(\mathbb{D})$, $\lim_{T \to \infty} \sum_{\xi \in \mathbb{D}_T} P_\xi p_\xi c_\xi^h = 0$, and by the Transversality Condition for $\mathbb{B}^h$, $\lim_{T \to \infty} \sum_{\xi \in \mathbb{D}_T} P_\xi q_\xi z_\xi^h = 0$, so that $x \in B^h(\boldsymbol{p}, \boldsymbol{P}, \boldsymbol{r}^1, \Theta)$.

($\Longleftarrow$) We show $\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}) \supset B^h(\boldsymbol{p}, \boldsymbol{P}, \boldsymbol{r}^1, \Theta)$. Pick $x^h \in B^h(\boldsymbol{p}, \boldsymbol{P}, \boldsymbol{r}^1, \Theta)$. We need to find portfolios z^h such that the sequential budget constraints are satisfied at each node and the Transversality Condition is satisfied. We define the portfolio z_ξ^h by the requirement that it brings enough wealth to the successors of ξ to finance the excess present value of expenditure over after-tax income on the subtrees originating at each of these nodes. In view of the assumption of complete markets such a portfolio exists and is defined by

$$\left[\sum_{\xi''\in\mathbb{D}(\xi')} P_{\xi''}p_{\xi''}\Big(c^h_{\xi''}-\frac{L^h_{\xi''}}{1+r^1_{\xi''}}\Big)+\gamma^h\sum_{\xi''\in\mathbb{D}(\xi')}P_{\xi''}\theta_{\xi''}-P_{\xi'}p_\xi L^h_\xi\right]_{\xi'\in\xi^+}=\Big[P\circ_{\xi^+}\hat{q}\Big]\,z^h_\xi,\quad\forall\xi\in\mathbb{D} \tag{30}$$

where $\Big[P\circ_{\xi^+}\hat{q}\Big]=\Big[P_{\xi'}\hat{q}^j_{\xi'}\Big]_{\substack{j\in\mathcal{J}\\ \xi'\in\xi^+}}$ is the $SG\times J$ matrix of present values of the payoffs of the J securities at the immediate successors ξ^+ of ξ. Let us show that with this choice of portfolio the sequential budget constraint is satisfied at each node. We begin with the initial node ξ_0. Premultiplying the SG equations (30) by $\mathbf{1}^\intercal\in\mathbb{R}^{SG}$ gives

$$\sum_{\xi\in\mathbb{D}\setminus\xi_0}\left(P_\xi p_\xi\Big(c^h_\xi-\frac{L^h_\xi}{1+r^1_\xi}\Big)+\gamma^h P_\xi\theta_\xi\right)-\Big(\sum_{\xi'\in\xi_0^+}P_{\xi'}\Big)p_{\xi_0}L^h_{\xi_0}=P_{\xi_0}q_{\xi_0}z^h_{\xi_0} \tag{31}$$

and since $x^h\in B^h(\boldsymbol{p},\boldsymbol{P},\boldsymbol{r}^1,\Theta)$ and $\sum_{\xi'\in\xi_0^+}P_{\xi'}=\frac{P_{\xi_0}}{1+r^1_{\xi_0}}$ it follows that

$$-P_{\xi_0}p_{\xi_0}\Big(c^h_{\xi_0}-\frac{L^h_{\xi_0}}{1+r^1_{\xi_0}}\Big)-\gamma^hP_{\xi_0}\theta_{\xi_0}+P_{\xi_0}w^h_0-\frac{P_{\xi_0}}{1+r^1_{\xi_0}}p_{\xi_0}L^h_{\xi_0}=P_{\xi_0}q_{\xi_0}z^h_{\xi_0}$$

namely

$$p_{\xi_0}c^h_{\xi_0}+\gamma^h\theta_{\xi_0}+q_{\xi_0}z^h_{\xi_0}=w^h_0$$

In the same way, for any any node $\tilde{\xi}$ with $t(\tilde{\xi})\geq 1$ premultiplying (30) by $\mathbf{1}^\intercal$ and using (ii) gives the analogue of (31) with $\mathbb{D}$ replaced by $D(\tilde{\xi})$ and ξ_0 replaced by $\tilde{\xi}$. Using (30) again to express $P_{\tilde{\xi}}\hat{q}_{\tilde{\xi}}z^h_{\tilde{\xi}^-}$ and substituting leads to

$$P_{\tilde{\xi}}\hat{q}_{\tilde{\xi}}z^h_{\tilde{\xi}^-}-P_{\tilde{\xi}}p_{\tilde{\xi}}\Big(c^h_{\tilde{\xi}}-\frac{L^h_{\tilde{\xi}}}{1+r^1_{\tilde{\xi}}}\Big)-\gamma^hP_{\tilde{\xi}}\theta_{\tilde{\xi}}+P_{\tilde{\xi}}L^h_{\tilde{\xi}^-}-\frac{P_{\tilde{\xi}}}{1+r^1_{\tilde{\xi}}}p_{\tilde{\xi}}L^h_{\tilde{\xi}}=P_{\tilde{\xi}}q_{\tilde{\xi}}z^h_{\tilde{\xi}}$$

so that the budget constraint at node $\tilde{\xi}$

$$p_{\tilde{\xi}}c^h_{\tilde{\xi}}+\gamma^hP_{\tilde{\xi}}\theta_{\tilde{\xi}}+q_{\tilde{\xi}}z^h_{\tilde{\xi}}=p_{\tilde{\xi}^-}L^h_{\tilde{\xi}^-}+\hat{q}_{\tilde{\xi}}z^h_{\tilde{\xi}^-}$$

is satisfied. It remains to show that the Transversality Condition is satisfied. To this end consider any node $\xi\in\mathbb{D}$ and for any date $T\geq t(\xi)$, consider the nodes of $\mathbb{D}_T(\xi)$ at date T in the subtree $\mathbb{D}(\xi)$. Using (30) and summing over the nodes of $\mathbb{D}_T(\xi)$ gives

$$\sum_{\xi'\in\mathbb{D}_T(\xi)}P_{\xi'}q_{\xi'}z^h_{\xi'}=\sum_{\substack{\xi''\in\mathbb{D}(\xi)\\ t(\xi'')\geq T+1}}\left(P_{\xi''}p_{\xi''}\Big(c^h_{\xi''}-\frac{L^h_{\xi''}}{1+r^1_{\xi''}}\Big)+\gamma^hP_{\xi''}\theta_{\xi''}\right)-\sum_{\xi'\in\mathbb{D}_T(\xi)}P_{\xi'}p_{\xi'^-}L^h_{\xi'^-} \tag{32}$$

By condition (iii) and (iv) all the tails of the series on the right side of (32) converge to zero, and thus $\sum_{\xi'\in\mathbb{D}_T(\xi)} P_{\xi'} p_{\xi'} z^h_{\xi'} \to 0$ when $T\to\infty$, so that $x^h \in \mathbb{B}^h(\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta})$.

Step 2. We show that, given price processes satisfying conditions (i)-(iii) of Step 1 and an aggregate consumption process $\boldsymbol{C} = \sum_{h\in\mathcal{H}} \boldsymbol{c}^h \in \ell^\infty(\mathbb{D})$, the government policy $(\boldsymbol{M},\boldsymbol{\theta},\boldsymbol{Z})$ is determined by the feasibility conditions and the fiscal rule, and it satisfies $\sum_{\xi'\in\mathbb{D}_T(\xi)} P_{\xi'} W_{\xi'} \to 0$ and $\sum_{\xi'\in\mathbb{D}_T(\xi)} P_{\xi'}\theta_{\xi'} \to 0$ as $T\to\infty$.

Since agents pay for the consumption good using money, the aggregate demand for money for transactions $p_\xi C_\xi = M_\xi,\ \xi\in\mathbb{D}$ is equal to the aggregate supply made available by the government. Given the short-term rate interest rate $\boldsymbol{r}^1 = (r^1_\xi)_{\xi\in\mathbb{D}}$ the seignorage $\frac{r^1_\xi M_\xi}{1+r^1_\xi}$ at each node is known. Let us show that the government's budget equation at each node and the Ricardian rule (α,β)

$$M_\xi + \theta_\xi + q_\xi Z_\xi = W_\xi \tag{33}$$

$$\frac{r^1_\xi M_\xi}{1+r^1_\xi} + \theta_\xi = \alpha_\xi W_\xi \tag{34}$$

$$\left(W_{\xi'}\right)_{\xi'\in\xi^+} = d_\xi \beta_\xi \tag{35}$$

where

$$W_\xi = M_{\xi^-} + \hat{q}_\xi Z_{\xi^-}$$

is the liability of the government at the beginning of node ξ, can be used to determine its portfolio and tax policy $(\boldsymbol{\theta},\boldsymbol{Z})$. Start at the initial node ξ_0. (33) and (34) determine θ_{ξ_0} and $k_0 = q_{\xi_0} Z_{\xi_0}$. Premultiplying (35) by the vector $P_{\xi_0^+} = (P_{\xi'})_{\xi'\in\xi_0^+}$ gives the equation

$$\left(\sum_{\xi'\in\xi_0^+} P_{\xi'}\right) M_{\xi_0} + P_{\xi_0} q_{\xi_0} Z_{\xi_0} = d_{\xi_0} \sum_{\xi'\in\xi_0^+} \beta_{\xi_0\xi'} \iff \frac{r^1_{\xi_0} M_{\xi_0}}{1+r^1_{\xi_0}} + P_{\xi_0} k_0 = d_{\xi_0} P_{\xi_0^+} \beta_0$$

which determines d_{ξ_0}. Then

$$\left(W_{\xi'}\right)_{\xi'\in\xi_0^+} = M_{\xi_0}\mathbf{1} + \left[\hat{q}_{\xi_0^+}\right] Z_{\xi_0}$$

determines Z_{ξ_0} since $\left[\hat{q}_{\xi_0^+}\right]$ is invertible. Note that

$$\sum_{\xi'\in\xi_0^+} P_{\xi'} W_{\xi'} = \frac{r^1_{\xi_0} M_{\xi_0}}{1+r^1_{\xi_0}} + P_{\xi_0} q_{\xi_0} Z_{\xi_0} = P_{\xi_0}(1-\alpha_{\xi_0}) W_0 \tag{36}$$

where the last inequality is obtained by substituting the value of θ_{ξ_0} given by (34) into the budget equation (33): thus the present value of the government's liabilities has decreased by the factor α_{ξ_0} by virtue of the Ricardian policy (34). Note also that since all the terms $(W_{\xi'})_{\xi' \in \xi_0^+}$ have the same sign ($\beta_{\xi_0} \gg 0$), they are of the same sign as W_0.

By induction we can use (33)-(35) to calculate $(\boldsymbol{\theta}_\xi, \boldsymbol{Z}_\xi)$ for all nodes ξ, showing that (36) is satisfied at each node and that the liabilities W_ξ always have the same sign, which is the sign of W_0. To establish the asymptotic properties, we assume that $W_0 > 0$, so that for all ξ, $W_\xi > 0$: if $W_0 < 0$ it suffices to reverse the inequalities in the analysis that follows and the same asymptotic results hold. It follows from (36) that

$$\sum_{\xi' \in \xi^+} P_{\xi'} W_{\xi'} \leq (1 - \underline{\alpha}) W_\xi$$

Applying this inequality recursively gives

$$\sum_{\substack{\xi'' \in \mathbb{D}(\xi) \\ t(\xi'') = t(\xi)+2}} P_{\xi''} W_{\xi''} \leq \sum_{\xi' \in \xi^+} (1 - \underline{\alpha})\, P_{\xi'} W_{\xi'} \leq (1 - \underline{\alpha})^2 P_\xi W_\xi$$

and moving forward $T - t(\xi)$ periods into the subtree $\mathbb{D}(\xi)$ gives

$$0 < \sum_{\xi'' \in \mathbb{D}_T(\xi)} P_{\xi''} W_{\xi''} \leq (1 - \underline{\alpha})^{T - t(\xi)} P_\xi W_\xi \tag{37}$$

Thus the Ricardian policy implies that the present value of the government's date T liabilities tend to zero when $T \to \infty$ on every subtree of $\mathbb{D}$.

Multiplying (34) by $P_{\xi'}$ for each node $\xi' \in \mathbb{D}_T(\xi)$ and forming the sum of these values gives

$$\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} = \sum_{\xi' \in \mathbb{D}_T(\xi)} \alpha_{\xi'} P_{\xi'} W_{\xi'} - \sum_{\xi' \in \mathbb{D}_T(\xi)} \frac{r^1_{\xi'}}{1 + r^1_{\xi'}} P_{\xi'} M_{\xi'}, \quad \xi \in \mathbb{D}$$

$$\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} = \sum_{\xi' \in \mathbb{D}_T(\xi)} \alpha_{\xi'} P_{\xi'} W_{\xi'} - \sum_{\xi' \in \mathbb{D}_T(\xi)} \frac{r^1_{\xi'}}{1 + r^1_{\xi'}} P_{\xi'} M_{\xi'}, \quad \xi \in \mathbb{D}$$

By (37) the first term on the right side tends to 0 as $T \to \infty$ and since $P_{\xi'} M_{\xi'} = P_{\xi'} p_{\xi'} C_{\xi'}$ by (iii) of Step 1 and $\boldsymbol{C} \in \ell^\infty(\mathbb{D})$, the second term tends to 0. Thus $\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} \to 0$ as $T \to \infty$.

Step 3. We show that an extensive-form equilibrium is a reduced-form equilibrium. Let

$$\Big(\Big((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}), (\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}}) \Big), \Big((\bar{\boldsymbol{x}}, \widetilde{\overline{\boldsymbol{m}}}, \bar{\boldsymbol{z}}), (\bar{\boldsymbol{y}}, \bar{\boldsymbol{L}}) \Big), \Big(\bar{\boldsymbol{p}}, \bar{\boldsymbol{\omega}}, \bar{\boldsymbol{\pi}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_p} \Big) \Big)$$

be an extensive- form equilibrium. The profit maximization in (v) of Definition 1 implies $\bar{\omega}_\xi = \bar{p}_\xi, \xi \in \mathbb{D}$. Since the government policy is Ricardian the asymptotic properties established in Step 2 hold. Thus if we define $\bar{\Theta} \equiv \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi$, all the conditions (i)-(iv) of Step 1 are satisfied, and $\mathbb{B}^h(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{\theta}}) = B^h(\bar{\boldsymbol{p}}, \bar{\boldsymbol{P}}, \bar{\boldsymbol{r}}^1, \bar{\Theta})$. Since $\bar{x}^h$ is optimal in $\mathbb{B}^h$, it is also optimal over B^h so that (1) in Definition 2 is satisfied. Defining the stochastic discount factor $\bar{\boldsymbol{\mu}}$ by $\bar{P}_\xi = \bar{B}_\xi \bar{\mu}_\xi, \xi \in \mathbb{D}$ and substituting into (ii) of Definition 1 gives (4) in Definition 2: it follows that the triple $\left(\left(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}, \bar{\Theta}\right), \bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}\right)$ satisfies all the conditions for a reduced-form equilibrium in Definition 2.

Step 4. We show that from a reduced-form equilibrium we can reconstruct the associated extensive-form equilibrium. Let $((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}, \bar{\Theta}), \bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})$ be a reduced-form equilibrium and let $\bar{\boldsymbol{q}}^j$ be defined by $\bar{P}_\xi \bar{q}^j_\xi = \sum_{\xi' \in \xi^+} \bar{\pi}_{\xi'} \hat{\bar{q}}^j_{\xi'}, j \in \mathcal{J}_p,\ \xi \in \mathbb{D}$ with $\hat{\bar{q}}^j_{\xi'} \equiv V^j_{\xi'}$. By assumption $\left[\hat{\bar{q}}_{\xi^+}\right]$ is invertible for all $\xi \in \mathbb{D}$. Given the properties of a reduced-form equilibrium, (i)-(iii) of Step 1 are satisfied and we can apply Step 2 to construct the government policy $(\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})$. To show that $\bar{\Theta} = \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi$, we sum the agents' budget equations in the reduced-form equilibrium and use the market clearing equations to obtain

$$\sum_{\xi \in \mathbb{D}} \frac{\bar{r}^1_\xi}{1 + \bar{r}^1_\xi} \bar{P}_\xi \bar{M}_\xi + \bar{\Theta} = \sum_{h \in \mathcal{H}} w^h_0 = W_0 \tag{38}$$

On the other hand multiplying the government's budget equation (33) at node ξ by $\bar{P}_\xi$ for all $\xi \in \mathbb{D}$ and summing over the whole event-tree leads to

$$\sum_{\xi \in \mathbb{D}} \frac{\bar{r}^1_\xi}{1 + \bar{r}^1_\xi} \bar{P}_\xi \bar{M}_\xi + \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi = W_0$$

which combined with (38) implies that $\bar{\Theta} = \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi$. Thus all the conditions (i)-(iv) of Step 1 are satisfied and $\mathbb{B}^h(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{\theta}}) = B^h(\bar{\boldsymbol{p}}, \bar{\boldsymbol{P}}, \bar{\boldsymbol{r}}^1, \bar{\Theta})$. Thus for each $h \in \mathcal{H}$, $\bar{\boldsymbol{x}}^h$ is optimal over $\mathbb{B}^h$ and the portfolio strategy $\bar{\boldsymbol{z}}^h$ which finances $\bar{\boldsymbol{x}}^h$ is given by (30). It remains to show that the financial markets clear, i.e. $\sum_{h \in \mathcal{H}} \bar{\boldsymbol{z}}^h = \bar{\boldsymbol{Z}}$. Summing the equations (30) at node ξ over the agents gives

$$\left[\sum_{\xi'' \in \mathbb{D}(\xi')} \bar{P}_{\xi''} \frac{\bar{r}^1_{\xi''} \bar{M}_{\xi''}}{1 + \bar{r}^1_{\xi''}} + \sum_{\xi'' \in \mathbb{D}(\xi')} \bar{P}_{\xi''} \bar{\theta}_{\xi''} - \bar{P}_{\xi'} \bar{M}_\xi\right]_{\xi' \in \xi^+} = [\bar{P} \circ_{\xi^+} \hat{\bar{q}}] \sum_{h \in \mathcal{H}} \bar{z}^h_\xi, \quad \forall \xi \in \mathbb{D} \tag{39}$$

On the other hand multiplying the government budget constraints (33) over the subtree $\mathbb{D}(\xi)$ by the corresponding node prices and, for each $\xi' \in \xi^+$, summing over the subtree $\mathbb{D}(\xi')$ leads to

$$\bar{P}_{\xi'}\bar{q}_{\xi'}\bar{Z}_\xi + \bar{P}_{\xi'}\bar{M}_\xi = \sum_{\xi''\in\mathbb{D}(\xi')} \bar{P}_{\xi''}\frac{\bar{r}^1_{\xi''}\bar{M}_{\xi''}}{1+\bar{r}^1_{\xi''}} + \sum_{\xi''\in\mathbb{D}(\xi')} \bar{P}_{\xi''}\bar{\theta}_{\xi''}, \quad \xi'\in\xi^+$$

which, combined with (39) implies that $\sum_{h\in\mathcal{H}} \bar{z}^h_\xi = \bar{Z}_\xi$ for all $\xi\in\mathbb{D}$. Thus all the properties of Definition 1 are satisfied, and the proof is complete. □

Proof of Proposition 4: ($\Longrightarrow$) Let $\left((\bar{\boldsymbol{B}},\bar{\boldsymbol{q}},\bar{\Theta}),\bar{\boldsymbol{x}},\bar{\boldsymbol{\mu}}\right)$ be a reduced-form equilibrium. The FOCs for the maximum problem (1) of Definition 2 for the agent imply that there exists $\lambda>0$ such that for all $\xi\in\mathbb{D}$

$$\bar{B}_\xi\,\delta^{t(\xi)}\,u_c(\bar{c}_\xi,\bar{\ell}_\xi) \;=\; \lambda\,\bar{P}_\xi\,\bar{p}_\xi = \lambda\,\bar{B}_\xi\,\bar{\mu}_\xi\,\bar{p}_\xi \tag{40}$$

$$\bar{B}_\xi\,\delta^{t(\xi)}\,u_\ell(\bar{c}_\xi,\bar{\ell}_\xi) \;=\; \lambda\,\frac{\bar{P}_\xi\,\bar{p}_\xi}{1+\bar{r}^1_\xi} = \lambda\frac{\bar{B}_\xi\,\bar{\mu}_\xi\,\bar{p}_\xi}{1+\bar{r}^1_\xi} \tag{41}$$

where, by Assumption $\mathcal{U}$ the consumption/leisure decision is always interior. Market clearing implies $\bar{\ell}_\xi = 1-\bar{c}_\xi$ and (a) follows by taking the ratio of (40) and (41). Replacing $\bar{\mu}_\xi$ by its value given in (40), (4) implies that (b1) is satisfied, and since (b2) is the same as (5) of Definition 2, a reduced-form equilibrium satisfies (a), (b1), (b2).

($\Longleftarrow$) Let $((\bar{\boldsymbol{B}},\bar{\boldsymbol{q}}),\bar{\boldsymbol{c}})$ satisfy (a), (b1), (b2). For all $\xi\in\mathbb{D}$ define $\bar{\ell}_\xi = 1-\bar{c}_\xi$, $\bar{x}_\xi = (\bar{c}_\xi, 1-\bar{c}_\xi)$, and $\bar{\mu}_\xi$, $\bar{P}_\xi$ (such that $\bar{P}_\xi = \bar{B}_\xi\bar{\mu}_\xi$) by (40). (a) implies that the FOCs (41) also hold. Since (a) holds, since the sequence $(\bar{r}^1_\xi)_{\xi\in\mathbb{D}}$ is bounded, and since by Assumption $\mathcal{U}$, $\dfrac{u_c(c,1-c)}{u_\ell(c,1-c)}\to\infty$ as $c\to 0$, the consumption sequence $(\bar{c}_\xi)_{\xi\in\mathbb{D}}$ is bounded away from zero and the sequence $(u_c(\bar{c}_\xi,\bar{\ell}_\xi))_{\xi\in\mathbb{D}}$ is bounded. Thus by (40) $(\bar{P}_\xi\,\bar{p}_\xi)_{\xi\in\mathbb{D}}\in\ell^1(\mathbb{D})$. Define $\bar{\Theta}$ by

$$\sum_{\xi\in\mathbb{D}}\frac{\bar{r}^1_\xi}{1+\bar{r}^1_\xi}\bar{P}_\xi\bar{p}_\xi\bar{c}_\xi + \bar{\Theta} = w_0$$

$\bar{\boldsymbol{x}}$ satisfies the budget equation in (1) of Definition 2 and the FOCs are satisfied, so (1) in Definition 2 is satisfied. Multiplying the two sides of (b1) by $\frac{\delta^{t(\xi)}}{\bar{p}_\xi}$gives (4) of Definition 2, so that $\left((\bar{\boldsymbol{B}},\bar{\boldsymbol{q}},\bar{\Theta}),\bar{\boldsymbol{x}},\bar{\boldsymbol{\mu}}\right)$ is a reduced-form equilibrium. □

Proof of Lemma 13: We first assume that $(\nu,a)\in\Delta^H_{++}\times\mathbb{R}^H_{++}$ are fixed and, to simplify notation, we omit these parameters as arguments of the functions. For $R>0$, $\Phi>0$, $h\in\mathcal{H}$, and C a large positive number, define the function $\tilde{x}^h:\mathbb{R}^2_{++}\to\mathbb{R}^2_{++}$ by

$$\tilde{x}^h(\Phi,R) = \operatorname{argmax}\left\{\nu^h u^h(c^h,\ell) - \Phi c^h - \Phi\frac{\ell^h}{R}\;\Big|\; 0\le c^h\le C,\; 0\le \ell^h\le e^h\right\}$$

By Assumption $\mathcal{U}$ and $\nu^h > 0$, the function which is maximized is strictly concave and has a unique maximum. By Assumption $\mathcal{U}$, for C large enough the maximum can not occur on the boundary of the constraint set so that the maximum is the solution of the FOCs (a1)-(a2). Let $\left(\tilde{c}^h(\Phi, R), \tilde{\ell}^h(\Phi, R)\right)$ denote the solution of the maximum problem viewed as function of (Φ, R). Since $u^h_{cc}u^h_{\ell\ell} - (u^h_{c\ell})^2 > 0$, the functions $\tilde{c}^h$ and $\tilde{\ell}^h$ are differentiable. To obtain a solution to (a1)-(a3) Φ must satisfy the market clearing equation

$$\sum_{h\in\mathcal{H}} \tilde{c}^h(\Phi, R) + \sum_{h\in\mathcal{H}} a^h \tilde{\ell}^h(\Phi, R) = \sum_{h\in\mathcal{H}} a^h e^h \tag{42}$$

By Assumption $\mathcal{U}$, when $\Phi \to 0$, $\tilde{c}^h(\Phi, R) \to \infty$ and the left side of (42) is greater than the right side. When $\Phi \to \infty$ both $\tilde{c}^h(\Phi, R)$ and $\tilde{\ell}^h(\Phi, R)$ tend to zero so that the left side of (42) is smaller than the right side. Thus it suffices to show that the functions $\tilde{c}^h$ and $\tilde{\ell}^h$ are strictly decreasing functions of Φ to show that equation (42) has a unique solution $\Phi(R)$. Differentiating the FOCs (a1)- (a2) gives

$$\begin{aligned}
\nu^h\, u^h_{cc}\frac{\partial \tilde{c}^h}{\partial \Phi} + \nu^h\, u^h_{c\ell}\frac{\partial \tilde{\ell}^h}{\partial \Phi} &= 1\\
\nu^h\, u^h_{\ell c}\frac{\partial \tilde{c}^h}{\partial \Phi} + \nu^h\, u^h_{\ell\ell}\frac{\partial \tilde{\ell}^h}{\partial \Phi} &= \frac{a^h}{R}
\end{aligned}$$

which implies

$$\frac{\partial \tilde{c}^h}{\partial \Phi} = \frac{u^h_{\ell\ell} - \frac{a^h u^h_{c\ell}}{R}}{\nu^h D^h}, \qquad \frac{\partial \tilde{\ell}^h}{\partial \Phi} = \frac{\frac{a^h u^h_{cc}}{R} - u^h_{c\ell}}{\nu^h D^h R}$$

where $D^h = u^h_{cc}u^h_{\ell\ell} - (u^h_{c\ell})^2$. By Assumption $\mathcal{U}$, $D^h > 0$ and the numerators of the fractions are negative, so that both $\tilde{c}^h$ and $\tilde{\ell}^h$ are decreasing functions of Φ: thus (42) has a unique solution $\Phi(R)$.

To sign $\frac{\partial \Phi}{\partial R}$, differentiating (42) gives

$$\sum_{h\in\mathcal{H}} \left(\frac{\partial \tilde{c}^h}{\partial \Phi} + a^h \frac{\partial \tilde{\ell}^h}{\partial \Phi}\right)\frac{\partial \Phi}{\partial R} = -\sum_{h\in\mathcal{H}} \left(\frac{\partial \tilde{c}^h}{\partial R} + a^h \frac{\partial \tilde{\ell}^h}{\partial R}\right) \tag{43}$$

The derivatives $\partial\tilde{c}^h/\partial R$ and $\partial\tilde{\ell}^h/\partial R$ can be found by differentiating the FOC (a1)- (a2)

$$\begin{aligned}
\nu^h\, u^h_{cc}\frac{\partial \tilde{c}^h}{\partial R} + \nu^h\, u^h_{c\ell}\frac{\partial \tilde{\ell}^h}{\partial R} &= 0\\
\nu^h\, u^h_{\ell c}\frac{\partial \tilde{c}^h}{\partial R} + \nu^h\, u^h_{\ell\ell}\frac{\partial \tilde{\ell}^h}{\partial R} &= \frac{-a^h\Phi}{R^2}
\end{aligned}$$

which gives

$$\frac{\partial \tilde{c}^h}{\partial R} = \frac{a^h \Phi u^h_{c\ell}}{\nu^h R^2 D^h} \geq 0, \quad \frac{\partial \tilde{\ell}^h}{\partial R} = -\frac{a^h \Phi u^h_{cc}}{\nu^h R^2 D^h} > 0$$

which by (43) implies $\frac{\partial \Phi}{\partial R} > 0$. Solving for $\frac{\partial \Phi}{\partial R}$ using (43) and substituting gives

$$\frac{\partial}{\partial R}\left(\frac{\Phi}{R}\right) = \frac{R\frac{\partial \Phi}{\partial R} - \Phi}{R^2} = \frac{-R\sum_{h\in\mathcal{H}}\left(\frac{\partial \tilde{c}^h}{\partial R} + a^h\frac{\partial \tilde{\ell}^h}{\partial R}\right) - \Phi\sum_{h\in\mathcal{H}}\left(\frac{\partial \tilde{c}^h}{\partial \Phi} + a^h\frac{\partial \tilde{\ell}^h}{\partial \Phi}\right)}{R^2\sum_{h\in\mathcal{H}}\left(\frac{\partial \tilde{c}^h}{\partial \Phi} + a^h\frac{\partial \tilde{\ell}^h}{\partial \Phi}\right)} \equiv \frac{N}{D} \tag{44}$$

The denominator D is negative. To sign the numerator we replace the partial derivatives of $\tilde{c}^h$ and $\tilde{\ell}^h$ by their values, which gives

$$N = \Phi\sum_{h\in\mathcal{H}}\frac{a^h u^h_{c\ell} - u^h_{\ell\ell}}{\nu^h D^h} > 0$$

so that $\frac{\partial}{\partial R}\left(\frac{\Phi}{R}\right) < 0$.

Reverting to the full notation, (42) defines the function $\Phi(\boldsymbol{\nu}, \boldsymbol{a}, R)$, and if the functions c^h and ℓ^h are defined by $c^h(\boldsymbol{\nu}, \boldsymbol{a}, R) = \tilde{c}^h(\Phi(\boldsymbol{\nu}, \boldsymbol{a}, R), R)$, $\ell^h(\boldsymbol{\nu}, \boldsymbol{a}, R) = \tilde{\ell}^h(\Phi(\boldsymbol{\nu}, \boldsymbol{a}, R), R)$, all the properties of Lemma 13 are satisfied for $(\boldsymbol{\nu}, \boldsymbol{a}, R) \in \Delta^H_{++} \times \mathbb{R}^H_{++} \times \mathbb{R}_{++}$. To show continuity with respect to $\boldsymbol{\nu}$ over the whole simplex, suppose that a sequence $(\boldsymbol{\nu}_n)_{n\geq 0} \in \Delta^H_{++}$ in the interior of the simplex converges to $\bar{\boldsymbol{\nu}} \in \Delta^H$ with $\bar{\boldsymbol{\nu}}^h = 0$. Since for some h', $\bar{\boldsymbol{\nu}}^{h'} \geq 1/H$ and (42) must hold, $\Phi(\boldsymbol{\nu}_n, \boldsymbol{a}, R)$ stays bounded away from zero and, since $\boldsymbol{\nu}^h_n \to 0$, $u^h_c(c^h(\boldsymbol{\nu}_n, \boldsymbol{a}, R), \ell^h(\boldsymbol{\nu}_n, \boldsymbol{a}, R))$ and $u^h_\ell(c^h(\boldsymbol{\nu}_n, \boldsymbol{a}, R), \ell^h(\boldsymbol{\nu}_n, \boldsymbol{a}, R))$ must tend to ∞, so that $(c^h(\boldsymbol{\nu}_n, \boldsymbol{a}, R), \ell^h(\boldsymbol{\nu}_n, \boldsymbol{a}, R))$ tend to 0. □

Proof of Proposition 14: Since for all $\eta \in \mathcal{S}\times\mathcal{G}$, $\boldsymbol{a}_\eta = (a^1_\eta, \ldots, a^H_\eta)$ is fixed, we omit it from the argument of the functions and let $c^h_\eta(\boldsymbol{\nu}, R), \ell^h_\eta(\boldsymbol{\nu}, R), \Phi_\eta(\boldsymbol{\nu}, R)$ denote the functions $c^h(\boldsymbol{\nu}, \boldsymbol{a}_\eta, R)$, $\ell^h(\boldsymbol{\nu}, \boldsymbol{a}_\eta, R)$, $\Phi(\boldsymbol{\nu}, \boldsymbol{a}_\eta, R)$ defined in Lemma 12. Since by (1) of Proposition 14

$$\widetilde{\Phi}_\eta(\boldsymbol{\nu}, 1) \geq \delta\sum_{\eta'\in\mathcal{S}\times\mathcal{G}} B_{\eta\eta'}\frac{\Phi_{\eta'}(\boldsymbol{\nu}, 1)}{1+\pi_{\eta'}} > \lim_{R\to\infty}\widetilde{\Phi}_\eta(\boldsymbol{\nu}, R)$$

there exists a solution $R^{\max}_\eta(\boldsymbol{\nu}) \geq 1$ to the equation (26), so that $R^{\max}_\eta \geq 1$. Thus

$$K = \left\{\boldsymbol{R} \in \mathbb{R}^{S\times G}_+ \mid \boldsymbol{1} \leq \boldsymbol{R} \leq \boldsymbol{R}^{\max}\right\}$$

is a non-empty compact convex subset of $\mathbb{R}^{S\times G}$. For $\boldsymbol{\nu} \in \Delta^H$ and $\boldsymbol{R} = (R_1, \ldots, R_{S\times G}) \gg 0$, let $\Theta(\boldsymbol{\nu}, \boldsymbol{R})$ be defined by the equation (obtained by summing the agents' budget equations)

$$\sum_{\eta\in\mathcal{S}\times\mathcal{G}}[I - \delta\boldsymbol{B}]_{\eta_0\eta}\Phi_\eta(\boldsymbol{\nu}, R_\eta)\Big(\sum_{h\in\mathcal{H}} c^h_\eta(\boldsymbol{\nu}, R_\eta)\Big)\frac{R_\eta - 1}{R_\eta} + \Phi_{\eta_0}(\boldsymbol{\nu}, R_{\eta_0})(\Theta - W_0) = 0 \tag{45}$$

$\Theta(\boldsymbol{\nu}, \boldsymbol{R})$ is the present value of the taxes needed to withdraw the government liabilities $W_0 = \sum_h w_0^h$ from the private sector when the seignorage revenue is given by the first term in (45). For each $h \in \mathcal{H}$ consider the function

$$\begin{aligned}\zeta^h \quad & (\boldsymbol{\nu}, \boldsymbol{R}) = \\ & \sum_{\eta \in \mathcal{S}\times\mathcal{G}} [I - \delta \boldsymbol{B}]_{\eta_0 \eta} \Phi_\eta(\boldsymbol{\nu}, R_\eta) \left(\frac{a_\eta^h (e^h - \ell_\eta^h(\boldsymbol{\nu}, R_\eta))}{R_\eta} - c_\eta^h(\boldsymbol{\nu}, R_\eta) \right) + \Phi_{\eta_0}(\boldsymbol{\nu}, R_{\eta_0})(w_0^h - \gamma^h \Theta(\nu, \boldsymbol{R}))\end{aligned}$$

which gives the excess of the present value of income over consumption for agent h when the vector of weights is $\boldsymbol{\nu}$ and the vector of returns is $\boldsymbol{R}$. Given the definition of $\Theta(\boldsymbol{\nu}, \boldsymbol{R})$, for all $\boldsymbol{\nu} \in \Delta^H$ and $\boldsymbol{R} \gg 0$, $\sum_{h\in\mathcal{H}} \zeta^h(\boldsymbol{\nu}, \boldsymbol{R}) = 0$.

Consider the map $\boldsymbol{\Psi}$: $\Delta^H \times K \to \mathbb{R}^H \times \mathbb{R}^{S\times G}$ defined by

$$\begin{aligned}\Psi_h(\boldsymbol{\nu}, \boldsymbol{R}) &= \frac{\nu^h + \max\{\zeta^h(\boldsymbol{\nu}, \boldsymbol{R}), 0\}}{1 + \sum_{h'\in\mathcal{H}} \max\{\zeta^{h'}(\boldsymbol{\nu}, \boldsymbol{R}), 0\}}, \qquad h \in \mathcal{H} \\ \Psi_\eta(\boldsymbol{\nu}, \boldsymbol{R}) &= \widetilde{\Phi}_\eta^{-1}\left(\boldsymbol{\nu}, \delta \sum_{\eta' \in \mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{\nu}, R_{\eta'}) \right), \qquad \eta \in \mathcal{S}\times\mathcal{G}\end{aligned} \tag{46}$$

where $\widetilde{\Phi}_\eta^{-1}(\boldsymbol{\nu}, \cdot)$ denotes the inverse of the decreasing function $R \to \widetilde{\Phi}_\eta(\boldsymbol{\nu}, R)$. Ψ_h increases the weight of agent h when the present value of his income exceeds the present value of his consumption, and decreases it otherwise. Ψ_η gives the return on the short term bond in state η which is such that the marginal cost of one unit of the bond is equal to its marginal benefit, when the vector of marginal utilities next period is $(\Phi_{\eta'}(\boldsymbol{\nu}, R_{\eta'}))_{\eta'\in\mathcal{S}\times\mathcal{G}}$. Since the function $R \to \Phi_\eta(\boldsymbol{\nu}, R)$ is increasing and $\lim_{R\to 0} \widetilde{\Phi}_\eta(\boldsymbol{\nu}, R) = \infty$, by (1) of Proposition 14

$$\lim_{R\to\infty} \widetilde{\Phi}_\eta(\boldsymbol{\nu}, R) < \delta \sum_{\eta'\in\mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1+\pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{\nu}, 1) \le \delta \sum_{\eta'\in\mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1+\pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{\nu}, R_{\eta'}) < \lim_{R\to 0} \widetilde{\Phi}_\eta(\boldsymbol{\nu}, R)$$

so that $\delta \sum_{\eta'\in\mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1+\pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{\nu}, R_{\eta'})$ is in the image of the function $R \to \widetilde{\Phi}_\eta(\boldsymbol{\nu}, R)$ and $\Psi_\eta(\boldsymbol{\nu}, \boldsymbol{R})$ is well defined. By construction $(\Psi_h(\boldsymbol{\nu}, \boldsymbol{R}))_{h\in\mathcal{H}}$ is in Δ^H. To show that $(\Psi_\eta(\boldsymbol{\nu}, \boldsymbol{R}))_{\eta\in\mathcal{S}\times\mathcal{G}}$ is in K, note that

$$\delta \sum_{\eta'\in\mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1+\pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{\nu}, R_{\eta'}) \le \delta \sum_{\eta'\in\mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1+\pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{\nu}, R_{\eta'}^{\max}) = \frac{\Phi_\eta(\boldsymbol{\nu}, 1)}{R_\eta^{\min}(\boldsymbol{\nu})} \le \widetilde{\Phi}_\eta(\boldsymbol{\nu}, 1) \tag{47}$$

where the equality comes from the definition of $R_\eta^{\min}(\boldsymbol{\nu})$, and the last inequality comes from (2) of Proposition 14 and the fact that $\Phi_\eta(\boldsymbol{\nu}, 1) = \widetilde{\Phi}_\eta(\boldsymbol{\nu}, 1)$ for all $\boldsymbol{\nu} \in \Delta^H$. Since $\widetilde{\Phi}_\eta^{-1}(\boldsymbol{\nu}, \cdot)$ is decreasing, (47) implies

$$\Psi_\eta(\boldsymbol{\nu}, \boldsymbol{R}) = \widetilde{\Phi}_\eta^{-1}\left(\boldsymbol{\nu}, \delta \sum_{\eta' \in \mathcal{S}\times\mathcal{G}} \frac{B_{\eta\eta'}}{1+\pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{\nu}, R_{\eta'})\right) \geq \widetilde{\Phi}_\eta^{-1}(\boldsymbol{\nu}, \widetilde{\Phi}_\eta(\boldsymbol{\nu}, 1)) = 1$$

so that $\Psi_\eta(\boldsymbol{\nu}, \boldsymbol{R}) \geq 1$, for all $\eta \in \mathcal{S} \times \mathcal{G}$. Thus Ψ is a continuous map from $\Delta^H \times K$ into itself and, by Brouwer's Theorem, has a fixed point $(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}})$. Let

$$\bar{q}_\eta^1 = \frac{1}{\bar{R}_\eta}, \quad \bar{x}_\eta^h = (c_\eta^h(\bar{\boldsymbol{\nu}}, \bar{R}_\eta), \ell_\eta^h(\bar{\boldsymbol{\nu}}, \bar{R}_\eta)),\ h \in \mathcal{H}, \quad \bar{\Phi}_\eta = \Phi_\eta(\bar{\boldsymbol{\nu}}, \bar{R}_\eta), \quad ,\ \eta \in \mathcal{S}\times\mathcal{G}, \quad \bar{\Theta} = \Theta(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}})$$

and let the prices of bonds of maturities $\tau = 2, \ldots, T$ be calculated recursively using equations (b1) in Proposition 12. Let us show that $((\boldsymbol{B}, \bar{q}, \bar{\Theta}), (\bar{\boldsymbol{\nu}}, \bar{x}, \bar{\Phi}))$ is a reduced-form equilibrium. From the construction of the functions (c^h, ℓ^h) in Lemma 13 and by the fixed-point property of $\bar{\boldsymbol{R}}$ in K, the equations (a1)-(a3), (b1)-(b2) of Proposition 12 are satisfied and it suffices to show that the budget equations (a4) hold.

Since $\sum_{h\in\mathcal{H}} \zeta^h(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}) = 0$, if $\zeta^h(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}) \neq 0$ for some agent. then there is one agent for which the value of excess income is strictly positive and $\sum_{h\in\mathcal{H}} \max\{\zeta^h(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}), 0\} > 0$, and there is another agent h' such that $\zeta^{h'}(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}) < 0$. For this agent the fixed point property

$$\bar{\nu}^{h'}(1 + \sum_{h\in\mathcal{H}} \max\{\zeta^h(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}), 0\}) = \bar{\nu}^{h'}$$

implies $\bar{\nu}^{h'} = 0$. This in turn implies $(c_\eta^{h'}(\bar{\boldsymbol{\nu}}, \bar{R}_\eta), \ell_\eta^{h'}(\bar{\boldsymbol{\nu}}, \bar{R}_\eta)) = 0$ for all $\eta \in \mathcal{S} \times \mathcal{G}$, so that

$$\zeta^{h'}(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}) = \sum_{\eta\in\mathcal{S}\times\mathcal{G}} [I - \delta\boldsymbol{B}]_{\eta_0\eta} \bar{\Phi}_\eta \frac{a_\eta^h e^h}{\bar{R}_\eta} - \bar{\Phi}_{\eta_0}(\gamma^{h'}\bar{\Theta} - w_0^{h'})$$

If $\gamma^{h'}\bar{\Theta} - w_0^{h'} \leq 0$ then it is not possible that $\zeta^{h'} < 0$. If $\gamma^{h'}\bar{\Theta} - w_0^{h'} > 0$, then since $\widetilde{\Phi}$ is decreasing and Φ is increasing

$$\zeta^{h'}(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}) \geq \sum_{\eta\in\mathcal{S}\times\mathcal{G}} [I - \delta\boldsymbol{B}]_{\eta_0\eta} \Phi_\eta(\bar{\nu}, R_\eta^{\max}) \frac{a_\eta^h e^h}{R_\eta^{\max}} - \Phi_{\eta_0}(\bar{\nu}, R_{\eta_0}^{\max})(\gamma^{h'}\bar{\Theta} - w_0^{h'}) > 0$$

by (3) of Proposition 14, contradicting the assumption $\zeta^{h'}(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}) < 0$. Thus $\zeta^h(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{R}}) = 0$ for all $h \in \mathcal{H}$ and (a4) holds. □

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