THE DUALITY OF THE PRICE AND TECHNOLOGY SETS

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This paper is concerned with the generalisation of Samuleson's factor-price frontier in the framework of efficient allocation of resources. Efficient prices dual to efficient allocations are defined in terms of the concept of normal profit and characterised by a theorem dual to Koopmans' theorem on efficient allocations. A Duality theorem (familar in convexity theory) shows that a knowledge of these prices leads to an alternative derivation of the technology set.

I. INTRODUCTION

1. The fundamental theorem of resource allocation relates a vector of prices to an efficient allocation. The theorem was first proved by Koopmans [1] for the static case, and was extended by Debreu [2] to efficient allocations over finite time, and by Malinvaud [3] and Radner [4] to efficient allocations over infinite time. The theorem is essentially an existence theorem, asserting the existence of a vector of prices associated with an efficient allocation, that maximises value over all allocations in the technology set. Usually there are many efficient allocations and each has at least one associated vector of prices. The principal object of the present paper is to show that these prices associated with the efficient allocations lie in a convex set that is dual to the original technology set in the sense that *it contains exactly the same information about the technology as the original technology set.* This set is called the *dual price set,* and the duality property is shown to hold for efficient allocations in the static case, as well as those with a finite or infinite time horizon

Mathematically the theorem is a well-known result in convexity theory and was originally proved by Minkowski [5] in the finite dimensional case: the extension to the infinite dimensional case is straightforward. The economic implications of the theorem do not seem to have been fully explored however. Thus Koopmans [1], assuming that the technology set is a convex cone, recognises that the prices lie in the dual convex cone but the economic importance of this result is not developed. The result furthermore is of particular interest in cases where the technology set *does not exhibit constant returns to scale* and where prices can more precisely control decentralised allocation of resources.

2. The price set is a generalisation of Samuelson's factor-price frontier [6], the frontier of primary good or factor prices associated with the efficient allocations of the primary goods, in an economy that produces a single aggregate output with many primary goods. This generalisation is less far-reaching in the dynamical case, however than the generalised factor-price frontier introduced by Bruno [7] and further analysed by Burmeister and Kuga [8] in the framework of a growth model with many capital goods.

Just as the technology set of the economy may be derived from the production sets (functions) of all the separate producing sectors whether individual firms or whole industries, so the price set of the economy may be derived from the price sets (functions) of all the separate production sectors, the price set of each sector being the dual of its production set. This duality between the production and *price sets of a sector* — or more accurately, the duality between its production and *cost functions*, since the sets are analytically represented and there is only one output and many inputs — was first exhibited by Samuelson [9] and in more detail by Shephard [10], and subsequently analysed more generally by Uzawa [11]. The price set of the economy thus also contains the information embodied in the sector cost functions of Samuelson and Shephard.

3. The definition of the price set leads to the concept of normal profit; prices may be classified as unfeasible, feasible but inefficient and efficient according as they generate under profit maximisation a profit that exceeds, is less than, or exactly equals normal profit. *Efficient prices* are of primary interest in establishing the equivalence of the price and technology sets and may be characterised by a theorem dual to Koopmans' theorem on efficient allocations: this theorem is a generalisation of the characterisation of competitive prices of primary goods in the familiar dual problem of linear programming.

While the price set is the dual of the technology set, the Duality theorem asserts that the technology set is the dual of the price set. This theorem is of great importance for empirical research, for it implies that the structure of the technology set can be determined from a knowledge of the price set. This method of determining the technology set has already been used in the pioneering study of Arrow, Chenery, Minhas, and Solow [12], where they determine both the sector cost functions and the factor-price frontier for nineteen different countries using two primary goods capital and labor.

4. In extending these ideas to allocations stretching over an infinite time horizon, we follow the original study of Malinvaud [3]. Since linear functionals cannot always be given a simple series representation in an infinite dimensional space, a problem arises in the interpretation of prices (Radner [4]). To circumvent this difficulty we confine the analysis to a Banach space (the space of convergent sequences) on which linear functionals can be given a series representation: there is good economic justification for using this space. While this is clearly not the most general framework for analysing

allocation over an infinite horizon, it is adequate for the present purposes, for it enables us to show one way in which the results in the static case may be extended to the dynamical case.

5. The mathematical analysis underlying the theory is extremely simple: the principal ideas on which the theory is based are the relation between a vector space and its dual (the space in which prices live), and the duality between points and hyperplanes induced by convex sets. All the theorems follow in a simple way from the theorem on the separation of disjoint convex sets known as the Hahn-Banach theorem (Dunford and Schwartz [13]). All the important ideas on the duality of convex sets, including the finite dimensional version of the Hahn-Banach theorem, may be found in the original paper of Minkowski [5].

II. Duality in the Static Case

6. Consider an economic system that uses m primary goods $H_1, \dots, H_m$ in amounts $y_1, \dots, y_m$ to produce n final goods $G_1, \dots, G_n$ in amounts $x_1, \dots, x_n$ during a given time period, and let $r_1, \dots, r_m$, and $p_1, \dots, p_n$ denote the prices of the primary and final goods during that period. The vector $(x, -y) = (x_1, \dots, x_n, -y_1, \dots, -y_m)$ may be considered an element of the vector space R^{n+m}. For any given vectors $(p, r) = (p_1, \dots, p_n, r_1, \dots, r_m)$, the expression

$$\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{m} r_i y_i \tag{1}$$

denotes a *valuation* assigned to the vector $(x, -y) \in R^{n+m}$. A valuation of this kind is a linear function defined on R^{n+m} which assigns a real number to each point in R^{n+m}: the collection of all linear functions of this kind on R^{n+m}, also called *linear functionals*, may be considered as elements of another vector space called the *dual space* $(R^{n+m})^*$. In a finite dimensional space every linear functional can be written in the form (1): thus each linear functional in $(R^{n+m})^*$ gives rise to a vector (p, r), so that the dual space $(R^{n+m})^*$ may be considered as the collection of all vectors of this form.

Every hyperplane in R^{n+m} not passing through the origin may be written in the form

$$\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{m} r_i y_i = px - ry = 1 \tag{2}$$

Such a hyperplane generates two closed half-spaces $\{(x, -y) \mid px - ry \leqq 1\}$ and $\{(x, -y) \mid px - ry \geqq 1\}$. Given (p, r), the hyperplane (2) is unique; there is thus a unique relation between non-zero linear functionals in $(R^{n+m})^*$ and hyperplanes in R^{n+m} not passing through the origin. This gives an important geometric way of interpreting linear functionals.

7. The collection of all pairs of the form $(x, -y) \in R^{n+m}$, $y \geqq 0$, such that x can be produced given y, is called the *technology set* of the economy, and is denoted by T. If the economy is composed of s separate sectors, then the collection of all pairs of the form $(x^k, -y^k) \in R^{n+m}$, $y^k \geqq 0$, $k = 1, \dots, s$ such that x^k can be produced by the k^{th} sector given y^k, is called the *production set* of the k^{th} sector, and is denoted by T^k. In this way the pair $(x, -y)$ is the combined activity of all s sectors,

$$(x, -y) = (\sum_{k=1}^{s} x^k, -\sum_{k=1}^{s} y^k), \qquad (x^k, -y^k) \in T^k$$

and

$$T = \sum_{k=1}^{s} T^k$$

where the latter denotes the usual vector sum of sets. The main structural property required of the technology and production sets $\{T; T^1, \dots, T^s\}$ is that they be *closed convex sets* in R^{n+m}.

A closed convex set can be characterised in two ways: either in terms of its boundary points or as the intersection of the closed half-spaces generated by supporting hyperplanes: a hyperplane is said to be a *supporting hyperplane* to a convex set T, if it contains at least one point of T, and T lies entirely in one of its halfspaces. The first representation views T as a collection of points in R^{n+m}: the second representation views T as a collection of points in the dual space $(R^{n+m})^*$, for each supporting hyperplane not passing through the origin generates a unique linear functional in $(R^{n+m})^*$ by the relation (2). If every supporting plane to T passes through the origin (if T is a cone) then we replace the incidence relation (2) by the relation

$$px - ry = 0$$

The collection of all supporting planes to T thus generate a collection of points in the dual space: the closed convex set formed in this way is called the *dual* T^* of the convex set T. Thus

$$(3) \qquad T^* = \{(p, r) \in (R^{n+m})^* \mid px - ry \leqq 1 \text{ for all } (x, -y) \in T\}$$

which if T is a cone, reduces to

$$(3') \qquad T^* = \{(p, r) \in (R^{n+m})^* \mid px - ry \leqq 0 \text{ for all } (x, -y) \in T\}$$

Consider the dual convex set T^*. Since T^* is a closed convex set it may also be characterised in two ways: either in terms of its boundary points or as the intersection of the closed half-spaces generated by its supporting hyperplanes: the second representation views T^* as a collection of points in the dual of the space $(R^{n+m})^*$, namely the *second dual space* $(R^{n+m})^{**} = R^{n+m}$, and this collection of points is a closed convex set called the *bidual* T^{**} of T. Thus

$$(3'') \qquad T^{**} = \{(x^{**}, -y^*) \in (R^{n+m})^{**} \mid x^{**}p - y^{**}r \leqq 1 \text{ for all } (p, r) \in T^*\}$$

and if T is a cone we replace 1 by 0 as in (3'). An important way of obtaining an analytic representation for T^* is to form the function

$$S_T(p, r) = \sup_{(x,-y)\in T} \{px - ry\}$$

called the *support function of* T, since it generates all the supporting hyperplanes of T,

$$T^* = \{(p, r) \in (R^{n+m})^* \mid S_T(p, r) \leqq 1\}$$

where 0 replaces 1 when T is a cone.

8. The technology set of the economy exhibits the alternative bundles of final goods available with a given sacrifice of the primary goods all such allocations $(x, -y) \in T$ are *feasiele.* The choice among feasible allocations will be determined by the criterion of efficiency due to Koopmans [1]: an allocation $(\bar{x}, -\bar{y}) \in T$ is *efficient* if there does not exist $(x, -y) \in T$ such that $(x, -y) \geq (\bar{x}, -\bar{y})$ where $x \geq \bar{x}$ implies $x_i \geqq \bar{x}_i$ all i and $x_j > \bar{x}_j$ for some j. The concept in particular assumes no saturation of preferences for any of the goods. Recall the following theorem due to Koopmans [1].

Theorem 1 (*Efficient Allocations*)

(i) If there exists a price $(p, r) > 0$ and $(\bar{x}, -\bar{y}) \in T$ such that

$$(4) \qquad p\bar{x} - r\bar{y} \geqq px - ry \qquad \text{for all } (x, -y) \in T$$

then $(\bar{x}, -\bar{y})$ is an efficient allocation.

(ii) If T is convex, and if $(\bar{x}, -\bar{y})$ is an efficient allocation, then there exists a price $(p, r) \geq 0$ such that (4) is satisfied.

The standard corollary asserts that the prices can also be used to achieve an efficient allocation of resources by a decentralisation of the allocation decision among the s separate sectors, for total profit $px - ry$ is maximised over T, if and only if each sector maximises profit $px^k - ry^k$ over T^k, since $T = \sum_{k=1}^{s} T^k$. This is perhaps the most important property of prices in the practical problem of resource allocation, for the existence of prices related to efficient allocations would not be so significant if they could not be used in practice to induce sectors to an efficient decentralised allocation of resources.

9. In order to show how the prices can be made to reveal the structure of the technology, we distinguish between two main types of technology sets: first those for which every non-zero efficient allocation generates *positive profit*, and second those for which every efficient allocation generates *zero profit*. The first type includes cases where the technology set is convex but does not exhibit constant returns to scale, while the second type is mainly concerned with cases where the technology set has both the property of convexity and constant returns to scale. Since the duality between prices and allocations breaks down for technology sets not containing the origin, we do not consider technology sets for which efficient points generate negative profit.

Theorem 1 associates with each efficient allocation $(\bar{x}, -\bar{y})$ a price (p, r) such that (4) is satisfied; thus for any $\lambda > 0$, $\lambda p\bar{x} - \lambda r\bar{y} \geqq \lambda px - \lambda ry$ for all $(x, -y) \varepsilon T$, so that any price $\lambda (p, r)$, $\lambda > 0$ leads under profit maximisation to the same efficient allocation. Now suppose the technology set is of the first type, then we can normalise the prices so that each non-zero efficient allocation generates unit profit,

$$p\bar{x} - r\bar{y} = 1 \tag{5}$$

In this way a single price is chosen along the ray $\lambda (p, r)$, $\lambda > 0$. If this normalisation is adopted, then

$$px - ry \leqq p\bar{x} - r\bar{y} = 1 \qquad \text{for all } (x, -y) \varepsilon T$$

so that by (3), (p, r) lies in the dual convex set T^*. Since the elements of T^* are prices, T^* may be called the *dual price set.* Thus if the technology set is of the first type, the normalised prices associated with efficient allocations in T lie in the dual price set T^*.

Geometrically it is useful to view the normalisation rule (5) as an incidence relation which induces a duality between points and hyperplanes in R^{n+m} and $(R^{n+m})^*$. For (5) associates with each non-zero point $(x^0, -y^0)$ in R^{n+m} a unique hyperplane $px^0 - ry^0 = 1$, not passing through the origin in $(R^{n+m})^*$ and conversely: similarily (5) associates with each non-zero point (p^0, r^0) in $(R^{n+m})^*$ a unique hyperplane $p^0x - r^0y = 1$, not passing through the origin in R^{n+m}, and conversely.

Now suppose the technology set is of the second type and in particular as in Koopmans' approach [1], is a convex cone, then it is not necessary to normalise prices in order to establish a duality between the price and technology sets. For Theorem 1 associates with each efficient allocation $(\bar{x}, -\bar{y})$ in T a price (p, r) such that

$$px - ry \leqq p\bar{x} - r\bar{y} = 0 \qquad \text{for all } (x, -y) \in T$$

and hence by (3'), (p, r) lies in the dual convex cone T^*, which may again be called the dual price set. Thus if the technology set is a cone all prices $\lambda (p, r)$, $\lambda > 0$ where (p, r) is a price associated with a non-zero efficient allocation, lie in the dual price set T^*.

In this case the incidence relation (5) is replaced by the incidence relation

$$px - ry = 0 \tag{6}$$

which in turn associates with each hyperplane passing through the origin in R^{n+m}, $p^0 x - r^0 y = 0$ a unique ray $\lambda (p^0, r^0)$, $\lambda > 0$ in $(R^{n+m})^*$ and conversely: similarily each hyperplane passing through the origin in $(R^{n+m})^*$, $px^0 - ry^0 = 0$ is associated with a unique ray $\lambda (x^0, -y^0)$, $\lambda > 0$ in R^{n+m} and conversely.

If the technology set is a combination of the first and second types, so that efficient points near the origin generate zero profit, and subsequent efficient points generate positive profit, then all prices generating positive profit are normalised using (5) while prices generating zero profit are restricted to line segments of the form $\lambda (p, r)$, $0 < \lambda < \lambda^0$ for some λ^0, so that T^* is a convex set.

As will be shown below (Theorem 2), if these normalisations are adopted, then for any convex technology set which contains the origin, the prices associated with efficient allocations can be made to reveal the structure of the technology set.

10. In order to retain the symmetry between allocations in T and prices in T^*, it seems natural that the normalised prices associated with efficient allocations should be efficient prices. If the technology set is of the first type, then (5) suggests that unit profit should be considered as the *normal profit* for an efficient allocation. We are thus led to the following definitions for prices in the dual space. Any non-negative price that generates under profit maximisation a profit exceeding normal profit is *unfeasible*; a non-negative price that generates a profit less than or equal to normal profit is *feasible*, while a price that leads to normal profit is *efficient*. Thus feasible but inefficient prices lead to less than normal profit. If the technology set is of the second type then zero profit is normal profit, and any price generating profit less than or equal to zero is feasible, while a price is efficient if it generates zero profit for some non-zero efficient allocation.

Feasible and efficient prices may then be characterised alternatively as follows; a price (p, r) is feasible provided $(p, r) \in T^*$ and a feasible price $(\bar{p}, \bar{r})$ is efficient if there does not exist a price $(p, r) \in T^*$ such that $(p, -r) \geq (\bar{p}, -\bar{r})$. There is thus a complete symmetry between feasible and efficient allocations in T, and feasible and efficient prices in T^*.

Given the normalisation rule (5) or the relation (6), Theorem 1 characterises an efficient allocation as an allocation that generates normal profit under a suitable price. There is a natural duàl to this theorem which characterises an efficient price as a price that generates normal profit under a suitable allocation.

Theorem 1* (*Efficient Prices*)

(i) If there exists an allocation $(x, -y) > 0$, and $(\bar{p}, \bar{r}) \varepsilon T^*$ such that

$$(7) \qquad \bar{p}x - \bar{r}y \geqq px - ry \qquad \text{for all } (p, r) \varepsilon T^*$$

then $(\bar{p}, \bar{r})$ is an efficient price.

(ii) If $(\bar{p}, \bar{r})$ is an efficient price, then there exists an allocation $(x, -y) \geq 0$ such that (7) is satisfied.

The proof is the same as for Theorem 1; we merely substitute prices for allocations, and replace T by T^*, noting that T^* is always convex.

While the theorem on efficient allocations characterises an ***efficient allocation as the solution of a maximum problem over*** T, the dual theorem on efficient prices characterises an ***efficient price as the solution of a maximum problem over*** T^*. A special case of these dual maximum problems is found in the familiar primal and dual problems of linear programming (as is shown below), the dual problem being that of imputing prices to the scarce primary goods.

11. The incidence relazions (5) and (6) allov os to draw some simple conclusions about the relation between efficient allocations and efficient prices for different types of technology sets.

Suppose T is of the first type. Consider in particular the ideal case where $\{T ; T^1, \ldots, T^s\}$ are strictly convex and contain the origin. In this case a single supporting hyperplane passes through each non-zero efficient point $(\bar{x}, -\bar{y})$ of T and gives rise to a unique efficient price. This price in turn generates a unique supporting hyperplane to each of the production sets T^k: each sector is thus induced to an allocation $(\bar{x}^k, -\bar{y}^k)$ such that $\sum_{k=1}^{s} (\bar{x}^k, -\bar{y}^k) = (\bar{x}, -\bar{y})$. Each efficient point of T can be attained by choosing the unique associated efficient price in T^* and requiring each sector to maximise profit. The price mechanism has complete control over the allocation of resources, for it can guide the economic system to any desired efficient allocation.

Now suppose T is of the first type but is not strictly convex. If T has hyperfaces, then a single supporting hyperplane passes through a given hyperface and the same efficient price can lead to any one of the efficient allocations in the hyperface. If T has vertices, then a whole collection of supporting hyperplanes passes through each vertex: in this case there are more than enough prices. At hyperedges although there is not control over allocations within the hyperedge, there are more than enough prices to lead to the given hyperedge. There are thus either too few or more than enough prices to control allocations.

Finally suppose T is of the second type. If $\{T ; T^1 , \dots , T^s\}$ are convex cones, as in Koopmans approach, then every efficient allocation $(\bar{x} , -\bar{y})$ lies in a supporting hyperplane which passes through the origin. The efficient price (or rather the ray of efficient prices) characterising this hyperplane is thus also the efficient price associated with any efficient allocation that lies on the ray $\lambda (\bar{x} , -\bar{y})$, $\lambda > 0$ through $(\bar{x} , -\bar{y})$. In this case decentralisation can lead to any allocation lying on the ray $\lambda (\bar{x} , -\bar{y})$, $\lambda > 0$; the price mechanism has lost its ability to control allocations. Although it may not have lost complete control, for it may still be possible to determine the proportions in which goods are used and produced, the problem is still serious for it implies that the price mechanism by itself is no longer able to completely control the decentralised allocation of resources [see Hicks [14] chapter 6].

12. The relation between T and T^* leads to a simple characterisation of technical progress in terms of the way it affects efficient prices. Technical progress enables a larger array of final goods to be produced with the same, or possibly smaller amount of primary goods. Thus if T denotes the original technology set, then a simple form of technical change transforms T into the set

$$T' = \{(\alpha_1 x_1 , \dots , \alpha_n x_n , -\beta_1 y_1 , \dots , -\beta_m y_m) \mid (x , -y) \in T , \ \alpha_i \geqq 1 , \ 0 < \beta_j \leqq 1 \quad i = 1 , \dots , n , \ j = 1 , \dots , m\}$$

If T is of the first type, then

$$T'^* = \{(p' , r') \in (R^{n+m})^* \mid \sum_{i=1}^{n} p'_i \alpha_i x_i - \sum_{j=1}^{m} r'_j \beta_j y_j \leqq 1 \text{ for all } (x , -y) \in T\} \tag{8}$$

and since

$$T^* = \{(p , r) \in (R^{n+m})^* \mid \sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} r_j y_j \leqq 1 \text{ for all } (x , -y) \in T\} \tag{9}$$

we see that

$$T'^* = \left\{(p , r) \in (R^{n+m})^* \mid p'_i = \frac{p_i}{\alpha_i} , \ r'_j = \frac{r_j}{\beta_j} , \ (p , r) \in T^*\right\}$$

implying that

$$T'^{*} = \left\{ \frac{p_1}{\alpha_1}, \ldots, \frac{p_n}{\alpha_n}, \frac{r_1}{\beta_1}, \ldots, \frac{r_m}{\beta_m} \mid (p, r) \in T^{*} \right\}$$

If T is a cone we merely replace 1 by 0 in (8) and (9).

Thus technical progress which acts so as to simultaneously increase the output of each final good G_i by a factor α_i, and to reduce the amount of each primary good H_j required by a proportion β_j, leads to a decrease in the efficient price of each final good by a factor $\frac{1}{\alpha_i}$ and an increase in the efficient price of each primary good by a factor $\frac{1}{\beta_j}$. An interesting special case arises when there is *Hicks-neutral* technical change (Hicks [15]). In this case $n = 1$, $\alpha_1 = 1$ and $\beta_1 = \ldots = \beta_m = \beta$, where $0 < \beta < 1$. All primary good prices are increased by the same proportion $\frac{1}{\beta} > 1$.

13. When the technology set is of the first type prices must be normalised in order to define T^{*}: the prices are normalised so as to generate unit profit. The ideal way of transforming the relative prices into absolute prices however, without breaking down the duality between T and T^{*}, is to link the prices to the money supply $m = Mv$, where M denotes the money stock and v the velocity of circulation. Such a transformation seems to be possible only in special cases however.

Suppose for example that each sector produces a single final good, and that the sector production sets are written in terms of smooth production functions

$$x_k = f^k(y_{1k}, \ldots, y_{mk}) \quad , \quad y_{ik} \geqq 0 \quad , \quad k = 1, \ldots, s$$

where y_{ik} denotes the amount of H_i used by sector k. If we assume that each production function is homogeneous of the same degree h, where $0 < h < 1$, then Euler's Theorem on homogeneous functions implies

$$\sum_{i=1}^{m} \frac{\partial f^k}{\partial y_{ik}} y_{ik} = h x_k , \qquad k = 1, \ldots, s$$

while profit maximisation requires

$$\left(p_k \frac{\partial f^k}{\partial y_{ik}} - r_i \right) y_{ik} = 0 \qquad \begin{matrix} i = 1, \ldots, m \\ k = 1, \ldots, s \end{matrix}$$

hence

$$hp_k x_k = p_k \sum_{i=1}^{m} \frac{\partial f^k}{\partial y'_{ik}} y_{ik} = \sum_{i=1}^{m} r_i y_{ik} , \qquad k = 1 , \dots , s$$

Since a proportion h of the revenue of the k^{th} sector is payment to the factors, the remainder $(1 - h) \, p_k x_k$ is its profit, which we denote by R_k : hence total profit R in the system is

$$(10) \qquad R = \sum_{k=1}^{s} R_k = (1 - h) \sum_{k=1}^{s} p_k x_k$$

ef $Sr (p , r)$ denotes the support function of the technology set T for this system, then $Sr (p , r) = R$. Now instead of normalising by setting $R = 1$, we may determine the level of prices by adding the monetary condition

$$\sum_{k=1}^{s} p_k x_k = m$$

then (10) implies that the price set, with absolute prices determined by the money supply, which we may call the *money-price set,* becomes

$$T^* = \{(p , r) \in (R^{n+m})^* \mid S_T (p , r) \leqq (1 - h) \, m\}$$

The duality between T and T^* is retained in this case if the incidence relation (5) is replaced by the incidence relation

$$px - ry = (1 - h) \, m$$

It does not seem to be possible in general however to obtain a simple relation between total profit R and the value of output $\sum_{k=1}^{s} p_k x_k$.

14. The principal motivation for introducing T^* and analysing in particular the efficient prices, is the fact that these efficient prices lead to an alternative way of characterising T . More precisely the result may be stated as follows:

Theorem 2 (*Duality*)

If the technology set T is a closed convex set in R^{n+m} , then $T^{**} = T$ if and only if $(0 , 0) \in T$.

Mathematically the theorem is due to Minkowski [5], section 8. A simple proof may be found in Gale [16] or Eggleston [17].

Note that just as in passing from T to T^* it is necessary to normalise prices if T is of the first type, so in passing from T^* to T^{**} it is necessary to normalise allocations. For Theorem 1^* associates with an efficient price $(\bar{p}, \bar{r})$ in T^* an allocation $(\bar{x}, -\bar{y})$ such that (7) is satisfied, thus for any $\lambda > 0$, $\lambda \bar{p}x - \lambda \bar{r}y \geqq \lambda px - \lambda ry$ for all $(p, r) \varepsilon T^*$ and $\lambda (x, -y)$, $\lambda > 0$ is associated with the same efficient price. If we normalise the allocations so that each non-zero efficient price generates normal profit, then

$$px - ry \leqq \bar{p}x - \bar{r}y = 1 \qquad \text{for all } (p, r) \varepsilon T^*$$

and $(x, -y)$ lies in the bidual T^{**} by (3''). If T is a cone then all allocations $\lambda (x, -y)$, $\lambda > 0$ lie in T^{**} and normalisation is not necessary.

The theorem states the conditions under which T may be recovered from T^*, given a knowledge of T^*. These conditions are fairly general. Thus the inclusion of the origin allows the possibility of inaction, while the closure condition essentially amounts to including efficient points as elements of T, since all efficient points are boundary points of T while boundary points that are not efficient points are not economically interesting. The convexity condition is the strongest assumption for it rules out both indivisibilities and increasing returns to scale.

The relation between T and T^* depends crucially on the location of T with respect to the origin. Thus if the origin is an internal point of T — a most unlikely occurrence since it implies the possibility of simultaneously acquiring final and primary goods — then T^* is bounded, while if the origin is a boundary point of T then T^* is unbounded: this is the most likely case. If the origin is not in T, which is economically possible, then duality breaks down: this is because T^* breaks up into two or more separate convex sets. A simple example illustrates this: consider the dual of any conic section, say a parabola; if the origin is an internal point of the parabola then the dual is an ellipse, if the origin is a boundary point then the dual is another parabola (this is essentially the example in Figure 1), while if the origin is not contained in the parabola then the dual is a hyperbola. The behavior of the dual depends on the number of support planes to the convex set that pass through the origin, for these planes are mapped into directions of infinity (directions in which the dual set is unbounded). Thus in the first case there are no directions of infinity, in the second case there is one, and in the third case there are two.

The duality between T and T^* induced by the incidence relation (5) is a special case of duality with respect to a quadratic form considered in projective geometry and corresponds to the case where the quadratic form is the unit sphere in R^{n+m}. The duality between conjugate functions (Fenchel [18]) is another example of duality with respect to a quadratic form that arises in the theory of valuation in dynamic economics — the quadratic form being the paraboloid of revolution (Magill [19]).

15. Just as we formed the price set T^* dual to T, so we may form the *sector price sets* T^{k*} dual to the sector production sets T^k, $k = 1, \ldots, s$. These sector price sets T^{k*} are a natural generalisation of the cost functions introduced by Samuelson [9] and Shephard [10] which we mentioned earlier.

The sector price set T^{k*} may be characterised by the support function

$$S_{T^k}(p, r) = \sup_{(x^k, -y^k) \varepsilon T^k} \{px^k - ry^k\} \qquad k = 1, \ldots, s$$

since

$$T^{k*} = \}(p, r) \varepsilon (R^{n+m})^* \mid S_{T^k}(p, r) \leqq 1\} \qquad k = 1, \ldots, s$$

where 0 replaces 1 when T^k is a cone. Since $T = \sum_{k=1}^{s} T^k$ total profit $px - ry$ is maximised over T if and only if sector profit is maximised over T^k, for $k = 1, \ldots, s$; hence the support function $S_T(p, r)$ of T is simply the sum of the support functions of the production sets T^k,

$$S_T(p, r) = \sum_{k=1}^{s} S_{T^k}(p, r)$$

so that

$$T^* = \{(p, r) \varepsilon (R^{n+m})^* \mid \sum_{k=1}^{s} S_{T^k}(p, r) \leqq 1\} \tag{11}$$

If the economic system has constant returns to scale in each sector, so that each T^k is a cone, then the relation between T^* and T^{k*} is very simple:

$$T^* = (\sum_{k=1}^{s} T^k)^* = \bigcap_{k=1}^{s} T^{k*}$$

a result which follows at once from the definition of T^{k*}. Theorem 2 then implies

$$(\bigcap_{k=1}^{s} T^{k*})^* = \sum_{k=1}^{s} T^k$$

16. A few simple examples will serve to illustrate the principal ideas of the present section.

Consider first an economy with a technology set of the first type, in particular the ideal case where $\{T; T^1, \ldots, T^s\}$ are strictly convex and contain the origin (the first example in section 11). Suppose the s sectors each produce one of the s final goods $G_1, \ldots, G_s$ with the aid of a single primary good H_1 (say labor) available in the amount y_1.

Then $x^k = (0, \dots, x_k, \dots, 0)$ and $y^k = y_{1k}$, $k = 1, \dots, s$ where y_{1k} denotes the amount of H_1 used by sector k. Suppose the sector production sets are

$$T^k = \{(x^k, -y^k) \mid (x^k, y^k) \in T_0^k\}, \qquad k = 1, \dots, s$$

where

$$T_0^k = \{(x^k, y^k) \mid x_k \leqq a_k(y_{1k})^{\alpha_k}, \ y_{1k} \geqq 0, \ a_k > 0, \ 0 < \alpha_k < 1\}$$
$$k = 1, \dots, s$$

then the technology set is

$$T = \{(x, -y) \mid (x, y) \in T_0\}$$

where

$$T_0 = \left\{(x, y) \mid \sum_{k=1}^{s} \left(\frac{x_k}{a_k y_1{}^{\alpha_k}}\right)^{\frac{1}{\alpha_k}} \leqq 1, \ y_1 \geqq 0\right\}$$

while the sector price sets are

$$T^{k*} = \left\{(p, r) \in (R^{s+1})^* \mid (1 - \alpha_k)\left(p_k a_k \left(\frac{\alpha_k}{r_1}\right)^{\alpha_k}\right)^{\frac{1}{1-\alpha_k}} \leqq 1, \ (p, r) \geqq 0\right\}$$
$$k = 1, \dots, s$$

hence by (11) the price set is

$$T^* = \left\{(p, r) \in (R^{s+1})^* \mid \sum_{k=1}^{s} (1 - \alpha_k)\left(p_k a_k \left(\frac{\alpha_k}{r_1}\right)^{\alpha_k}\right)^{\frac{1}{1-\alpha_k}} \leqq 1, \right.$$
$$\left. (p, r) \geqq 0\right\}$$

It is easy to verify that $T^{k**} = T^k$, and $T^{**} = T$. It is also evident that in this system prices can exactly control decentralised allocation of resources.

T and T^* are shown in Figure 1 for the case $s = 1$. The unit circle is drawn so that T^* can be verified to be the polar reciprocal of T. While Theorem 1 associates $(p_1{}^0, r_1{}^0)$ with the efficient allocation $(x_1{}^0, -y_1{}^0)$, Theorem 1* associates $(x_1{}^0, -y_1{}^0)$ with the efficient price $(p_1{}^0, r_1{}^0)$. The money-price set in this case becomes

$$T^* = \left\{(p_1, r_1) \in R^{2*} \mid \left(p_1 a_1 \left(\frac{\alpha_1}{r_1}\right)^{*}\right)^{\frac{1}{1-\alpha_1}} \leqq m, \ p_1 \geqq 0 \ r_1 \geqq 0\right\}$$

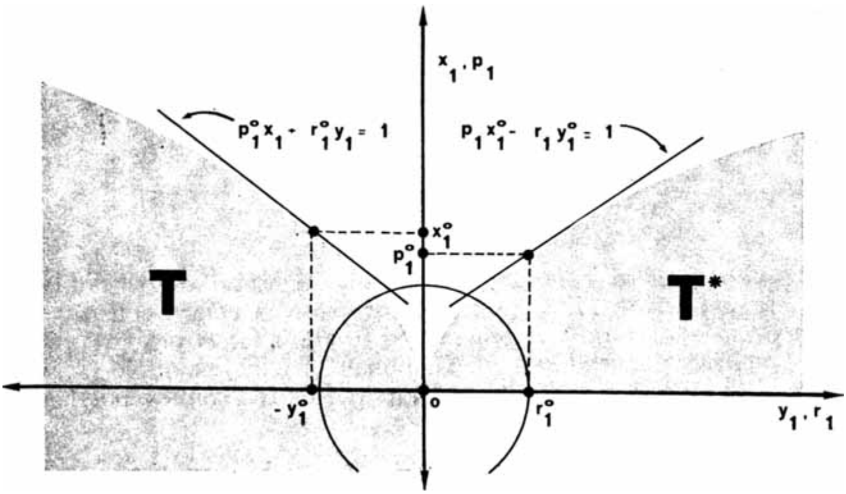

Figure I. - The technology set T and the dual price set T*.

As a natural generalisation of this example we may allow each of the sectors to use m primary goods to generate the s final goods. Thus $x^k = (0, \ldots, x_k, \ldots, 0)$ and $y^k = (y_{1k}, \ldots, y_{mk})$, $k = 1, \ldots, s$. The sector production sets become

$$T^k = \{(x^k, -y^k) \mid (x^k, y^k) \in T_0^k\} \qquad k = 1, \ldots, s$$

where

$$T_0^k = \{(x^k, y^k) \mid x_k \leqq a_k \prod_{i=1}^{m} (y_{ik})^{\alpha_{ik}}, \; y_{ik} \geqq 0, \; a_k > 0,$$
$$0 < \sum_{i=1}^{m} \alpha_{ik} < 1\} \qquad k = 1, \ldots, s$$

In this case it is difficult to obtain a general expression for T. The sector price sets are

$$T^{k*} = \Big\{(p, r) \in (R^{s+m})^* \mid (1 - \sum_{i=1}^{m} \alpha_{ik}) \left(a_k p_k \prod_{i=1}^{m} \left(\frac{\alpha_{ik}}{r_i}\right)^{\alpha_{ik}}\right)^{\frac{1}{1-\sum_{i=1}^{n} \alpha_{ik}}} \leqq ,$$
$$(p, r) \geqq 0\Big\} \qquad k = 1, \ldots, s$$

while the price set is

$$T^* = \left\{(p\,,\,r) \,\varepsilon\, (R^{s+m})^* \;\middle|\; \sum_{k=1}^{s} \left(1 - \sum_{i=1}^{m} \alpha_{ik}\right) \left(a_k p_k \prod_{i=1}^{m} \left(\frac{\alpha_{ik}}{r_i}\right)^{\alpha_{ik}}\right)^{\frac{1}{1-\sum_{i=1}^{m}\alpha_{ik}}} \leqq 1\,,\; (p\,,\,r) \geqq 0\right\}$$

As an example of an economy with a technology set of the second type, in particular where $\{T\,;\,T^1\,,\,\ldots\,,\,T^s\}$ are convex cones, consider the basic Leontief system [20] generalised to allow for m primary goods. This system is a special case of Koopmans' original example [1] with inputs and outputs explicitly separated. T is the convex polyhedral cone in R^{n+m},

$$T = \left\{(z\,,\,-y) \;\middle|\; \begin{bmatrix} z \\ -y \end{bmatrix} \leqq \begin{bmatrix} I - A \\ -B \end{bmatrix} x\,,\; x \geqq 0\right\}$$

where x_j is the gross output and z_j the net output of $G_j\,,\, j = 1\,,\,\ldots\,,\,n$; A is the matrix of elements a_{ij} denoting the number of units of G_i required to produce one unit of G_j ; B is the matrix of elements b_{ij} denoting the number of units of H_i required to produce one unit of G_j. If we introduce the matrix

$$C = \begin{bmatrix} I - A \\ -B \end{bmatrix}$$

and let $c^k\,,\; k = 1\,,\,\ldots\,,\,n$ denote the columns of C then sector production sets become

$$T^k = \left\{(z^k\,,\,-y^k) \;\middle|\; \begin{bmatrix} z^k \\ -y^k \end{bmatrix} \leqq x_k c^k\,,\; x_k \geqq 0\right\} \qquad k = 1\,,\,\ldots\,,\,n$$

and $T = \sum_{k=1}^{n} T^k$. The sector price sets are then

$$T^{k*} = \{(p\,,\,r) \,\varepsilon\, (R^{n+m})^* \mid (p\,,\,r)\, c^k \leqq 0\,,\; (p\,,\,r) \geqq 0\} \qquad k = 1\,,\,\ldots\,,\,n$$

while the price set for the entire economy is the dual cone

$$T^* = \bigcap_{k=1}^{n} T^{k*} = \{(p\,,\,r) \,\varepsilon\, (R^{n+m})^* \mid (p\,,\,r)\, C \leqq 0\,,\; (p\,,\,r) \geqq 0\}$$

It is easy to verify that $T^{k**} = T^k$, and $T^{**} = T$.

If we let $A = 0$ in this system, then we obtain the standard linear programming system. The present approach to duality helps to clarify the well-known duality between the primal and the dual problems. For consider the technology set

$$T = \{(z\,,\,-y) \mid \begin{bmatrix} z \\ -y \end{bmatrix} \leqq \begin{bmatrix} I \\ -B \end{bmatrix} x\,,\ x \geqq 0\}$$

which is again a convex polyhedral cone in R^{n+m}. Associated with each efficient allocation in T is a vector of prices $(p\,,\,r) > 0$, $(p\,,\,r) \in T^*$, where

$$T^* = \{(p\,,\,r) \in (R^{n+m})^* \mid (p\,,\,r) \begin{bmatrix} I \\ -B \end{bmatrix} \leqq 0\,,\ (p\,,\,r) \geqq 0\}$$

The primal and dual problems in linear programming are then the maximum problems over T and T^* which characterise the efficient allocations in T and the efficient prices in T^* according to Theorem 1 and Theorem 1*. The asymmetry between the primal and the dual arises because the vector of final goods prices p and the endowment of primary goods y are fixed. Thus the problem which characterises an efficient allocation in T,

$$\max_{(z,\,-y) \in T} \{pz - ry\}$$

becomes with y fixed,

$$\max\ px\,,\ \text{subject to}\ Bx \leqq y\,,\ x \geqq 0\,,$$

while the problem which characterises an efficient price in T^*,

$$\max_{(p,\,r) \in T^*} \{pz - ry\}$$

becomes with p fixed,

$$\min\ ry\,,\ \text{subject to}\ rB \geqq p\,,\ r \geqq 0\,.$$

If $(p^0\,,\,r^0)$ is one of the efficient prices in T^* associated with the efficient allocation $(z^0\,,\,-y^0)$ in T, then

$$\max_{(z,\,-y) \in T} \{p^0 z - r^0 y\} = p^0 z^0 - r^0 y^0 = \max_{(p,\,r) \in T^*} \{p z^0 - r y^0\}$$

and

$$p^0 z^0 - r^0 y^0 = 0$$

since the support function of a cone must pass through the origin. Thus the optimal value of the primal equals the optimal value of the dual.

The economic emphasis in the use of linear programming in economic theory seems to have been misplaced. For efficient allocations are clearly the primary concept, the maximisation of profit (or the value of output when primary goods are fixed) being only of secondary interest — indeed it is only of interest because it leads to efficient allocations. Furthermore in order to retain the symmetry between efficient allocations and efficient prices, the dual maximum problems should be considered over the entire technology set T and the entire price set T^* : fixing the endowment of the primary goods y confines the primal to a maximum problem over only a cross-section of T, while fixing the price p of the final goods confines the dual to a maximum problem over only a cross-section of T^*.

III. Duality in the dynamic case

17. In the following sections we consider very briefly the simplest way of extending the preceeding ideas to the dynamical case. In particular since allocations over a finite horizon may be viewed as a special instance of allocations over an infinite horizon, we confine the analysis to allocations over an infinite horizon.

To analyse production we use the framework introduced by Malinvaud [3]. Thus we consider an economic system which produces and consumes n goods $G_1, \dots, G_n$ which may serve as both inputs and outputs. The economy is made up of s distinct sectors; the activity of the k^{th} sector at time t transforms the input $y^k{}_t$ into the output $x^k{}_{t+1}$ provided

$$(12) \qquad (y^k{}_t \, , \, x^k{}_{t+1}) \; \varepsilon \; T^k{}_t \, , \qquad \begin{array}{l} k = 1, \dots, s \\ t = 1, 2, \dots \end{array}$$

where $T^k{}_t$ is the production set of the k^{th} sector for the period t. The activity of the economy is simply the combined activity of all the sectors: thus the input $y_t = \sum_{k=1}^{s} y^k{}_t$ is transformed into the output $x_{t+1} = \sum_{k=1}^{s} x^k{}_{t+1}$ provided

$$(13) \qquad (y_t \, , \, x_{t+1}) \; \varepsilon \; T_t \, , \qquad t = 1, 2, \dots$$

where $T_t = \sum_{k=1}^{s} T^k{}_t$. The production sets $\{T_t \, ; \, T^1{}_t, \dots, T^s{}_t\}$ are assumed to be convex sets in R^{2n}. The output x_{t+1} can be used during the period $t + 1$ either for consumption or as an input for production:

this if z_t denotes the amount withdrawn for consumption or *net output* at time t, then

$$(14) \qquad z_t = x_t - y_t, \qquad t = 1, 2, \ldots$$

where the output inherited in the first period x_1 is given as an initial condition. Since T_t, $t = 1, 2, \ldots$ are convex sets, the collection of all net output sequences $z = (z_1, z_2, \ldots)$ satisfying (13) and (14) form a convex set Z, called the net output technology set, in the space of infinite sequences in R^n. Similarily since $T^k{}_t$, $t = 1, 2, \ldots$ and $k = 1, \ldots, s$ are convex sets the collection of sector net output sequences $z^k = (z^k{}_1, z^k{}_2, \ldots)$ satisfying (12) and

$$z^k{}_t = x^k{}_t - y^k{}_t \qquad \begin{matrix} k = 1, \ldots, s \\ t = 1, 2, \ldots \end{matrix}$$

form convex sets Z^k, $k = 1, \ldots, s$ called sector net output production sets. As in the static case we say that z is feasible if $z \varepsilon Z$ and $\bar{z} \varepsilon Z$ is efficient if there does not exist $z \varepsilon Z$ such that $z \geq \bar{z}$.

18. As in the finite dimensional case, closed hyperplanes lead to an alternative characterisation of a convex set. Thus if X is a normed linear space, and X^* its dual and if Z is a convex set in X, then the *dual convex set* Z^* is defined as

$$Z^* = \{p \varepsilon X^* \mid p(z) \leqq 1 \text{ for all } z \varepsilon Z\}$$

and the *bidual* as

$$Z^{**} = \{z^{**} \varepsilon X^{**} \mid z^{**}(p) \leqq 1 \text{ for all } p \varepsilon Z^*\}$$

where 0 replaces 1 if Z is a cone. In the infinite dimensional case a problem of interpretation arises however, for if X is a space of infinite sequences in R^n, $p \varepsilon X^*$, $z \varepsilon X$, then $p(z)$ which denotes a valuation assigned to the infinite sequence z may not always be given an exact economic interpretation, for it may not be possible to represent the valuation as a series $p(z) = \sum_{t=1}^{\infty} p_t z_t$ (Radner [4]).

To circumvent this difficulty we shall consider the space of infinite sequences which converge to a limit, $\lim_{t \to \infty} z_t = z_0$, where $z_0 \varepsilon R^n$. If we introduce the norm

$$\| z \|_\infty = \sup_t \| z_t \|_\infty \text{ where } \| z_t \|_\infty = \max_i | z_{it} |$$

then the space of infinite sequences in R^n becomes the normed linear space of bounded sequences (l_∞), while the convergent sequences (c)

form a subspace of (l_∞) . [Brackets distinguish the space of sequences in R^n from the standard space of sequences in R^1]. Since the sequence of infinite matrices

$$\begin{aligned} E_1 &= [I_1 , O , \dots] \\ E_2 &= [O , I_2 , O , \dots] \\ &\vdots \\ E_t &= [O , O , \dots , I_t , O , \dots] \\ &\vdots \end{aligned}$$

and

$$E = [I_1 , I_2 , \dots , I_t , I_{t+1} , \dots]$$

where each I_t , $t = 1 , 2 , \dots$, denotes an identity matrix of order n , forms a basis for (c), and every $z \varepsilon (c)$ can be written as

$$z = z_0 E + \sum_{t=1}^{\infty} (z_t - z_0) E_t$$

if $p \varepsilon (c)^*$ is a continuous linear functional on (c) and if

$$p (E) = \hat{p} , \; p (E_t) = p_t , \qquad t = 1 , 2 , \dots$$

then

$$p (z) = z_0 p (E) + \sum_{t=1}^{\infty} (z_t - z_0) p (E_t) = z_0 \hat{p} + \sum_{t=1}^{\infty} (z_t - z_0) p_t$$

where $\sum_{t=1}^{\infty} \sum_{i=1}^{n} | p_{it} | < \infty$, thus

$$(15) \qquad p (z) = p_0 z_0 + \sum_{t=1}^{\infty} p_t z_t = \sum_{t=0}^{\infty} p_t z_t$$

where $p_0 = \hat{p} - \sum_{t=1}^{\infty} p_t$, so that every continuous linear functional on (c) can be given a (unique) series representation. Since the valuations $\hat{p}$, p_t , $t = 1 , 2 , \dots$ must refer to a common focal date, which for convenience we may take as the initial period, p_t , $t = 1 , 2 , \dots$ denote the present values of one unit of each commodity available at time t , while $\hat{p}$ denotes the present value of a perpetual annuity of one unit of each commodity and (15) may be interpreted as the present value of the infinite sequence $z \varepsilon (c)$, $z_0 \hat{p}$ being the present value of being perpetually at the « steady state » level z_0

towards which the sequence converges and $\sum_{t=1}^{\infty} (z_t - z_0) p_t$ being the deduction for the present value of the «loss» incurred by the short fall of z_t from z_0 at each instant.

There is good economic justification for using this space. For the convergence condition essentially imposes a boundary condition on the net output sequence indefinitely far in the future: economic systems with the Turnpike property or systems that tend to a steady state will naturally satisfy such a boundary condition.

If we introduce the norm

$$\| p \|_1 = \sum_{t=0}^{\infty} \| p_t \|_1 \text{ where } \| p_t \|_1 = \sum_{i=1}^{n} | p_{it} |$$

then $(c)^*$ is congruent to the space of summable sequences

$$(l_1) = \{(p_0, p_1, \ldots) \mid \sum_{t=0}^{\infty} \| p_t \|_1 < \infty\}$$

and we write $(c)^* \simeq (l_1)$. Furthermore $(c)^{**} \simeq (l_\infty)$. Even though the space (c) is not reflexive however $[(c)^{**} \neq (c)]$ the duality between Z and Z^* carries over to the infinite dimensional case.

19. Efficient allocations may be characterised by a theorem analogous to Theorem 1. Thus Radner's theorem [4] associates with each efficient allocation $z \varepsilon Z \subset (l_\infty)$ a non-negative $(p(z) \geqq 0, z \geqq 0)$ non-zero continuous linear functional $p \varepsilon (l_\infty)^*$ such that

$$p(\bar{z}) \geqq p(z) \qquad \text{for al} \infty \; z \varepsilon Z \subset (l)$$

(15) therefore implies that we may associate with each efficient allocation $\bar{z} \varepsilon Z \subset (c)$, an infinite sequence of present values $(p_0, p_1, \ldots) \geq 0$, $(p_0, p_1, \ldots) \varepsilon (l_1)$ such that

$$\sum_{t=0}^{\infty} p_t \bar{z}_t \geqq \sum_{t=0}^{\infty} p_t z_t \qquad \text{for all } z \varepsilon Z \subset (c)$$

If Z is a technology set of the first type, we may normalise the present values as in the static case so as to generate unit profit,

$$\sum_{t=0}^{\infty} p_t z_t \leqq \sum_{t=0}^{\infty} p_t \bar{z}_t = 1 \qquad \text{for all } z \varepsilon Z \subset (c)$$

and $(p_0, p_1, \ldots)$ lies in the dual convex set Z^* which may again be called the *dual price set*. If Z is a cone normalisation is not necessary for $(p_0, p_1, \ldots)$ automatically lies in the dual convex cone Z^*.

The *sector price sets* Z^{k*} dual to the sector production sets Z^k are an immediate generalisation from the static case

$$Z^{k*} = \{p \,\varepsilon\, (c)^* \mid \sum_{t=0}^{\infty} p_t z^k{}_t \leqq 1 \text{ for all } z^k \,\varepsilon\, Z^k\} \qquad k = 1, \dots, s$$

0 replacing 1 if Z^k is a cone.

The theorem on efficient prices analogous to Theorem 1* can only be established if additional assumptions are made: in particular since the non-negative orthant in (l_1) has an empty interior the Hahn-Banach theorem does not apply unless Z^* is assumed to have a non-empty interior. The Duality theorem however extends without difficulty to the dynamical case; although the theorem holds in any normed linear sequence space, we confine the statement to the space of convergent sequences, since it is in this case that the theorem may be given an exact economic interpretation. The proof is a straightforward generalisation of [16] and [17].

Theorem 3 (*Duality*)

If the technology set Z is a closed convex set in (c), then $Z^{**} = Z$ if and only if $0 \,\varepsilon\, Z$.

Proof: Let $\langle p, z \rangle$ denote $p(z)$ when p is a linear functional and $z(p)$ when z is a linear functional. Suppose $0 \,\varepsilon\, Z$. If $z \,\varepsilon\, Z$, then for all $p \,\varepsilon\, Z^*$, $\langle p, z \rangle \leqq 1$ which implies $z \,\varepsilon\, Z^{**}$. Suppose $z \notin Z$, then $z \neq 0$, since $0 \,\varepsilon\, Z$. Since Z is closed, $\inf_{z^0, Z} \| z - z^0 \|_\infty = \delta > 0$.

Since the sphere $S_z = \left\{ z^0 \,\varepsilon\, (c) \mid \| z^0 - z \| < \frac{\delta}{2} \right\}$ has a non-empty interior and is convex, and $S_z \cap Z = \emptyset$, by the Hahn-Banach Theorem there exists a non-zero continuous linear functional $p \,\varepsilon\, (c)^*$ such that

$$\sup_{z^0 \varepsilon Z} \langle p, z^0 \rangle \leqq 1 \leqq \inf_{z^1 \varepsilon S_z} \langle p, z^1 \rangle$$

Thus $\langle p, z^0 \rangle \leqq 1$ for all $z^0 \,\varepsilon\, Z$ and hence $p \,\varepsilon\, Z^*$, and while $\langle p, z \rangle > 1$ for some $p \,\varepsilon\, Z^*$ implies $z \notin Z^{**}$. Hence $Z = Z^{**}$. Suppose $0 \notin Z$, then $Z \neq Z^{**}$ since $0 \,\varepsilon\, Z^{**}$, which completes the proof.

As in the static case the relation between $\{Z; Z^1, \dots, Z^s\}$ and $\{Z^*; Z^{1*}, \dots, Z^{s*}\}$ is particularily simple when each of the single period production sets $T^k{}_t$, $t = 1, 2, \dots$ and hence each Z^k is a convex cone, for

$$Z^* = (\sum_{k=1}^{s} Z^k)^* = \bigcap_{k=1}^{s} Z^{k*}$$

while Theorem 3 implies

$$Z^{**} = (\bigcap_{k=1}^{s} Z^{k*})^{*} = \sum_{k=1}^{s} Z^{k} = Z .$$

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