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A COMOMENT CRITERION FOR THE CHOICE OF RISKY INVESTMENT BY FIRMS*

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This article uses Taylor series expansions and the assumption of small risks to derive a comoment criterion that firms should maximize so that the resulting equilibrium is Pareto optimal. This is done in two models of production under uncertainty: the state-of-nature ($\mathcal{SN}$) model in which the firms' outputs depend on states of nature and financial markets are complete with respect to these states of nature and the probability ($\mathcal{P}$) model in which the firms' risky outputs are modeled by their joint probabilities and financial markets span the outcome space of the firms. The comoment criterion provides a unifying framework for the two equilibrium models of production under uncertainty, has the merit of being based on information which is readily available to firms, and provides greater insight than the theoretical criterion into the risk characteristics of its profit stream that a firm should focus on when choosing its investment plan.

1. INTRODUCTION

This article studies the simplest one-good two-period general equilibrium model of a production economy under uncertainty. The focus is on understanding how firms should evaluate their risks so that the resulting choice of investment leads to a Pareto-optimal allocation.

Just as there are two alternative ways of describing a random variable, so there are two alternative ways of describing a production economy under uncertainty. The first approach views a random variable as a map from a probability space to the real line, and this corresponds to the approach of Arrow–Debreu. The model describes how uncertainty in production can be traced to a set of states of nature (primitive causes) with fixed objective probabilities, which when combined with the input decisions of firms serves to explain the realized outputs: We refer to this approach as the *state-of-nature* ($\mathcal{SN}$) *model*. The second approach, which models a random variable through the distribution function it induces on the range, leads to a less ambitious causal description of the stochastic nature of the economy, in which the primitive causes are left unspecified and the model describes how the input decisions of firms influence the probability distribution of their outputs: We refer to this approach as the *probability* ($\mathcal{P}$) *model*. From a mathematical point of view, the two approaches to modeling random variables can be shown to be equivalent.[2] However, from an economic point of view, although the underlying phenomenon of risky production is the same, the two approaches lead to equilibrium models that are different because they are associated with financial contracts with different characteristics.

The Arrow–Debreu model assumes that agents have a complete understanding of the primitive causes, represented by the states of nature, and hence can trade contracts based on these states. Standard general equilibrium theory emphasizes that to obtain a Pareto-optimal outcome a complete set of such state-contingent contracts is required. However, a model based on the assumption that agents trade state-contingent contracts fits poorly with the observed nature of the securities used to share production risks (equity, risky corporate bonds, options on equity, etc.), which are typically based on the outcomes of the firms and not on primitive states. By

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[2] This is the content of the Kolmogorov extension theorem: see e.g., Grimmett and Stirzaker (1992, pp. 343–346).

contrast the $\mathcal{P}$ model seeks no explanation of primitive causes, and thus leads directly to the postulate that contracts are based on outcomes instead of primitive causes. The $\mathcal{P}$ model can only be used if the primitive causes that explain randomness in production do not directly affect the preferences of the consumers (investors) in the economy. Under this assumption we showed in Magill and Quinzii (2009) that if the financial markets permit full spanning of the firms' outcomes, then a Pareto-optimal allocation can be achieved, even though the markets are incomplete with respect to any underlying state-of-nature space. Markets based on primitive states are difficult to operate, because the states on which the contracts would need to be based are often not verifiable by third parties. The $\mathcal{P}$ model suggests that such contracts are not really needed: It suffices to have markets based on outcomes.

However, to get to a Pareto-optimal outcome with this simplified market structure, firms need to maximize a more complicated objective function than in the $\mathcal{SN}$ model. For the rich structure of markets in the $\mathcal{SN}$ model induces a present-value price, or social valuation, of income for each state, and it suffices that firms maximize the present value of their profit evaluated with these prices to obtain a Pareto-optimal allocation. Trying to mimic this approach for the $\mathcal{P}$ model, by using present-value prices for outcomes, does not, however, lead to an efficient outcome. As shown in Magill and Quinzii (2009), the criterion that emerges from a normative analysis is that each firm should maximize its contribution to expected social welfare, and firms can make use of market prices to construct an approximation of this function.

Even though the criteria for the firms in these two models appear very different, they must in fact induce firms to evaluate risks in a similar way since, after all, a Pareto-optimal allocation in the more detailed $\mathcal{SN}$ model is also Pareto-optimal in the associated $\mathcal{P}$ model. The objective of this article is to provide a way of expressing the two criteria that makes clear that the risk considerations involved in a firm's maximization problem are actually the same in the two models.

In much of the finance literature the intuition for understanding risky securities has been obtained from the mean–variance model, which summarizes the random variables by their first and second moments and comoments—means, variances, and covariances. The mean–variance model—the two-period or the continuous-time Brownian motion version—is still dominant in finance, but the difficulty of fitting it to the data on prices and portfolios has led some researchers to pay attention to higher order moments and comoments, up to order three or four—skewness, co-skewness, kurtosis, and co-kurtosis. These higher-order moments contain additional information that can be relevant for explaining the valuation of securities and the structure of the portfolios that agents hold.[3]

The first author to explore how the valuation of a risky income stream by an agent with expected utility preferences can be expressed in terms of moments and comoments was Rubinstein (1973). He used a Taylor expansion of the agent's marginal utility around mean consumption to express the agent's valuation of the security as a sum of moments and comoments of the random payoff with the agent's consumption stream. The idea was extended to an equilibrium setting by Kraus and Litzenberger (1976, 1983), who stopped the expansion at terms of third order: Such an analysis, which can be extended to terms of all orders, leads to an expression for the equilibrium price of the security as the sum of its expected discounted payoff plus a weighted sum of its comoments with aggregate output, where the coefficients can be considered as the "prices" of the associated comoments. In this article, we exploit the Taylor series expansion approach to transform the criterion of a firm into a moment/comoment expression, under the

[3] It was recognized in the 1970s that taking into account preference for positive skewness could help explain the observed lack of diversification of most investors' portfolios—for diversification typically destroys skewness: See Simkowicz and Beedles (1973) and, for a more recent empirical study of investors' lack of diversification and its relation to skewness, see Mitton and Vorkink (2007). As far as security prices are concerned, Harvey and Siddique (2000) have shown that security returns, which are not well explained by the mean–variance CAPM, are better explained when the co-skewness of the returns with the market portfolio is taken into account.

assumption that the risks of the economy are not too large, so that the Taylor series expansion is valid.[4]

However, simply replacing the marginal utility of the representative agent in the $\mathcal{SN}$ model or the social welfare function in the $\mathcal{P}$ model by its Taylor expansion around mean aggregate output does not lead to a "natural" objective function for a firm to maximize in order to be led to its optimal choice of investment: The reason for this is that there are terms that the firm must take as given—the aggregate output in the $\mathcal{SN}$ model or the distribution of aggregate output in the $\mathcal{P}$ model—which in part depend on its own action. In the $\mathcal{SN}$ model, this is just the manifestation of the familiar problem with the competitive assumption that firms must take prices—here the present-value prices—as given: However, the present-value prices depend on aggregate output, which in turn depends on the firm's own production.

In order to avoid this difficulty, we derive, for each model, a function that has the same first-order condition as the original criterion but in which the only terms that the firm takes as given are the actions of all other firms and the coefficients of the pricing formula. Maximizing this function, which we call the *comoment criterion* of the firm, corresponds to "Nash/competitive" behavior by the firm, where the Nash part consists in taking the production decisions of the other firms as given, and the competitive part consists in taking the comoment prices as given. This greatly weakens the assumption of myopia by firms required in the original criterion.

The striking feature of this transformation is that the two objective functions reduce to the same comoment criterion. In the $\mathcal{SN}$ model firms maximize the criterion by choosing how much to produce in each state (subject to the technology constraints), whereas in the $\mathcal{P}$ model firms maximize the same criterion by choosing their $\mathcal{P}$ distribution over the outcomes. This formalizes the idea that the risk considerations that a firm must assess in determining its optimal choice of investment—what is the expected payoff, what is the variability, is there large upside potential or possibility of large losses, and so forth, and how these characteristics are related to the corresponding characteristics of other firms—are the same regardless of the model chosen to represent the stochastic nature of the economy.

This gives another way of understanding why markets contingent on exogenous states of nature may not really be needed. For, although firms may well find it useful to perform a scenario analysis to understand the probabilistic nature of the environment they face, by laying out all possible contingencies they might encounter in the future, such contingencies will typically be too firm specific, or too difficult to describe to third parties with sufficient precision, to permit a market for income contingent on their occurrence to operate. The $\mathcal{P}$ model shows that prices for income in the different scenarios are not needed—it suffices for firms to understand the statistical consequences of these scenarios and the way the market evaluates moments and comoments of income streams to arrive at their optimal choice of investment.

A drawback of the comoment criterion is that, being derived from a Taylor series expansion, it involves infinitely many comoment terms whose prices need to be known for a firm to evaluate its investment plan. It should be noted, however, that since the series converges, the sum can be approximated by a finite number of terms, and the smaller the aggregate risks and the firms' idiosyncratic risks, the smaller the number of terms needed to obtain a good approximation. If the intuition behind the asset pricing literature that extends the CAPM model to a mean–variance–skewness model is a guide, an approximation obtained by neglecting terms of order higher than the third or fourth order may already provide a good approximation if the risks in the economy are not too large.

In Section 2, we present the two models of production under uncertainty, the $\mathcal{SN}$ model and the $\mathcal{P}$ model, and derive the criterion for each firm that leads to a Pareto-optimal allocation. Section 3 derives the general comoment formula for security prices and relates it to the recent literature on mean–variance–skewness asset pricing. Section 4 derives the comoment formula

[4] Judd and Guu (2001) use Taylor expansions around the no-risk equilibrium to calculate an equilibrium with incomplete markets for an economy with small risks, yielding interesting insights on the role of the agents' preferences for higher moments for the equilibrium prices and trades.

for the $\mathcal{SN}$ model and relates it to a result of Stiglitz (1972) on the suboptimality of investment in the CAPM model. Section 5 derives the comoment formula for the $\mathcal{P}$ model, and Section 6 offers some concluding remarks.

2. STATE-OF-NATURE AND PROBABILITY MODELS OF PRODUCTION

Consider a two-period ($t = 0, 1$) finance economy with I consumer-investors and K firms. Each firm $k \in K$[5] makes an investment $a_k \in \mathbb{R}_+$ at date 0, which leads to a random output at date 1: The two models differ by the way the date 1 random output is characterized.

2.1. *State-of-Nature ($\mathcal{SN}$) Model.* Uncertainty is modeled by S states of nature with exogenously fixed probabilities $\rho = (\rho_s)_{s\in S}$. The set of feasible production plans of firm k is represented by a differentiable, increasing, quasi-convex transformation function $T_k : \mathbb{R}_- \times \mathbb{R}^S_+ \to \mathbb{R}$. A production plan with input $y^k_0 = -a_k$ and output $y^k = (y^k_1, \dots, y^k_S)$ in the possible states of nature at date 1 is feasible if $T_k(-a_k, y^k) \leq 0$.[6] In order to avoid boundary solutions we assume that

$$\lim_{y^k_0 \to 0} \frac{\partial T_k}{\partial y^k_0} > 0, \quad \lim_{y^k_s \to 0} \frac{\partial T_k}{\partial y^k_s} = 0, \quad \forall s \in S.$$

2.2. *Probability ($\mathcal{P}$) Model.* Each firm $k \in K$ has a fixed set of possible outcomes $\{y^k_1, \dots, y^k_{S_k}\}$ at date 1, ranked in increasing order, and y^k denotes the random variable with fixed support $\{y^k_1, \dots, y^k_{S_k}\}$. An outcome for the economy is a realization of the date 1 output for each firm, $y_s = (y^1_{s_1}, \dots, y^K_{s_K})$, indexed by the element $s = (s_1, \dots, s_K)$ of the set $S = S_1 \times \cdots \times S_K$, which is called the outcome space. The firms' choices of investment $a = (a_1, \dots, a_K)$ at date 0 determine the joint probability $\rho(a) = (\rho_s(a))_{s\in S}$ of the firms' random outcomes at date 1. Let $Y = \sum_{k\in K} y^k$ denote the random aggregate output and let

$$G(\eta, a) = \sum_{\{s\in S \mid Y_s \geq \eta\}} \rho_s(a)$$

denote the upper cumulative distribution function for Y: We assume that $\int_0^\eta G(t, a)dt$ is increasing and concave in a for all $\eta \leq Y_{\max}$. This implies that investment is productive, in the sense that an increase in any a_k leads to a second-order stochastic dominant shift in the distribution of the aggregate output. Concavity in a_k implies that there are stochastic decreasing returns to scale for firm k's investment. The assumption of joint concavity is needed to obtain the equivalent of the First Welfare Theorem for this economy (see Magill and Quinzii, 2009).

REMARK. When analyzing the investment decision of firm k it will be convenient to use the notation $s = (s_k, s_{-k})$, $a = (a_k, a_{-k})$, and $y = (y^k, y^{-k})$, where '$-k$' stands for the firms other than k.

2.3. *Consumption Sector.* The consumption sector is the same in both models. The I consumer–investors are the initial owners of the K firms and they trade on security markets to share the production risks: To simplify the analysis we assume that these production risks are the only risks to which agents are exposed. Thus the initial endowment of agent i consists of an amount ω^i_0 of income at date 0 and ownership shares $(\delta^i_k)_{k\in K}$ of the firms, with $\delta^i_k \geq 0$, $\sum_{i\in I} \delta^i_k = 1$

[5] In order to simplify notation we use the same symbol to denote a set and the number of elements in the set: Thus K denotes both the number of firms and the set of all firms.

[6] It might have been more intuitive to represent the production possibilities of firm k by S concave increasing functions $a_k \to y^k_s(a_k)$, $s = 1, \dots, S$, describing the outcome of investment a_k in each state of nature. A transformation function is more general since it allows for substitution among the outputs across the different states at date 1, and it does not complicate the analysis. Quasi-convexity of T_k is equivalent to convexity of firm k's production set.

for all $k \in K$. Agents have no endowment income at date 1. Let $x^i = (x^i_0, x^i_1) = (x^i_0, (x^i_s)_{s\in S})$ denote agent i's random consumption stream, where S denotes the set of states of nature in the $\mathcal{SN}$ model or the firms' outcomes (the outcome space) in the $\mathcal{P}$ model. Each agent is assumed to have expected utility preferences of the form

$$u^i(x^i) = u^i_0(x^i_0) + \sum_{s\in S} \rho_s u^i_1(x^i_s), \tag{1}$$

where u^i_0, u^i_1 are smooth, increasing, strictly concave functions.

2.4. *Security Markets.* There are J securities, which consist of two types: those in positive supply (the K equity contracts of the firms) and those in zero net supply (such as bonds and options on equity). The firms' production plans, consisting of the vectors (a^*_k, y^{*k}) satisfying $T_k(-a^*_k, y^{*k}) \leq 0$ in the $\mathcal{SN}$ model or the investment a^*_k leading to the probability distribution $\rho(a^*)$ in the $\mathcal{P}$ model, are assumed to be known by the consumer–investors. Thus agents can correctly anticipate the payoffs of the securities and the probability distribution of the payoffs.

In order to define an exchange equilibrium with fixed production plans using common notation for the two models, we let

- $\rho^* = \rho$ in the $\mathcal{SN}$ model
- $\rho^* = \rho(a^*)$ in the $\mathcal{P}$ model,
- $y^* = (y^{1*}, \dots, y^{K*})$, the firms' choices of production feasible with investments $a^* = (a^*_1, \dots, a^*_K)$ in the $\mathcal{SN}$ model,
- $y^* = (y^{1*}, \dots, y^{K*}) = (y^1, \dots, y^K)$ the fixed outcomes of the firms in the $\mathcal{P}$ model,
- $V^* = [V^j_s(y^*_s)]_{\substack{j\in J\\ s\in S}}$ the payoff matrix in the $\mathcal{SN}$ model,
- $V^* = [V^j(y^*_s)]_{\substack{j\in J\\ s\in S}}$ the payoff matrix in the $\mathcal{P}$ model,

where in the $\mathcal{SN}$ model the payoff $V^j_s : \mathbb{R}^K \to \mathbb{R}$ of security j in state s can depend both on the state of nature and on the firms' outputs, whereas in the $\mathcal{P}$ model the payoff $V^j : \mathbb{R}^K \to \mathbb{R}$ of security j can only depend on the realized outputs of the firms. For both models we assume that the first K securities are the equity of the firms, and the remaining securities are in zero net supply. Let q_j denote the price of security j and let $q = (q_j)_{j\in J}$ denote the vector of security prices. $z^i = (z^i_j)_{j\in J}$ denotes the portfolio of securities purchased or sold by agent i and $z = (z^i)_{i\in I}$ denotes the vector of portfolios of the agents. Finally $x = (x^i)_{i\in I}$ denotes the vector of consumption streams of the I agents.

2.5. *Equilibrium with Fixed Production Plans.* We can now define an exchange equilibrium on the security markets for fixed and known production plans for the firms.

Definition 1. (x^*, z^*, q^*) is an exchange equilibrium with fixed production plans (a^*, y^*) if

(i) for each $i \in I$, x^{i*} maximizes $u^i_0(x^i_0) + \sum_{s\in S} \rho^*_s u^i_1(x^i_s)$ subject to

$$x^i_0 = \omega^i_0 + (\hat{q}^* - a^*)\delta^i - q^* z^i, \quad x^i_1 = V^* z^i, \quad z^i \in \mathbb{R}^J,$$

(ii) $\sum_{i\in I} z^{i*}_j = 1,\ j = 1, \dots K, \quad \sum_{i\in I} z^{i*}_j = 0,\ j > K,$

where $\hat{q}^* = (q^*_j)^K_{j=1}$ denotes the vector of equity prices.

Remark. We assume that the investment a^*_k of firm k is financed by the initial shareholders: This is without loss of generality since the Modigliani-Miller theorem on the irrelevance of the choice of financing policy holds for this economy. The market clearing condition (ii) on the security market combined with the agents' budget equations in (i) implies the feasibility of the equilibrium allocation since

$$\sum_{i\in I} x_0^{i*} = \sum_{i\in I} \omega_0^i - \sum_{k\in K} a_k^*, \quad \sum_{i\in I} x_1^{i*} = V^* \sum_{i\in I} z^{i*} = \sum_{k\in K} y^{k*}.$$

In order to avoid the introduction of multipliers for the nonnegativity constraints on agents' consumption streams, we assume either that the marginal utility of consumption tends to infinity when consumption tends to zero (e.g., power or log utility functions) or, if the marginal utility is defined at zero, that consumption is not restricted to be nonnegative. Of course the first case is the most realistic, but polynomial utilities are convenient for constructing simple examples.

In order that an allocation (x^*, a^*, y^*) be Pareto optimal, two conditions must be satisfied: The distribution $x^* = (x^{i*})_{i\in I}$ of the available resources $(\sum_{i\in I} \omega_0^i - \sum_{k\in K} a_k^*, \sum_{k\in K} y^{k*})$ among consumers must be optimal, and firms must choose their production plans (a^*, y^*) optimally. We assume that the financial markets assure an optimal distribution of resources (income streams) among the agents and focus attention on the second condition.

Assumption EE (Efficiency of Exchange). *If (x^*, z^*, q^*) is an exchange equilibrium with fixed production plans (a^*, y^*), then there exists $\pi^* \in \mathbb{R}^S_{++}$ such that*

$$\pi_s^* = \frac{\rho_s^* u_1^{i\prime}(x_s^{i*})}{u_0^{i\prime}(x_0^{i*})} \quad \forall s \in S,\ \forall i \in I. \tag{2}$$

When Assumption EE is satisfied, all agents have the same marginal valuation π_s^* at date 0 for one additional unit of income in state or outcome s at date 1, and π_s^* is called the present value of income in state/outcome s. The simplest way in which EE is satisfied is when there are markets for contracts that pay one unit in state/outcome s and zero otherwise. Such contracts are called "Arrow securities." Insurance contracts resemble Arrow securities, but they are rare for production risks. Even if Arrow securities are not traded, the existing financial contracts may be sufficiently rich to include S independent securities, in which case rank $V^* = S$. Then, since the first-order conditions $q^* = \pi^{i*} V^*$ must be satisfied for each agent in order that (i) of Definition 1 holds, where π^{i*} denotes the personal present-value vector of agent i, if V^* is invertible, the vectors π^{i*} are equalized and EE holds. If the rank condition holds, we say that there are "complete markets" in the $\mathcal{SN}$ model and that there is "complete spanning" in the $\mathcal{P}$ model: We do not use the same terminology for the two models because, in any state-of-nature representation of a $\mathcal{P}$ model, there must be more states than outcomes, so that financial contracts based on outcomes are not complete in the state-of-nature sense (see Magill and Quinzii, 2009).

EE can also be satisfied with a less demanding assumption on the set of financial contracts, provided that consumers have sufficiently similar preferences: If all agents have utility functions u_1^i satisfying linear risk tolerance (LRT) $(\frac{u_1^{i\prime}(x)}{u_1^{i\prime\prime}(x)} = \alpha_i + \beta_i x)$ with the same slope coefficient $(\beta_i = \beta, \forall i \in I)$, then it is sufficient that agents trade the firms' equity contracts and a riskless bond[7] for Assumption EE to be satisfied. The equity of the firms must be traded so that agents can get rid of their initial risks, exchanging their initial holdings for a share of the market portfolio. If they have differences in risk tolerance due to differences in the coefficients α_i, they must have access to a riskless bond to attenuate or leverage the risk of the market portfolio.[8]

[7] The riskless bond is not needed if in addition the intercepts α_i are the same for all agents.

[8] This case is particularly interesting in the $\mathcal{P}$ model because it makes it possible to dispense with the assumption of a discrete outcome space. Also LRT economies serve as a benchmark for understanding when derivative securities are needed to achieve optimal risk sharing. Judd and Guu (2001) study a simple economy with two investors and three securities, a riskless bond, risky equity, and an option on equity, and calculate the equilibrium for small risks by using Taylor expansions around the zero-risk case. They show that there is nonzero trade on the option, to terms of first order, only if the two agents have distinct coefficients of skewness tolerance, a coefficient defined in their paper involving the derivatives of the utility functions up to third order. It is easy to verify that all utility functions in the same LRT class (a given value of β) have the same skewness coefficient, so that in an LRT economy, if derivative securities were present, agents would not trade them.

Because all agents have the same probability estimate ρ^*, the equality of the agents' present-value vectors implies the equality of their *stochastic discount factors* (SDF). Let μ^* denote the common SDF defined by

(3) $$\mu_s^* = \frac{u_1^{i\prime}\left(x_s^{i*}\right)}{u_0^{i\prime}\left(x_0^{i*}\right)}, \quad \forall\, s \in S,\ \forall\, i \in I.$$

The equality of the agents' stochastic discount factors imply that for each $s \in S$ the allocation $x_s^* = (x_s^{i*})_{i\in I}$ is the solution of the social welfare maximum problem

$$\max_{x_s \in \mathbb{R}^I} \left\{ \sum_{i\in I} \frac{1}{u_0^{i\prime}\left(x_0^{i*}\right)} u_1^i\left(x_s^i\right) \,\middle|\, \sum_{i\in I} x_s^i = Y_s^* \right\},$$

where Y_s^* denotes the aggregate output in state or outcome s. Let $\Phi^*(\eta)$ denote the associated *convolution function* of the date 1 utilities defined by

(4) $$\Phi^*(\eta) = \max_{\xi \in \mathbb{R}^I} \left\{ \sum_{i\in I} \alpha_i^* u_1^i(\xi_i) \,\middle|\, \sum_{i\in I} \xi_i = \eta \right\} \quad \text{with} \quad \alpha_i^* = \frac{1}{u_0^{i\prime}\left(x_0^{i*}\right)}.$$

It is readily shown that the function Φ^* has the two properties (see, e.g., Magill and Quinzii, 1996, p. 163)

(5) $$\Phi^*\left(Y_s^*\right) = \sum_{i\in I} \alpha_i^* u_1^i\left(x_s^{i*}\right), \quad \Phi^{*\prime}\left(Y_s^*\right) = \mu_s^*,$$

where Φ^* is the utility function of the "representative agent" at the equilibrium. From the first-order conditions for the optimal choice of a portfolio in (i) of Definition 1, it follows that the equilibrium prices of the securities satisfy

(6) $$q^* = \pi^* V^* = E^*(\mu^* V^*) = E^*(\Phi^{*\prime}(Y^*) V^*),$$

where E^* denotes the expectation operator with respect to the probability ρ^*, and where the last equality comes from the property (5) of the convolution function.

2.6. *Optimal Choice of Production Plans.* We can now characterize the production plans (a^*, y^*) that lead to Pareto optimality of the combined consumption–production allocation.

PROPOSITION 1. *Let (a^*, y^*) be a choice of production plans for the firms and (x^*, z^*, q^*) an associated exchange equilibrium satisfying Assumption EE, with present-value vector $\pi^* = (\pi_s^*)_{s\in S}$ and SDF $\mu^* = (\mu_s^*)_{s\in S}$. The consumption–production plan (x^*, a^*, y^*) is a Pareto-optimal allocation for the production economy if and only if*

(i) *in the $\mathcal{SN}$ model, for each $k \in K$, (a_k^*, y^{k*}) maximizes*

(7) $$M(a_k, y^k) = \sum_{s\in S} \pi_s^* y_s^k - a_k = E(\mu^* y^k) - a_k$$

subject to $T_k(-a_k, y^k) \leq 0$;

(ii) *in the $\mathcal{P}$ model, for each $k \in K$, a_k^* maximizes*

(8) $$V(a_k) = \sum_{s\in S} \rho\left(a_k, a_{-k}^*\right) \Phi^*(Y_s) - a_k.$$

PROOF. (i) is the standard result of the Arrow–Debreu theory that, with convex production sets, Pareto optimality is equivalent to firms maximizing profit at prices π^* collinear to the agents' gradients at the consumption allocation x^*. (ii) is derived in Magill and Quinzii (2009) from the first-order conditions for Pareto optimality: In the $\mathcal{P}$ model firms must maximize their contribution to social welfare measured in units of date 0 consumption.[9] ■

The forms (7) and (8) of the firms' criteria are the most convenient for establishing existence of equilibrium of the production economy: The equilibrium of the $\mathcal{SN}$ model is a standard Arrow–Debreu equilibrium that is known to exist under the convexity assumptions on preferences and technology given above. An equilibrium of the $\mathcal{P}$ model is shown to exist in Magill and Quinzii (2009) if in addition to the assumptions made above, the market values of the firms are nonnegative or taxes are used to subsidize firms with negative market values.

The criterion (7) of the $\mathcal{SN}$ model has the apparent merit of simplicity. It is linear in the firm's production plan and, if firms know the present-value prices π^*, the only additional information that each firm needs in order to make an optimal choice of production plan is its own technology T_k: The prices π^* do all the coordination of information required for efficiency. The problem is that in practice the prices needed to implement this criterion cannot be found, directly or indirectly. It is difficult to find securities whose payoff is based on exogenous events (states of nature) that explain firms' profits: For a market requires an objective description of the contingency to be traded on, whose occurrence must be verifiable by third parties. The only examples of contracts based on states of nature that come to mind are the recently developed weather related futures and options, introduced on the Chicago Mercantile Board in 1999, and expanded to include hurricane risks in 2005: These markets, however, cover a very small part of the production risks. Most markets for risk sharing in production are based on outcomes, and markets based on outcomes are never complete for the underlying states of nature.

The criterion (8) of the $\mathcal{P}$ model is more complex than the market-value criterion (7). It is a nonlinear function of the firm's production plan and requires that firms know the function Φ^*—which in turn requires that they know agents' preferences: Markets, it would seem, have lost their role of providing firms with the requisite information to guide production decisions. In Magill and Quinzii (2009) we argue that if the prices π^*, and hence the SDF μ^*, can be recovered from the financial market equilibrium, then the stochastic discount factor can be "integrated" to obtain an approximation of the function Φ^*, since $\Phi^{*\prime}(Y_s) = \mu_s^*$. In this way Φ^* can be recovered from the present-value prices π^*.

In principle, π^* can be deduced from the observed security prices q^* of the underlying exchange equilibrium (x^*, z^*, q^*) on the financial markets. But this may be quite a challenging task. Obtaining π^* by inverting the pricing relation (6), $q^* = \pi^* V^*$, requires discretizing the range of possible values of the profit streams of all firms, calculating the corresponding payoffs of all the risky securities on the market to construct the matrix V^*, and then inverting it. Although inverting large matrices has become computationally feasible, to the best of our knowledge no empirical paper on asset pricing has ever attempted such a calculation, suggesting that it is unlikely that firms would attempt to calculate π^* from security prices in this way. It seems thus worthwhile to find a form for criteria (7) and (8) that would permit firms to obtain a good approximation of the optimal choice on investment by using information that is more readily obtained from the financial markets.

Empirical papers in finance that focus on explaining the prices of individual firms' securities typically base their analysis on the CAPM formula, which can be viewed as the first step in a "moment approach" to asset pricing. The CAPM formula prices the discounted mean of a security return and its covariance (beta) with the market. These are the first two terms that appear if the SDF $\Phi^{*\prime}(Y_s)$ in the pricing formula (6) is developed in a Taylor series expansion. More recently empirical studies have introduced the third-order comoment, i.e., the co-skewness

[9] The derivation of the criterion V in (8) was motivated by the observation in Magill and Quinzii (2008) that if firms maximize the analogue of the market-value criterion $\sum_{s\in S} \rho_s(a_k, a_{-k}^*)\mu_s^* y_s^k - a_k$ under the assumptions of the $\mathcal{P}$ model, then the resulting allocation is (generically) not Pareto optimal.

of the securities' rates of return with the market. We will apply this "moment" approach to the firms' criteria by expanding the functions $\Phi^*(Y_s)$ and $\Phi^{*\prime}(Y_s)$ that appear in the criteria (7) and (8) in Taylor series expansions. We will find that there are three advantages to this approach: First, a "moment" representation of the criteria (7) and (8) expresses in a more intuitive way how a firm should choose its production plan given the risk characteristics of its profit stream and the way it relates to the profit streams of other firms; second, the "prices" that the firm needs to know to maximize the criterion that we derive are prices of comoments that can be deduced from the security prices by simple regression; third, the two objectives (7) and (8), which appear as different criteria in Proposition 1, reveal themselves to be the same function in the moment representation.

Before exploring the moment approach to the objectives of the firms, we first derive the moment formula for the prices of the securities in an exchange equilibrium (x^*, z^*, q^*).

3. MOMENT FORMULA FOR SECURITY PRICES

Two properties of the convolution function Φ^* in (4) are needed to obtain a moment representation of the pricing formula (6) and the criteria (7) and (8): Φ^* must be smooth and the Taylor series expansion of $\Phi^*(Y_s^*)$ must converge to $\Phi^*(Y_s^*)$ in a neighborhood of the mean aggregate output. The assumption that the agents' date 1 utility functions are smooth and strictly concave ensures that Φ^* is smooth, a necessary condition for writing the Taylor expansion series, but it does not guarantee that the series converges and coincides with the value of the function.

Let $A = \{a = (a_1, \dots, a_K) \in \mathbb{R}_+^K \mid \sum_{k\in K} a_k \leq \sum_{i\in I} \omega_0^i\}$ denote the set of feasible date 0 investments by the firms.

ASSUMPTION TS. *For all $a^* \in A$, y^* feasible given a^*, and $s \in S$*

$$\Phi^*(Y_s^*) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \Phi^{*(n)}(\bar{Y}^*)(Y_s^* - \bar{Y}^*)^n,$$

$$\Phi^{*\prime}(Y_s^*) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \Phi^{*(n+1)}(\bar{Y}^*)(Y_s^* - \bar{Y}^*)^n,$$

where $Y_s^* = \sum_{k\in K} y_s^{*k}$, $\bar{Y}^* = E(Y^*)$ in the $\mathcal{SN}$ model and $\bar{Y}^* = E_{a^*}(Y)$ in the $\mathcal{P}$ model.

EXAMPLE. Assumption TS is an assumption on both the preferences and the technology, for it requires the functions Φ^* and $\Phi^{*\prime}$ to be analytic at $\bar{Y}^*$, which is an assumption on the preferences, and it requires that Y_s^* is in the radius of convergence around $\bar{Y}^*$, which limits the variability of aggregate output. In order to see how stringent this restriction is, consider the familiar case where the consumer sector of the economy can be summarized by a representative agent with a CRRA utility function

$$U(Y) = \frac{1}{1-\alpha}\left(Y_0^{1-\alpha} + \delta E(Y^{1-\alpha})\right)$$

for $\alpha > 0$: When $\alpha = 1$, the Bernoulli index is $\ln(Y)$ and the reasoning is the same as with $\alpha \neq 1$. If (a^*, y^*) is feasible and $Y_0^* = \sum_{i\in I} \omega_0^i - \sum_{k\in K} a_k^*$, then

$$\Phi^*(Y_s^*) = \frac{\delta}{1-\alpha}\frac{(Y_s^*)^{1-\alpha}}{(Y_0^*)^{-\alpha}}, \quad \forall s \in S,$$

and the Taylor series expansion of $\Phi^*(Y^*_s)$ around $\bar{Y}^*$ is of the form $\sum_{n=0}^{\infty} v_n$ with

$$v_n = \delta\big(Y^*_0\big)^{\alpha}(-1)^{n+1}\,\frac{\alpha(\alpha+1)\ldots(\alpha+n-2)}{n!}(\bar{Y}^*)^{-\alpha-n+1}\big(Y^*_s-\bar{Y}^*\big)^n,\ n\geq 2.$$

v_0 and v_1 are easily calculated, but they do not influence the convergence of the series.

$$\left|\frac{v_{n+1}}{v_n}\right| = \frac{\alpha+n-1}{n+1}\,\frac{|Y^*_s-\bar{Y}^*|}{\bar{Y}^*}$$

so that the series converges if $\frac{|Y^*_s-\bar{Y}^*|}{\bar{Y}^*} < 1$, namely, if $Y^*_s \in (0,\ 2\bar{Y}^*)$.[10] The terms in the Taylor expansion are of the order of $(\frac{Y^*_s-\bar{Y}^*}{\bar{Y}^*})^n$, so that the smaller the fluctuations in aggregate output around its mean, the faster the terms in the Taylor expansion become negligible. Since in practice variations in aggregate output of the order of 20% are considered very large, the limit placed by Assumption TS of a 100% variation in aggregate output around its mean does not seem too restrictive.

Under Assumption TS, the pricing formula (6) for an exchange equilibrium implies that for all $j \in J$

(9)
$$\begin{aligned} q^*_j &= E^*(\Phi^{*\prime}(Y^*)((V^{j*}-\bar{V}^{j*})+\bar{V}^{j*}))\\ &= \frac{\bar{V}^{j*}}{1+r^*} + E^*\left(\sum_{n=1}^{\infty}\frac{1}{n!}\Phi^{*(n+1)}(\bar{Y}^*)(Y^*-\bar{Y}^*)^n(V^{j*}-\bar{V}^{j*})\right), \end{aligned}$$

where the interest rate r^* is defined by the price of the riskless income stream, $\frac{1}{1+r^*} = E^*(\Phi^{*\prime}(Y^*))$, and where $\bar{V}^{j*} = E^*(V^{j*})$.

For a pair of random variables (x, y) with means $\bar{x}$ and $\bar{y}$ and a pair of nonnegative integers (m, n) the central comoment (m, n) is defined by

(10)
$$\text{como}^{(m,n)}(x, y) = E((x-\bar{x})^m(y-\bar{y})^n).$$

For $(m, n) = (1, 1)$ the comoment is the covariance; for $(m, n) = (2, 1)$ the comoment is the co-skewness of y with respect to x (we adopt the convention of the finance literature that the squared deviation is in the first variable); for $(m, n) = (3, 1)$ the comoment is the co-kurtosis of y with respect to x. For $(m, n) = (2, 0)$ the comoment is the variance of x, for $(m, n) = (3, 0)$ it is the (unnormalized) skewness of x, and for $(m, n) = (4, 0)$ it is the (unnormalized) kurtosis of x. The pricing formula introduces the comoments $(m, 1)$ for $m \geq 1$, and we will see that the moment representation of the criteria (7) and (8) introduces all comoments (m, n) with $m \geq 0$, $n > 0$.

With this notation, the pricing formula (9) can be written as

(11)
$$q^*_j = \frac{\bar{V}^{j*}}{1+r^*} + \sum_{n=1}^{\infty} c^*_{n+1}\text{como}^{(n,1)}(Y^*, V^{j*}),\quad \text{with}\quad c^*_{n+1} = \frac{1}{n!}\Phi^{*(n+1)}(\bar{Y}^*).$$

Equation (11) expresses how the random income stream V^{j*} is priced in equilibrium; the risk premium $\frac{\bar{V}^{j*}}{1+r^*} - q^*_j$ depends on the way the purely risky part $V^{j*} - \bar{V}^{j*}$ of the income stream comoves with the purely risky part $Y^* - \bar{Y}^*$ of aggregate output, as expressed by the c^*_{n+1}-weighted sum of the comoments $\text{como}^{(n,1)}(Y^*, V^{j*})$, c^*_{n+1} reflecting the weight attached by the social welfare function to the comoment of order $n+1$ at $\bar{Y}^*$.

[10] A more abstract proof consists of noting that the power function in the complex plane is analytic except at zero, with a radius of convergence equal to the distance to the closest singularity, which is at zero.

Because the infinite series in (11) converges, comoments of high order can be neglected. As shown in the example above, the smaller the fluctuations in aggregate risk, the faster the terms become negligible. In the finance literature the valuation expression (in rate-of-return form) is typically restricted to the first three or four terms. If we neglect the comoments of order 5 or more, the price q_j^* can be expressed as

$$q_j^* = \frac{\bar{V}^{j*}}{1+r^*} + c_2^* \operatorname{cov}(Y^*, V^{j*}) + c_3^* \text{co-skew}(Y^*, V^{j*}) + c_4^* \text{co-kurt}(Y^*, V^{j*})$$

with $c_2^* = \Phi^{*\prime\prime}(\bar{Y}^*)$, $c_3^* = \frac{1}{2}\Phi^{*\prime\prime\prime}(\bar{Y}^*)$, $c_4^* = \frac{1}{6}\Phi^{*(4)}(\bar{Y}^*)$. The coefficients c^* can be expressed as functions of the preferences of the agents, their weights in the social welfare function (their wealth), and the derivatives of their shares $x^i(\bar{Y}^*)$ in the solution of the allocation problem (4) at $\bar{Y}^*$. As shown in more detail in the Appendix, taking derivatives of the function Φ^* defined in (4) and using the first-order conditions gives

$$c_2^* = \Phi^{*\prime\prime}(\bar{Y}^*) = \sum_{i\in I} \alpha_i^* u_1^{i\,\prime\prime}(x^i(\bar{Y}^*))(x^{i\prime}(\bar{Y}^*))^2$$

$$c_3^* = \frac{1}{2}\Phi^{*\prime\prime\prime}(\bar{Y}^*) = \frac{1}{2}\sum_{i\in I} \alpha_i^* u_1^{i\,(3)}(x^i(\bar{Y}^*))(x^{i\prime}(\bar{Y}^*))^3.$$

In an efficient allocation of the random aggregate output, each agent's consumption stream $x^i(Y)$ is comonotone with aggregate output, so that $x^{i\prime}(\bar{Y}^*) > 0$, $i \in I$. Thus if agents are risk averse, $(u^{i\prime\prime} < 0)$, and have preference for positive skewness $(u^{i\prime\prime\prime} > 0)$—or dislike negative skewness, i.e., are "prudent"—then $c_2^* < 0$ and $c_3^* > 0$. The term $c_4^* = \frac{1}{6}\Phi^{*(4)}(\bar{Y}^*)$ does not necessarily inherit the sign of the fourth derivative of the agents' utility functions since

$$\Phi^{*(4)}(\bar{Y}^*) = \sum_{i\in I} \alpha_i^* \left(u_1^{i\,(4)}(x^i(\bar{Y}^*))(x^{i\prime}(\bar{Y}^*))^4 + 3u_1^{i\,(3)}(x^i(\bar{Y}^*))(x^{i\prime}(\bar{Y}^*))^2 x^{i\,\prime\prime}(\bar{Y}^*)\right).$$

Thus even if $u_1^{i\,(3)} > 0$ and $u_1^{i\,(4)} < 0$, since $\sum_{i\in I} x^{i\,\prime\prime}(\bar{Y}^*) = 0$, the sign of $\Phi^{*(4)}$ is ambiguous if the sharing rule is not linear ($x^{i\,\prime\prime}(\bar{Y}^*) \neq 0$ for some i).

Formula (11) with agent-specific coefficients was first derived in the finance literature by Rubinstein (1973), and by Kraus and Litzenberger (1976, 1981) as an equilibrium formula truncated at the comoment of order 3, generalizing the CAPM formula to incorporate the effect of preference for skewness. Pricing formulaes in finance are typically expressed in return form, which is more convenient for empirical analysis. If R^{j*} denotes the return on security j, defined by $R^{j*} = \frac{V^{j*}}{q_j^*}$, if $R^{M*} = \frac{Y^*}{q_M^*}$ with $q_M^* = \sum_{j=1}^k q_j^*$ denotes the return on the market portfolio of all equity, and $R^* = 1 + r^*$ denotes the return on a riskless bond, then (11) can be written as

$$\text{(12)}\quad E^*(R^{j*}) - R^* = -\sum_{n=1}^{\infty} \gamma_{n+1}^* \text{como}^{(n,1)}(R^{M*}, R^{j*}), \quad \text{with} \quad \gamma_{n+1}^* = (q_M^*)^n c_{n+1}^*(1+r^*).$$

If for some $\bar{n}$ the terms of order $n > \bar{n}$ are negligible and the number of securities exceeds $\bar{n}$, then in principle the coefficients γ_{n+1}^* and thus c_{n+1}^* can be obtained by linear regression of the excess returns of the securities on the comoments of these returns with the market portfolio.

Recently there has been a revival of interest in the three comoment version of the pricing formula, since adding a preference for skewness over and above mean–variance preferences can help to explain "puzzles" in observed investors' portfolios and in the pricing of assets that cannot be explained by the standard mean–variance model. An analysis of portfolio holdings of 60,000 investors at a large discount brokerage firm in the period 1991–96 revealed that the majority of investors hold very undiversified portfolios, typically with only a few securities (Mitton and

Vorknik, 2007). This study served to confirm results of earlier studies (e.g., Blume and Friend, 1975), which found that the diversification in investors' portfolios is far less than that prescribed by CAPM. Although a number of explanations for this phenomenon have been proposed, the simplest and most natural from a theoretical point of view is based on the observation that many investors have a preference for positive skewness, and with such preferences optimal portfolios are less diversified than those that are optimal with mean–variance preferences: See Briec et al. (2007) for an analysis of the efficient frontier of mean–variance–skewness portfolios.

The standard mean–variance model that truncates the excess return formula (12) at the co-variance term does a poor job of explaining observed excess returns in the postwar period (see, e.g., Fama and French, 1992). Harvey and Siddique (2000) have shown that taking into account the co-skewness term in (12) greatly improves the fit of the model: Typically securities with higher than average expected returns (e.g., small company stocks or those with high book to market value) have negative co-skewness with the market portfolio, whereas those with lower than average expected returns (e.g., large companies) have positive co-skewness.

4. MOMENT REPRESENTATION OF CRITERION FOR THE STATE-OF-NATURE MODEL

We are now in a position to derive the moment representation of a firm's criterion that leads to Pareto optimality in the $\mathcal{SN}$ model. In order to express the criterion in a natural way we decompose the aggregate output into the contribution y^k of firm k and the contribution $Y^{-k} = \sum_{k' \neq k} y^{k'}$ of the other firms: $Y = y^k + Y^{-k}$.

PROPOSITION 2. *Let (a^*, y^*) be a choice of production plans by the firms and (x^*, z^*, q^*) an associated exchange equilibrium. The consumption–production plan (x^*, a^*, y^*) is a Pareto-optimal allocation of the production economy in which Assumptions EE and TS hold if and only if for each firm $k \in K$ the production plan (a_k^*, y^{k*}) is an extremum of the comoment criterion*

$$\tilde{M}(a_k, y^k) = \frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} \frac{c_{n+1}^*}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} \operatorname{como}^{(n+1-j,\, j)}(Y^{-k*}, y^k) - a_k \tag{13}$$

subject to the constraint $T_k(-a_k, y^k) \leq 0$, where the coefficients c_{n+1}^* are those of the pricing formula (11): $c_{n+1}^* = \frac{1}{n!}\Phi^{*(n+1)}(\bar{Y}^*), n \geq 1$.

PROOF. By Proposition 1 (x^*, a^*, y^*) is Pareto optimal if and only if, for each firm $k \in K$, the plan (a_k^*, y^{k*}) maximizes

$$\begin{aligned} M(a_k, y^k) &= E(\mu^* y^k) - a_k = E(\mu^*)E(y^k) + E(\mu^*(y^k - E(y^k)) - a_k \\ &= \frac{E(y^k)}{1+r^*} + E(\mu^*(y^k - E(y^k)) - a_k \end{aligned}$$

subject to the constraint $T_k(-a_k, y^k) \leq 0$. Thus the first-order conditions, which are necessary and sufficient for optimality,

$$D_{y^k} M(a_k^*, y^{k*}) = \lambda_k D_{y^k} T_k(-a_k^*, y^{k*}), \quad 1 = \lambda_k \frac{\partial}{\partial y_0^k} T_k(-a_k^*, y^{k*}), \tag{14}$$

where $D_{y^k}M = (\frac{\partial M}{\partial y^k_s})_{s\in S}$, must be satisfied for some $\lambda_k > 0$. Since by (5) $\mu^*_s = \Phi^{*\prime}(Y^*_s), s \in S$, developing $\Phi^{*\prime}(Y^*_s)$ in Taylor series about the mean $\bar{Y}^*$ gives

(15)
$$\begin{aligned} M(a_k, y^k) &= \frac{E(y^k)}{1+r^*} + E\left(\sum_{n=1}^{\infty} c^*_{n+1}(Y^* - \bar{Y}^*)^n(y^k - E(y^k))\right) - a_k \\ &= \frac{E(y^k)}{1+r^*} + E\left(\sum_{n=1}^{\infty} c^*_{n+1}(Y^{-k*} - \bar{Y}^{-k*} + y^{k*} - \bar{y}^{k*})^n(y^k - E(y^k))\right) - a_k \\ &= \frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} c^*_{n+1}\sum_{j'=0}^{n}\binom{n}{j'}E[(Y^{-k*} - \bar{Y}^{-k*})^{n-j'}(y^{k*} - \bar{y}^{k*})^{j'}(y^k - E(y^k))] - a_k \end{aligned}$$

with $c^*_{n+1} = \frac{1}{n!}\Phi^{*(n+1)}(\bar{Y}^*)$, for $n \geq 1$. Note that when differentiating $M(a_k, y^k)$ with respect to y^k (to check that the FOCs (14) are satisfied) the firm must take the term $(y^{k*} - \bar{y}^{k*})^{j'}$, which comes from the development of $(Y^* - \bar{Y}^*)^n$, as given, while differentiating only the term $y^k - E(y^k)$: This is the competitive assumption for this model. It is, however, an awkward expression for a firm to "maximize," since it must only take into account a part of the terms involving its actions. A more natural expression can be obtained by noting that the product terms in (15) satisfy

$$\begin{aligned} &D_{y^k}E[(Y^{-k*} - \bar{Y}^{-k*})^{n-j'}(y^{k*} - \bar{y}^{k*})^{j'}(y^k - E(y^k))]_{y^k=y^{k*}} \\ &= \frac{1}{j'+1}D_{y^k}E[(Y^{-k*} - \bar{Y}^{-k*})^{n-j'}(y^k - E(y^k))^{j'+1}]_{y^k=y^{k*}} \end{aligned}$$

so that the FOCs (14) are satisfied if and only if

(16)
$$D_{y^k}\tilde{M}(a^*_k, y^{k*}) = \lambda_k D_{y^k}T_k(-a^*_k, y^{k*}), \quad 1 = \lambda_k\frac{\partial}{\partial y^k_0}T_k(-a^*_k, y^{k*}),$$

where

$$\tilde{M}(a_k, y^k) = \frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} c^*_{n+1}\sum_{j=1}^{n+1}\frac{1}{j}\binom{n}{j-1}E[(Y^{-k*} - \bar{Y}^{-k*})^{n-j+1}(y^k - E(y^k))^j]$$

with $j = j' + 1$ in (15). Since $\frac{1}{j}\binom{n}{j-1} = \frac{1}{n+1}\binom{n+1}{j}$, $\tilde{M}(a_k, y^k)$ is the function in (13). ■

It may be useful to rephrase the idea underlying the derivation of criterion $\tilde{M}$. We know that in the $\mathcal{SN}$ model if a firm's plan (a^*_k, y^{k*}) is optimal, then it maximizes market value, which, in view of (11), can be written as

(17)
$$M(a_k, y^k) = \frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} c^*_{n+1}\text{como}^{(n,1)}(Y^{-k*} + y^{k*}, y^k) - a_k.$$

The Arrow–Debreu criterion in Proposition 1(i), usually interpreted as the "perfectly competitive" criterion, requires that the firm take the present-value prices π^*_s as given. When expressed in comoment form, this price taking requirement is equivalent to taking the coefficients $(c^*_{n+1})_{n\geq 1}$ and the aggregate output $Y^* = Y^{-k*} + y^{k*}$ as given. The coefficients $(c^*_{n+1})_{n\geq 1}$ can be considered as the "prices" of the comoments in the pricing formula (11), and it seems natural for firms to take these price coefficients as given. Taking Y^* as given, however, is more delicate, since the firm's decision variable y^k is part of the aggregate output. The idea of Proposition 2 is to find an alternative expression for the objective such that the firm takes the actions Y^{-k*} of all other firms and the coefficients $(c^*_{n+1})_{n\geq 1}$ as given, and can treat all terms involving its own actions as completely under its control. This is achieved by first separating the terms involving firm k's actions (a_k, y^k) from the production Y^{-k*} of other firms (Equation (15) in the proof above) and then noting that if in (15) all terms involving y^k (with or without a star) are differentiated, then the weights on the comoment terms need to be adjusted to obtain the correct first-order conditions. Once the criterion is transformed to the form (13), it can be used even if firm k is aware of its influence on aggregate output, so that it may be appropriate for a firm that is nonnegligible on its market—even though noncompetitive behavior on the goods market may be an impediment to obtaining a first-best allocation.

In order to understand the transformation of the criterion into the form (13) and its usefulness, consider the simplest case where agents have mean–variance preferences, firm k's technology can be described by a production function $y^k(a_k) = (y^k_s(a_k))_{s\in S}$, and its output is independent of the outputs of other firms. Then (17) reduces to

$$M(a_k, y^k) = \frac{E(y^k)}{1+r^*} + c_2^* \operatorname{cov}(y^{k*}, y^k) - a_k \tag{18}$$

and, if a_k^* is optimal, it must satisfy the FOC

$$\frac{E(y^{k\prime}(a_k^*))}{1+r^*} + c_2^* \operatorname{cov}(y^{k*}, y^{k\prime}(a_k^*)) - 1 = 0. \tag{19}$$

If in (18) y^{k*} is replaced by y^k and both terms in the covariance are differentiated, then the coefficient on the covariance term will end up being twice that in (19), the correct FOC. Thus the function that gives the correct FOC *when all terms involving firm k's production are differentiated* is obtained by dividing the coefficient c_2^* by 2, leading to the criterion

$$\tilde{M}(a_k, y^k) = \frac{E(y^k)}{1+r^*} + \frac{c_2^*}{2} \operatorname{var}(y^k) - a_k. \tag{20}$$

If the analysis were to begin with the firm's value expressed via the comoment pricing formula (11) instead of starting with Proposition 1(i), which gives the abstract form of the market-value criterion, then market-value maximization might be interpreted as the maximization of

$$\frac{E(y^k)}{1+r^*} + \sum_{n=1}^{\infty} c^*_{n+1} \operatorname{como}^{(n,1)}(Y^{-k*} + y^k, y^k) - a_k \tag{21}$$

instead of (17). Indeed in a well-known paper analyzing production decisions in a mean–variance framework, Stiglitz (1972) took it for granted that (21) (with $c^*_{n+1} = 0$ for $n \geq 2$) is the natural criterion for a competitive firm. In particular, when studying the case where the firms' outputs are independent (one of the cases considered by Stiglitz), he adopted the criterion

$$\frac{E(y^k)}{1+r^*} + c_2^* \operatorname{var}(y^k) - a_k$$

with a weight on the variance term that is twice the weight in (20), the correct criterion from the normative point of view. He reached the controversial conclusion that when the CAPM is generalized to a setting with production, a competitive system of markets leads to a misallocation of investment since the resulting equilibrium is not Pareto optimal. The reason why Stiglitz arrived at this conclusion should now be clear: He thought that to obtain a competitive criterion it suffices to take a "price-taking" assumption. However, he did not adopt the right notion of "price-taking behavior," for Pareto optimality requires that the present-value prices π^* be taken as given; replacing this, as Stiglitz did, by the weaker requirement that firms take the comoment prices c^* as given does not lead (without transformation) to Pareto optimality.

Market-value maximization, which is widely regarded in the finance literature as a universal criterion for firms, appears rather straightforward and unambiguous in the original Arrow–Debreu formulation (7) using state prices. However, as shown by the paper of Stiglitz, when the model within which the analysis is carried out begins naturally with a moment/comoment expression for market value—as is the case with the CAPM formula—the interpretation of market-value maximization is more subtle, since the apparently natural interpretation (21) does not respect the price-taking assumption required by the Arrow–Debreu analysis. The transformation to the criterion $\tilde{M}$ avoids these problems.

One consequence of transforming a firm's criterion into the comoment form $\tilde{M}$ is that convexity of the maximum problem of a firm may be lost, since $\tilde{M}$ may not be a concave function of y^k. For example in the mean–variance–skewness model ($c^*_{n+1} = 0$ for $n \geq 3$), the transformed criterion is

$$\tilde{M}(a_k, y^k) = \frac{E(y^k)}{1+r^*} + c_2^* \left(\text{cov}(Y^{-k*}, y^k) + \frac{1}{2}\text{var}(y^k)\right)$$
$$+ c_3^* \left(\text{coskew}(Y^{-k*}, y^k) + \text{coskew}(y^k, Y^{-k*}) + \frac{1}{3}\text{skew}(y^k)\right) - a_k.$$

Although in the mean–variance model ($c_2^* < 0, c_3^* = 0$) $\tilde{M}(a_k, y^k)$ is concave in y^k, if agents have a preference for skewness ($u^{i'''} > 0$) and weigh the mean–skewness trade-off sufficiently relative to the mean–variance trade-off, then the terms coskew(y^k, Y^{-k*}) and skew(y^k) may make $\tilde{M}$ a nonconcave function of y^k.

Because the criterion $M(a_k, y^k)$ is linear and the production set represented by T_k is convex, the FOCs (14) are necessary and sufficient for maximizing M subject to the technology constraint: Since any extremum of $\tilde{M}$ subject to the technology constraint satisfies (14), it is an optimal choice. The cost of the nonconcavity of (13) is that not all the Pareto-optimal allocations can be found by firms maximizing the objective functions $\tilde{M}(a_k, y^k)$, $k \in K$. The situation here is analogous to using a weighted sum of agents' utility functions to find a Pareto-optimal allocation in the standard GE model, when agents' utility functions are quasi-concave but not concave. Any extremum of the weighted sum of utility functions (the social welfare function) subject to the feasibility constraints satisfies the FOCs for Pareto optimality and, in a convex economy, is a Pareto-optimal allocation, but it is not necessarily a maximum of the social welfare function.

5. MOMENT REPRESENTATION OF CRITERION FOR THE PROBABILITY MODEL

In the $\mathcal{P}$ model s denotes a realization of the firms' outputs $y_s = (y^1_{s_1}, \ldots, y^K_{s_K})$, with aggregate output $Y_s = \sum_{k\in K} y^k_{s_k}$. The firms' choices of investment $a = (a_1, \ldots, a_K)$ at date 0 determine the probability distribution $(\rho_s(a))_{s\in S}$ of the firms' outcomes at date 1. If a^* is a vector of investment for the K firms and if (x^*, z^*, q^*) is an associated exchange equilibrium satisfying Assumption EE, in which agents correctly anticipate the probability distribution $\rho(a^*)$, then agents will have the same stochastic discount factor $\mu^* = \Phi^{*\prime}(Y)$, so that x^*_s is the optimal distribution of the aggregate output Y_s among consumers for the social welfare function Φ^*. Proposition 1 then asserts that a necessary condition for the combined production–consumption plan (a^*, x^*) to be Pareto optimal is that the investment a^*_k of each firm k maximizes the expected contribution of firm k to social welfare

$$V(a_k) = E_{(a_k, a^*_{-k})}\Phi^*(Y) - a_k,$$

where $E_{(a_k,a^*_{-k})}$ denotes the expectation operator for the probability distribution $\rho(a_k, a^*_{-k})$. Our objective is to find an alternative way of expressing this criterion, which is more easily implementable by a firm, i.e., which can be expressed in terms of variables that are readily observable. As in the previous section, we find this expression by applying a Taylor series expansion to Φ^*: This leads to an expression involving comoments between the production of firm k and the production of the other firms, which firm k's choice of investment a_k influences by changing the probability of the outcomes instead of by changing quantities as in the previous section. The resulting criterion will have the same form as the criterion $\tilde{M}$ in Proposition 2 if we assume that the probabilities of the outcomes of the firms other than k are not influenced by firm k's choice of investment, so that a_k has no external effect on the other firms.

ASSUMPTION NE. *For all $a \in A$ and each $k \in K$, the marginal probability of y^{-k}*

$$\sum_{s_k \in S_k} \rho_{(s_k,s_{-k})}(a_k, a_{-k}), \quad \forall\, s_{-k} \in S_{-k}$$

does not depend on a_k.

Note that Assumption NE does not imply independence among the firms' outcomes: Assumption NE is satisfied when firms' outcomes are correlated but conditionally independent (see Magill and Quinzii, 2009).

In order to indicate that the comoments between y^k and the output Y^{-k} of other firms depend on the choice a_k of firm k given the investment a_{-k} of the other firms, we write

$$\text{como}^{(m.n)}(Y^{-k}, y^k; a_k, a_{-k}) = E_{(a_k,a_{-k})}(Y^{-k} - E_{a_{-k}}(Y^{-k}))^m(y^k - E_{a_k}(y^k))^n.$$

PROPOSITION 3. *Let a^* be choice of investment by the firms, (x^*, z^*, q^*) an associated exchange equilibrium, and let Assumptions EE, TS, and NE be satisfied. The consumption–investment plan (x^*, a^*) is Pareto optimal if and only if, for each firm $k \in K$, the investment a^*_k is an extremum of the comoment criterion*

$$\tilde{V}(a_k) = \frac{E_{a_k}(y^k)}{1+r^*} + \sum_{n=1}^{\infty} \frac{c^*_{n+1}}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} \text{como}^{(n+1-j,j)}\left(Y^{-k}, y^k; a_k, a^*_{-k}\right) - a_k, \tag{22}$$

where the coefficient c^*_{n+1} are those of the pricing formula (11): $c^*_{n+1} = \frac{1}{n!}\Phi^{*(n+1)}(\bar{Y}^*)$, $n \geq 1$.

PROOF. Under Assumption TS, for all $s \in S$, $\Phi^*(Y_s)$ coincides with its Taylor expansion series around $\bar{Y}^*$. Since a^*_k maximizes (8), the first-order condition

$$\frac{\partial}{\partial a_k} E_{(a_k,a^*_{-k})} \left(\Phi^*(\bar{Y}^*) + \sum_{n=1}^{\infty} \frac{\Phi^{*(n)}(\bar{Y}^*)}{n!}(Y - \bar{Y}^*)^n \right)_{a_k = a^*_k} - 1 \leq 0, \quad \text{with equality if } a^*_k > 0$$

must be satisfied, where $\Phi^*(Y_s)$ has been developed in Taylor series. The first term can be omitted, since for all a, $E_a(\Phi^*(\bar{Y}^*)) = \Phi^*(\bar{Y}^*)$ and is independent of a_k. Expanding $(Y - \bar{Y}^*)^n$ into the terms that depend on the output of firm k and those that depend on the output of all other firms and using the convention that for any random variable x, $\bar{x}^*$ denotes the expectation under the probability distribution $\rho(a^*)$ gives

$$\frac{\partial}{\partial a_k} E_{(a_k,a^*_{-k})} \left(\sum_{n=1}^{\infty} \frac{\Phi^{*(n)}(\bar{Y}^*)}{n!} \sum_{j=1}^{n} \binom{n}{j} (Y^{-k} - \bar{Y}^{-k*})^{n-j}(y^k - E_{a^*_k}(y^k))^j \right)_{a_k = a^*_k} - 1 \leq 0, \tag{23}$$

where by Assumption NE the term with index $j = 0$ has been omitted, since the distribution of Y^{-k} is not influenced by a_k. The function in (23) whose derivative with respect to a_k must be nonpositive at a_k^* (and zero if $a_k^* > 0$) can be used to check whether the choice a_k^* is optimal, but it does not provide an objective function that the firm can maximize to find its optimal investment, for the decision a_k^* must already be known to evaluate $E_{a_k^*}(y^k)$ and only some of the effect of the action a_k must be taken into account. In order to transform (23) into the first-order condition for an objective function for firm k, we need to find a function $\tilde{V}(a_k)$ in which no prior choice a_k^* appears, with the property that if $\frac{d}{da_k}\tilde{V}(a_k) = 0$ for $a_k = a_k^*$, then (23) is satisfied. In order to obtain the appropriate transformation note that

$$(24)\qquad \frac{\partial}{\partial a_k} E_{(a_k,a_{-k}^*)}(Y^{-k} - \bar{Y}^{-k*})^{n-j}(y^k - E_{a_k^*}(y^k))^j \Big|_{a_k=a_k^*}$$
$$= \frac{\partial}{\partial a_k} E_{(a_k,a_{-k}^*)}(Y^{-k} - \bar{Y}^{-k*})^{n-j}(y^k - E_{a_k}(y^k))^j$$
$$+ j\ E_{(a_k,a_{-k}^*)}(Y^{-k} - \bar{Y}^{-k*})^{n-j}(y^k - E_{a_k}(y^k))^{j-1}\frac{d}{da_k}E_{a_k}(y^k)\Big|_{a_k=a_k^*}.$$

Substituting (24) into (23), we find that

(25)
$$\frac{\partial}{\partial a_k}\sum_{n=1}^{\infty}\frac{\Phi^{*(n)}(\bar{Y}^*)}{n!}\sum_{j=1}^{n}\binom{n}{j}\mathrm{como}^{(n-j,j)}(Y^{-k}, y^k; a_k, a_{-k}^*)$$
$$+\left[\sum_{n=1}^{\infty}\frac{\Phi^{*(n)}(\bar{Y}^*)}{n!}\sum_{j=1}^{\infty} j\binom{n}{j} E_{(a_k,a_{-k}^*)}(Y^{-k} - \bar{Y}^{-k*})^{n-j}(y^k - E_{a_k}(y^k))^{j-1}\right]\frac{d}{da_k}E_{a_k}(y^k) - 1 \le 0$$

holds at a_k^* if and only if (23) is satisfied. Consider the term in square brackets that multiplies $\frac{\mathrm{d}}{\mathrm{d}a_k}E_{a_k}(y^k)$. Since $j\binom{n}{j} = n\binom{n-1}{j-1}$, when it is evaluated at a_k^* it can be written as

$$(26)\qquad \Phi^{*\prime}(\bar{Y}^*) + \sum_{n=2}^{\infty}\frac{\Phi^{*(n)}(\bar{Y}^*)}{n!}n\sum_{j=1}^{n}\binom{n-1}{j-1}E_{a^*}(Y^{-k} - \bar{Y}^{-k*})^{n-1-(j-1)}(y^k - \bar{y}^{k*}))^{j-1}.$$

Setting $j' = j - 1$ in the second sum in (26) gives

$$\sum_{j'=0}^{n-1}\binom{n-1}{j'}E_{a^*}(Y^{-k} - \bar{Y}^{-k*})^{n-1-j'}(y^k - \bar{y}^{k*})^{j'} = E_{a^*}(Y - \bar{Y})^{n-1}$$

so that (26) reduces to

$$\Phi^{*\prime}(\bar{Y}^*) + \sum_{n=2}^{\infty}\frac{\Phi^{*(n)}(\bar{Y}^*)}{(n-1)!}E_{a^*}(Y - \bar{Y})^{n-1} = E_{a^*}(\Phi^{*\prime}(Y)) = \frac{1}{1+r^*}.$$

Now consider the first term in (25) consisting of the sum of the comoments. For $n = 1$ the term is

$$\mathrm{como}^{(0,1)}(Y^{-k}, y^k; a_k, a_{-k}^*) = E_{(a_k,a_{-k}^*)}(y^k - E_{a_k}(y^k)) = 0$$

so that this term can be omitted, the summation beginning at $n = 2$. Thus the index can be shifted from n to $n+1$ to begin with $n = 1$ and, since $c^*_{n+1} = \frac{\Phi^{*(n+1)}(\bar{Y}^*)}{(n)!}$, (25) can be written as $\frac{\mathrm{d}}{\mathrm{d}a_k}\tilde{V}(a_k) \leq 0$ where $\tilde{V}(a_k)$ is the criterion in Proposition 3. ∎

Proposition 1(ii) tells us that in order to obtain a Pareto optimum in the $\mathcal{P}$ model each firm must choose its investment a_k to maximize the expected value of its contribution to social welfare $V(a_k) = E_{(a_k, a^*_{-k})}\Phi^*(Y) - a_k$, which, in particular, implies that the firm must "know" the social welfare function Φ^*. This looks like a very demanding requirement, since, in essence, it requires that firms know the utility functions $(u^i)_{i\in I}$ and the distribution of income summarized by the coefficients $(\alpha^*_i)_{i\in I}$. Even though the model assumes that there are sufficiently rich financial markets, in the representation V of a firm's criterion, the informational role of prices—by which they convey the requisite information to each firm to make its optimal investment decision—seems to be lost. The comoment representation $\tilde{V}$ of the firm's criterion gives a simpler way of expressing the criterion when assumption TS holds, exhibiting the market information that can be used by a firm to infer the function Φ^* and make its (socially) optimal choice of investment: In addition to the probability distribution $\rho(a_k, a_{-k})$ and the investment decisions a^*_{-k} of the other firms, firm k needs to know the derivatives of the social welfare function Φ^* at the mean aggregate output, namely, the pricing coefficients $c^*_{n+1} = \frac{1}{n!}\Phi^{*(n+1)}(\bar{Y}^*)$. If the risks are not too high and a good approximation for both the security prices and the firms' criterion can be obtained by keeping only a finite number of terms, then the needed coefficients c^*_{n+1} can be deduced (by regression) from the security prices using (11). In this way "markets" once again provide firms with the requisite information. Both the criteria V and $\tilde{V}$ require that firms know enough about Φ^* in an appropriate neighborhood of the mean aggregate output $\bar{Y}^*$. The real advantage of the comoment representation $\tilde{V}$ is that it provides a simple way of obtaining an approximation of Φ^* in a neighborhood of $\bar{Y}^*$ by recovering the value of the derivatives at $\bar{Y}^*$ from the prices; a truncated Taylor series expansion then provides an approximate value of $\Phi^*(Y)$.

6. CONCLUSION

The striking feature of the two criteria, $\tilde{M}$ for the $\mathcal{SN}$ model and $\tilde{V}$ for the $\mathcal{P}$ model, is that they are the same function of the comoments, the only difference being the way a firm's investment influences the function—through quantities in the $\mathcal{SN}$ model (in which probabilities are fixed), and through probabilities in the $\mathcal{P}$ model (in which outcomes are fixed). This is reassuring, since it means that the theoretical prescription for the way corporate firms should choose their investment to maximize social welfare is in an important sense independent of the model chosen to represent the stochastic nature of the economy and the associated financial markets, provided optimal risk sharing is attainable.

Another way of understanding why the same comoment criterion emerges from the two alternative ways of modeling uncertainty is to note that in both models a firm's optimal investment maximizes its contribution to expected social welfare. In the $\mathcal{P}$ model, this is immediate, since it is precisely what the V criterion (8) requires. In the $\mathcal{SN}$ model, a firm chooses its investment to maximize its market value

$$\pi^* y^k - a_k = E(\Phi^{*\prime}(Y^*)y^k) - a_k, \tag{27}$$

and, if we simplify the firm's technology so that its output is directly a function $(y^k_s(a_k))_{s\in S}$ of its investment, then the FOC for maximizing (27) is

$$E\left(\Phi^{*\prime}(Y^*)y^{k\prime}(a^*_k)\right) - 1 = 0.$$

But this is also the FOC for maximizing the firm's contribution to expected social welfare

$$E\left(\Phi^*(Y^{-k*} + y^k(a_k))\right) - a_k.$$

Thus since in the two models firm k's optimal investment maximizes the same social welfare function—albeit by different channels—the comoment criteria $\tilde{M}$ and $\tilde{V}$ end up being the same expression.[11]

The comoment criterion thus seems to be a robust criterion but is less simple to express than "market value maximization." The positive side, however, is that it brings out more clearly the risk characteristics that a firm should take into account—expected profit, its variability, its upside potential and downside risks, and so forth, its comovements with the profits of other firms—in evaluating its investment decision. Furthermore the weights that a firm should attribute to the trade-offs between the moments and comoments of different orders can readily be obtained from the market.

For both the $\mathcal{SN}$ and the $\mathcal{P}$ models we have chosen a representation for the firms' technologies that leads to a simple derivation of the comoment criterion—a transformation function T_k for each firm in the $\mathcal{SN}$ model and a joint probability distribution $\rho(a)$ for the firms' outcomes in the $\mathcal{P}$ model. The results can be extended to other representations of the firms' technologies. For the $\mathcal{SN}$ model, even if "states of nature" or primitive causes that influence a firm's profit outcomes could be known, it is unlikely that a firm could vary its production in each state independently, as implied by the increasing differentiable transformation function T_k. The proof of Proposition 2 (and the objective $\tilde{M}$) does not, however, depend on the number of constraints that limit the production possibilities of firm k and would go through if, for example, the firm could invest in several projects with fixed risk characteristics or at the extreme if its technology was described by a production function $y^k(a_k) = (y_s^k(a_k))_{s\in S}$.

For the $\mathcal{P}$ model, we have assumed that each firm chooses a single parameter a_k that can be thought of as a scale parameter, but the possibility of a choice of technique has not been taken into account. A more developed model in which NE is satisfied can be obtained by assuming that the probability distribution of each firm's output depends on a vector of parameters ν^k chosen by the firm (choice of technique) and on a vector of exogenous shocks $\gamma = (\gamma_1, \ldots, \gamma_m)$, which can be either economy wide or sectoral and can affect firms or subgroups of firms; conditional on the value of the exogenous shocks, the firms' probability distributions are independent. The choice of parameters ν^k that affect the expected value, variance, and, more generally, the moments of the firms' outputs given γ, has associated with it a cost $a_k = C_k(\nu^k)$, which is incurred at date 0. Given the choice of parameters $\nu = (\nu^1, \ldots, \nu^K)$ by each of the firms, the joint probability distribution of firms' outcomes is given by

$$\rho_s(\nu) = \int \rho^1_{s_1}(\nu^1 | \gamma) \ldots \rho^K_{s_K}(\nu^K | \gamma) \mathrm{d}H(\gamma),$$

where $H(\cdot)$ is the distribution function for the economy-wide and sectoral shocks γ. For a vector of choices $(\nu^{k*})_{k\in K}$ to be optimal, each firm $k \in K$ must choose ν^{k*} that maximizes

$$E_{(\nu^k,\, \nu^{-k*})}\Phi(Y) - C_k(\nu^k),$$

[11] The proof of Proposition 3 could probably be adapted to cover the two models at the same time. However, since most readers are likely to be more familiar with the $\mathcal{SN}$ model than the $\mathcal{P}$ model and the market-value criterion is well accepted in the economics and finance literature, a separate treatment of the two models seems clearer and more helpful.

and the same reasoning as in Proposition 3 leads to the criterion

$$\tilde{V}(\nu^k) = \frac{E_{\nu^k}(y^k)}{1+r^*} + \sum_{n=1}^{\infty} \frac{c^*_{n+1}}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} \text{como}^{(n+1-j,j)}(Y^{-k}, y^k; \nu^k, \nu^{-k*}) - C_k(\nu^k).$$

The analysis of this article depends strongly on the assumption that all agents in the economy have access to a sufficiently rich array of financial markets to permit an optimal allocation of risk bearing. It might be argued that the last 30 years of financial innovation (derivative securities and the like) have greatly enriched the risk sharing opportunities offered by the financial markets so that the idealized assumption of complete spanning may be useful as a first approximation. However, for reasons of moral hazard and adverse selection, some agents may not have complete access to these markets. In each corporation there are stakeholders who cannot fully diversify their risks—managers and workers for reasons of moral hazard or large shareholders for signaling reasons. It would be interesting to extend the analysis of this article to take into account this limited access to risk diversification by some stakeholders of the corporations. We conjecture that in this case firms should maximize a criterion of a similar nature to the one studied in this article, but with larger coefficients on the moments involving the firms' idiosyncratic risks, reflecting the increased importance of these risks for social welfare. Whether a simple and intuitive formula can be found in the case of imperfect diversification of some of the firms' stakeholders is left as a topic for future research.

APPENDIX

A. *Derivatives of the Social Welfare Function.* Consider the function $\Phi(\eta)$ defined in (4)

$$\Phi(\eta) = \max \left\{ \sum_{i \in I} \alpha_i u_1^i(\xi_i) \;\middle|\; \xi_i \in \mathbb{R}_+,\ \sum_{i \in I} \xi_i = \eta \right\},$$

where u_1^i is increasing, strictly concave, and smooth and satisfies $\lim u_1^{i\prime}(\xi_i) = \infty$ if $\xi_i \to 0$. The star symbol in (4), which refers to the particular equilibrium where the function is evaluated, is omitted for simplicity.

The solution $(\xi_i(\eta))_{i \in I}$ of the maximum problem and the Lagrange multiplier $\lambda(\eta)$ associated with the constraint are defined by the system of equations

(A.1) $$\alpha_i u_1^{i\prime}(\xi_i) = \lambda, \quad \forall\, i \in I,$$

(A.2) $$\sum_{i \in I} \xi_i = \eta \implies \sum_{i \in I} \xi_i' = 1, \quad \sum_{i \in I} \xi_i^{(n)} = 0,\ \forall n \geq 2,$$

where, to simplify, we omit the argument η of the functions $\xi_i(\eta)$ and $\lambda(\eta)$, adopting this simplified notation for the calculation of the derivatives of Φ. The envelope theorem implies that $\Phi' = \lambda$. Thus $\Phi^{(n)} = \lambda^{(n-1)}$. Differentiating (A.1) gives

(A.3) $$\lambda' = \alpha_i u^{i\prime\prime}(\xi_i)\, \xi_i', \quad \forall\, i \in I.$$

Multiplying by ξ_i' for all i and summing leads, in view of (A.2), to

$$\Phi'' = \sum_{i \in I} \alpha_i u^{i\prime\prime}(\xi_i)(\xi_i')^2.$$

Differentiating (A.3) leads to

$$\text{(A.4)} \qquad \lambda'' = \alpha_i\big(u_1^{i(3)}(\xi_i)(\xi_i')^2 + u^{i''}(\xi_i)\xi_i''\big), \quad \forall\, i \in I.$$

Multiplying by ξ_i' for all i and summing leads, in view of (A.2) and (A.3), to

$$\lambda'' = \sum_{i\in I}\alpha_i\big(u_1^{i(3)}(\xi_i)(\xi_i')^3\big) + \lambda'\sum_{i\in I}\xi_i''$$

and, since $\sum_{i\in I}\xi_i'' = 0$,

$$\Phi^{(3)} = \sum_{i\in I}\alpha_i u_1^{i(3)}(\xi_i)(\xi_i')^3.$$

Differentiating (A.4) leads to

$$\lambda^{(3)} = \alpha_i\big(u_1^{i(4)}(\xi_i)(\xi_i')^3 + 3u_1^{i(3)}(\xi_i)\,\xi_i'\,\xi_i'' + u^{i''}(\xi_i)\xi_i^{(3)}\big), \quad \forall\, i \in I.$$

Multiplying by ξ_i' for all i and summing leads, in view of (A.2) and (A.3), to

$$\lambda^{(3)} = \sum_{i\in I}\alpha_i\big(u_1^{i(4)}(\xi_i)(\xi_i')^4 + 3u_1^{i(3)}(\xi_i)\,(\xi_i')^2\,\xi_i''\big) + \lambda'\sum_{i\in I}\xi_i^{(3)}.$$

Since the last term vanishes.

$$\Phi^{(4)} = \sum_{i\in I}\alpha_i\big(u_1^{i(4)}(\xi_i)(\xi_i')^4 + 3u_1^{i(3)}(\xi_i)\,(\xi_i')^2\,\xi_i''\big).$$

Using (A.4), $\Phi^{(4)}$ can be expressed equivalently as

$$\Phi^{(4)} = \sum_{i\in I}\alpha_i\big(u_1^{i(4)}(\xi_i)(\xi_i')^4 - 3u^{i''}(\xi_i)(\xi_i'')^2\big).$$

If the signs of the derivatives of u_1^i alternate: $u_1^{i'} > 0, u_1^{i''} < 0, u_1^{i(3)} > 0, u_1^{i(4)} < 0$, then the sign of $\Phi^{(4)}$ is ambiguous unless $\xi_i'' = 0$, for all i.

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