

Term structure and forward guidance as instruments of monetary policy

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Abstract This paper studies a simple monetary model with a Ricardian fiscal policy in which equilibria are indeterminate if monetary policy consists solely of a rule for fixing the short-term interest rate. We introduce explicitly into the model the agents' expectations of inflation which create the indeterminacy and show that there are two types of policies—a term structure rule or a forward guidance rule for the short rate—which lead to determinacy. The first consists in fixing the interest rates on a family of bonds of different maturities as function of realized inflation; the second consists in fixing the short-term interest rate and the expected values of the short-term interest rate for a sequence of periods into the future as a function of realized inflation. If the monetary authority chooses an inflation process that satisfies conditions derived in the paper and applies one of these rules, it anchors agents' expectations to this process, in the sense that it is the unique inflation process compatible with equilibrium when the interest rates or expected future values of the short rate are those specified by the term structure or forward guidance rule.

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1 Introduction

The possible indeterminacy of inflation expectations in forward-looking rational expectations models over an open-ended future when a monetary authority uses an interest rate rule was first raised by [Sargent and Wallace \(1975\)](#). The price level today depends on agents' consumption-savings decisions, and these in turn depend on their anticipations of the price level in the future: this induces a forward-looking dynamics which, if not tied down by a condition at infinity, has a continuum of possible solutions. Whether or not a transversality condition ties down the equilibrium depends on the assumption on fiscal policy. If the fiscal policy always adapts itself to the level of the government's debt to ensure that the debt does not grow faster than the interest rate—a policy which [Sargent \(1982\)](#) referred to as a Ricardian policy—then the transversality condition does not tie down a unique path since every path automatically satisfies the transversality condition, and there is a continuum of equilibria.

An extensive literature subsequently emerged¹, which studies indeterminacy of equilibrium models with Ricardian fiscal policy. Much of the literature is based on a local analysis around a steady state of an underlying nonlinear system: such an analysis only gives a valid approximation to the nonlinear system for paths that stay in a neighborhood of the steady state. In standard New Keynesian models, an active monetary policy, by which the short-term nominal interest rate is raised by more than the increase in inflation, leads to a unique path of the linearized system which stays close to the steady state, and this approximate equilibrium is selected as the basis for policy analysis (see [Woodford 2003](#)). However, while using a local analysis with active monetary policy selects an equilibrium, it does not eliminate the existence of other equilibria.²

It thus seems worthwhile to explore alternative approaches by which monetary policy can lead to determinacy of equilibrium. The anticipatory mechanism creating multiple equilibria—namely that the consumption-savings decision today that creates current inflation depends on agents' expectations of inflation in the future—makes clear that the many different equilibria arise from the many different self-fulfilling beliefs regarding future inflation. Our approach consists in modeling explicitly the expectations of inflation which can be self-fulfilling and asks whether a monetary authority can determine a unique equilibrium by choosing a specific expectations process and, by a suitable choice of its monetary policy instruments or procedures, can

¹ For a non-exhaustive list of references, see [McCallum \(1981, 1983, 2003\)](#), [Leeper \(1991\)](#), [King \(2000\)](#), [Woodford \(2003\)](#), [Walsh \(2003\)](#).

² A series of papers ([Schmitt-Grohé and Uribe 2000](#); [Benhabib et al. 2001a,b](#)) have pointed out that the local determinacy result with an active monetary policy is sensitive to the way preferences and technology are modeled and that the nonlinear system of equations describing equilibrium can give rise to a continuum of equilibria although there is local determinacy around a particular steady state.

make this process the only possible expectations process compatible with equilibrium: when this is possible, we say that the monetary authority can *anchor* agents' expectations of inflation. If the sole instrument of the monetary authority is the short-term nominal interest rate, then it can only tie down the mean of the probability distribution of the inflation rates next period. Since there are many probability distributions with the same mean, agents' expectations of inflation are indeterminate, and to tie down the full probability distribution, more instruments are required.

We study two types of policy instruments or procedures for anchoring agents' expectations. The first consists in extending the traditional policy of fixing the short-term interest rate to a policy of fixing the interest rates (yields to maturity) on government bonds of several maturities, which we call a *generalized interest rate rule* or a *term structure rule*. Analyzing the expectations processes that can be anchored by such a rule, we show that it is not possible to anchor expectations that inflation will always be at target or will return immediately to target if there is a deviation, even if the real side is deterministic. More generally i.i.d. expectations cannot be anchored. As [McCallum \(1981\)](#) pointed out, reducing indeterminacy of equilibrium requires using feedback rules. The interest rate rules compatible with i.i.d. expectations are constant and hence do not provide the requisite feedback. Thus, expectations must vary systematically with inflation which, as we shall see, amounts to permitting some permanence in the inflation expectations process.

The second type of policy consists of a forward guidance rule for the future short-term interest rate: such a rule associates with each possible current inflation rate, the short-term interest rate, and the future short-term interest rate which is expected to prevail for a sequence of T periods into the future: we call this a *forward guidance rule* or an *expected future interest rate rule*. From a mathematical point of view, this type of rule is approximately equivalent to a term structure rule, since modulo a term premium, long-term interest rates are averages of expected future short-term rates.³ Analyzing expectations that can be anchored by such a forward guidance rule leads to essentially the same conclusions as those obtained with a term structure rule.

Interestingly, the above two approaches, which are naturally suggested by the theoretical model, are closely related to recent innovations in monetary policy by the Federal Reserve and other central banks: quantitative easing, which seeks to influence the long-term bond prices and forward guidance on (communication of) the expected path of the future short-term policy rate ([Bernanke 2011](#)).

An important motivation for the present paper was the analysis of [Nakajima and Polemarchakis \(2005\)](#) (NP), which analyzes the indeterminacy of monetary equilibrium with a Ricardian fiscal policy and a short-term interest rate rule for the monetary authority, using the methods of general equilibrium theory. As in equilibrium theory, they count the "degree of indeterminacy" of equilibrium.⁴ [Adao et al. \(2010\)](#) noted

³ The importance of the mutual dependence between the long-term interest rates and agents' expectations of future short-term rates has been emphasized by [Goodfriend \(1991, 1993, 1998\)](#). Goodfriend notes that the term structure can be used by the monetary authority to discover the private sector's expectations of future inflation and future short-term rates. In essence, we reverse this logic and assume that the monetary authority fixes either the term structure or the future expected short-term rates to anchor agents' expectations.

⁴ The method was introduced by [Balasko and Cass \(1989\)](#) and [Geanakoplos and Mas-Colell \(1989\)](#).

that in the NP model with S exogenous “states of nature” at each date, fixing the prices of bonds of S maturities could determine the equilibrium. Our model differs from the model in these two papers in the way uncertainty is modeled. Instead of taking as primitive a set of states of nature with fixed probabilities on which agents base their actions and the monetary authority bases its policy,⁵ we take as primitive the possible inflation rates which can be realized and assume that the monetary authority bases its policy and the agents base their actions on these observable inflation rates. The probability distribution on the inflation rates is the endogenous variable determined in equilibrium. In spirit, our model is an endogenous probability version of a “sunspot” model, which is alternative to the model of [Cass and Shell \(1983\)](#) which models sunspots as primitive states of nature. Our approach has the advantage that it allows monetary policy to be made a function of a simple observable variable—the realized inflation—rather than a function of the myriad contingencies both fundamental and “sunspot” which may serve to explain the realized inflation.

The paper is organized as follows. Section 2 presents the simplest deterministic economy with a cash-in-advance constraint and studies the determinacy of equilibrium, showing uniqueness with a non-Ricardian fiscal policy and one degree of indeterminacy with a Ricardian policy. Section 3 shows that with a Ricardian policy, the degree of indeterminacy increases when stochastic expectations are taken into account. Introducing an explicit model of expectations of inflation, we give conditions under which a term structure rule leads to determinacy. Section 4 studies the alternative monetary policy—a forward guidance rule for the short rate—which, under appropriate conditions, also leads to determinacy. Section 5 shows how the analysis of the previous sections can be extended to a production economy subject to real shocks, and in Sect. 6, we show how the analysis can be applied to model a policy of inflation targeting when monetary policy consists either of a term structure rule or a forward guidance rule for the short rate. Section 7 concludes.

2 Deterministic exchange economy

We begin with the simplest model of an exchange economy with a representative agent and a monetary–fiscal authority financing an exogenously given debt inherited from the past and examine whether the monetary authority can tie down the price level, i.e., the purchasing power of money. Although in an exchange economy with a representative agent nominal variables do not have a real effect, as is standard in monetary theory we study the determinacy of equilibrium in the exchange model and then check that the properties obtained extend to more general models in which the nominal interest rate and /or inflation has a real effect on output. Consider therefore a simple deterministic exchange economy with a (composite) good and a representative agent. The agent has a constant endowment stream $(e_0, e_1, \dots, e_t, \dots) = (e, e, \dots, e, \dots)$ and additively separable preferences

⁵ For a discussion of the conceptual difficulties raised by seeking to make monetary policy a function of states of nature see [Drèze and Polemarchakis \(2001\)](#).

$$U(c_0, c_1, \dots, c_t, \dots) = \sum_{t=0}^{\infty} \delta^t u(c_t), \tag{1}$$

over consumption streams $(c_0, c_1, \dots, c_t, \dots)$, with $c_t \in \mathbb{R}_+$. Money must be used to buy consumption c_t , with the usual timing of the cash-in-advance (CIA) model: an agent cannot directly consume his endowment but must use money to buy his consumption from another agent: the money obtained from the sale of his endowment is obtained after the opportunity to purchase consumption goods and so must be carried into the next period. There is a monetary authority which can increase or decrease the amount of money in circulation in the private sector by buying or issuing one-period nominal bonds; there is also a fiscal authority which imposes taxes to be paid in money (or makes transfers to the private sector). For simplicity we omit government expenses and assume that the government has a debt at date 0 to the private sector inherited from the past. The evolution of the government debt as a function of the monetary and fiscal (tax) policy is what distinguishes a Ricardian from a non-Ricardian policy.

The timing of the transactions (which allows us to follow how money is exchanged for goods and assets) is such that financial markets open at the beginning of each period and agents pay taxes, then agents buy goods with their money balances and finally at the end of the period receive money from the sale of their endowment. This gives them money balances which they transfer to the next period. Let z_t denote the amount of the one-period government bond purchased by the representative agent in period t and let $q_t = \frac{1}{1+r_t}$ denote its price, where r_t is the nominal interest rate in period t . The agent pays θ_t in taxes (if $\theta_t > 0$) or receives a transfer (if $\theta_t < 0$). Let p_t denote the dollar price of one unit of the good in period t , then $\tilde{m}_t = p_t c_t$ units of money must be kept to purchase the amount c_t of the good for consumption in period t . The money balances $m_t = p_t e$ earned from the sale of the endowment in period t must then be carried over into the next period $t + 1$. Focusing on a setting where the nominal interest rate is always positive, the agent’s transactions and money holdings in period t must satisfy

$$\begin{cases} \tilde{m}_t + \theta_t + q_t z_t = m_{t-1} + z_{t-1}, & t = 0, 1, \dots \\ p_t c_t = \tilde{m}_t \\ m_t = p_t e \end{cases}$$

which can be summarized as

$$p_t c_t + \theta_t + q_t z_t = p_{t-1} e + z_{t-1}, \quad t = 0, 1, \dots, \tag{2}$$

with (p_{-1}, z_{-1}) or equivalently (m_{-1}, z_{-1}) being exogenously given. The agent is to make a sequence of consumption–portfolio choices $(c_t, z_t)_{t \geq 0}$ which maximizes (1) subject to (2). The date t budget constraint induces a multiplier λ_t , which is the marginal utility at date 0 of (a promise to deliver) one dollar at date t . The necessary and sufficient conditions for a consumption–portfolio sequence to maximize (1) subject to (2) are given by

$$\delta^t u'(c_t) = \lambda_t p_t, \quad t = 0, 1, \dots \quad (3)$$

$$\lambda_t q_t = \lambda_{t+1}, \quad t = 0, 1, \dots \quad (4)$$

$$\lim_{T \rightarrow \infty} \left(\frac{\lambda_T}{\lambda_0} \right) q_T z_T = 0, \quad (5)$$

Equation (3) is the FOC for c_t and defines λ_t ; (4) is the FOC for z_t expressing equality of the marginal cost and marginal benefit of an additional unit of the bond; (5) expresses the transversality condition—an asymptotic property of the agent's portfolio asserting that the agent does not allow himself to be a lender, nor seeks to be a borrower, at infinity. For the government, we do not assign a specific objective function, but rather focus on the feasibility of monetary–fiscal policies and their consequences for the determinacy of equilibrium. In period t , the government chooses the price of the bond q_t , the taxes θ_t , the amount Z_t of the bond to issue, and the quantity of money M_t subject to its budget equation

$$M_t - M_{t-1} + \theta_t + q_t Z_t = Z_{t-1}, \quad t = 0, 1, \dots,$$

with $M_{-1} + Z_{-1}$ denoting its initial liabilities, where $Z_{-1} = z_{-1}$ and $M_{-1} = m_{-1}$. Thus the government can increase the money supply, use taxes or issue debt to pay off the debt Z_{t-1} inherited from $t - 1$. Equilibrium on the goods market, bond, and money markets requires

$$c_t = e, \quad Z_t = z_t, \quad M_t = m_t, \quad t = 0, 1, \dots$$

Since the monetary authority that fixes the interest rate must accommodate the private sector's demand for money and bonds, the equilibrium conditions $Z_t = z_t$ and $M_t = m_t$ are automatically satisfied.

Equations (3) and (4) imply that at equilibrium

$$q_t = \frac{\delta^{t+1} u'(c_{t+1})}{p_{t+1}} \frac{p_t}{\delta^t u'(c_t)} = \frac{\delta}{1 + \pi_{t+1}},$$

since $c_t = e$, where $\pi_{t+1} = \frac{p_{t+1}}{p_t} - 1$ is the inflation rate in period $t + 1$. Thus the constant consumption stream implies that the Fisher relation takes the form

$$\frac{1}{1 + r_t} = q_t = \frac{\delta}{1 + \pi_{t+1}}. \quad (6)$$

If the monetary authority fixes (q_0, \dots, q_t, \dots) or equivalently the interest rates (r_0, \dots, r_t, \dots) , then the inflation rates $(\pi_1, \dots, \pi_{t+1}, \dots)$ are determined so that $p_1 = (1 + \pi_1)p_0, \dots, p_t = (1 + \pi_1) \dots (1 + \pi_t)p_0$. But is p_0 determined? This depends on the fiscal policy and there are two cases.

Case I Taxes do not adjust to the current nominal government debt. The interpretation usually given is that the rate at which output is taxed is exogenously given. The real tax τ_t is some given proportion β_t of real output $\tau_t = \beta_t e$, $0 < \beta_t < 1$ and the nominal

tax is $\theta_t = p_t \tau_t$ for all $t = 0, 1, \dots$. We discuss this case assuming that $\theta_t = p_t \beta_t e$ for all $t \geq 0$.

If the Fisher relation (6) holds, then the consumption stream $c_t = e$ for all $t \geq 0$ satisfies the FOC (3) and (4) for all sequences of multipliers $(\lambda_t)_{t \geq 0}$ such that

$$u'(e) = \lambda_0 p_0, \quad \lambda_t = \lambda_{t-1} q_{t-1}, \quad t \geq 1.$$

Since as we have seen the equilibrium conditions on money and bonds are automatically satisfied, restrictions additional to the market-clearing conditions $c_t = e$ and the Fisher relation (6) to determine an equilibrium can only come from the budget equations (2) and the transversality condition (5). The Eq. (2) can be used sequentially, beginning with $t = 0$, to determine the equilibrium portfolio $(z_t)_{t \geq 0}$. A particularly simple case, which illustrates the general property, occurs when the monetary authority chooses a constant nominal interest rate

$$r_t = r, \quad t \geq 0, \quad \delta(1 + r) = 1, \tag{7}$$

so that $\pi_{t+1} = 0, t \geq 0$. Thus, the nominal interest rate is equal to the real interest rate and there is no inflation. Multiplying the budget equation (2) at date t by q^t (where $q = \frac{1}{1+r}$) and noting that since there is no inflation $(p_t - p_{t-1})e = 0$ gives the sequence of present values

$$\begin{aligned} p_0 e + \theta_0 + q z_0 &= z_{-1} + m_{-1} \\ q \theta_1 + q^2 z_1 &= q z_0 \\ q^2 \theta_2 + q^3 z_2 &= q^2 z_1 \\ \dots \\ q^t \theta_t + q^{t+1} z_t &= q^t z^{t-1} \\ \dots \end{aligned}$$

which when summed to date T implies

$$\sum_{t=0}^T q^t \theta_t + q^{T+1} z_T = z_{-1} + m_{-1} - p_0 e.$$

Since by (4), $\frac{\lambda_T}{\lambda_0} = q^T = \frac{1}{(1+r)^T}$ and since $\theta_t = p_t \beta_t e = p_0 \beta_t e$

$$\frac{\lambda_T}{\lambda_0} q z_T = \frac{q z_T}{(1+r)^T} = m_{-1} + z_{-1} - p_0 e \left(1 + \sum_{t=0}^T \frac{\beta_t}{(1+r)^t} \right).$$

Thus, the transversality condition (5) implies that the only value of p_0 for which there is an equilibrium is given by $p_0 = \frac{m_{-1} + z_{-1}}{(1 + \sum_{t=0}^{\infty} \frac{\beta_t}{(1+r)^t})e}$: the price level p_0 at date 0 equates the present value of the infinite stream of future real taxes to the total current real liabilities $(m_{-1} + z_{-1})/p_0$ of the government. A tax policy that takes

the infinite stream $(\tau_t)_{t \geq 0}$ of future real taxes as exogenously given is referred to as a *non-Ricardian* tax policy, and this approach to determining the price level is often called the fiscal theory of the price level.⁶

Case II Taxes are adjusted to accommodate the current nominal government debt. The idea is that tax policy prevents government debt from growing indefinitely in the manner of a Ponzi scheme. A simple way of implementing such a tax policy—which captures the essential elements of the general setting—is to set $\theta_t = \alpha Z_{t-1}$, with $0 < \alpha < 1$ so that taxes in period t are used to pay back a fraction α of the debt Z_{t-1} carried into the current period.

To compare the outcome obtained in this case with the outcome in Case I, we continue to assume that the monetary authority sets the nominal interest rate equal to the real interest rate [as in (7)] so that inflation is zero in every period: $\pi_{t+1} = 0$, $t \geq 0$. Once again the equations that determine the equilibrium are the market-clearing equations $c_t = e$ for all t , the Fisher relation (6), to which are added the budget equations (2) and the transversality condition (5). As before the budget equations (2) can be used recursively, beginning with $t = 0$, to calculate the agents' portfolio $(z_t)_{t \geq 0}$. Market clearing and zero inflation imply $(p_t - p_{t-1})e = 0$; this combined with the tax policy $\theta_t = \alpha_t Z_{t-1}$ and $Z_t = z_t$ implies that the date t budget equation (2) for $t \geq 1$ reduces to

$$\frac{z_t}{1+r} + \alpha z_{t-1} = z_{t-1} \iff z_t = (1-\alpha)(1+r)z_{t-1}.$$

Thus $z_T = (1-\alpha)^T(1+r)^T z_0$ so that the transversality condition (5)

$$\lim_{T \rightarrow \infty} \left(\frac{\lambda_T}{\lambda_0} \right) q z_T = \frac{z_T}{(1+r)^{T+1}} = (1-\alpha)^T \frac{z_0}{1+r} \rightarrow 0,$$

it is satisfied for any z_0 . Any pair (p_0, z_0) satisfying the date 0 budget equation

$$p_0 e + q z_0 = M_{-1} + (1-\alpha)Z_{-1},$$

gives an equilibrium. Since the transversality condition is automatically satisfied for any z_0 , the price level p_0 indeterminate. Thus in Case II, equilibrium prices $(p_t)_{t \geq 0} = (p_0, p_0, \dots)$ are *indeterminate*.

When Case II holds—tax policy always ensures that the government's debt does not grow indefinitely like a Ponzi scheme—fiscal policy is said to be *Ricardian*. A Ricardian policy formalizes the idea of a responsible government whose fiscal policy does not let the debt grow without bound, and it provides the reference case for much of monetary theory. This is the case we study in this paper.

⁶ There is an extensive literature in monetary theory discussing assumptions on fiscal and monetary government policies and their consequence for the determination of the price level: see in particular [Sargent and Wallace \(1981\)](#), [Leeper \(1991\)](#), [Sims \(1994\)](#), [Woodford \(1994\)](#), [Woodford \(1995\)](#), and [Cochrane \(2001\)](#). For a general equilibrium model with Ricardian policy and determinate equilibria see [Dubey and Geanakoplos \(2006\)](#). [Bloise et al. \(2005\)](#) show that the determinacy result with a non-Ricardian policy may not be robust to slight modifications of the assumed underlying fiscal policy.

As shown in the two cases studied above, given a sequence of consumption satisfying the market-clearing equations and a sequence of prices, there is always a portfolio strategy which solves the budget equations (2). Since the portfolio of the government is the mirror image of that of the representative agent, a Ricardian policy implies that such a portfolio strategy necessarily satisfies the transversality condition. Thus, the equations that determine the equilibrium reduce to the market-clearing equations for goods, $c_t = e_t$ and the Fisher relation $q_t = \frac{\delta}{1+\pi_{t+1}}$ at each date $t \geq 0$. Given a Ricardian rule, all the other variables can be recovered as a function of p_0 . The property that in a Ricardian equilibrium, the equilibrium equations reduce to the market-clearing equation and the Fisher relation continues to hold when the equilibrium prices are stochastic, as we show in the next section.

3 Stochastic equilibria and term structure rule

In the deterministic model, the difference between a Ricardian and a non-Ricardian policy may appear trivial—after all when the monetary authority sets the nominal interest rate it does tie down inflation [that is the content of the Fisher equation (6)]: all that is missing is the determination of the initial price level p_0 . However, this apparently innocent indeterminacy opens the door to a much more pervasive indeterminacy of agents’ beliefs regarding the future course of inflation, which arises when fiscal policy is Ricardian, but does not arise when the policy is non-Ricardian.

To see this consider the same economy as in the previous section with the same monetary policy which consists of setting the nominal interest rate equal to the real interest rate, $r_t = r = \frac{1}{\delta} - 1$, and the same Ricardian fiscal policy $\theta_t = \alpha Z_{t-1}$.⁷ Suppose, for example, that agents have come to have beliefs at each date t that inflation next period can take one of the three values $\{\pi_l, \pi_m, \pi_h\}$ with probabilities $\{B_l, B_m, B_h\}$, which are consistent with the nominal interest rate rule of the monetary authority

$$\frac{1}{1+r} = \delta \left[\frac{B_l}{1+\pi_l} + \frac{B_m}{1+\pi_m} + \frac{B_h}{1+\pi_h} \right] \iff \frac{B_l}{1+\pi_l} + \frac{B_m}{1+\pi_m} + \frac{B_h}{1+\pi_h} = 1, \tag{8}$$

so that the expected purchasing power of money next period is the same as today. Equation (8) is the Fisher equation for economy in which $c_t = e$ at every date, and agents are uncertain about the purchasing power of money next period. Since the market clearing and the Fisher equation at each date characterize an equilibrium, there exists a stochastic equilibrium in which consumption is constant and inflation follows an i.i.d. process taking the values $\{\pi_l, \pi_m, \pi_h\}$ with probabilities $\{B_l, B_m, B_h\}$ in which agents’ expectations are self-fulfilling. Since any beliefs satisfying (8) generate an equilibrium, there is a continuum of equilibria in which the interest rate rule no longer determines the inflation process.

⁷ It is less immediate to check that the transversality condition is satisfied in the stochastic case (see Magill and Quinzii 2013).

Suppose the central bank recognizes that there are many stochastic equilibria associated with any short-term interest rate rule it may choose, since the short-term interest rate ties down expected inflation next period, but leaves undetermined all other characteristics (moments) of the probability distribution characterizing agents' beliefs. To tie down these probability distributions, i.e., to really anchor agents' expectations, more instruments will be needed. Our goal is to study the stochastic processes of beliefs which can be induced as agents' expectations if the central bank can control a sufficient number of instruments, which, in this section, we take to be the prices of nominal government bonds of different maturities.

We present a framework in which the different stochastic equilibria can be parameterized by probability distributions, or more precisely by Markov processes on the space of possible inflation rates. The idea is that agents' beliefs about the future course of inflation and the price that they are willing to pay for government bonds of different maturities are intimately related, so that if the central bank can control the prices of the bonds, then it can influence agents' expectations when these expectations are a priori indeterminate.

The monetary authority is assumed to choose an inflation process B that it wants to induce agents in the private sector to adopt as their beliefs. To simplify the analysis, the process is taken to be Markovian, so that the beliefs about inflation next period only depend on current realized inflation. The instruments that the monetary authority uses to direct agents' expectations are the prices, or equivalently the yields to maturity (interest rates) on a family of bonds of maturities $1, \dots, T$. Thus, for each inflation rate π , the monetary authority chooses the interest rates $r(\pi) = (r^1(\pi), \dots, r^T(\pi))$ that are equilibrium interest rates when the inflation process is B . It then adopts $r(\pi)$ as its rule for fixing the interest rates when the current inflation is π . The question that we study is the following: When is this generalized interest rate rule compatible with only *one* inflation process (which then necessarily must be B) so that there is a unique self-fulfilling equilibrium associated with the monetary policy $r(\pi)$ and a Ricardian fiscal policy? When there is only one inflation process associated with $r(\pi)$, we say that the beliefs B are *anchored* by the generalized interest rate rule.

To obtain such a uniqueness result, we consider Markov processes on a finite set of inflation rates. Such processes may be considered as discrete approximations of processes with continuous support as shown in Fig. 1. Thus, at each date, agents' beliefs about inflation next period, which are characterized by a probability distribution conditional on current inflation, are approximated by a discrete probability distribution on a finite set $\Pi = \{\pi_1, \dots, \pi_S\}$. A Markov process of beliefs is then represented by a Markov matrix $B = [B_{ss'}]_{s, s' \in S}$ where $B_{ss'}$ is the probability that inflation is (lies in the interval summarized by) $\pi_{s'}$ next period when current inflation is π_s . To express the idea that agents can have beliefs that express genuine uncertainty about next period inflation, we assume $S > 1$.

Figure 1 shows a distribution on an infinite support where the upper and lower end points of the discrete approximation represent the two unbounded tails of the distribution. In a Ricardian framework of the type we consider, where the generalized interest rate rule is typically made to drive expectations toward an inflation target (see Sect. 6), there are no forces at work to cause unbounded inflation or deflation. Thus, there is no loss of generality in restricting agents' expectations to be on a bounded interval.

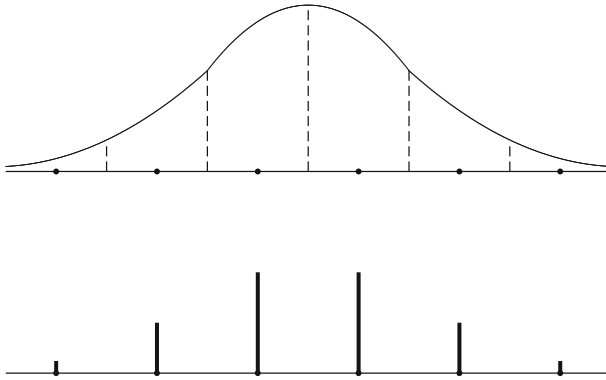


Fig. 1 Approximating a continuous random variable by a discrete random variable: the support of the density function is partitioned into a finite number of intervals and each interval is represented by a value (typically the midpoint, except at the tails where the interval may be infinite) to which is assigned the probability that the random variable lies in the corresponding interval. The finer the partition and the slimmer the tail, the better the approximation

The model that we present is different from the model which is usually adopted to study the role that agents' expectations play in an equilibrium model, namely the sunspot model. Such a model takes as given a probability space (Ω, \mathbb{P}) consisting of the set Ω of “sunspot states” which can occur with exogenously given probabilities $\mathbb{P}(\omega)$, the discretization being made at this level since the set Ω is usually taken to be finite. These sunspot states do not influence the characteristics of the economy—the agents' Bernoulli utility index u , their endowments e , and the technology when there is production—but agents believe that the equilibrium variables (prices and quantities) depend on the sunspot states—and their beliefs turn out to be self-fulfilling. Trying to implement a monetary policy that leads to determinacy of equilibrium in such a setting would be difficult since the monetary policy would need to depend on the sunspot states and such states, which represent all possible causal factors which can influence agents' beliefs, would be numerous and difficult to identify. For this reason, we explore a different approach, expressing agents' beliefs by probability distributions on the possible inflation rates that can be realized: monetary policy can then be made a function of observable realized inflation rate and agents' probability assessments become the endogenous variables that can be influenced by monetary policy.

To study the uniqueness of equilibrium, we extend the model of Sect. 2 to incorporate agents' uncertainty about future inflation, summarized by the Markov matrix B . As before we assume that the representative agent has a constant endowment stream. The utility function is now

$$E^B \sum_{t=0}^{\infty} \delta^t u(c_t),$$

where B is the stochastic process for inflation. The uncertainty about the purchasing power of money gives a role for bonds of different maturities for spanning purposes so we assume that government bonds of maturities $\tau = 1, \dots, T$ can be traded, denoting

by q_t^τ the (random) price at date t of the bond of maturity τ . Let $q_t = (q_t^1, \dots, q_t^T)$ denote the vector of bond prices at date t . We assume that all bonds are zero-coupon bonds so that the payoff at date $t + 1$ of a bond of maturity τ purchased at date t is $q_{t+1}^{\tau-1}$, i.e., the price of a $(\tau - 1)$ -bond next period. Let $\hat{q}_{t+1} = (1, q_{t+1}^1, \dots, q_{t+1}^{T-1})$ denote the vector of payoffs of the bonds at date $t + 1$. The agent now chooses a portfolio $z_t = (z_t^1, \dots, z_t^T)$ of the bonds at each date, and the date t budget equation (2) of Sect. 2 becomes

$$p_t c_t + \theta_t + q_t z_t = p_{t-1} e + \hat{q}_t z_{t-1}. \tag{9}$$

Eliminating the multipliers induced by the budget constraints (9), the first-order conditions for the agent’s choice of consumption and portfolio become

$$q_t^\tau = E_t^B \left(\delta \frac{u'(c_{t+1})}{u'(c_t)} \frac{q_{t+1}^{\tau-1}}{1 + \pi_{t+1}} \right), \quad \tau = 1, \dots, T, \quad t = 0, 1, \dots, \tag{10}$$

with the transversality condition at each date t , $E_t^B \Delta_t^{t'} q_{t'} z_{t'} \rightarrow 0$, as $t' \rightarrow \infty$ where

$$\Delta_t^{t'} = \prod_{\tau=t}^{t'-1} \left(\frac{1}{1 + r_\tau} \right), \tag{11}$$

is the present value at date t of a promise to pay one dollar at date t' .

The government acts in a way similar to that in Sect. 2 except that now instead of choosing the short-term bond price (interest rate) it chooses the prices $q_t = (q_t^1, \dots, q_t^T)$ of a family of bonds of maturities $\tau = 1, \dots, T$ at each date, accommodating the private sector demand by issuing appropriate amounts of money M_t and a portfolio $Z_t = (Z_t^1, \dots, Z_t^T)$ of the bonds. Its overall policy $(q_t, M_t, Z_t, \theta_t)$ must satisfy the budget equation

$$M_t + \theta_t + q_t Z_t = M_{t-1} + \hat{q}_t Z_{t-1}, \quad t = 0, 1, \dots, \tag{12}$$

at each date. In addition, we assume that the fiscal policy is Ricardian in the sense that for all t , $E_t^B \Delta_t^{t'} q_{t'} Z_{t'} \rightarrow 0$ as $t' \rightarrow \infty$, where $\Delta_t^{t'}$ is given by (11). As before the market-clearing conditions are given by

$$c_t = e, \quad Z_t = z_t, \quad M_t = p_t c_t, \quad t = 0, 1, \dots, \tag{13}$$

where the latter two equations express the fact that the government’s issues of money and bonds must accommodate private sector demand. When (13) is satisfied, the representative agent’s budget constraint is the mirror image of that of the government, and since the transversality condition is automatically satisfied for the government, it also holds for the representative agent. Thus, the equations that determine an equilibrium reduce to the market-clearing equation for the good $c_t = e$ for all $t \geq 0$, and the FOC’s for the bonds (10).

If the monetary authority is to implement a given Markov process B for inflation by anchoring agents' expectations to B , then it must set the prices of the bonds so as to be commensurate with the representative agent's FOC's for the optimal choice of the portfolio of bonds of maturities $\tau = 1, \dots, T$, for each current inflation rate $s = 1, \dots, S$

$$q_s^\tau = \delta \sum_{s'=1}^S \frac{B_{ss'}}{1 + \pi_{s'}} q_{s'}^{\tau-1}, \quad s \in S, \quad \tau = 1, \dots, T, \tag{14}$$

where we have used the fact that in equilibrium $c_t = e$ for all t so that pricing is risk neutral. If the bond prices satisfy (14), we say that they are compatible with the inflation process B . Note that fixing the price of a zero-coupon bond of maturity τ is equivalent to fixing the τ -period interest rate (yield to maturity) since $q_s^\tau = 1/(1 + r_s^\tau)^\tau$, so that a generalized interest rate rule that consists in choosing the interest rates as functions of realized inflation can equivalently be described as a bond pricing rule for choosing bond prices as functions of current inflation, and the latter is often more convenient in view of (14).

Suppose the monetary authority seeks to implement the Markov inflation process B and uses equations (14) to determine its bond pricing rule. Consider all the rational expectations equilibria, which can be generated by this bond pricing rule. If there is another Markov matrix \tilde{B} such that

$$q_s^\tau = \delta \sum_{s'=1}^S \frac{\tilde{B}_{ss'}}{1 + \pi_{s'}} q_{s'}^{\tau-1}, \quad s = 1, \dots, S, \quad \tau = 1, \dots, T,$$

then the inflation process \tilde{B} with consumption $c_t = e$ for all t is another rational expectations equilibrium associated with the same bond pricing rule. Thus, to obtain a unique equilibrium⁸ for the bond pricing rule, we need to be sure that the system of Eq. (14) viewed as a system of linear equations in the unknowns $B = [B_{ss'}]_{s,s' \in S}$ with fixed coefficients $q = [q_s^\tau]_{s \in S, \tau=1, \dots, T}$ (determined by the bond pricing rule) has a unique solution.

A necessary condition for this is that (14) consists of $S \times S$ independent equations, which implies that the interest rate rule must involve bonds of $T = S$ maturities.⁹ To characterize the conditions for independence, let us introduce the notation

$$D_{ss'} = \frac{\delta B_{ss'}}{1 + \pi_{s'}}, \tag{15}$$

⁸ Note that since the prices $(p_t)_{\geq 0}$ only enter the equations through the inflation rate π_{t+1} , the price p_0 is necessarily undetermined. Thus when we define uniqueness of equilibrium and mean uniqueness of the inflation process, the general price level still being indeterminate.

⁹ By taking into account the condition $B\mathbf{1} = \mathbf{1}$ which adds S equations that the coefficients $B_{ss'}, s, s' \in S$ must satisfy, one can obtain slightly weaker conditions (requiring one less instrument) than the conditions (R1), (R2), and (R3) of Propositions 1, 3 and 5. However, since (R1), (R2), and (R3) are easier to interpret, we have chosen not to use the Markov condition $B\mathbf{1} = \mathbf{1}$.

where $D_{ss'}$ is the present value in inflation state s of a promise to pay one dollar if inflation is s' next period. Equation (14) gives a simple recursive way of calculating the prices of the bonds as functions of $[D_{ss'}]_{s,s' \in S}$. Since the one-period bond promises to pay one dollar in every state s' that can arise next period

$$q_s^1 = \sum_{s'=1}^S D_{ss'} 1, \quad s \in S. \quad (16)$$

This gives the price of the one-period bond in each possible state s today. Since a two-period bond becomes a one-period bond next period, the payoff of the two-period bond in state s' is $q_{s'}^1$, and the price of the two-period bond when the current state is s is given by

$$q_s^2 = \sum_{s'=1}^S D_{ss'} q_{s'}^1, \quad s \in S. \quad (17)$$

Proceeding recursively in this way, the price of the τ -period bond in state s can be obtained

$$q_s^\tau = \sum_{s'=1}^S D_{ss'} q_{s'}^{\tau-1}, \quad s \in S, \quad (18)$$

once the prices $q_{s'}^{\tau-1}$ of the $(\tau - 1)$ -period bond in each state s' have been calculated.

Writing the pricing Eqs. (16)–(18) in matrix form will quickly reveal the invertibility condition which must be satisfied in order that there is a unique Markov matrix B which satisfies (14). Let D denote the $S \times S$ matrix of present values defined by (15), let $q^\tau = (q_1^\tau, \dots, q_S^\tau)'$ denote the column vector of prices of the τ -period bond for the S possible values of current inflation, and let $\mathbf{1} = (1, \dots, 1)'$ denote the S -vector of sure payment of one dollar in each of the possible inflation states next period (i.e., the payoff stream promised next period by a one-period bond in any state s). Then (16)–(18) can be written as

$$[q^1, q^2, \dots, q^T] = D[\mathbf{1}, q^1, \dots, q^{T-1}]. \quad (19)$$

Viewing (19) as a system of linear equations in $[D_{ss'}]$, the solution is unique if and only if $T = S$ and the matrix $[\mathbf{1}, q^1, \dots, q^{T-1}]$ is invertible, or equivalently the vectors $\mathbf{1}, q^1, \dots, q^{T-1}$ are linearly independent. Since by (15), there is a one-to-one relation between $[B_{ss'}]$ and $[D_{ss'}]$, and hence between B and D , the uniqueness of D is equivalent to the uniqueness of B . Thus, we have shown the following proposition.

Proposition 1 *A Markov matrix B represents expectations which can be anchored if the bond prices of maturities $\tau = 1, \dots, S - 1$ which are compatible with B are such that the matrix of bond payoffs satisfies*

$$\text{rank} [I, q^1, \dots, q^{T-1}] = S. \tag{R1}$$

This result, while established in the simplified setting of a constant endowment-exchange economy, can (as we show in Sect. 5) be extended to much more general settings. To understand the restrictions on an inflation expectations process implied by Proposition 1, we begin by exhibiting some inflation processes which cannot be anchored. The first and simplest is the expectation that, whatever inflation is today, it will revert to the steady state $\pi^* = 0$ next period. To see this, consider the example of Sect. 3 in which the real rate of interest is 2% ($1/\delta = 1.02$) and there are three possible inflation rates $\pi_1 = -1\%$, $\pi_2 = -0\%$, $\pi_3 = 1\%$. Suppose the inflation process B is Markov with $B_{ss'} = 0$ if $s' \neq 2$, $B_{ss'} = 1$ if $s' = 2$, which implies that inflation reverts to 0 after any deviation. Then, the bond prices (interest rates) associated with B are given by

$$q_s^1 = \frac{1}{(1+r_s^1)} = \delta \sum_{s'} \frac{B_{ss'}}{1+\pi_{s'}} = \frac{\delta}{1+\pi^*} = \delta \implies q_s^1 = \delta,$$

i.e., $r_s^1 = 2\%$ for $s = 1, 2, 3$

$$q_s^2 = \frac{1}{(1+r_s^2)^2} = \delta \sum_{s'} \frac{B_{ss'} q_{s'}^1}{1+\pi_{s'}} = \frac{\delta^2}{1+\pi^*} = \delta^2 \implies q_s^2 = \delta^2,$$

i.e., $r_s^2 = 2\%$ for $s = 1, 2, 3$

so that the matrix in (R1) is

$$\begin{bmatrix} 1 & \delta & \delta^2 \\ 1 & \delta & \delta^2 \\ 1 & \delta & \delta^2 \end{bmatrix}$$

which has rank $1 < 3$. More generally, any Markov process on a set of inflation rates π_1, \dots, π_S which is i.i.d. has identical rows (i.e., $B_{ss'}$ independent of s) and hence yields bond prices which are identical across the different inflation states $q_s^T = q_{s'}^T, s' \neq s$, so that the rank of the matrix of bond prices in (R1) is one. Thus, fixing the prices of sufficiently many bonds does not suffice to tie down expectations: the bond prices must also differ sufficiently across the different inflation states for the rank condition to be satisfied.

As these examples show, the property that the bond prices q vary sufficiently across the states for (R1) to be satisfied imposes restrictions on the inflation process B which the monetary authority can seek to anchor, in particular the restriction that B be non-trivially Markov, i.e., that the expectations of future inflation depend on the currently realized inflation. Actually, the next proposition shows that the conditional probabilities of future inflation must differ systematically from one inflation state to another.

Proposition 2 *A necessary condition for an expectations matrix B to yield bond prices satisfying (R1) is that B is invertible.*

Proof We show that the vectors $(\mathbf{1}, q^1, \dots, q^{T-1})$ are in the range of D . For the bond prices, this is immediate since $q^\tau = Dq^{\tau-1}$, $\tau = 1, \dots, T-1$. For the vector $\mathbf{1}$, note that $B\mathbf{1} = \mathbf{1}$ implies $D \text{diag}[\frac{1+\pi}{\delta}]\mathbf{1} = \mathbf{1}$, where $\text{diag}[\frac{1+\pi}{\delta}]$ denotes the diagonal matrix with diagonal elements $\frac{1+\pi}{\delta}$. Thus, $\mathbf{1}$ is the image of the vector $\text{diag}[\frac{1+\pi}{\delta}]\mathbf{1}$. By Proposition 1, the vectors $(\mathbf{1}, q^1, \dots, q^{T-1})$ are linearly independent and since they are in the range of D , D must be of rank S . Since $D = B \text{diag}[\frac{\delta}{1+\pi}]$ and $1 + \pi_s > 0$ for all s , D is invertible if and only if B is invertible. \square

Proposition 2 leads to an intuitive interpretation of the rank condition (R1) on the bond prices. Since the matrix of payoffs $[\mathbf{1}, q^1, \dots, q^{T-1}]$ is invertible and we just showed that this implies that D is invertible, it follows that when (R1) is satisfied the matrix of bond prices $[q^1, \dots, q^T]$ is invertible. In order that the rows

$$\left[q_s^1, q_s^2, \dots, q_s^T \right] = \left[\frac{1}{1+r_s^1}, \frac{1}{(1+r_s^2)^2}, \dots, \frac{1}{(1+r_s^T)^T} \right], \quad s \in S,$$

are linearly independent, the term structure of interest rates $r_s = (r_s^1, r_s^2, \dots, r_s^T)$ must be systematically different when current inflation s varies. Propositions 1 and 2 show that what is needed to eliminate indeterminacy is a monetary policy which provides a feedback rule between inflation and the term structure of interest rates. This result is in essence a generalization of the idea originally introduced by McCallum (1981) that a feedback rule can eliminate the indeterminacy of equilibrium first exhibited by Sargent and Wallace (1975) for Ricardian models. In our model the term structure of interest rates serves as an instrument for conveying to the agents in the private sector the inflation process chosen by the monetary authority. To be sufficiently informative, i.e., to determine a unique process of inflation, the rule must be a true “feedback rule,” which chooses different term structures of interest rates for different realized inflation rates. As we have seen, when the term structure is the same for every current inflation rate, the information is not sufficient.

From the theoretical point of view, our model is a rational expectations model in which the term structure rule is applied by the government, i.e., the model assumes commitment on the part of the monetary authority. The advantage of a term structure rule is that the commitment is immediate to verify—agents can easily check that the interest rates correspond to the announced rule. Although this may seem to make the policy too “inflexible,” we show in Sect. 5, where we introduce production and real shocks, that the rule can be made conditional on the shocks affecting the real side without requiring more instruments than the number of possible inflation rates (i.e., the same number as in this section).

Is such a policy rule of fixing the term structure of interest rates—i.e., the prices of a fixed number of government bonds as a function of the realized inflation—a policy rule which would be feasible for a central bank to implement? Translated literally to the institutional framework in the USA, the model would require changing the way government bonds are traded: currently, the finance department of the government (the Treasury) chooses the quantities of bonds of different maturities to auction and the demand determines the prices in the auction. The model would suggest that the Treasury sells the bonds of different maturities at the prices chosen by the monetary

authority (the Fed), selling the quantities that the “market” wants to buy at these prices. Taxes then adjust to clear the government budget constraint. It might be difficult to change the current operating procedure to implement our proposed policy in this way.

An alternative approach, which respects the separation of the monetary and finance branches of the government, is that the monetary authority participates in the trade on the bond market, influencing the demand and thus the price of the bonds. This is close to what has been attempted in several episodes in the USA and other countries and is now practiced by several central banks (USA, UK) under the label of “quantitative easing,” policies that “alter the scale and the composition of their balance sheets” (Bernanke 2011). These policies are still controversial and are typically used with the goal of influencing real activity through the real interest rate rather than influencing expectations of inflation as in our model. However, since the way a nominal interest rate translates into a real interest rate depends on expectations of inflation, directly or indirectly there is a need to control expectations of inflation.

A concern often expressed is that a monetary policy which consists of fixing the prices of a given number of bonds and accommodating the demand for these bonds by the private sector could become “very expensive” if agents did not behave as anticipated. This concern sometimes takes the form of asking what happens “out of equilibrium.” Note first that our assumption of a Ricardian monetary/fiscal policy implies that the intertemporal (present-value) budget constraint of the government is satisfied on any path $(q_t, M_t, Z_t, \theta_t)$ satisfying the period-by-period government budget equations (12). A Ricardian fiscal rule often considered in the literature is defined by

$$\frac{M_t}{1 + r_t} + \theta_t = \alpha_t(M_{t-1} + q_t Z_{t-1}), \quad \alpha_t < 1,$$

that is, at each date t , seignorage plus taxes (LHS) reimburse a proportion α_t of the liabilities of the government to the private sector (RHS). Thus, if it happened that at date $t - 1$ the liabilities have increased, then at date t the taxes would increase. Even if in practice taxes cannot be adjusted so often or so readily, the fundamental assumption behind a Ricardian monetary/fiscal policy is that an increase in liabilities would be followed by a correction by increasing taxes at some subsequent date.

It is actually unlikely that such an increase in taxes would be needed to cover increased liabilities incurred by implementing the bond pricing policy. Since the bond prices are consistent with one probability belief (namely the one chosen by the monetary authority), they do not offer arbitrage opportunities (by the fundamental theorem of finance). Thus if an agent had beliefs about inflation which differ from the one chosen by the monetary authority, he/she could not make a sure gain by playing (investing) against the government: the agent could at best perceive the possibility of a *gain in expected value*. Since there are no arbitrage opportunities, realizing such an expected gain necessarily involves losses for some realizations. To create problems for the monetary authority, agents anticipating expected gains would have to trade on a large scale and this would mean that they would have to have “deep pockets” to sustain the possibility of large losses. Rather than seeking to maintain their own beliefs against those of the monetary authority, typical Fed watchers are more likely to seek to adapt their beliefs to those implied by the monetary authority’s bond price policy.

In the next section we show that if for institutional or other reasons it is difficult for a central bank to use a term structure policy, there is an alternative policy—which we call a forward guidance rule—which can be used to anchor agents’ expectations of inflation. This policy consists in continuing to use the short-term interest rate and replacing the prices of long-term bonds as policy instruments by communication of the path of the expected value of the future short-term interest rates for a sequence of T periods into the future. To be successful, this policy must be applied in a systematic way and agents in the private sector must believe in the announcements of the central bank.

4 Forward guidance rule

If a monetary policy based on a term structure of interest rates of the kind outlined above has not been used until now on a regular basis, most central banks have felt the need to complement their short-term interest rate policy by a periodic announcement of the path of future expected short-term interest rates. Central banks that practice inflation targeting regularly post “fan charts” with the path of expected values for inflation, the short-term interest rate and output. The Federal Reserve, which follows an informal (recently made official) policy of inflation targeting, has begun to make announcements of the path of the expected future short-term rate and is considering making such announcements a systematic policy. It is instructive to use our model to compare this practice, which [Bernanke \(2011\)](#) has referred to as “forward guidance about the future path of policy rates,” with a monetary policy based on a term structure rule like the one studied in the previous section.

We formalize the policy of announcing the path of the expected future (short-term) interest rate by a rule which associates with each inflation rate a sequence of future expected short-term bond prices, which amounts to using the approximation $E_t(q_{t+\tau}^1) = E_t\left(\frac{1}{1+r_{t+\tau}^1}\right) \approx 1 - E_t(r_{t+\tau}^1)$. A monetary policy based on a rule for *paths of short-term interest rates* is then of the form $(B, [q_s^1, E_s(q_\tau^1), \tau = 1, \dots, T]_{s \in S})$ where q_s^1 is the short-term interest rate (bond price) given current inflation s , $E_s(q_\tau^1)$ is the expected future short-term interest rate τ periods in the future (given s), for a sequence of T periods into the future $\tau = 1, \dots, T$. As before, the vector of short-term bond prices q^1 must be compatible with the expectations B

$$q_s^1 = \delta \sum_{s' \in S} \frac{B_{ss'}}{1 + \pi_{s'}}, \quad s \in S,$$

as well as the expected prices τ periods ahead

$$E_s(q_\tau^1) = \sum_{s' \in S} B_{ss'}^\tau q_{s'}^1, \quad s \in S, \quad \tau = 1, \dots, T,$$

where B^τ denotes the τ th power of B . Suppose the monetary authority has a policy $(B, q^1, (E^\tau)_{\tau=1, \dots, T})$ where $q^1 = (q_s^1)_{s \in S}$ and $E^\tau = (E_s^\tau)_{s \in S}$ with $E_s^\tau = E_s(q_\tau^1)$. We

say that $(q^1, (E^\tau)_{\tau=1,\dots,T})$ anchors B if there is only one Markov matrix B compatible with $(q^1, (E^\tau)_{\tau=1,\dots,T})$. This is equivalent to requiring that the system of equations

$$\begin{aligned} \delta \sum_{s' \in S} \frac{B_{ss'}}{1 + \pi_{s'}} &= q_s^1, \quad s \in S \\ \sum_{s' \in S} B_{ss'}^\tau q_{s'}^1 &= E_s^\tau, \quad s \in S, \quad \tau = 1, \dots, T, \end{aligned} \tag{20}$$

has a unique solution $B = [B_{ss'}]_{s,s' \in S}$. As written, this system of equations is nonlinear in B since the coefficients of the matrix B^τ are polynomials of degree τ in $B_{ss'}$. However, given the form of this system of equations, it can be transformed into an equivalent linear system. Writing the equations in matrix form, i.e., taking all values $s \in S$ simultaneously gives

$$B \frac{\delta}{1 + \pi} = q^1, \quad Bq^1 = E^1, \quad B^2q^1 = E^2, \dots, \quad B^Tq^1 = E^T,$$

where $\frac{\delta}{1 + \pi}$ is the column vector with components $\frac{\delta}{1 + \pi_s}$, $s \in S$. Substituting Bq^1 into the equations for expectations of bond prices 2 periods ahead gives $BE^1 = E^2$; in the same way, substituting B^2q^1 for E^2 gives $BE^2 = E^3$, and by successive substitutions $BE^{\tau-1} = E^\tau$. Thus, the nonlinear system of equations (20) is equivalent to the system of linear equations

$$B \left[\frac{\delta}{1 + \pi}, q^1, E^1, \dots, E^{T-1} \right] = \left[q^1, E^1, \dots, E^T \right] \tag{20'}$$

If the monetary policy is consistent, then B is a solution of this system of equations. For B to be the only solution, (20') must be a system of $S \times S$ equations in which the matrix of coefficients is invertible.

Proposition 3 *The Markov expectations matrix B is uniquely determined by a forward guidance rule (q, E^1, \dots, E^T) for the expected future short-term interest rates if $T = S - 1$ and*

$$\text{rank} \left[\frac{\delta}{1 + \pi}, q^1, E^1, \dots, E^{T-1} \right] = S. \tag{R2}$$

Equation (R2) is slightly different from (R1) in the previous section.¹⁰ As we have seen, i.i.d expectations are never compatible with (R1): however, i.i.d expectations are

¹⁰ A knowledge of the term structure of interest rates is equivalent to the knowledge of the associated family of forward rates. Even with risk neutral pricing as in Sects. 3 and 4, the forward rates differ from the expected value of the corresponding short-term rates, so that a term structure rule and an expected future short-rate rule are not equivalent. To see this, let q_s^1 and q_s^2 be the prices at date t in inflation state s of the bonds maturing, respectively, in 1 and 2 periods. Then

compatible with (R2) in the special case $S = 2$. For then the condition reduces to

$$\det \begin{bmatrix} \frac{\delta}{1+\pi_1} & q_1^1 \\ \frac{\delta}{1+\pi_2} & q_2^1 \end{bmatrix} \neq 0$$

and this condition is satisfied by an i.i.d process for which $q_1^1 = B_1 \frac{\delta}{1+\pi} = q_2^1 = B_2 \frac{\delta}{1+\pi}$ (with B_s denoting row s of B) provided $\pi_1 \neq \pi_2$. The case $S = 2$, or a high/low discrete distribution of inflation rates, seems however too crude to provide a useful approximate model of inflation expectations. As shown in the next proposition, this is the only case where i.i.d expectations satisfy condition (R2).

Proposition 4 *A necessary condition for B to lead to a path of expected future short-term interest rates satisfying (R2) is that rank $B \geq S - 1$.*

Proof The $S - 1$ vectors $q^1 = B \frac{\delta}{1+\pi}$, $E^1 = Bq^1, \dots, E^{S-2} = BE^{S-3}$ are in the range of B and must be linearly independent by (R2). Thus, the range of B must be at least of dimension $S - 1$. □

The rank condition (R2) is a little harder to interpret than (R1). As we saw in the previous section, (R1) implies that B must be invertible and we conjecture that, generically in the inflation rates (π_1, \dots, π_s) , all invertible matrices B will satisfy (R1). Condition (R2) only requires that rank $B \geq S - 1$ and it is straightforward to construct examples of matrices B (e.g., 3×3) with rank $S - 1$ which satisfy (R2) and thus do not satisfy (R1). Interpreting the requirement that the rank of B be at least $S - 1$ and that $\{q^1, E^1, \dots, E^{S-2}\}$ and $\frac{\delta}{1+\pi}$ are not linearly dependent is less immediate. However, the spirit of the conditions in Propositions 1–4 are the same: the rank condition (R2) requires that the probability distributions for inflation next period depend in a systematic way on current inflation and that the expected future path of inflation must be different in the different inflation states in order that at least $S - 1$ rows $\left[\frac{1}{1+\pi_s}, q_s^1, E_s^1, \dots, E_s^T \right]$ be linearly independent.

Footnote 10 continued

$$q_s^2 = \frac{1}{(1+r_s^1)(1+r_s^f)} = E_s \left(\frac{\delta q_{t+1}^1}{1+\pi_{t+1}} \right),$$

where the first equality is the definition of the forward rate r_s^f between periods $t + 1$ and $t + 2$, and the second equality is (14). Since $q_s^1 = E_s \left(\frac{\delta}{1+\pi_{t+1}} \right)$

$$q_s^2 = q_s^1 E_s(q_{t+1}^1) + \text{cov}_s \left(q_{t+1}^1, \frac{\delta}{1+\pi_{t+1}} \right).$$

Knowing (q_s^1, q_s^2) is not equivalent to knowing $(q_s^1, E_s(q_{t+1}^1))$ because of the covariance term. In terms of the forward rate, using the approximation $\frac{1}{1+r} = 1 - r$, the relation can be written as

$$r_s^f = E_s(r_{t+1}^1) - \text{cov}_s \left(q_{t+1}^1, \frac{\delta}{1+\pi_{t+1}} / E_s \left(\frac{\delta}{1+\pi_{t+1}} \right) \right).$$

The most natural property of the Markov expectations that permits the rank conditions (R1) or (R2) to be satisfied is to allow for some degree of permanence in the inflation process. If the CB wants to induce expectations that inflation will return to a target, permanence in the mean-reverting Markov matrix will create inertia, with different paths of expected future inflation, expected future interest rates, and different term structures of interest rates for different realized inflation rates.

5 Extending model to production economy

In this section, we show how the previous analysis can be extended to more realistic models of economies in which goods are produced, the technology is affected by real shocks, and production decisions are affected by the nominal interest rate. To keep the analysis simple, we do not go all the way to a New Keynesian model, which has the most realistic description of the feedback between nominal and real variables, adhering instead to the simpler flexible price model of production introduced by Lucas and Stokey (1987) and used by Schmitt-Grohé and Uribe (2000) and Nakajima and Polemarchakis (2005) to study the indeterminacy of equilibrium when monetary–fiscal policy is Ricardian. This amounts to changing the description of the private sector of the economy in the following way. Instead of a constant endowment e of the good, the representative agent has a constant endowment e of labor which is normalized to $e = 1$. The consumption good is produced from labor with the constant-returns technology $y = aL$, where a is subject to real shocks. Let $G = \{1, \dots, G\}$ denote the possible shocks¹¹ and let $(a_g)_{g \in G}$ denote the resulting productivities of labor. We assume that the shocks to productivity are driven by an exogenously given Markov process with transition matrix $A = [A_{gg'}]_{g, g' \in G}$ where the matrix A is known by all agents in the economy. The assumption of constant returns implies that the structure of the production sector does not matter—however, we make the standard assumption of the cash-in-advance model that an agent cannot transform his/her labor into output but needs to work for a firm and is paid a salary at the end of the period, too late to buy consumption goods within the period. This timing creates a cash-in-advance constraint in the purchase of the good. Apart from this change in the way the consumption good enters the economy (through production rather than as an exogenous endowment), the model is the same as in the previous sections. We focus on the case where monetary policy involves fixing the term structure of interest rates, since this is the reference case of the earlier sections: the analysis can be readily adapted to a policy of announcing the future path of the expected short-term interest rate.

As in Sect. 2, the representative agent’s transactions on the goods and financial markets must satisfy sequential budget equations, which here can be written

$$\begin{aligned}
 p_0 c_0 + \theta_0 + q_0 z_0 &= W_0 \\
 p_t c_t + \theta_t + q_t z_t &= w_{t-1} L_{t-1} + \hat{q}_t z_{t-1}, \quad t = 1, 2, \dots,
 \end{aligned}
 \tag{21}$$

¹¹ For simplicity, we use the same letter for the number and the set of shocks.

where $W_0 = m_1 + \hat{q}_0 z_{-1}$ denotes the agent’s inherited wealth at date 0. At each date the agent gets income $w_{t-1}L_{t-1}$ (money carried over from the previous period) where w_{t-1} denotes the wage rate and L_{t-1} the labor sold in period $t - 1$. The agent buys a portfolio $z_t = (z_t^1, \dots, z_t^T)$ of the bonds of maturities $1, \dots, T$ at prices $q_t = (q_t^1, \dots, q_t^T)$ and receives the income $\hat{q}_t z_{t-1}$ from the portfolio acquired in the previous period, where $\hat{q}_t = (1, q_t^1, \dots, q_t^{T-1})$ denotes the vector of payoffs of the bonds purchased at $t - 1$, which depends on the bond prices at date t .

The agent seeks a sequence of consumption/leisure choices which maximizes

$$E \sum_{t=0}^{\infty} \delta^t u(c_t, \ell_t),$$

subject to being achievable through labor choices $(L_t)_{t \geq 0}$ and portfolio choices z_t satisfying (21). The function u is an increasing, concave, and differentiable function of current consumption c_t and leisure ℓ_t , where $\ell_t = 1 - L_t$. The expectation E is taken with respect to the stochastic process of agents’ beliefs. Monetary policy consists in fixing the prices $q_t = (q_t^1, \dots, q_t^T)$ of the bonds as a function of the currently realized inflation and productivity shock (s, g) . Current inflation is determined by the amount $q_t z_t$ that the agent saves (or borrows) since this determines the amount of money $m_t = m_{t-1} + \hat{q}_t z_{t-1} - q_t z_t - \theta_t$ that the agent lays aside to spend on the goods market. This savings decision in turn depends on the agent’s expectations of future inflation, future taxes, and future wages, and hence of the future shocks to productivity since, with constant returns to scale, the wage is equal to productivity: $w_{t+\tau} = a_{t+\tau}$ for $\tau \geq 1$.

The setting is now substantially richer than in Sect. 3: agents have to form beliefs $\Gamma_{\eta\eta'}$ about the transition probabilities from the current inflation–productivity pair characterized by $\eta = (s, g)$ to the possible inflation–productivity pairs next period $\eta' = (s', g')$, where we retain the assumption that the inflation expectations have a Markov structure. We assume that the productivity shocks are independent of inflation, giving the following structure to agents’ expectations:

Assumption MT (Markov transitions) Given the current inflation–productivity state $\eta = (s, g)$, the agents’ expectations of inflation–productivity next period are given by the joint probability distribution

$$\Gamma_{\eta,\eta'} = B_{ss'}^g A_{gg'}, \quad \eta, \eta' \in S \times G.$$

B^g is the matrix of expectations of the distributions of inflation rates next period, which can depend on the current productivity shock g . These expectations are induced by the monetary authority, which may want to induce differential expectations for different shocks g in order to use the nominal interest rate to counter the effect of the productivity shock.¹² The transition matrix A for real shocks is exogenously given and is independent of the realized inflation.

¹² The dependence of the expectations process, and thus of the interest rate rule, on the productivity shock g is the analogue in this model of the term involving the “output gap” in a Taylor rule.

The role of monetary policy is to anchor agents' expectations of inflation and perhaps partially to undo the effect of the productivity shocks on production through an interest rate policy q . In each period, the government accommodates the agents' demands for bonds and money by issuing $Z_t = (Z_t^1, \dots, Z_t^T)$ bonds of maturities $1, \dots, T$ and setting the money supply M_t in such a way that $Z_t = z_t$ and $M_t = m_t = p_t c_t$. We assume that the fiscal policy is Ricardian, i.e., the choice of taxes θ_t ensures that for any policy $(M_t, Z_t, q_t)_{t \geq 0}$, the present value of the government debt $q_t Z_t$ tends to zero when t tends to infinity.¹³

As in Sect. 2, the characterization of equilibrium can be reduced to two collections of equations

- (1) market clearing on the good and the labor markets
- (2) first-order conditions for the representative agent and the representative firm.

Explicit market-clearing equations for the bonds and money are not needed since the government accommodates the demand for bonds and money. Solving the Eq. (21) from date 0 forward, there always exist portfolios satisfying the budget constraints (21) for any given consumption and tax processes. The Ricardian policy then implies that the transversality condition is satisfied. Thus, an equilibrium is characterized by the equations

- (i) $c_t = a_t L_t, \quad L_t + \ell_t = 1, \quad t \geq 0$
- (ii) $\delta^t u_c(c_t, \ell_t) = \lambda_t p_t, \quad \delta^t u_\ell(c_t, \ell_t) = E_t(\lambda_{t+1} w_t), \quad w_t = a_t, \quad t \geq 0$
- (iii) $\lambda_t q_t^\tau = E_t(\lambda_{t+1} q_{t+1}^{\tau-1}), \quad q_{t+1}^0 = 1, \quad \tau = 1, \dots, T, \quad t \geq 0$

where (ii) and (iii) are the first-order conditions for the representative consumer and firm. Eliminating the multipliers λ_t and solving (i) as $\ell_t = 1 - \frac{c_t}{a_t}$, these equations reduce to

$$(ii)' \quad \frac{u_c\left(c_t, 1 - \frac{c_t}{a_t}\right)}{u_\ell\left(c_t, 1 - \frac{c_t}{a_t}\right)} = \frac{1 + r_t^1}{a_t}, \quad t \geq 0$$

$$(iii)' \quad u_c\left(c_t, 1 - \frac{c_t}{a_t}\right) q_t^\tau = \delta E_t \left(\frac{u_c\left(c_{t+1}, 1 - \frac{c_{t+1}}{a_{t+1}}\right)}{1 + \pi_{t+1}} q_{t+1}^{\tau-1} \right), \quad \tau = 1, \dots, T, \quad t \geq 0$$

¹³ For the purpose of this paper, the exact form of the fiscal policy does not matter, as long it is Ricardian. An example of a Ricardian policy which is often considered (Benhabib et al. 2001a; Nakajima and Polemarchakis 2005) is the following

$$\theta_t = \alpha M_{t-1} \left(1 + \frac{\hat{q}_t Z_{t-1}}{M_{t-1}} \right) - \frac{r_t^1 M_t}{1 + r_t^1}, \quad 0 < \alpha < 1.$$

Since in equilibrium $M_{t-1} = p_{t-1} c_{t-1}$ is the nominal value of aggregate consumption which is equal to aggregate output, each period the government raises revenue equal to a fraction α of nominal aggregate output, increased or reduced by the factor $\frac{\hat{q}_t Z_{t-1}}{M_{t-1}}$ which is the ratio of the debt to GDP. The revenue is the sum of the indirect seignorage tax $\frac{r_t^1 M_t}{1 + r_t^1}$ and the direct tax θ_t levied on the private sector. It is shown in Magill and Quinzii (2013) that in a stochastic environment this fiscal policy ensures that for every subtree starting at every date event of the economy, the present value of the government debt at date t tends to zero when t tends to infinity.

Suppose the monetary authority wants to anchor the inflation expectations $(B^g)_{g \in G}$ in the private sector. The goal of this section is to study which expectations can be anchored, and how many instruments (bond prices) the monetary authority needs to control: a similar analysis could be carried out for a policy of announcing the path of future expected short-term interest rates.

To this end, we consider stationary solutions of the equilibrium equations (ii)' and (iii)', namely $(c, r^1) = (c_\eta, r_\eta^1)_{\eta \in S \times G}$ satisfying

$$(ii)'' \quad \frac{u_c\left(c_\eta, 1 - \frac{c_\eta}{a_g}\right)}{u_\ell\left(c_\eta, 1 - \frac{c_\eta}{a_g}\right)} = \frac{1 + r_\eta^1}{a_g}, \quad \eta \in S \times G$$

$$(iii)'' \quad u_c\left(c_\eta, 1 - \frac{c_\eta}{a_g}\right) \frac{1}{1 + r_\eta^1} = \delta \sum_{\eta'=(s',g') \in S \times G} B_{ss'}^g A_{gg'} \frac{u_c\left(c_{\eta'}, 1 - \frac{c_{\eta'}}{a_{g'}}\right)}{1 + \pi_{s'}}, \quad \eta \in S \times G$$

where we have only written the pricing equation for the short-term bond ($\tau = 1$) in (iii)': the idea is to focus on solving (ii)'' and (iii)'' first: this will give us the short-term bond prices, and from these, we can calculate the long-term bond prices using the remaining equations in (iii)'. The equations (ii)'' define optimal consumption in state η as a function of the short-term interest rate r_η^1 , i.e., $c_\eta(r_\eta^1)$ for each $\eta \in S \times G$. Substituting this expression into (iii)'' gives a system of $S \times G$ nonlinear equations for determining the equilibrium short-term interest rate $r^1 = (r_\eta^1)_{\eta \in S \times G}$ and this interest rate must be nonnegative in every state $r_\eta^1 \geq 0$. This implies that some restrictions must be imposed on the expectations $(B^g)_{g \in G}$. To see why, recall that the Fisher equation can be written approximately as

$$r_\eta^1 = r_\eta^{\text{real}} + E_\eta(\pi_{\eta'}),$$

where η' denotes a state at date $t + 1$. If the shocks to productivity imply that consumption is expected to fall, then the real interest rate can become negative and expectations of inflation must be sufficiently high to compensate for this. In the same way, if there can be deflation and expectations are such that the expectation of inflation on the right side is negative, then it may not be possible to find a nonnegative equilibrium interest rate for such expectations. Conditions for the existence of a nonnegative short-term interest rate $r^1 = (r_\eta^1)_{\eta \in S \times G}$ satisfying (iii)'' are given in Magill and Quinzii (2013). The problem of possible non-existence of equilibrium interest rates satisfying (ii)'' and (iii)'' arises from the presence of the zero lower bound on the nominal interest rate and imposes a restriction on the permissible expectations of inflation, requiring that enough weight be placed on inflation (as opposed to deflation) to ensure that the resulting nominal interest rate is nonnegative.

From now on, we take as given that the pair $[(B^g)_{g \in G}, A]$ is such that (ii)'' and (iii)'' have a solution $(c, r^1) = (c_\eta, r_\eta^1)_{\eta \in S \times G}$ with $r_\eta^1 \geq 0$ for all η . For this solution let

$$\phi_\eta = u_c\left(c_\eta, 1 - \frac{c_\eta}{a_g}\right), \quad \eta \in S \times G,$$

denote the marginal utility of consumption when the current state is $\eta = (s, g)$, and let $q_\eta^1 = \frac{1}{1+r_\eta}$ denote the short-term bond price compatible with the expectations $(B^g)_{g \in G}$. The first-order conditions (iii)' for bond holdings of all the maturities can be written as

$$q_\eta^\tau = \delta \sum_{\eta' \in S \times G} \frac{B_{ss'}^g A_{gg'} \phi_{\eta'}}{1 + \pi_{s'}} \frac{\phi_{\eta'}}{\phi_\eta} q_{\eta'}^{\tau-1}, \quad \eta \in S \times G, \quad \tau = 1, \dots, T. \tag{22}$$

Given the price $q^1 = (q_\eta^1)_{\eta \in S \times G}$ of the one-period bond, (22) permits us to calculate the price $q^2 = (q_\eta^2)_{\eta \in S \times G}$ of the two-period bond and so on until we obtain the T vectors (q^1, q^2, \dots, q^T) of the bond prices across the inflation–productivity states $\eta \in S \times G$, which are compatible with the expectations $(B^g)_{g \in G}$. Having found the equilibrium bond prices compatible with $(B^g)_{g \in G}$, we now ask whether these are the only expectations compatible with these equilibrium prices. We say that $(B^g)_{g \in G}$ is anchored by the term structure (q^1, q^2, \dots, q^T) if $(B^g)_{g \in G}$ is the only family of expectations of inflation which solves (22). Formally, this requires that the system of equations (22) viewed as a system of linear equations in the unknowns $[(B_{ss'}^g)_{s, s' \in S}, g \in G]$, taking (A, π, ϕ, q) as fixed, has a unique solution. Since the expectations involve $G \times S \times S$ unknowns, we need the same number of equations. For each maturity τ , there are $S \times G$ equations which determine the vector $(q_\eta^\tau)_{\eta \in S \times G}$. Thus we need $T = S$ maturities—perhaps surprisingly the same as in the simple exchange economy without productivity shocks.

The key to obtaining the appropriate restrictions on the expectations matrices $(B^g)_{g \in G}$ to obtain uniqueness is to form appropriate averages over the productivity shocks g' next period in (22). Consider first the average marginal utility of income next period with respect to g' , given that the current shock is g , if inflation next period is s'

$$\hat{\phi}_{s'}^g = \sum_{g' \in G} A_{gg'} \phi_{s'g'}, \quad s' \in S, \quad g \in G.$$

Equation (22) can be written as

$$q_{sg}^\tau = \delta \sum_{s' \in S} \frac{B_{ss'}^g}{1 + \pi_{s'}} \sum_{g' \in G} \frac{A_{gg'} \phi_{s'g'}}{\sum_{g' \in G} A_{gg'} \phi_{s'g'}} \frac{\hat{\phi}_{s'}^g}{\phi_{sg}} q_{s'g'}^{\tau-1}. \tag{23}$$

This suggests replacing the objective probabilities $A_{gg'}$ by the risk neutral probabilities (the “equivalent martingale measure”)

$$\rho_{s'}^g(g') = \frac{A_{gg'} \phi_{s'g'}}{\sum_{g' \in G} A_{gg'} \phi_{s'g'}}, \quad g' \in G, \quad s' \in S, \quad g \in G,$$

$\rho_{s'}^g(g')$ is risk neutral since the marginal utility $\phi_{s'g'}$ is incorporated into the probability of the productivity shock g' next period and is conditional on the current shock being

g and on inflation next period being s' . This measure leads naturally to average values of the bond prices with respect to productivity shocks

$$E_{s'}^g(q^\tau) = \sum_{g' \in G} \rho_{s'}^g(g') q_{s'g'}^\tau, \quad s' \in S, \quad g \in G, \quad \tau = 1, \dots, T.$$

The first-order conditions (23) can now be expressed as

$$q_{sg}^\tau = \delta \sum_{s' \in S} \frac{B_{ss'}^g}{1 + \pi_{s'}} \frac{\hat{\phi}_{s'}^g}{\phi_{sg}} E_{s'}^g(q^{\tau-1}), \quad s \in S, \quad g \in G, \quad \tau = 1, \dots, T. \quad (24)$$

Since these are just present-value equations linking the price of a τ -bond to its payoff next period, which is the price of a $\tau - 1$ -bond, we should expect that the equations can be written in a form reminiscent of (19) in Sect. 3. This can be seen by introducing for each $g \in G$ the two diagonal matrices

$$D_1^g = \text{diag} \left(\frac{1}{\phi_{sg}} \right)_{s \in S}, \quad \widehat{D}_2^g = \text{diag} \left(\frac{\delta \widehat{\phi}_{s'}^g}{1 + \pi_{s'}} \right)_{s' \in S},$$

and the present-value matrix

$$\widehat{D}^g = D_1^g B^g \widehat{D}_2^g,$$

where the marginal utilities next period in \widehat{D}_2^g have been averaged over the possible real shocks. The present-value matrix \widehat{D}^g is the generalization of (15) in Sect. 3 to the case where the stochastic discount factor is not constant because the consumption varies with both the nominal interest rate and the productivity shock. For each $g \in G$ and each τ , (24) consists of S equations for determining the prices of the bond of maturity τ in each inflation state s .

Expressing these equations in matrix form for the bonds of maturities $1, \dots, T$ gives the matrix equation

$$\left[q_g^1, q_g^2, \dots, q_g^T \right] = \widehat{D}^g \left[\mathbf{1}, E^g(q^1), \dots, E^g(q^{S-1}) \right], \quad g \in G. \quad (25)$$

Viewing (25) as a system of linear equations in $[\widehat{D}_{ss'}^g]$ for each $g \in G$, the solution is unique if and only if $T = S$ and the matrix $[\mathbf{1}, E^g(q^1), \dots, E^g(q^{S-1})]$ is invertible. Since the matrices D_1^g and \widehat{D}_2^g are diagonal matrices with positive entries, there is a one-to-one relation between \widehat{D}^g and B^g so that uniqueness of \widehat{D}^g for each $g \in G$ implies uniqueness of B^g for each $g \in G$. Thus we have shown

Proposition 5 *The family of Markov matrices $[B^g, g \in G]$ represents inflation expectations, which can be anchored by a term structure policy $[q_g^\tau = (q_{sg}^\tau)_{s \in S}, \tau =$*

$1, \dots, T, g \in G$] if the compatible bond prices, averaged over the productivity shocks, satisfy

$$\text{rank}\left[\mathbf{1}, E^g(q^1), \dots, E^g(q^{S-1})\right] = S, \quad \text{for each } g \in G. \tag{R3}$$

Condition (R3) is essentially the same as the rank condition (R1) for an exchange economy. In each case, bond prices fluctuate because of the endogenous effect of expectations of inflation. Here, however, bond prices also fluctuate because of the exogenous fluctuations in productivity, which lead to fluctuations in the real rate of interest. As in Proposition 1, the matrix $[\mathbf{1}, E^g(q^1), \dots, E^g(q^{S-1})]$ is an $S \times S$ matrix and row s' , $[1, E_{s'}^g(q^1), \dots, E_{s'}^g(q^{S-1})]$ gives the payoffs in period $t + 1$ of the bonds of maturities $1, \dots, S$ purchased in period t . $E_{s'}^g(q^\tau)$ is the average over the real shocks of the price of the τ -bond in inflation state s' . (R3) requires that these average prices vary systematically with s' so that the rows of the matrix $[\mathbf{1}, E^g(q^1), \dots, E^g(q^{S-1})]$ are linearly independent. Since for each row of the matrix the bond prices are averaged over the shocks g' using the same probability distribution, the variations across the rows can only come from the systematic variation induced by the expectations of inflation $[B_{ss'}^g]$. Thus, for each $g \in G$, R_3 imposes essentially the same condition on B^g as the rank condition (R1).

6 Example of inflation targeting

The rank conditions (R1), (R2), or (R3) are not practical to enter as constraints for the choice of an inflation process, which maximizes a social welfare criterion because they are “open” conditions: the set of invertible matrices is an open set. To see how the previous analysis can be made operational, consider an economy in which a central bank has a target rate π^* and seeks to induce expectations that inflation reverts to the target rate whenever current inflation deviates from π^* . For simplicity, we take the model of Sect. 5 with $G = 1$, i.e., no real shock. However, since the nominal interest rate affects the supply of labor, inflation has real effects, and the real interest rate fluctuates.

Suppose the interval of inflation rates is represented by $\Pi = \{-0.02, -0.01, \dots, 0.06\}$. The increments are 0.01 so that Π consists of 9 inflation rates indexed by $s = 1, \dots, 9$. Suppose in addition that the target rate is $\pi^* = 0.02$, corresponding to the index $s^* = 5$. We know from the analysis of the paper that it is not possible to anchor expectations of an immediate reversion to π^* , so the central bank chooses a mean-reverting expectations matrix B of the following form

for $s = 1$	$B(1, 1) = \frac{b}{s^*},$	$B(1, 2) = 1 - \frac{b}{s^*},$	$B(1, \sigma) = 0$ otherwise	
for $1 < s < s^*$	$B(s, s - 1) = \frac{ab}{s^* - s + 1},$	$B(s, s) = \frac{(1-a)b}{s^* - s + 1},$	$B(s, s + 1) = 1 - \frac{b}{s^* - s + 1},$	$B(s, \sigma) = 0$ otherwise
for $s = s^*$	$B(s^*, s^* - 1) = a,$	$B(s^*, s^*) = 1 - 2a,$	$B(s^*, s^* + 1) = a,$	$B(s^*, \sigma) = 0$ otherwise
for $s^* < s < 9$	$B(s, s - 1) = \frac{b}{s - s^* + 1},$	$B(s, s) = \frac{(1-a)b}{s - s^* + 1},$	$B(s, s + 1) = \frac{ab}{s - s^* + 1},$	$B(s, \sigma) = 0$ otherwise
for $s = 9$	$B(9, 8) = 1 - \frac{b}{9 - s^* + 1},$	$B(9, 9) = \frac{b}{9 - s^* + 1},$	$B(9, \sigma) = 0$ otherwise	

Whenever current inflation deviates from the target π_s^* , it is drawn in the direction of the target (from s to $s + 1$ if $s < s^*$, and from s to $s - 1$ if $s > s^*$) with a probability which increases with the deviation $|s - s^*|$ from the target. The intensity of the reversion to the target is determined by the parameter b : the smaller b , the higher the probability that inflation moves toward the target. For example, for $s = 2$, the probability of the transition to $s = 3$ is $1 - \frac{b}{4}$, and for $s = 3$ the probability of transition to $s = 4$ is $1 - \frac{b}{3}$. In each case, the remaining probability ($\frac{b}{4}$ or $\frac{b}{3}$ respectively) is divided between staying at the current inflation (with probability $(1 - a)\frac{b}{4}$ or $(1 - a)\frac{b}{3}$) and transitioning away from the target (with probability $\frac{ab}{4}$ or $\frac{ab}{3}$). When the target is reached, inflation next period deviates either up or down with probability a : the higher a , the greater the probability of moving away from target. Thus a is a “noise” parameter which makes the process fluctuate more.

In order that B be a Markov matrix, we must have $0 \leq a \leq 1/2$ and $0 \leq b \leq 2$. Since the probability of moving toward the target decreases when a and b increase, if the criterion was solely the rapidity of the return to the target, the ideal value of the parameters would be $(a, b) = (0, 0)$. But as discussed below, for these parameter values, the process B can not be anchored.

Suppose the preferences are given by $u(c, \ell) = (c^{1-\alpha} + \beta \ell^{1-\alpha}) / (1 - \alpha)$ with $\alpha = 3$, $\beta = 0.4$ and with discount factor $\delta = 0.98$. Since the target inflation rate $\pi^* = 2\%$ is sufficiently high, an equilibrium exists for all parameter values satisfying $0 \leq a \leq \frac{1}{2}$, $0 \leq b \leq 2$. Let $r(a, b) = (r_1(a, b), \dots, r_9(a, b))$ denote the short-term interest rates satisfying the equilibrium equations (ii)'' and (iii)'' in Sect. 5 for the parameter values (a, b) : from this, we obtain the vector of marginal utilities $\phi(a, b)$ and the bond prices can be recursively calculated using (24), leading to the two 9×9 matrices consisting of the bond prices and their payoffs at the next date

$$Q(a, b) = [q^1(a, b), \dots, q^9(a, b)], \quad \widehat{Q}(a, b) = [\mathbf{1}, q^1(a, b), \dots, q^8(a, b)].$$

Solving the equations (ii)'' and (iii)'' using MATLAB for the parameter values $(a, b) = (0.1, 0.1)$ and $(a, b) = (0.3, 0.3)$ gives the associated 9 term structures of interest rates for the 9 possible current inflation rates shown in Fig. 2i, ii respectively. In each figure, a curve represents the term structure or yield curve $(r_s^r)_{\tau=1}^9$ for a given current inflation rate π_s : the top curve corresponds to $\pi_s = 6\%$ and the lowest curve to $\pi_s = -2\%$. For both pairs of parameter values, the yield curve $(r_s^r)_{\tau=1}^9$ viewed as a function of current inflation π_s increases with π_s , the increase in the yield of the short-term bond typically being greater than that on a long-term bond. Thus, when the inflation process is mean reverting to a target the associated term structure policy that ensures determinacy is in essence a “generalized Taylor rule,” the interest rate on each maturity increasing whenever current inflation increases. In Fig. 2i, the term structure rises fast if there is deflation or low inflation ($\pi_s \leq 0\%$) and decreases fast when current inflation is high ($\pi_s \geq 4\%$), leading agents to anticipate a fast return to the target inflation rate of 2%. In Fig. 2ii, for each current inflation rate, the term structure is flatter, leading agents to anticipate more permanence and a more sluggish return to the target rate.

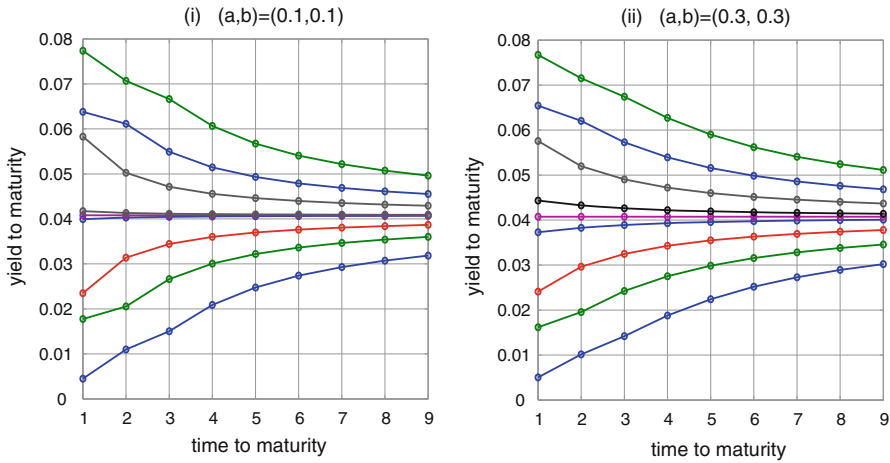


Fig. 2 Term structure rule: term structures associated with the expectations matrix B in the example. **i** $(a, b) = (0.1, 0.1)$; **ii** $(a, b) = (0.3, 0.3)$. Each curve represents the yield curve $(r_s^\tau)_{\tau=1}^9$ for a possible current inflation rate π_s

If the matrix $\widehat{Q}(a, b)$ is invertible, the present-value matrix $D = D_1(a, b)BD_2(a, b)$ (with the notation of Sect. 5 can be uniquely recovered by solving the 9×9 system of equations

$$D \widehat{Q}(a, b) = Q(a, b). \tag{26}$$

Standard linear algebra requires that $\det(\widehat{Q}(a, b)) \neq 0$ for $\widehat{Q}(a, b)$ to be invertible. This, however, is not a meaningful criterion from the point of view of numerical analysis. Since several operations are involved in calculating \widehat{Q} (including solving a fixed point), \widehat{Q} can only be calculated approximately with the precision of a computer. Even if the “true” value of the determinant is zero, the calculated value will be a “small number,” but a small determinant is not necessarily indicative of a lack of invertibility.¹⁴ The criterion in numerical analysis is that D can be recovered with sufficient precision by solving (26), and the precision of the solution depends on the *condition number* $\kappa(\widehat{Q}(a, b))$ of the matrix $\widehat{Q}(a, b)$.¹⁵

If we replace the condition of invertibility of $\widehat{Q}(a, b)$ by the requirement that $\kappa(\widehat{Q}(a, b)) \leq \kappa^*$ where κ^* is an appropriate critical value, then we can determine the optimal parameters (a, b) from the point of view of a central bank seeking to stabilize inflation around the target π^* . Given the symmetry of the Markov matrix B around s^* , the associated invariant measure $\rho(a, b)$ puts equal weight on inflation

¹⁴ There are matrices with a very small determinant which are easily invertible: for example if $M = \varepsilon I$ where I is the $n \times n$ identity matrix, then $\det(M) = \varepsilon^n$ can be arbitrarily small.

¹⁵ If x is the solution of $Ax = b$ and $\tilde{x} = x + e$ is the solution of $A\tilde{x} = \tilde{b}$ with $\tilde{b} = b + r$, then $\eta = \frac{\|e\|/\|x\|}{\|r\|/\|b\|}$ measures the elasticity of the solution (x) with respect to the ‘data’ (b) and is a measure of the “sensitivity” of the solution. The condition number $\kappa(A) = \|A\| \|A^{-1}\|$ of A bounds the sensitivity of the solution, $\frac{1}{\kappa(A)} \leq \eta \leq \kappa(A)$. See Judd (1998) and Trefethen and Bau (1997).

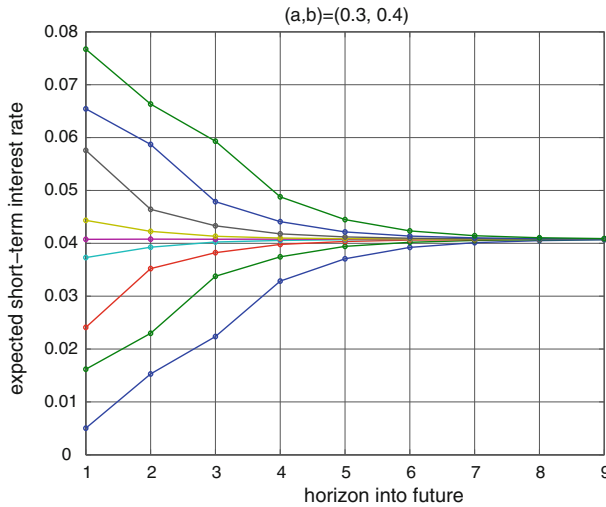


Fig. 3 Forward guidance rule: paths of expected future short-term rates for the 9 possible current inflation rates, the lower curve corresponding to -2% deflation, the upper curve to 6% inflation

rates equidistant from π^* , so that $E^\rho(\pi) = \pi^*$. The smaller the standard deviation $\sigma^\rho(\pi)$ the greater the proportion of time the inflation process spends close to the target, and the better the inflation process from the perspective of the central bank. The smaller the values of the parameters (a, b) , the smaller $\sigma^\rho(\pi)$, and the faster the convergence of the anticipated probability distribution $\rho^{(t)}$ of inflation in t periods.

However, as shown in Fig. 2, the smaller the parameters (a, b) , the closer the yield curves for different inflation rates; in particular for $(a, b) = (0.1, 0.1)$, the yield curves for the inflation rates in the vicinity of the target ($s = 4, 5, 6$) are virtually indistinguishable. As a result, the matrix $\widehat{Q}(a, b)$ is badly conditioned for small (a, b) : the condition number is of the order of 10^{13} for $(a, b) = (0.1, 0.1)$. Thus, there is a trade-off between the conditioning of $\widehat{Q}(a, b)$ and the variability of the inflation process around the target. In addition, there is a trade-off between increasing a (more noise) or increasing b (slower return to target) to decrease the condition number. If we follow common usage and require that the condition number of $\widehat{Q}(a, b)$ is less than $\kappa^* = 10^{10}$, then we find numerically that the standard deviation of $\rho(a, b)$ is minimized for the parameter value $(a, b) = (0.3, 0.3)$, which corresponds to the yield curves of Fig. 2ii. The standard deviation of the associated invariant measure $\rho(0.3, 0.3)$ which is equal to 0.74% is relatively small.

If instead of a term structure rule the monetary authority uses a forward guidance rule, the analysis is similar except that the condition number of the matrix

$$\widehat{EQ}(a, b) \equiv \left[\frac{\delta}{1 + \pi}, q^1(a, b), E^1(q^1(a, b)), \dots, E^7(q^1(a, b)) \right],$$

is somewhat higher than that of $\widehat{Q}(a, b)$. The value of (a, b) which minimizes the standard deviation $\sigma^\rho(\pi)$, while ensuring that the condition number of $\widehat{EQ}(a, b)$ is less than 10^{10} , is $(a, b) = (0.3, 0.4)$. The corresponding paths of the expected short-term rates are shown in Fig. 3.

7 Conclusion

A recurrent theme in discussions of monetary policy is the idea that an important role of a central bank is to anchor agents' expectations of inflation, because any expectations once adopted can become self-fulfilling. This idea can be formalized in a rational expectations model only if there is an inherent indeterminacy of equilibrium due to the self-fulfilling nature of the expectations. Such indeterminacy arises naturally in monetary models in which fiscal policy is Ricardian, and we have used the simplest example of such a model as the framework for our analysis.

The innovation of the paper is to incorporate explicitly into the model the set of possible expectations of inflation which agents can potentially adopt, and to show that by an appropriate choice of its monetary policy—either a term structure rule or a forward guidance rule for the short rate—the monetary authority can lead the economy to the unique equilibrium of its choice.

The analysis has focused on the determinacy of equilibrium and on characterizing the inflation processes which can be anchored. The spirit of Propositions 1–5 that provide formal statements of these conditions is that an inflation process that can be anchored must fluctuate across the different inflation rates and do so with some permanence. We showed using an example how the analysis can be applied to selecting a mean-reverting process around a target, which spends as much time as possible close to target: however the target rate and the minimum variance criterion were not derived “from first principles.” Thus, the paper can be viewed as a first step toward a richer analysis, which simultaneously studies not only the determinacy but also the optimality of the monetary policy. A more complete analysis along these lines would require the introduction of some frictions to give a real effect to inflation, for example, the price rigidities studied in the New Keynesian models. Extending the analysis to a richer model which permits an optimal inflation process to be derived, which can also be anchored, is left for future research.

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