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STABILITY OF REGULAR EQUILIBRIA AND THE CORRESPONDENCE PRINCIPLE FOR SYMMETRIC VARIATIONAL PROBLEMS*

By MICHAEL J. P. MAGILL AND JOSÉ A. SCHEINKMAN[1]

1. INTRODUCTION

This paper gives a complete characterization of the local stability of regular equilibria for a relatively broad class of dynamic systems arising in mathematical economics. The completeness of the characterization depends upon an assumption of *symmetry* on the integrand of the basic variational problem.

We show that there is a single scalar valued function which first locates all the equilibria. The *equilibria* are precisely the positions where the function attains an *extremum*. The nature of the local stability properties of each *regular equilibrium point* is then determined by the nature of the extremum at that point. In particular if the function attains a *local maximum* (*minimum*), then the equilibrium is *locally asymptotically stable* (*unstable*).[2] We thus obtain a remarkably complete picture of the behavior of the dynamic system in a neighborhood of each regular equilibrium point (Sections 2, 3).

To cope with the long-run evolution of a dynamic system along the lines suggested by Magill [1977a] and to further extend the analysis, we introduce a vector of *exogenous parameters* into the integrand of the basic variational problem and consider the family of extremum problems induced in this way. We introduce an *equilibrium manifold* in the *parameter-state space* which reveals the equilibria for each value of the exogenous parameter.[3] We show that the assumption of symmetry implies that the eigenvalues of the dynamic system are *real*. This in turn implies that the local stability properties of regular equilibria may be deduced from simple geometric properties of the equilibrium manifold.

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[2] This result is closely related to Lagrange's classical theorem on stability: *if the potential function of a conservative dynamical system attains a minimum* (*maximum*) *at a position of equilibrium then the motion in a neighborhood of this equilibrium point is stable* (*unstable*). See Lagrange [1888, pp. 69–76] and Liapunov [1967, pp. 62, 377–386].

[3] The idea of considering the potential function of a dynamical system as a function not only of the state of the system but also of an exogenous parameter was introduced by Poincaré who simultaneously laid the foundations for the *theory of bifurcation of equilibrium* [1952, pp. 43–55].

Thus *critical equilibria* at which an exchange of stability can take place are those for which the tangent hyperplane to the equilibrium manifold projects degenerately onto the parameter space. We show that such critical parameter values form a closed set of measure zero in the parameter space. The analysis here is closely related to the earlier important work of Debreu [1970].

We show in addition that if a rank and boundary condition on the *parameter* are satisfied, then the equilibrium manifold consists of a *unique connected component*, while if a rank and boundary condition on the *state* are satisfied then there is a *unique equilibrium point of the same degree of stability* for each feasible parameter value. This analysis is contained in Section 5, which may also be viewed as a preliminary attack on the problem of bifurcation of equilibrium. In a complete analysis we expect the bifurcation diagram for the equilibria of the dynamic system to be revealed by the bifurcation diagram of the basic scalar valued function that characterizes the equilibria. This would lead to a remarkable unification of the basic theory of the dynamic system.

Samuelson's *Correspondence Principle* [1947] is concerned with the relationship between the stability of equilibrium and infinitesimal changes in the equilibrium as the exogenous parameters are varied. As Brock [forthcoming] has recently pointed out, the difficulties associated with obtaining meaningful theorems in Arrow-Debreu-McKenzie general equilibrium theory on the basis of the Correspondence Principle arise in essence from two causes:[4] the first is the Sonnenschein [1972]-Mantel [1974]-Debreu [1974] theorem, by which, under standard axioms on the utility functions of agents, any continuous vector valued function from R^n to R^n can be generated as a market excess demand function for an economy with $n+1$ commodities, while the second is the fact that general equilibrium theory lacks any explicit dynamics. In a large class of dynamic optimization problems the latter difficulty is automatically eliminated, while the former may be circumvented by imposing sufficient structure on the dynamic system. With this in view Brock [forthcoming] and Burmeister-Long [1977] have recently proposed a rehabilitation of Samuelson's Correspondence Principle for a class of dynamic optimization problems. Indeed Burmeister-Long viewed the Correspondence Principle as a vehicle for resolving certain paradoxes connected with the *Cambridge Controversy* in capital theory and the associated *Hahn problem*, if the latter is interpreted within the framework of a centralized economy.[5] As Brock points out, such a move to rehabilitate the Correspondence Principle was implicit in the earlier work of Lucas [1967] and Mortensen [1973] in the context of an adjustment cost model of the firm and in the Liviatan-Samuel-

[4] See also Arrow and Hahn [1971], Gordon and Hines [1970], Quirk and Saposnik [1946].

[5] There is a danger of confusing *two distinct causes of instability of equilibrium*: in the first case the equilibrium is unstable because the *optimality (transversality) conditions* are *not* satisfied, in the second case the equilibrium is unstable when the transversality conditions *are* satisfied. Hahn doubts that a decentralized market system provides a mechanism by which the transversality condition associated with convergence to the steady state is satisfied. This is the first case. But we are in essence interested in the second case.

son [1969] analysis of a one-sector model of aggregated growth with joint production.

Brock [forthcoming] and Magill [1977a, Section 5] showed how *sufficient conditions* for local stability of regular equilibria lead to qualitative statements about infinitesimal changes in such equilibria with respect to changes in the parameters.[6] In Section 4 we show, under the assumption of symmetry, that such results may be deduced directly from the *local stability or instability of each regular equilibrium point.*[7] Such a result is possible because the basic matrix that characterizes the local stability properties of regular equilibria is precisely the matrix that determines the qualitative changes in the equilibrium with respect to the parameters. We apply this result to an integrand in which the parameters enter additively to obtain a generalization of the earlier results of Mortensen [1973]. We show furthermore that it is precisely at critical equilibria that the Correspondence Principle breaks down.

Liviatan-Samuelson [1969] characterized the local stability of each regular equilibrium in a one-sector model of growth in terms of the derivative of the own-rate of return of the capital good at the equilibrium point. We generalize this result to the n-dimensional case by showing that the local stability of each regular equilibrium may be characterized by the Jacobian of the vector of own-rates of return at the equilibrium point.

This paper is essentially an extension of the earlier results of Magill [1977a, Section 5] and the related paper of Scheinkman [1978]. All of this work may in turn be viewed as part of the extensive recent research by Brock-Magill [forthcoming], Brock-Scheinkman [1976], Cass-Shell [1976], McKenzie [1976], Magill [1977a, 1977b], Rockafellar [1976], Samuelson [1972] and Scheinkman [1976, 1978], that attempts to develop a general theory of the dynamic systems that arise in mathematical economics.

2. SYMMETRIC VARIATIONAL PROBLEMS

Let $\mathscr{K}$ denote a compact convex subset of R^n, $n \geqq 1$. We consider a vector of *exogenous parameters* $\alpha = (\beta, \delta) \in \mathscr{A} = \mathscr{A}_\beta \times \mathscr{A}_\delta \subseteq R^s$, $s \geqq 1$,

DEFINITION. For fixed $(k_0, \alpha) \in \mathscr{K} \times \mathscr{A}$, the class of absolutely continuous functions

$$k(t) = k(t; k_0, \alpha): [0, \infty) \longrightarrow \mathring{\mathscr{K}} \tag{1}$$

where $\mathring{\mathscr{K}}$ denotes the interior of $\mathscr{K}$, is called the class of *feasible paths* and is denoted by $\mathscr{F}$.

[6] See also Arrow-Hahn [1971, Chapter 10].

[7] The results of this paper thus suggest that the negative conclusions of Arrow and Hahn with respect to the Correspondence Principle [1971, pp. 320–321] seem to have been somewhat overstated since we are able, at least in the symmetric case, to make qualitative statements about comparing equilibria directly from the assumption of local stability (instability).

We consider real valued functions

$$L(k, \dot{k}; \beta): \mathring{\mathscr{K}} \times R^n \times \mathscr{A}_\beta \longrightarrow R \tag{2}$$

where $L \in C^r$, $r \geqq 2$, which satisfy the following:

ASSUMPTION 1 (Concavity). *$L(\cdot, \beta)$ is a concave function in $(k, \dot{k})$ for all $(k, \dot{k}) \in \mathscr{K} \times R^n$, for all $\beta \in \mathscr{A}_\beta$.*

We consider feasible paths (1) induced by (2) through the following

VARIATIONAL PROBLEM. *Find $k(t) \in \mathscr{F}$ such that*

$$(\mathscr{P}_\alpha) \qquad \sup_{k(t) \in \mathscr{F}} \int_0^\infty L(k(\tau), \dot{k}(\tau); \beta) e^{-\delta\tau} d\tau.$$

DEFINITION. $\mathscr{P}_\alpha$ is called a *symmetric variational problem* if

$$L_{k\dot{k}}(k, \dot{k}; \beta) = L(k,_{\dot{k}k} \dot{k}; \beta) \quad \text{for all} \quad (k, \dot{k}) \in \mathring{\mathscr{K}} \times R^n, \quad \text{for all} \quad \beta \in \mathscr{A}_\beta. \tag{3}$$

The Euler-Lagrange equations for $\mathscr{P}_\alpha$ are given by

$$L_{\dot{k}\dot{k}}\ddot{k} + L_{\dot{k}k}\dot{k} - (L_k + \delta L_{\dot{k}}) = 0. \tag{4}$$

DEFINITION. A path $k(t) \in \mathscr{F}$ with $\dot{k}(t) \equiv 0$ for all $t \in [0, \infty)$, which satisfies (4), is called an *equilibrium point* (*stationary state*).

ASSUMPTION 2 (Existence of equilibrium point). *For every $\alpha = (\beta, \delta) \in \mathscr{A}$ there exists at least one $k^* \in \mathring{\mathscr{K}}$ satisfying*

$$L_k(k^*, 0; \beta) + \delta L_{\dot{k}}(k^*, 0; \beta) = 0. \tag{5}$$

DEFINITION. $\mathscr{E} = \{(k^*, \alpha) \mid L_k(k^*, 0; \beta) + \delta L_{\dot{k}}(k^*, 0; \beta) = 0\}$ is called the *equilibrium set* for the variational problem $\mathscr{P}_\alpha$.

DEFINITION. Let $(k^*, \alpha) \in \mathscr{E}$. The *local coordinates* around an equilibrium point $k^* = k^*(\alpha)$ are defined by the transformation

$$x = k - k^*. \tag{6}$$

The *variational equations* associated with (4) in a neighborhood of an equilibrium point $k^* = k^*(\alpha)$ are given in terms of the local coordinates (6) by

$$L^*_{\dot{k}\dot{k}}\ddot{x} + (L^*_{\dot{k}k} - L^*_{k\dot{k}} - \delta L^*_{\dot{k}\dot{k}})\dot{x} - (L^*_{kk} + \delta L^*_{\dot{k}k})x = 0 \tag{7}$$

where the asterisk denotes evaluation at $(k^*, 0)$ for a fixed $\alpha \in \mathscr{A}$. For a symmetric variational problem (7) reduces to

$$L^*_{\dot{k}\dot{k}}\ddot{x} - \delta L^*_{\dot{k}\dot{k}}\dot{x} - (L^*_{kk} + \delta L^*_{\dot{k}k})x = 0. \tag{8}$$

DEFINITION. An *equilibrium point* $k^* = k^*(\alpha)$, where $(k^*, \alpha) \in \mathscr{E}$, is said to be *regular* (*critical*) if $\lambda_i \neq 0$, $i=1,\dots,2n$ ($\lambda_i = 0$ for some i) where λ_i is a root of the *characteristic polynomial*

$$D(\lambda_i) = |L^*_{\dot{k}\dot{k}}\lambda_i^2 + (L^*_{k\dot{k}} - L^*_{\dot{k}k} - \delta L^*_{\dot{k}\dot{k}})\lambda_i + (L^*_{kk} + \delta L^*_{\dot{k}k})| = 0. \tag{9}$$

A *parameter value* $\alpha \in \mathscr{A}$ is *regular* (*critical*) if all (at least one) of the associated equilibria $k^*(\alpha)$ are regular (is critical).

DEFINITION. We let $\mathscr{E}^r$ ($\mathscr{E}^c$) denote the *set of regular* (*critical*) *equilibria* in $\mathscr{E}$. Similarly we let $\mathscr{A}^r$ ($\mathscr{A}^c$) denote the *set of regular* (*critical*) *parameter values* in $\mathscr{A}$.

PROPOSITION 1 (Regular equilibria). *Let* $k^* = k^*(\alpha)$ *where* $(k^*, \alpha) \in \mathscr{E}$ *then* $k^* \in \mathscr{E}^r$ ($\mathscr{E}^c$) *if and only if*

$$\Delta = |L_{kk}(k^*, 0; \beta) + \delta L_{\dot{k}k}(k^*, 0; \beta)| \neq 0 \ (= 0). \tag{10}$$

PROOF. Let $\lambda_1, \dots, \lambda_{2n}$ denote the roots of (9) then for some $\gamma \neq 0$,

$$\frac{D(\lambda)}{\gamma} = (\lambda_1 - \lambda)\cdots(\lambda_{2n} - \lambda)$$

thus

$$\Delta = (-1)^n \frac{D(0)}{\gamma} = (-1)^n \lambda_1 \cdots \lambda_{2n} \neq 0 \ (= 0)$$

according as $k^* \in \mathscr{E}^r$ ($\mathscr{E}^c$) □

Remark. An equilibrium point k^* is said to be *isolated* if there is a neighborhood of k^* containing no other equilibrium points than k^*. (10) and the implicit function theorem imply that *regular equilibria are isolated.*

Remark. An equilibrium point $k^*(\alpha)$ is a point of intersection of the n, $(n-1)$-dimensional surfaces

$$L_{k_i}(k^*, 0; \beta) + \delta L_{\dot{k}_i}(k^*, 0; \beta) = 0, \qquad i = 1,\dots, n \tag{11}$$

in R^n. *The equilibrium point* $k^*(\alpha)$ *is regular if and only if the gradients*

$$(L^*_{k_ik_1} + \delta L^*_{\dot{k}_ik_1}, \dots, L^*_{k_ik_n} + \delta L^*_{\dot{k}_ik_n}) \qquad i = 1,\dots, n \tag{12}$$

are well-defined at $k^*(\alpha)$ *and are linearly independent.* If the surfaces (11) are smooth, an equilibrium point $k^*(\alpha)$ is critical if and only if the gradients (12) are linearly dependent. At such an equilibrium point one of the eigenvalues λ_i in (9) is zero.

Remark. Let $\alpha \in \mathscr{A}^r$ and let $\mu(\Delta^+(\alpha))$ [$\mu(\Delta^-(\alpha))$] denote the number of equilibria, at the parameter value α for which $\Delta > 0$ [$\Delta < 0$]. Let C denote a subset of $\mathscr{K}$ with boundary ∂C. The following classical theorem of Kronecker-Poincaré [1951, Chapter 18] gives very general conditions under which the existence of at least one regular equilibrium point is assured. *Suppose* $\alpha \in \mathscr{A}^r$ *and suppose there exists a subset* $C \subset \mathscr{K}$ *which is diffeomorphic to an open ball of radius* $r > 0$ *in* R^n, *if there are no equilibria on* ∂C *and if*

$$v(\bar{k})'(L_k(\bar{k}, 0; \beta) + \delta L_{\dot{k}}(\bar{k}, 0; \beta)) < 0 \qquad \text{for all} \quad \bar{k} \in \partial C$$

where $v(\bar{k})$ denotes the outward normal of ∂C, then

$$\mu(\Delta^+(\alpha)) - \mu(\Delta^-(\alpha)) = (-1)^n.$$

There is thus at least one regular equilibrium point.

ASSUMPTION 3 (Strong concavity at equilibrium point). *For each $(k^*, \alpha) \in \mathscr{E}$*

$$\begin{bmatrix} L^*_{kk} & L^*_{k\dot{k}} \\ L^*_{\dot{k}k} & L^*_{\dot{k}\dot{k}} \end{bmatrix} \tag{13}$$

is negative definite.

Under Assumption 3, (8) may be written as

$$\ddot{x} - \delta\dot{x} - (L^*_{\dot{k}\dot{k}})^{-1}(L^*_{kk} + \delta L^*_{\dot{k}k})x = 0. \tag{14}$$

If we let $v = \dot{x}$ then (14) reduces to the first order system

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ (L^*_{\dot{k}\dot{k}})^{-1}(L^*_{kk} + \delta L^*_{\dot{k}k}) & \delta I \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{15}$$

$\lambda \in \mathscr{C}$ and $(w, z) \in \mathscr{C}^n \times \mathscr{C}^n$ are an eigenvalue and associated eigenvector of (15) if and only if $z = \lambda w$ and

$$(-L^*_{\dot{k}\dot{k}})^{-1}(L^*_{kk} + \delta L^*_{\dot{k}k})w = \lambda(\delta - \lambda)w. \tag{16}$$

The analysis of the stability of equilibrium will depend in a crucial way on the information contained in equation (16).

3. STABILITY OF REGULAR EQUILIBRIA

Consider the real-valued function

$$\phi(k^*, \alpha) : \mathscr{K} \times \mathscr{A} \longrightarrow R$$

induced by the following line integral

$$\phi(k^*, \alpha) = \Gamma\int_{\bar{k}}^{k^*} [L_k(k, 0; \beta) + \delta L_{\dot{k}}(k, 0; \beta)]' dk \tag{17}$$

where Γ denotes the line-segment joining $\bar{k}$ and k^* for some fixed $\bar{k} \in \mathscr{K}$ and any $k^* \in \mathscr{K}$.

LEMMA 1 (Existence of equilibrium potential). *If $L(\cdot\,; \beta) \in C^2$ and if there exist constants $\underline{M}_j < \overline{M}_j$, $j = 1, \ldots, n$ such that $L_{\dot{k}k}(k, 0; \beta)$ is symmetric for all $k \in \mathscr{K}_M = \{k \in R^n | \underline{M}_j < k_j < \overline{M}_j, j = 1, \ldots, n\} \subseteq \mathscr{K}$ for all $\beta \in \mathscr{A}_\beta$, then there exists a function $\phi(k^*, \alpha)$, given by (17), such that*

$$\phi_k(k^*, \alpha) = L_k(k^*, 0; \beta) + \delta L_{\dot{k}}(k^*, 0; \beta) \tag{18}$$

for all $k^* \in \mathscr{K}_M$, *for all* $\beta \in \mathscr{A}_\beta$.

PROOF. The result follows at once from the standard theorem for the existence of a potential function (Apostol [1957, pp. 293–297]) since $L_{kk}(k, 0; \beta)$ is symmetric for all $k \in \mathscr{K}_M$, for all $\beta \in \mathscr{A}_\beta$. □

Remark. In the analysis that follows we assume for simplicity that $\bar{\mathscr{K}}_M = \mathscr{K}$. *Thus for a symmetric variational problem the equilibrium points* $k^* = k^*(\alpha)$ *associated with a given value of the exogenous parameter* $\alpha \in \mathscr{A}$, *are the extrema of an equilibrium potential function* $\phi(k^*, \alpha)$ *viewed as a function of* k^*.

In view of (18), (16) becomes

$$(-L_{kk}^*)^{-1}\phi_{kk}^* w = \lambda(\delta - \lambda)w \tag{19}$$

where $\phi_{kk}^* = \phi_{kk}(k^*, \alpha)$ is a symmetric matrix. We need the following

LEMMA 2. *Let* A, B *be* $n \times n$ *symmetric matrices with real coefficients,* B *positive definite, then* BA *has real eigenvalues and* BA *has the same number of positive, negative and zero eigenvalues, respectively, as* A.

PROOF. Since B is positive definite, its positive definite square root $\sqrt{B} = C$ exists, so that BA is similar to $C^{-1}BAC = CAC$. Thus BA has the same eigenvalues as CAC whose eigenvalues are real. Let

$$B_\gamma = \gamma I + (1 - \gamma)B$$

then B_γ is positive definite for $0 \leq \gamma \leq 1$, $B_0 = B$, $B_1 = I$. Let $C_\gamma = \sqrt{B_\gamma}$ then as before $B_\gamma A$ is similar to $C_\gamma A C_\gamma$, and $\text{rank}(B_\gamma A) = \text{rank}(A)$ for $0 \leq \gamma \leq 1$. Hence $\text{rank}(C_\gamma A C_\gamma) = \text{rank}(A)$ for $0 \leq \gamma \leq 1$. Since a continuous change in coefficients leads to a continuous change in the eigenvalues of a symmetric matrix, the difference between the number of positive and the number of negative eigenvalues (the *signature* of the matrix) only changes when some eigenvalue changes its sign. But if an eigenvalue changes sign it must pass through zero, leading to a change in the rank of the matrix. Thus $\text{signature}(C_1 A C_1) = \text{signature}(A)$ implying that BA has the same number of positive (negative) ((zero)) eigenvalues as A. □

Applying Lemma 2 to (19) we find that $\nu = \lambda(\delta - \lambda)$ is real for each eigenvalue λ of (15). This leads at once to the following

LEMMA 3. *If* λ *is an eigenvalue of* (15) *and Assumption* 3 *holds, then* λ *is real.*[8]

PROOF. The variational equations (15) are the Euler-Lagrange equations for the problem, $\max \int_0^\infty L^\circ(x, \dot{x})e^{-\delta t}dt$, where $L^\circ(x, \dot{x})$ is the quadratic form generated by the matrix (13). If we introduce the mirage variable $\hat{x} = e^{-\frac{\delta}{2}t} x$ (Magill

[8] For a statement of more general conditions under which the eigenvalues in (9) are real see Magill [1977b].

[1977a, Section 4]), then the problem becomes, $\max \int_0^\infty \hat{L}^\circ(\hat{x}, \dot{\hat{x}})dt$ where $\hat{L}^\circ(\hat{x}, \dot{\hat{x}})$ is negative definite. Now if λ is an eigenvalue of (15), then $\mu = \lambda - \frac{\delta}{2}$ is an eigenvalue of the mirage system. But Assumption 3 and the result of Levhari-Liviatan [1972] imply $\mathrm{Re}(\mu) \neq 0$. Hence $\mathrm{Re}(\lambda) \neq \frac{\delta}{2}$. Since $\nu = \lambda(\delta - \lambda)$ is real, the result follows. □

DEFINITION. The *degree of stability* of (15) is the number of negative eigenvalues of (15).

PROPOSITION 2 (Characterization of stability by $\phi(k^*, \alpha)$). (i) *The degree of stability of* (15) *at* $(k^*, \alpha) \in \mathscr{E}$ *is equal to the number of negative eigenvalues of* $\phi_{kk}(k^*, \alpha)$. (ii) *The number of zero eigenvalues of* (15) *is equal to the number of zero eigenvalues of* $\phi_{kk}(k^*, \alpha)$. (iii) *The number of positive eigenvalues of* $\phi_{kk}(k^*, \alpha)$ *is equal to the number of positive eigenvalues of* (15) *minus n.*

PROOF. Let $\lambda_1, \ldots, \lambda_k$ denote the k negative eigenvalues of (15), so that $\delta - \lambda_i > 0$ and $\lambda_i(\delta - \lambda_i) < 0$, $i = 1, \ldots, k$. Thus for each λ_i there is a negative eigenvalue of $(-L_{kk}^*)^{-1}\phi_{kk}^*$. By Lemma 2 the number of negative eigenvalues of $(-L_{kk}^*)^{-1} \cdot \phi_{kk}^*$ is the same as the number of negative eigenvalues of ϕ_{kk}^*. Conversely if μ_i, $i = 1, \ldots, k'$ are the negative eigenvalues of ϕ_{kk}^*, then there exist eigenvalues ν_i, $i = 1, \ldots, k'$ which are the negative eigenvalues of $(-L_{kk}^*)^{-1}\phi_{kk}^*$. But $\nu_i = \lambda_i(\delta - \lambda_i)$ for some eigenvalue λ_i of (15). Hence $\lambda_i(\delta - \lambda_i) < 0$. By Kurz [1968], λ_i and $\delta - \lambda_i$ are both eigenvalues of (15). By Lemma 3 either λ_i or $\delta - \lambda_i$ is negative. The proofs of (ii) and (iii) are identical. □

Remark. Proposition 2 implies that *the stability properties of regular equilibria*, $(k^*, \alpha) \in \mathscr{E}^r$, *are determined by* $\phi_{kk}^*(k^*, \alpha)$. In particular we have the following

COROLLARY. *If* $(k^*, \alpha) \in \mathscr{E}^r$, *then* $\phi(k^*, \alpha)$ *attains*
(i) *a local maximum if and only if* $k^*(\alpha)$ *is locally asymptotically stable*
(ii) *a local minimum if and only if* $k^*(\alpha)$ *is completely unstable.*

We can in fact obtain an even more precise picture of the motion in a neighborhood of a regular equilibrium point. To this end we introduce the following

DEFINITION. Let A, B be $n \times n$ matrices, B positive definite, symmetric. We say that ν_i is an *eigenvalue of A in the metric of B* and $w^i \in R^n$, $w^i \neq 0$ is an associated eigenvector if $Aw^i = \nu_i B w^i$.

Let $\nu_1, \ldots, \nu_n$ and $w^1, \ldots, w^n$ denote the eigenvalues and associated eigenvectors of ϕ_{kk}^* in the metric of $(-L_{kk}^*)$.

DEFINITION. $\nu_1, \ldots, \nu_n$ will be called the *curvature coefficients of* ϕ_{kk}^* since they yield measures of the curvature of ϕ_{kk}^* in the directions $w^1, \ldots, w^n$, in the metric induced by $(-L^*_{kk})$.

Remark. (19) implies that the curvature coefficients of ϕ^*_{kk} are related to the eigenvalues of the variational equations (15) in the following way

$$\nu_i = \lambda_i(\delta - \lambda_i) \qquad i = 1,\dots, n \tag{20}$$

so that

$$\lambda_i = \frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 - \nu_i} \qquad i = 1,\dots, n. \tag{21}$$

It is clear from (16) and (19) that the eigenvectors for the system (15) may be derived from the eigenvectors $w^1,\dots, w^n$ of ϕ^*_{kk} in the metric of $(-L^*_{kk})$. But as is well-known, if we assume without loss of generality that $\nu_1 < \cdots < \nu_n$, then $\nu_1,\dots, \nu_n$ *and* $w^1,\dots, w^n$ *may be characterized as the solutions of the following sequence of n constrained extremum problems* (Gantmacher [1960, pp. 317–320])

$$\begin{aligned} \nu_i &= \min_{x\in L_i} \frac{x'\phi^*_{kk}x}{x'(-L^*_{kk})x} = \frac{w^{i\prime}\phi^*_{kk}w^i}{w^{i\prime}(-L^*_{kk})w^i} \qquad i = 1,\dots, n \\ L_i &= \{x \in R^n,\ x \neq 0 \mid x'(-L^*_{kk})w^j = 0,\ j = 1,\dots, i-1\} \end{aligned} \tag{22}$$

(21) *and* (22) *give the most complete general expression for the eigenvalues of the system* (15) *that we can expect to obtain.* When $n=1$ we obtain the standard explicit expression for the eigenvalues, with ν reducing to

$$\nu = \frac{\phi^*_{kk}}{(-L^*_{kk})}$$

since there is then no minimization to be carried out.

It is well-known from the theory of *pencils of quadratic forms*[9] (Gantmacher [1960, pp. 310–312]) that the $n \times n$ matrix of eigenvectors $W=[w^1 \cdots w^n]$, where $w^1,\dots, w^n$ denote n column vectors, may be chosen in such a way that

$$W'(-L^*_{kk})W = I, \quad W'\phi^*_{kk}W = N, \quad N = \begin{bmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_n \end{bmatrix}.$$

The nonsingular transformation to principal coordinates $y=(y_1,\dots, y_n)$

$$x = Wy$$

reduces the variational equations (8) *to the normal form*

$$W'(-L^*_{kk})W\ddot{y} - \delta W'(-L^*_{kk})W\dot{y} + W'\phi^*_{kk}Wy = \ddot{y} - \delta\dot{y} + Ny = 0$$

so that the equations of motion break up into n separate one-sector models

[9] The reader is referred to the paper of McKenzie [1968] for an interesting related use of the theory of pencils of quadratic forms.

$$(23)\qquad \ddot{y}_i - \delta\dot{y}_i + v_i y_i = 0 \quad\text{or}\quad \begin{bmatrix} \dot{y}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -v_i & \delta \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} \qquad i = 1,\dots, n$$

where the latter equations represent (15) *in principal coordinates.* (21) are just the eigenvalues for the n one-sector systems (23). *In view of Assumption* 3 *and the assumption of regularity, each one-sector system in* (23) *has one of two phase portraits in the space* $(\dot{y}_i, y_i)$, *a saddlepoint if* $v_i<0$ *and an unstable node if* $v_i>0$. *If* $v_i<0$ *the standard transversality condition shows that* $\dot{y}_i=\lambda_i y_i$, *where* $\lambda_i<0$, *is the locally optimal path.*[10] When $v_i>0$ it seems reasonable to conjecture that $\dot{y}_i=\lambda_i y_i$ is again a locally optimal path, for one of the two positive λ_i in (21), but *global* considerations of the phase portrait outside a neighborhood of the equilibrium point are needed to check that the associated path satisfies a transversality condition (see Liviatan-Samuelson [1969, Section 5]).

Remark. In a model of capital accumulation with n capital goods (Magill [1977a, Section 6]) $x'\phi^*_{kk}x$ is a measure of the *loss* induced by the deviation x of the state from the equilibrium point. $x'(-\phi^*_{kk})x$ is thus a measure of the *benefit* generated by the equilibrium point. $x'(-L^*_{\dot{k}\dot{k}})x$, on the other hand, is a measure of the *cost* of reaching the equilibrium point. (22) shows in a very precise way the *benefit-cost* calculations that underlie the local motion in the neighborhood of an equilibrium point.

4. THE CORRESPONDENCE PRINCIPLE

Samuelson's Correspondence Principle [1947] relates the local stability properties of the dynamic system to the local properties of $\mathscr{E}^r$. In view of Proposition 1 and (18)

$$\mathscr{E}^r = \{(k^*, \alpha) \in \mathscr{E} \mid |\phi_{kk}(k^*, \alpha)| \neq 0\}.$$

Thus by the implicit function theorem if $(\bar{k}^*, \bar{\alpha}) \in \mathscr{E}^r$ there exists a neighborhood $\mathscr{N}$ of (k^*, α) and a C^1 function $k^*(\alpha)$ such that $(k, \alpha) \in \mathscr{N} \cap \mathscr{E}^r$ if and only if $k=k^*(\alpha)$ and

$$(24)\qquad \left.\frac{dk^*}{d\alpha}\right|_{\bar{\alpha}} = [\phi_{kk}(\bar{k}^*, \bar{\alpha})]^{-1}\phi_{k\alpha}(\bar{k}^*, \bar{\alpha}).$$

In view of Proposition 2 and (24), the stability properties of (15) are closely related to the properties of $\dfrac{dk^*}{d\alpha}$. The propositions that follow will attempt to explore this relationship without, however, any pretense to completeness.

PROPOSITION 3. *Let* $L(k, \dot{k}; \beta)=u(k, \dot{k})-wk-p\dot{k}$, $\alpha=(w, p)$, *then*

$$(25)\qquad \frac{dk^*}{dw},\quad \frac{dk^*}{dp},\quad \frac{dk^*}{d(w+\delta p)}$$

[10] See for example Magill [1977a].

are symmetric matrices, with as many negative eigenvalues as the degree of stability of (15).

PROOF. The result follows from (24) and Proposition 2. □

COROLLARY. *At a regular equilibrium the matrices* (25) *are negative* (*positive*) *definite if and only if the equilibrium point is locally asymptotically stable* (*completely unstable*).

Remark. This is a generalization of part of Theorem 1 of Mortensen [1973]. The proof of the following result is also immediate.

PROPOSITION 4. *If k^* is a regular equilibrium and if k^* is locally asymptotically stable* (*completely unstable*) *then*

$$(-L_{\dot{k}}(k^*, 0; \beta))' \frac{dk^*}{d\delta} < 0 \ (> 0).$$

Remark. In capital theory the shadow prices of the capital goods are given by $p = -L_{\dot{k}}(k, \dot{k}; \beta)$. Thus Proposition 4 tells us that if a regular equilibrium is locally asymptotically stable (completely unstable), a rise in the rate of interest causes a fall (rise) in a price weighted average of the capital stocks. The index $p^{*\prime} \frac{dk^*}{d\delta}$ was introduced by Burmeister-Turnovsky [1972] who attempted to relate it to stability properties of an equilibrium. Of course when $n = 1$, the sign of this index is precisely related to the stability of an equilibrium. It is clear however that such an index cannot be used to characterize the stability of equilibria in the general case $n > 1$, since it is in essence a scalar index. For symmetric variational problems we will introduce a matrix index which precisely reveals the stability properties of regular equilibria. To this end we make

ASSUMPTION 4 (Strict free disposal). *At every equilibrium point $p^* = -L_{\dot{k}}(k^*, 0; \beta)$ is a strictly positive vector.*

DEFINITION. We call

$$r_i(k) = \frac{L_{k_i}(k, 0; \beta)}{-L_{\dot{k}_i}(k, 0; \beta)} \qquad i = 1, \ldots, n$$

the (*static*) *own-rates of return.*

Remark. Under Assumption 4 the own-rates of return are well-defined in a neighborhood of an equilibrium point.

Let $$r_{ij}(k) = \frac{\partial r_i(k)}{\partial k_j} \text{ and } r(k) = \begin{bmatrix} r_{11}(k) \cdots r_{1n}(k) \\ \vdots \qquad \vdots \\ r_{n1}(k) \cdots r_{nn}(k) \end{bmatrix}.$$

PROPOSITION 5. *Under Assumptions* 1–4 *the degree of stability of* (15) *at an equilibrium point k^* is equal to the number of negative eigenvalues of the* *matrix $r(k^*)$, furthermore the number of zero eigenvalues of* (15) *is equal to the number of zero eigenvalues of $r(k^*)$.*

PROOF. It follows from $r_{ij}(k^*)=\dfrac{\phi_{ij}(k^*)}{p_i^*}$, $i, j=1,\dots, n$ that

$$\begin{bmatrix} r_{11}(k^*)\dots r_{1n}(k^*) \\ \vdots \qquad \vdots \\ \vdots \qquad \vdots \\ r_{n1}(k^*)\cdots r_{nn}(k^*) \end{bmatrix} = \begin{bmatrix} \frac{1}{p_1^*} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{p_n^*} \end{bmatrix} \begin{bmatrix} \phi_{11}(k^*)\cdots\phi_{1n}(k^*) \\ \vdots \qquad \vdots \\ \vdots \qquad \vdots \\ \phi_{n1}(k^*)\cdots\phi_{nn}(k^*) \end{bmatrix}. \tag{26}$$

Since the first matrix on the right-hand side is positive definite, we can show as in the previous section that the number of negative (zero) eigenvalues of $r(k^*)$ is the same as the number of negative (zero) eigenvalues of $\phi_{kk}(k^*)$. The result then follows from Proposition 2. □

Remark. Proposition 2 may be interpreted as asserting that the degree of stability of an equilibrium point is characterized by the following *price-weighted own-rate of return matrix*

$$\begin{bmatrix} p_1^* r_{11}(k^*) \cdots p_1^* r_{1n}(k^*) \\ \vdots \qquad\qquad \vdots \\ p_n^* r_{n1}(k^*) \cdots p_n^* r_{nn}(k^*) \end{bmatrix}. \tag{27}$$

Remark. If a regular equilibrium point k^* is locally asymptotically stable (completely unstable) then the own-rates of return satisfy

$$r_{ii}(k^*) < 0 (> 0) \qquad\qquad i = 1,\dots, n.$$

Remark. The characterization of the stability of equilibrium in terms of (26) or (27) may be viewed as a natural generalization to the case of an arbitrary number of capital goods of the earlier result of Liviatan-Samuelson [1969] for the case $n=1$.

Remark. Consider the Kronecker-Poincaré theorem of Section 2. Let C be replaced by the subset

$$\{k \in \mathscr{K} \mid 0 < \underline{k}_i < k_i < \bar{k}_i,\ i = 1,\dots, n\}$$

and let the condition $v'(L_k+\delta L_{\dot{k}})<0$ be replaced by

$$\left.\begin{aligned} \rho_j(k_1,\dots, \underline{k}_j,\dots, k_n) > 0 \\ \rho_j(k_1,\dots, \bar{k}_j,\dots, k_n) < 0 \end{aligned}\right\} \quad \text{for all} \quad k_i \in (\underline{k}_i, \bar{k}_i), \quad i \neq j \tag{28}$$

where $$\rho_j(k) = r_j(k) - \delta, \qquad\qquad j = 1,\dots, n$$

then there exists at least one regular equilibrium point $k^* \in \mathring{\mathscr{K}}$ for which $\rho_j(k^*)=0$ and $k_j^* \in (\underline{k}_j, \bar{k}_j)$, $j=1,\dots, n$.

(28) is a natural economic condition on the net own-rates of return $\rho_j(k)$. *For each capital good j, $\rho_j(k)$ must be positive (negative) when the endowment*

of this capital good is sufficiently small (large), independent of the endowments of the other capital goods $(i \neq j)$.

5. THE EQUILIBRIUM MANIFOLD

The analysis of a dynamic system whose trajectories are bounded reduces in essence to an analysis of its ω-limit sets. To the extent that equilibria are important components of the ω-limit sets, an analysis of the equilibria is an important ingredient towards understanding the dynamic system. With this in mind we develop certain general properties of the equilibrium set $\mathscr{E}$ which, under a rank condition, becomes an equilibrium manifold. *The main qualitative results are not restricted to cases where the equilibria are derivable as the extrema of a potential function. It is only when stability considerations enter that this assumption is required.*

The equilibrium manifold is of especial interest for a dynamic system whose eigenvalues are *real*, for when the eigenvalues are *complex* the stability of equilibrium can change by a pair of complex conjugate eigenvalues crossing the imaginary axis, and such a change of stability cannot be deduced from geometric properties of the equilibrium manifold, since stability can change without passing through a critical parameter value.

ASSUMPTION 5 (Rank condition). *The matrix*

$$[\phi_{kk}(k^*, \alpha) \quad \phi_{k\alpha}(k^*, \alpha)]$$

has rank n *for all* $(k^*, \alpha) \in \mathscr{E}$.

Remark. The following is a result of the implicit function theorem: *under Assumption 5,* $\mathscr{E}$ *is a* C^{r-1} *s-dimensional manifold.* Thus $\mathscr{E}$ cannot have the form shown in Figure 1.

PROPOSITION 6. *Under Assumption 5,* $\mathscr{A}^c$ *is a set of measure zero in* $\mathscr{A}$.

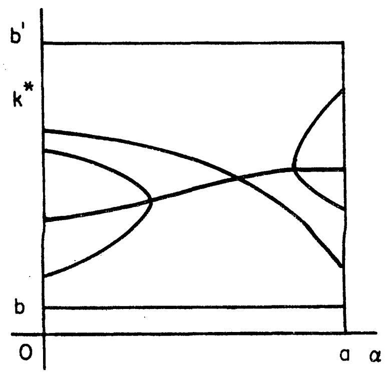

FIGURE 1

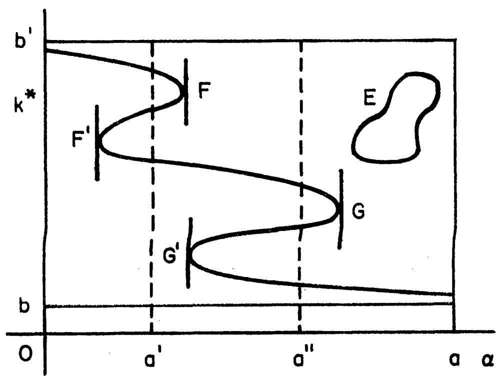

FIGURE 2

PROOF. Assumption 5 asserts that $\phi_k(k^*, \alpha)$ is transversal to 0. The result then follows from the Transversal Density Theorem (Abraham-Robbin [1967, p. 48]; Dierker [1974, p. 91]). □

PROPOSITION 7. *Let A be a subset of $\mathscr{A}$ with the property that $(k^*, \alpha) \in \mathscr{E}$, $\alpha \in A$ implies $k^* \in K \subseteq \mathring{\mathscr{K}}$, for K compact, then $\mathscr{A}^c$ is closed in A.*

PROOF. Consider $\alpha_n \in \mathscr{A}^c \cap A$, $\alpha_n \to \bar{\alpha}$, then there exist $k_n^* \in K$ such that $\phi_k(k_n^*, \alpha_n) = 0$, $|\phi_{kk}(k_n^*, \alpha_n)| = 0$. Since K is compact there exists a subsequence $k_{n_j}^* \to \bar{k}^*$ and since ϕ_k, ϕ_{kk} are continuous $\phi_k(\bar{k}^*, \bar{\alpha}) = |\phi_{kk}(\bar{k}^*, \bar{\alpha})| = 0$ so that $\bar{\alpha} \in \mathscr{A}^c$. □

PROPOSITION 8. *Let $A \subseteq \mathscr{A}$ be as in Proposition 7. If $\alpha \in \mathring{A} \cap \mathscr{A}^r$, then there exist a neighborhood $\mathscr{N}(\bar{\alpha})$ and m, C^{r-1} functions $\psi^1(\alpha), \ldots, \psi^m(\alpha)$, where $\psi^i(\alpha): \mathscr{N}(\bar{\alpha}) \to \mathscr{K}$, such that for all $\alpha \in \mathscr{N}(\bar{\alpha})$, $\mathscr{E}(\alpha)$ consists of the m distinct points $\psi^1(\alpha), \ldots, \psi^m(\alpha)$.*

PROOF. The result follows from the implicit function theorem. □

Remark. For a compact subset of parameters (such as the interval $[0, a]$ in Figure 2) such that all equilibria associated with these parameter values lie in a compact set (say the interval $[b, b']$) the equilibrium manifold must consist of finitely many connected components. As yet nothing prevents the existence of a closed curve of equilibria E as in Figure 2. It is also clear that in general $\mathscr{E}$ need not consist of a unique connected component for arbitrary parameter sets (as is evident by selecting the subset $[a', a'']$ in Figure 2). A special condition must also be imposed to eliminate the folds F, F', G, G' in Figure 2. Proposition 9 will eliminate closed curves such as E, while Propositions 10 and 11 give conditions under which the latter two properties hold.

PROPOSITION 9. *Let $A \subseteq \mathscr{A}$ be as in Proposition 7, $\mathring{A} \neq \emptyset$ and let* $\operatorname{rank}(\phi_{k\alpha}) = n$, *then no connected component of $\mathscr{E}$ is compact in $\mathring{\mathscr{K}} \times \mathring{A}$.*

PROOF. Suppose E is a connected component of $\mathscr{E}$ which is a compact subset of $\mathring{\mathscr{K}} \times \mathring{A}$. Let $\bar{k}$ be such that there exists an $\bar{\alpha}$ with $(\bar{k}, \bar{\alpha}) \in E$ and $\|\bar{k}\| = \sup\{\|k\| \mid (k, \alpha) \in E$, for some $\alpha \in \mathring{A}\}$. Such a $\bar{k}$ must exist since E is compact and projection preserves compactness. Since $\phi_{k\alpha}(\bar{k}, \bar{\alpha})$ has rank n there exist neighborhoods $\mathscr{N}(\bar{\alpha})$, $\mathscr{M}(\bar{k})$ and a map $\psi: \mathscr{M}(\bar{k}) \to \mathscr{N}(\bar{\alpha})$ such that for all $k \in \mathscr{M}(\bar{k})$ if $\alpha = \psi(k)$ then $(k, \alpha) \in \mathscr{E}$. Let $k_n \to \bar{k}$ be a sequence such that $\|k_n\| > \|\bar{k}\|$ and $k_n \in \mathscr{M}(\bar{k})$. Since ψ is C^{r-1}, $(k_n, \psi(k_n)) \to (\bar{k}, \bar{\alpha})$. Since E is a connected component of $\mathscr{E}$, for large n, $(k_n, \psi(k_n)) \in E$ which is a contradiction. □

PROPOSITION 10 (Unique component). *Let A be an open ball of finite radius $r > 0$ with $\bar{A} \subseteq \mathscr{A}$. If there exists*

(i) *a compact set $K \subseteq \mathring{\mathscr{K}}$ such that $\alpha \in \bar{A}$ and $(k, \alpha) \in \mathscr{E}$ implies $k \in K$*

(ii) *a function $\psi: \partial A \to K$ such that $(k, \alpha) \in \mathscr{E}$ if and only if $k = \psi(\alpha)$*

and if

(iii) $\operatorname{rank} \phi_{k\alpha}(k, \alpha)=n$ *for all* $(k, \alpha) \in \mathscr{E}$ *with* $\alpha \in \bar{A}$,
then $\mathscr{E} \cap (\mathscr{K} \times \bar{A})$ *has a unique connected component.*

PROOF. Note that ψ in (ii) must be continuous, for if $\alpha_n \to \bar{\alpha}$, since $\psi(\alpha_n) \in K$ there exists a subsequence $\psi(\alpha_{n_j}) \to \bar{k} \in K$. By continuity of ϕ_k, $(\bar{k}, \bar{\alpha}) \in \mathscr{E}$. Hence $\bar{k}=\psi(\bar{\alpha})$. Thus if $s>1$, $\mathscr{E} \cap (\mathscr{K} \times \partial A)$ has a unique connected component that is the graph of a continuous function ψ which is defined on a connected set. For $s=1$, $\mathscr{E} \cap (\mathscr{K} \times \partial A)$ consists of two points.

Suppose W is a connected component of $\mathscr{E} \cap (\mathscr{K} \times \bar{A})$ such that there exists $\alpha \in A$, $k \in K$ with $(k, \alpha) \in W$. In view of Proposition 9, W is not compact in $\mathring{\mathscr{K}} \times A$. By the continuity of ϕ_k and compactness of K if $\alpha_n \in A$, $\alpha_n \to \alpha \in A$ and $k_n \to k$, $(k_n, \alpha_n) \in W$, then $(k, \alpha) \in W$. Hence there exists a sequence $(k_n, \alpha_n) \in W$ with $\alpha_n \to \bar{\alpha} \in \partial A$. By the continuity of ϕ_k and compactness of K there exists $\bar{k} \in K$ with $(\bar{k}, \bar{\alpha}) \in W \subset \mathscr{E}$. Hence by (ii), $\bar{k}=\psi(\bar{\alpha})$. Thus for $s>1$ any connected component possesses a point in common with the unique connected component at the boundary. Hence there exists only one such connected component.

When $s=1$, let $\underline{k}=\inf\{k \mid (k, \alpha) \in W, \alpha \in \bar{A}\}$ then by the same reasoning as in Proposition 9, $(\underline{k}, \underline{\alpha}) \in W$ for some $\underline{\alpha} \in \partial A$. Similarly let $\bar{k}=\sup\{k \mid (k, \alpha) \in W, \alpha \in \bar{A}\}$, then $(\bar{k}, \bar{\alpha}) \in W$ for some $\bar{\alpha} \in \partial A$. By (iii) $\underline{k} \neq \bar{k}$. Hence W must contain the two points in $\mathscr{E} \cap (\mathscr{K} \times \partial A)$. □

The following result is a straightforward extension to the case of an arbitrary number of parameters of Brock's Jacobian condition for uniqueness (Brock [1973, Theorem 1]).

PROPOSITION 11 (Unique equilibrium). *Under the conditions of Proposition 8, if there exists*

(i) *a compact connected subset* $B \subseteq \mathring{A}$

(ii) $\bar{\alpha} \in B$ *such that* $k(\bar{\alpha})$ *is unique with* $(k(\bar{\alpha}), \bar{\alpha}) \in \mathscr{E}$

and if

(iii) $\operatorname{rank} \phi_{kk}(k, \alpha)=n$ *for all* $(k, \alpha) \in \mathscr{E}$ *with* $\alpha \in B$,

then for any $\alpha \in B$ *there exists* $k(\alpha)$, *with* $(k(\alpha), \alpha) \in \mathscr{E}$. *Furthermore* $k(\alpha)$ *is unique and depends* C^{r-1} *on* α.

PROOF. By (iii) every $\alpha \in B$ is regular. By Proposition 8, since B is compact there exists a finite collection $\mathscr{N}(\alpha_1), \ldots, \mathscr{N}(\alpha_n)$ that covers B. Since B is connected the number m of solutions $\psi^1(\alpha), \ldots, \psi^m(\alpha)$ in Proposition 8 is the same for all $\alpha \in \mathscr{N}(\alpha_j)$ for $j=1, \ldots, n$. The result follows from (ii). □

DEFINITION. An equilibrium point is *hyperbolic* if $\operatorname{Re}(\lambda_i) \neq 0$, $i=1, \ldots, 2n$.

Remark. The theorem of Hartman-Grobman [1960] asserts that *two hyperbolic equilibria of the same degree of stability are locally topologically equivalent* (i.e., there exist neighborhoods of each equilibrium point and a homeomorphism that transforms the trajectories around one equilibrium point into trajectories around the other equilibrium point).

DEFINITION. Let $\alpha, \alpha' \in \mathscr{A}^r$. We say that α and α' have *$\mathscr{E}$ equivalent phase portraits* if the number of distinct equilibria is the same and there exists an ordering of the equilibria such that $k^{*i}(\alpha)$ has the same degree of stability as $k^{*i}(\alpha')$.

Remark. By Proposition 8 *if $\alpha, \alpha' \in \mathscr{N}(\bar{\alpha})$ then α and α' have $\mathscr{E}$ equivalent phase portraits.* Another way of saying this is as follows: a critical parameter value $\bar{\alpha}$ is a parameter value such that in an arbitrarily small neighborhood of $\bar{\alpha}$ there may exist parameter values α and α' with nonequivalent phase portraits. If in an arbitrarily small neighborhood of $\bar{\alpha}$ there exist parameter values α and α' such that the phase portraits are nonequivalent, then $\bar{\alpha}$ is said to be a *bifurcation point*. *Propositions* 6 *and* 7 *establish that the critical parameter values and hence the bifurcation points are a negligible set in the parameter space $\mathscr{A}$. Proposition* 11 *gives conditions under which the phase portraits are $\mathscr{E}$ equivalent for all parameter values $\alpha \in B$, the unique equilibrium point having the same degree of stability for all $\alpha \in B$.*

Remark. If we knew more about the family of potential functions $\phi(k^*, \alpha)$ generated by a given family of integrands $e^{-\delta t}L(k, \dot{k}; \beta)$ then it might be feasible to use certain *generic properties* to establish more precise results about the nature of the equilibrium manifold $\mathscr{E}$ and its associated set of bifurcation points. Thus for example in the case where there are 5 or less parameters *catastrophe theory* might be used to assert that the bifurcation sets consist of smooth surfaces in $\mathscr{A}$ glued together in an appropriate way (Zeeman [1976]; Bröcker [1975, Chapters 15, 17]).

The reader is referred to the paper of Magill [1977a] for a simple example of the application of catastrophe theory that involves a highly simplified version of the results of this section. The example uses the Liviatan-Samuelson one sector model with joint production. The same paper and the paper of Scheinkman [1978] contain several other examples to which the theory of the present paper is applicable. In particular when $n=1$ the symmetry condition is automatically satisfied and the ω-limit sets consist solely of equilibrium points. For this case we thus have a complete theory (Magill [1977a, Section 5]).

6. FINAL REMARKS

In this paper we have established a fairly complete picture of the *local* phase portrait of a dynamic system generated by a symmetric variational problem, for all values of the underlying vector of parameters $\alpha \in \mathscr{A}$. We have related, at least in a preliminary way, the nature of these phase portraits to properties of the equilibrium manifold $\mathscr{E}$.

When $n \geqq 2$ it is well-known however that ω-limit sets are not restricted to equilibrium points. Thus when $n=2$, ω-limit sets may consist of *equilibria, closed orbits* or *limit continua* (closed curves formed by a combination of orbits and equilibria). For this reason when $n \geqq 2$ the local phase portrait for the equilibria may give a very incomplete picture of the global phase portrait of the dynamic system. We have shown however, that the dynamic systems generated by *symmetric variational problems* behave in the neighborhood of each equilibrium point like *gradient systems*. It is well-known that the ω-limit sets of gradient systems consist solely of equilibria. Scheinkman [1978] established that *if* $L(k, \dot{k})$ *is additively separable*[11] *then there exists a* $k^* \in \mathring{\mathscr{K}}$ *which is the* ω-*limit set of all positive semi-trajectories.* It would be interesting to establish general conditions under which the ω-limit sets of a symmetric variational problem consist solely of equilibrium points.

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