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Nonshiftable capital, affine price expectations and convergence to the Golden Rule

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Abstract

This paper studies an overlapping generations (OLG) model with production under the assumption that capital investment is completely irreversible: installed capital cannot be transformed back into consumption good nor transferred from one firm to another. Since firms cannot be dismantled at each generational change without loosing their value, their ownership is transmitted from generations to generations through a stock market. The paper shows that the financial price of a firm can be lower than the replacement value of its capital without creating arbitrage or dampening the incentives to invest. This possibility changes the long-run behavior of the equilibrium, but only for economies with underaccumulation. In the stock market dynamics these economies have two steady-states, the Diamond steady-state and the Golden Rule. The Diamond steady stable is locally saddle-point stable and can be reached by only one trajectory on which the financial price and replacement value of firms coincide at all times. All other trajectories on which there is a discount on equity converge (when they converge) to the Golden Rule which is locally stable: the discount on equity has the same effect as an increase of the savings of the young, which lowers the interest rate, and increases investment and wages at the next generation, a virtuous cycle which leads to the efficient long-run steady-state. On all these trajectories the equity prices are larger than the fundamental value of future dividends and thus include a bubble component.

JEL classification: E 44

Keywords: Overlapping generations; Firm specificity of capital; Affine price expectations; Golden Rule; Equity prices; Bubble

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1. Introduction

This paper addresses the following question: does the stock market influence the process of capital accumulation? If exchanging ownership of firms on a stock market is equivalent to exchanging the ownership of their capital on a capital goods market, then introducing a stock market will not affect the predictions of the real models of capital accumulation—the Ramsey model (1928) if agents are infinitely lived, or Diamond's overlapping generations (OLG) model (1965) if agents are finitely lived. The assumption that ownership of firms is transferred through the stock market rather than the capital goods market can lead to a different outcome only if there is a friction which makes the stock market into a financial entity distinct from the real capital goods market.

The friction that we study in this paper is the *nonshiftability* or *firm specificity* of installed capital which makes it costly, if not impossible, to detach part of the tangible or intangible capital of a firm to sell it on a (second-hand) market for capital goods. To quote Tobin (1998, p. 147) "The various physical assets of a business enterprise are often designed, installed and used in complex combinations specific to the technology. It is costly or impossible to detach and move individual assets or to apply them to alternative purposes." We take this observation to the theoretical limit by assuming that capital, once installed, is nonshiftable: it cannot be transformed back into a consumption good or transferred for use by other firms.

Under this assumption, when capital is durable, firms must be long lived and if transferred, must be kept intact in their entirety. If, as we shall assume, economic agents are short lived, then there is a need for a market which makes such transfers possible, and this is one of the important roles of the stock market: each firm becomes a separate legal entity which issues equity shares to its future income stream, and ownership of firms can be transferred in perfectly divisible amounts across an indefinite succession of finite-lived shareholders, while retaining in perpetuity the full physical and organizational entity of the firm.[1]

We are thus led to study the role of the stock market as an instrument for transferring firms in the setting of the standard Diamond model to which we add the friction that capital once installed in a firm cannot subsequently be sold (i.e. has a zero price) on the market for current output. We do not assume any frictions on "new" investment: thus the financial value of a firm cannot exceed its replacement cost, for the young agents could always recreate the capital of the firm out of current output if it were less expensive to do so, and the firm would not sell at its current equity price. However, and this is the important point, the assumption that previously installed capital is nonshiftable, permits the equity price to be less than the replacement cost without creating arbitrage opportunities.

If the equity price of a firm does not necessarily coincide with the replacement value of its capital, then it has to be determined by a rate of return condition: while old agents have no choice but to sell their firms on the equity market, young agents have several choices. They can decide whether to invest in equity or bonds and whether or not to invest new

[1] Blackstone (1765) in his *Commentaries on the Laws of England* (Book I, Chapter XVIII), referred to "perpetual succession" as the "very end of incorporation: for there cannot be a succession forever without an incorporation." He explained "it has been found necessary when it is for the advantage of the public to have any particular rights kept on foot and continued, to constitute artificial persons, who may maintain a perpetual succession, and enjoy a kind of legal immortality. These artificial persons are called . . . *corporations*."

capital in their firms. Absence of arbitrage requires that the rate of return on each of these "investments" is the same. Since the rate of return on equity depends on the dividends that agents anticipate for next period and on the price that they expect to receive when they sell their shares, to close the model we need to introduce an assumption regarding agents' expectations of future equity prices. The assumption that we adopt in this paper is that agents have *affine price expectations*, i.e. they expect that the equity price of a firm will be equal to the replacement value of its capital less a lump-sum discount. We show that such expectations are compatible with an equilibrium in which all rates of return are equalized and investment is positive, provided that the discount grows at the rate of interest and does not become too large—in a sense made precise in Section 2.

The affine pricing of equity leads to an interesting new mechanism by which the stock market influences investment, especially for the class of economies regarded by many economists (see, for e.g. Abel et al., 1989) as empirically the most relevant, namely, those characterized by underaccumulation. For in such economies the savings of the young are scarce and in the standard Diamond model, do not suffice to lead the economy to the Golden Rule: as the term of 'underaccumulation' suggests, the Diamond equilibrium, although dynamically efficient is not long-run efficient. However when there is a discount on the equity prices of firms, this discount—no matter how small—frees some of the scarce savings of the young and enables them to be used to purchase new investment rather than paying for previously installed capital. Although the investment behavior of firms in our model is the same as in Diamond's model, it is "as if" there were more savings in the economy (thanks to the discount), so that the equality "savings = investment" occurs at a lower interest rate than in the Diamond equilibrium. As a result there is more investment, and hence more output, wages and savings in the next period, and this virtuous cycle fuels a sufficient increase in investment to lead the economy to the Golden Rule—the efficient steady-state—rather than to the Diamond steady-state.

The idea that frictions, or adjustment costs, may importantly influence the process of capital accumulation has a long tradition in economics (Lucas, 1967; Gould, 1968; Uzawa, 1969; Kydland-Prescott, 1982) and some authors have derived from such adjustment costs the existence of a Tobin's q different from 1 (Hayashi, 1982; Basu, 1987; Abel, 1999). These papers study the effects of adjustment costs in installing new capital, and typically assume that these costs are convex. Our approach is different in that it focuses on the adjustment costs that would need to be incurred if previously installed capital were to be put to an alternative use, rather than on the cost of installing new capital. As a consequence in our model, Tobin's q is always less than or equal to one.

Since the model has explicit equity and bond markets, in Section 4 we analyze the properties of the equilibria from the financial perspective. Models of financial economies over an open-ended future are mainly models with infinite-lived agents in which equilibria have the property that the price of a security in positive supply is equal to the discounted sum of its future stream of dividends (its *fundamental value*). Although this property is considered as 'normal', it is known since the work of Tirole (1985) that it does not always hold in an OLG model. Tirole exhibited a variant of Diamond's model in which a security with a zero dividend has a positive price, so that its price exceeds its fundamental value (its price is said to have a *bubble component*). However in Tirole's model such a bubble component can arise only in economies with overacccumulation, leading many economists to believe that bubbles can exist only in economies in which equilibria are inefficient. In our model, the equilibrium price of equity has a bubble component even in economies with underaccumulation, as soon that there is a positive discount on equity. The price of equity is equal to its fundamental value only for an equilibrium of an economy with underaccumulation for which the initial discount on equity is zero, in which case the equilibrium coincides with the Diamond equilibrium. Since the equilibria of an economy with underaccumulation in which equity prices have a positive discount relative to replacement value converge to the Golden Rule and are both dynamically and long-run efficient, the presence of a bubble component on equity is not in this case a sign of inefficiency.

The paper is organized as follows. Section 2 describes the model and introduces the concept of a stock market equilibrium. The asymptotic properties of such an equilibrium are studied in Section 3. The comparison between firms' market values and the discounted sums of their dividends is the subject of Section 4.

2. The stock market model

Consider a standard OLG model with production: at each date t, N_t young agents are born who live for two periods, t and $t+1$, and each of these agents is endowed with 1 unit of labor when young, having no initial resources when old. Agents of all generations are identical, with the same endowment (1 unit of labor when young) and the same preferences, represented by a utility function $u(c_0^t, c_1^t)$ over consumption streams $c^t = (c_0^t, c_1^t)$, where $c_s^t, s = 0, 1$, represents the consumption at date $t+s$ of an agent born at date t. The population is assumed to grow at the exogenous rate n $(n \geq 0)$, i.e. $N_{t+1} = (1+n)N_t$.

On the production side, there is a collection of J firms $(j = 1, \dots, J)$, each firm producing at each date t an all-purpose good—which we will call the output—from capital and labor, with the time-invariant technology $Y_t^j = F(K_t^j, L_t^j)$, where the function F is the same for all firms and is smooth, concave, strictly increasing and homogeneous of degree 1. The output of firms can be used either directly for consumption or to create new capital, where it takes one unit of the good to produce one unit of new capital for any firm. Capital in each firm is durable and depreciates at the rate $\beta(0 < \beta < 1)$, and needs to be installed one period before it is used: thus the capital K_t^j used by firm j is the capital that it has carried over from date $t-1$.

The model that we introduce differs from that of Diamond (1965) by the assumption that capital once installed in a firm cannot be "unbolted" and transformed back into the homogeneous current output or transferred to another firm, without incurring significant adjustment costs—which for simplicity we take to be infinite. Thus once capital has been installed in a firm, it cannot be used for consumption, nor can it be used for new investment (i.e. additional capital) by any other firm: in short, it is sunk in the firm. As indicated in Section 1, this assumption is designed to capture the fact that many resources invested in a firm have to be adapted in a way which is firm specific to make the whole production process function smoothly and efficiently. Since the precise way in which the resources have been adapted typically makes them inappropriate for use by other firms, such installed capital has limited value on a resale market. For example, software written specifically for a firm and incorporating its specific needs may be very expensive—it consumes a great deal of labor not used for producing the consumption good—but has essentially no resale value. Even those capital goods which have a resale value, for example, plant and machinery, usually have a low value on the used capital market relative to their replacement cost, since significant "adjustment costs" have to be incurred to adapt them for use by other firms. To capture this phenomenon in a simple way, we study the theoretical limit in which the installed capital of a firm is completely firm specific, so that no part of it has a positive resale value on the second-hand market.

In such an economy capital accumulation will only take place if the market structure permits firms to be infinitely lived. Invested capital has no value if the firm is liquidated, and has value only if the firm retains its identity as an income generating unit in the economy. The natural market structure which permits short-lived agents to transfer ownership of long-lived firms from one generation to the next is an equity market for ownership shares of firms. Thus to have a market structure consistent with the firm specificity of capital, we assume that each firm is a corporation with an infinite life whose ownership shares are transmitted from one generation to the next through the stock market. Let Q_t^j denote the equity price of firm j at date t.

At each date t, in addition to the stock market, there are three other markets: a market for current output, a labor market, and a bond market. Since this is a real (as opposed to a monetary) model, the price of a unit of current output is normalized to be 1. Let w_t denote the wage rate at date t on the labor market on which the (homogeneous) services of labor supplied by the young generation are sold to the firms. The bond market provides firms with a source of external funds for financing investment which is an alternative to issuing new equity shares, and gives young agents a way of borrowing and lending. Let r_{t+1} denote the interest rate on a loan from date t to date $t+1$ and let $(1, (Q_t^j)_{j=1}^J, w_t, r_{t+1})$ denote the vector of prices on these four markets at date t $(t = 0, 1, \ldots)$.

2.1. Affine price expectations

The assumption that capital is firm specific can be viewed as a strengthened version of the assumption of nonshiftability of capital across sectors. As LeRoy (1983) pointed out, the assumption of nonshiftability of capital across sectors was the accepted framework of analysis of the classical economists from Adam Smith to Marshall and is essential for a proper appreciation of Keynes' (1936) theory of investment. Authors who formalized the assumption of nonshiftability of capital across sectors (e.g. Ryder, 1969; LeRoy, 1983) assumed that capital once in a given sector cannot be transferred to another sector. Capital goods can however be shifted from one firm to another within a given sector. As a result the price of existing capital must be the same for all firms in a same sector and equal to the price of the new capital goods in this sector, but the prices can differ across sectors.

We make the stronger assumption that capital once installed cannot be shifted from one firm to another, even if they belong to the same sector—since for simplicity we study only a one-sector economy. One can think of firms operated at different locations, with the same ability to produce output (income) from a given value of capital and a given quantity of labor. While new capital can be installed at any location, once installed the capital cannot be transferred from one location to another. In particular there is no rental of capital and all firms install and own their own capital. Since there is no market on which used capital goods can be sold per unit and since capital goods can only be sold indirectly through the transfer of ownership of the firms in which they are embodied, the (equity) price of a firm does not need to be equal to the quantity of capital.

Let ξ_t^j denote the installed capital of firm j at date t when it is to be sold on the equity market. ξ_t^j is the result of past capital accumulation and is equal to the accumulated sum of past investments, once depreciation has been taken into account. Since we assume that one unit of good can be transformed into one unit of capital, ξ_t^j is also the *replacement cost* of firm j: by this we mean that agents could recreate a firm equivalent to firm j by purchasing ξ_t^j units of good on the current output market.[2] If firms can be recreated in this way, then the equity price of firm j cannot exceed its replacement cost. On the other hand, the equity price can be below ξ_t^j without creating an arbitrage opportunity since, after buying the firm, agents cannot turn around and recover ξ_t^j since there is no market for installed capital. If all we know is that the equity price must be less than or equal to the replacement cost of the firm, how can it be determined? To complete the story we need to introduce an assumption about agents' expectations. Buying the shares of firm j at date t gives the right to make the production decision for the firm—in particular the investment decision—to receive the profit of the firm next period and since agents are two-period lived, sell the equity next period. Thus the price that agents are prepared to pay for the firm at date t and the investment decision that will be made in the firm depends on the expectation of the equity price at date $t+1$ and on the way this price is influenced by the investment decision. The objective of the paper is to show that if agents at date t have expectations of the form

$$Q_{t+1}^j(\xi_{t+1}^j) = \max(\xi_{t+1}^j - V_{t+1}^j, 0), \quad V_{t+1}^j \geq 0 \tag{1}$$

for the equity price at date $t+1$ then under suitable conditions there exist equilibria with self-fulfilling expectations and positive investment. Actually we will show that investment is positive only if the anticipated (and realized) price is positive, so that price expectations—and equilibrium prices—take the simpler form

$$Q_{t+1}^j(\xi_{t+1}^j) = \xi_{t+1}^j - V_{t+1}^j, \quad V_{t+1}^j \geq 0 \tag{2}$$

which is an affine function of firm j's capital stock. We will call the resulting equilibrium an equilibrium *with affine price expectations*. Such expectations include the standard case for which $V_{t+1}^j = 0$ at all dates and the price of the firm is equal to its replacement cost. If $V_{t+1}^j > 0$ the price of firm j is less than its replacement cost and we call V_{t+1}^j the *discount* on the equity of firm j.

The assumption that capital, once installed, cannot be transformed back into the consumption good, nor transferred to other firms, has two consequences: the first is that investment

[2] Thus we assume that if a firm consists of both tangible and intangible capital, then both types of capital can be reproduced. If there is some capital which cannot be replaced at any (or only at a very large) cost due to some special knowledge or some first-mover advantage, then ξ_t^j, taken as the accumulated sum of past investments, can be less than the replacement cost of the firm, and the equity price could be more than ξ_t^j. In this paper we do not consider such non-reproducible capital. Note that it will follow from the analysis of Section 3 that for economies with underaccumulation, removing the assumption of reproducibility does not permit any new equilibria other than that considered in this paper.

must be non-negative—the irreversibility constraint; the second is that it opens up the possibility of affine price expectations. The irreversibility constraint can be incorporated into Diamond's model and, when binding, has consequences for equity prices, which have been studied by Huffman (1986) in a stochastic Diamond model. In this paper we consider only equilibria where investment is positive at all dates: we thus by-pass the effect of the irreversibility constraint. This amounts to restricting attention to economies for which the initial level of capital is low. Our goal is to focus on the second effect—on the existence of equilibria with affine price expectations and on their long-run properties.

2.2. Corporation's decision problem

Firms are owned by the equity holders and are managed so as to maximize the payoff to the current owners. Suppose the stock market opens at date t, after production has taken place and capital has depreciated: young agents buy the shares of firm j, endowed with a capital $(1-\beta)K_t^j$, from the old for the price Q_t^j and decide on the investment $I_t^j \geq 0$ to be made. The date t investment is chosen so as the maximize the net present value of investment

$$-I_t^j + \frac{1}{1+r_{t+1}}[F((1-\beta)K_t^j + I_t^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j$$
$$+\max\{(1-\beta)^2K_t^j + (1-\beta)I_t^j - V_{t+1}^j, 0\}] \quad (3)$$

anticipating the next-period labor decision L_{t+1}^j and the effect of the investment I_t^j on the resale price Q_{t+1}^j of equity next period, where the anticipated price is given by (1).

To eliminate the max operator in expression (3) we need to consider two cases:

(i) $(1-\beta)^2K_t^j - V_{t+1}^j < 0$
(ii) $(1-\beta)^2K_t^j - V_{t+1}^j \geq 0$

In case (i) the first units of investment, up to $\bar{I}$ such that $(1-\beta)^2K_t^j + (1-\beta)\bar{I} - V_{t+1}^j = 0$, do not increase the resale value of the firm, which stays equal to zero. It follows that if investment is positive the resulting increase in the equity price will not fully reflect the (depreciated) value of the new investment. In this case it is shown in the proof of Proposition 1 that the optimal strategy is not to invest in the firm.

Proposition 1. *If the anticipated price for equity at date $t+1$ is given by (1), and if $V_{t+1}^j > (1-\beta)^2K_t^j$, then the optimal solution to the investment problem (3) at date t is $I_t^j = 0$.*

Proof. See Appendix A. □

In case (ii), for any value of I_t^j, the expected price is $(1-\beta)^2K_t^j + (1-\beta)I_t^j - V_{t+1}^j \geq 0$: since V_{t+1}^j is a constant which does not affect the solution to the maximum problem (3), the choice of investment is the same as in the standard Diamond model. A solution of this problem is a solution to the FOC

$$\begin{aligned} &F'_K((1-\beta)K_t^j + I_t^j, L_{t+1}^j) \le r_{t+1} + \beta, \quad \text{with equality if } I_t^j > 0, \quad t \ge 0 \\ &F'_L((1-\beta)K_t^j + I_t^j, L_{t+1}^j) = w_{t+1}, \qquad t \ge 0 \end{aligned} \tag{4}$$

The homogeneity of degree one of F implies that F'_K and F'_L are homogenous of degree 0, so that (4) only determines a capital–labor ratio. Since we have assumed that all firms have the same production function, the optimal capital–labor ratio will be the same for all firms. If we let $k = K/L$ denote the capital–labor ratio and let $f(\cdot)$ denote the production function per unit of labor

$$f(k) = F(k, 1)$$

then by the homogeneity of F, $F'_K(K, L) = f'(k)$, $F'_L(K, L) = f(k) - kf'(k)$. Whether or not firms undertake positive investment in equilibrium depends on the sequence of equilibrium prices $(w_t, r_{t+1})_{t\ge 0}$ and whether or not there exists a sequence of capital–labor ratios $(k_t)_{t\ge 0}$ satisfying

$$\begin{aligned} &f'(k_{t+1}) = r_{t+1} + \beta, \qquad t \ge 0 \\ &f(k_t) - k_t f'(k_t) = w_t, \quad t \ge 0 \end{aligned} \tag{5}$$

and which in addition satisfies

$$K_{t+1} = k_{t+1}N_{t+1} > (1-\beta)K_t = (1-\beta)k_t N_t, \quad t \ge 0 \tag{6}$$

Recall that N_t is the quantity of labor supplied at date t, since each agent is assumed to supply inelastically one unit of labor when young. Thus conditions (5) and (6) must be entered in the definition of an equilibrium with positive investment, to be defined shortly, and will ensure, as long as $V_{t+1}^j < (1-\beta)^2 K_t^j$, that for $j = 1, \dots, J$ there exists a solution $I_t^j > 0$ to the maximum problem of firm j for all $t \ge 0$ and that the FOC (4) are satisfied with equality. From now on we consider only equilibria which satisfy (5) and (6) and

$$V_{t+1}^j \le (1-\beta)^2 K_t^j, \quad t \ge 0, \quad j = 1, \dots, J \tag{7}$$

The criterion (3) for the choice of investment at date t suggests that the shareholders directly finance the investment, receiving the output of the firm (net of labor costs) plus the resale value of their equity in the next period. Such a method of financing is not especially realistic. However if the firm finances its investment by one-period borrowing, then the decision criterion is unchanged since (3) can be written as

$$\begin{aligned} \frac{1}{1+r_{t+1}}[&F((1-\beta)K_t^j + I_t^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j \\ &-(1+r_{t+1})I_t^j + Q_{t+1}^j((1-\beta)K_{t+1}^j)] \end{aligned}$$

This corresponds to the sum of the dividend D_{t+1}^j and the capital value Q_{t+1}^j, where

$$D_{t+1}^j = F(K_{t+1}^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j - (1+r_{t+1})I_t^j \tag{8}$$

is the dividend received by the shareholders of firm j. This more realistic method of financing thus leads to the same investment decision. More generally it can be shown that the Modigliani–Miller theorem holds for this economy: the real outcome (firms' production and agents' consumption) is independent of the mode of financing, provided the borrowing of firms does not lead the firms to bankruptcy. To simplify the exposition, we will assume up to Section 4 that firms finance their investment using one-period loans which are reimbursed the following period.

2.3. Agent's maximum problem

The representative young agent born at date t purchases a portfolio of securities

$$(z_t, \theta_t^1, \ldots, \theta_t^J)$$

consisting of an amount z_t of bonds and a share θ_t^j of firm j (for $j = 1, \ldots, J$), so as to maximize lifetime utility $u(c^t)$ subject to the budget constraints

$$\left.\begin{aligned} c_0^t &= w_t - z_t - \sum_{j=1}^{J} \theta_t^j Q_t^j \\ c_1^t &= z_t\,(1 + r_{t+1}) + \sum_{j=1}^{J} \theta_t^j (D_{t+1}^j + Q_{t+1}^j) \end{aligned}\right\} \quad \forall t \geq 0 \tag{9}$$

where D_t^j denotes the dividend paid by firm j at date t. The agent takes the current prices $(1, (Q_t^j)_{j=1}^J, w_t, r_{t+1})$ as given, and correctly anticipates the next-period dividends and prices of the firms $(D_{t+1}^j, Q_{t+1}^j)_{j=1}^J$. The maximum problem of the agent has a solution if and only if the no-arbitrage condition between the stock and the bond market

$$Q_t^j = \frac{1}{1 + r_{t+1}} (D_{t+1}^j + Q_{t+1}^j), \quad j = 1, \ldots, J, \quad \forall t \geq 0 \tag{10}$$

holds for the equity price of each firm. In view of (8) and Euler's theorem, D_{t+1}^j can be written as

$$D_{t+1}^j = K_{t+1}^j F_K'(K_{t+1}^j, L_{t+1}^j) + L_{t+1}^j F_L'(K_{t+1}^j, L_{t+1}^j) - w_{t+1} L_{t+1}^j - (1 + r_{t+1}) I_t^j$$

which, on a trajectory for which the FOC (4) are satisfied with equality reduces to

$$D_{t+1}^j = K_{t+1}^j (r_{t+1} + \beta) - (1 + r_{t+1}) I_t^j$$

If, in addition, agents' affine price expectations given by (2) are realized in equilibrium then

$$\frac{D_{t+1}^j + Q_{t+1}^j}{Q_t^j} = \frac{K_{t+1}^j (r_{t+1} + \beta) - (1 + r_{t+1}) I_t^j + (1 - \beta) K_{t+1}^j - V_{t+1}^j}{(1 - \beta) K_t^j - V_t^j}$$

and (10) holds if and only if

$$V_{t+1}^j = (1 + r_{t+1}) V_t^j, \quad t \geq 0 \tag{11}$$

When (11) is satisfied for all $j = 1, \ldots, J$, the rate of return on the bond and each of the equity contracts is the same, and the agent is indifferent between investing in any firm or investing in the bond market: all that matters is the total sum invested in the capital markets, namely, the agent's total savings s_t. When (10) holds the budget Eq. (9) can be written as

$$\begin{aligned} c_0^t &= w_t - s_t \\ c_1^t &= s_t(1 + r_{t+1}) \end{aligned} \tag{12}$$

where

$$s_t = z_t + \sum_{j=1}^{J} \theta_t^j Q_t^j \tag{13}$$

The maximizing behavior of the agent is summarized by the savings function $s : \mathbb{R}_t^2 \to \mathbb{R}$ defined by

$$s(r_{t+1}, w_t) = w_t - c_0(r_{t+1}, w_t)$$

where $(c_0(r, w), c_1(r, w))$ is the solution of the problem of maximizing $u(c_0, c_1)$ subject to the budget Eq. (12), or equivalently the solution of the problem

$$\max_{(c_0, c_1) \in \mathbb{R}_+^2} \left\{ u(c_0, c_1) | c_0 + \frac{c_1}{1 + r} = w \right\}$$

Assumption $\mathcal{C}$. The utility function $u(c_0, c_1)$ is smooth, increasing, strictly quasi-concave and such that the induced savings function $s(r, w)$ satisfies

(a) $s'_w(r, w) > 0, \quad \forall (r, w) \gg 0$
(b) $s'_r(r, w) \geq 0, \quad \forall (r, w) \gg 0$

(a) is the assumption that consumption in the second period is a normal good, while (b) implies that when the interest rate increases, the substitution effect dominates the income effect, so that savings increase.

2.4. Equilibrium

A *stock market equilibrium with affine price expectations and positive investment* is defined as a sequence of prices $(w_t, r_{t+1}, (Q_t^j)_{j=1}^J)_{t \geq 0}$, non-negative expected discounts $(V_t^j)_{t \geq 0}$, production–investment decisions $(L_t^j, Y_t^j; I_t^j, K_{t+1}^j)_{t \geq 0}$ for each of the J corporations, and consumption–savings portfolio decisions $(c^t, s_t, z_t, (\theta_t^j)_{j=1}^J)_{t \geq 0}$ for the sequence of representative consumers born at each date $t \geq 0$ such that

(i) the equity price of each firm is given by $Q_t^j = (1 - \beta)K_t^j - V_t^j$ (expectations are fulfilled);
(ii) the discounts $(V_t^j)_{t \geq 0}$ satisfy (7) and (11) for all $t \geq 0$ (discounts do not discourage investment and there is no arbitrage between bond and equity);
(iii) each firm maximizes its market value (condition (4) with equality and $K_{t+1}^j > (1 - \beta)K_t^j$) (positive investment);

(iv) each consumer maximizes lifetime utility subject to the budget constraints (9); and
(v) the output, labor and financial (bond, equity) markets clear at every date $t \geq 0$.

By Walras Law the output market clears once the labor market ($\sum_{j=1}^{J} L_t^j = N_t$) and financial markets clear. Given the indeterminacy of the bond–equity portfolio choice of consumers when (11) holds, market clearing on the bond and equity markets

$$N_t z_t = \sum_{j=1}^{J} I_t^j, \qquad N_t \theta_t^j = 1, \quad j = 1, \ldots, J$$

only requires that the financial markets clear in aggregate:

$$N_t s_t = \sum_{j=1}^{J} I_t^j + \sum_{j=1}^{J} Q_t^j \tag{14}$$

The young agents must buy the equity of the firms from the old and finance new investment. Given the pricing of equity (i) and the evolution of capital $K_{t+1}^j = (1-\beta)K_t^j + I_t^j$, (14) holds if and only if

$$N_t s_t = \sum_{j=1}^{J} (K_{t+1}^j - V_t^j)$$

As we will see shortly, to be determined an equilibrium needs two sets of initial conditions: the initial capital stocks (K_0^j) and as often in OLG models in which agents trade a security whose price must satisfy a forward-looking rate-of-return condition, an initial condition on the price of equity. This initial condition can be taken as the date 0 discounts $(V_0^j)_{j=1}^J$, and interpreted as the discounts that the agents, old at date 0, expected when they bought the equity at date −1. Alternatively, if one does not want to involve in the concept of equilibrium expectations formed at a date which is not explicitly in the model, the initial condition can be taken as the discount $(V_1^j)_{j=1}^J$ expected by the young agents born at date 0. The date 0 discounts are then deduced by the requirement that $V_1^j = (1 + r_1)V_0^j$, which implies that the expected rate of return on equity at date 0 is equal to the interest rate at this date. Since it is formally equivalent to take $(V_1^j)_{j=1}^J$ or $(V_0^j)_{j=1}^J$ as exogenous, for the ease of the dynamic analysis in the next section, we will take $(V_0^j)_{j=1}^J$ as the initial condition on the price of equity.

To reduce the analysis to the study of the aggregate economy we study only balanced-growth equilibria in which firms have at all times the same relative sizes and stock market values. Consider therefore initial conditions $(K_0^j, V_0^j) = \mu_j(K_0, V_0)$ with $\mu_j > 0$ and $\sum_{j=1}^{J} \mu_j = 1$. If the sequence of prices $(w_t, r_{t+1})_{t\geq 0}$, aggregate discounts $(V_t) \geq 0$ and labor–investment decisions $(L_t, I_t)_{t\geq 0}$ satisfy (4), (7) and (11), then $(V_t^j, L_t^j, I_t^j) = \mu_j(V_t, L_t, I_t)$ also satisfy (4), (7) and (11), so that for each firm j, (L_t^j, I_t^j) is optimal, its market value is positive and the return on its equity is r_{t+1}. Thus the maximizing behavior of individual firms can be summarized by the optimal choice of aggregate capital and labor.

Equilibrium on the labor market, which can be expressed by $L_t = N_t$, is satisfied if we require the capital–labor ratio to be equal to the per-capita capital stock $k_t = K_t/N_t$. Using lower-case letters (k_t, i_t, v_t) to denote per-capita capital, investment and discount, a balanced-growth equilibrium can be summarized in the following per-capita aggregate form.

Definition 1. A path of savings, capital accumulation, wages, security prices and discounts $((s_t, i_t, k_{t+1}), (w_t, r_{t+1}, q_t), (v_{t+1}))_{t\geq 0}$, with initial conditions (k_0, v_0), is a *stock market equilibrium with affine expectations and positive investment* if the following conditions are satisfied for all $t \geq 0$

$$\begin{array}{ll} \text{(i) } q_t = (1-\beta)k_t - v_t & \text{(v) } f'(k_{t+1}) = \beta + r_{t+1} \\ \text{(ii) } (1+n)v_{t+1} = (1+r_{t+1})v_t & \text{(vi) } s(r_{t+1}, w_t) = (1+n)k_{t+1} - v_t \\ \text{(iii) } 0 \leq (1+n)v_{t+1} \leq (1-\beta)^2 k_t & \text{(vii) } (1+n)k_{t+1} = (1-\beta)k_t + i_t \\ \text{(iv) } f(k_t) - k_t f'(k_t) = w_t & \text{(viii) } i_t > 0 \end{array} \tag{E}$$

(i) is the condition that affine price expectations are realized in equilibrium, while condition (ii) ensures that the rate of return on equity is the same as on the bond, and (iii) ensures that each firm has incentives to undertake positive investment. When these conditions hold, (iv) and (v) characterize the maximizing behavior of firms, while (vi) summarizes the maximizing behavior of consumers and equilibrium on the financial markets at each date. The consumption of the agents, while not explicitly given in Definition 1, can be derived from the sequential budget Eq. (12) for all agents born at date 0 or thereafter, and is given by the initial condition

$$c_1^{-1} = (1+n)(f(k_0) - w_0 + (1-\beta)k_0 - v_0) = (1+n)(k_0 f'(k_0) + (1-\beta)k_0 - v_0)$$

for the old agents at date 0.

Since condition (viii) requires investment to be positive, if the initial stock of capital is large, an equilibrium in the sense of Definition 1 may not exist. Since our objective is to study the process of 'capital accumulation' (rather than de-cumulation), and how the discount on equity affects this process, we will restrict attention to economies with a low initial level of capital.

3. Dynamics of stock market equilibrium

In this section we study the long-run dynamics of a stock market equilibrium with affine price expectations and positive investment. There are two kinds of equilibria: the *zero discount equilibria* for which $v_t = 0, t \geq 0$, and the *positive discount equilibria* for which $v_t > 0, t \geq 0$. For economies in which agents are sufficiently willing to save (economies with overaccumulation, to be defined shortly) the long-run behavior of the two types of equilibria is the same. However for economies in which savings are sufficiently scarce (economies with underaccumulation)—and it is often argued that this is the more realistic case—the long-run behavior is significantly different: zero discount equilibria converge to the Diamond steady-state while positive discount equilibria converge to the Golden Rule.

Thus, as we shall see, in this case zero discount equilibria are *long-run inefficient* while positive discount equilibria are *long-run efficient*.

3.1. Zero discount (Diamond) equilibria

If the initial discount on equity is zero, $v_0 = 0$, then condition (ii) of Definition 1 implies that $v_t = 0$ for all t: as a result conditions (ii) and (iii) can be omitted. Conditions (iv) and (v) which characterize the optimizing behavior of firms can be used to define the wage and interest rate (w_t, r_{t+1}) as function of the capital–labor ratio (k_t, k_{t+1}), and substituting these functions into (vi) gives the first-order difference equation

$$\Phi(k_{t+1}, k_t) \equiv (1+n)k_{t+1} - s(r(k_{t+1}), w(k_t)) = 0, \quad \forall t \geq 0 \qquad (\mathrm{E_D})$$

$(\mathrm{E_D})$ is the basic "investment = savings" equation of the classic Diamond model (1965). We call a trajectory $(k_t)_{t\geq 0}$ satisfying $(\mathrm{E_D})$ with initial condition k_0 a *Diamond equilibrium*. Such a trajectory gives a zero-discount equilibrium provided investment is positive at each date (condition (viii)).

Since the properties of a stock market equilibrium in which there is a discount on the equity prices of firms (equilibrium with $v_0 > 0$) depend in an essential way on the properties of the underlying Diamond equilibrium ($v_0 = 0$), we recall briefly the requisite properties of such an equilibrium.

A *Diamond steady-state* k_D is a solution of the equation

$$(1+n)k_\mathrm{D} - s(r(k_\mathrm{D}), w(k_\mathrm{D})) = 0 \qquad (15)$$

For general preferences and technology (u, F, β, n) there can be several non-trivial steady-states and the dynamics $(\mathrm{E_D})$ can exhibit complex behavior. We restrict attention to economies $\mathcal{E}(u, F, \beta, n)$ for which there is a unique positive steady-state k_D and every solution of $(\mathrm{E_D})$ converges to k_D: as noted by Galor and Ryder (1989), the standard Assumption $\mathcal{C}$ on preferences combined with Inada (and the usual concavity and homogeneity) conditions on F do not suffice to give this property. Assumption $\mathcal{C}$(b) and the concavity of F imply that there exists a unique solution

$$k_{t+1} = \phi(k_t) \qquad (\mathrm{E'_D})$$

to the Eq. $(\mathrm{E_D})$. By the implicit function theorem, ϕ is differentiable. An additional assumption is needed to ensure that the graph of ϕ cuts the diagonal with a positive slope at a unique $k_\mathrm{D} > 0$. The following condition—which is less restrictive and simpler to verify than the one given by Galor and Ryder (1989)—is sufficient.[3]

Assumption $\mathcal{S}$. Define $S(k) = s(r(k), w(k))$. The function $S(k)/k$ is decreasing for all $k > 0$, $\lim_{k\to 0^+} S(k)/k > 1+n$, and $\lim_{k\to +\infty} S(k)/k < 1$.

The property $S(k)/k$ decreasing is equivalent to $\log(S(k)/k)$ decreasing and this is equivalent to the elasticity of S being less than one ($\eta_S = \mathrm{d}S/S/\mathrm{d}k/k < 1$): a given percentage

[3] A similar assumption was used by Weil (1987). For the sake of completeness we prove in Appendix A that Assumption $\mathcal{S}$ implies uniqueness and global stability of the Diamond steady-state.

increase in the capital stock k gives rise to a smaller percentage increase in savings S. Although this assumption is a joint assumption on preferences and technology, it can be decomposed into separate assumptions on the consumption and the production sides. For example, it holds if

- u is homothetic and satisfies Assumption $\mathcal{C}$
- f is such that $w(k)/k$ is a decreasing function with $\lim_{k\to 0^+} w(k)/k = \infty$ and $\lim_{k\to\infty} w(k)/k = 0$

These conditions are satisfied if both u and F are CES with elasticity of substitution greater than or equal to 1—which includes Cobb–Douglas utility and production functions.

Proposition 2. *Under Assumptions $\mathcal{C}$ and $\mathcal{S}$, the Diamond steady-state capital k_{D} is globally stable for the dynamics* ($\mathrm{E_D}'$): *for any initial condition $k_0 > 0$, the per-capita capital stock on a Diamond equilibrium trajectory converges to $k_{\mathrm{D}} > 0$.*

Proof. See Appendix A. □

There are a number of different criteria which can be used to evaluate the efficiency of OLG economies. One is the usual Pareto criterion which, in OLG economies, is often called dynamic efficiency: we recall briefly the definition for the case where agents in all generations are identical and treated equally, so that the allocations can be defined in per-capita terms.

Definition 2. An allocation $(c_1^{-1}, (c^t, i_t, k_t)_{t\geq 0})$, where i_t and k_t denote per-capita investment and capital, is *feasible* for the economy $\mathcal{E}(u, F, \beta, n)$ if for all $t \geq 0$

$$c_0^t + \frac{1}{1+n}c_1^{t-1} + i_t = f(k_t), \qquad (1+n)k_{t+1} = (1-\beta)k_t + i_t$$

A feasible allocation $(c_1^{-1}, (c^t, i_t, k_t)_{t\geq 0})$ is *Pareto optimal* or *dynamically efficient* if there does not exist another feasible allocation $(\tilde{c}_1^{-1}, (\tilde{c}^t, \tilde{i}_t, \tilde{k}_t)_{t\geq 0})$ with the same initial capital ($\tilde{k}_0 = k_0$) such that $\tilde{c}_1^{-1} \geq c_1^{-1}, u(\tilde{c}^t) \geq u(c^t)$ for $t \geq 0$, with at least one strict inequality.

For an allocation which converges to a steady-state one can also study the long-run efficiency of the allocation, which will hold if the limiting steady-state is optimal in the set of feasible steady-states.

Definition 3. A steady-state allocation (c_0, c_1, i, k) is *feasible* if

$$\left.\begin{array}{l} c_0 + \dfrac{1}{1+n}c_1 + i = f(k) \\ (1+n)k = (1-\beta)k + i \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} c_0 + \dfrac{1}{1+n}c_1 = f(k) - (n+\beta)k \\ i = (n+\beta)k \end{array}\right.$$

A feasible steady-state allocation (c_0, c_1, i, k) is *steady-state optimal* if there does not exist another feasible steady-state allocation $(\tilde{c}_0, \tilde{c}_1, \tilde{i}, \tilde{k})$ such that $u(\tilde{c}_0, \tilde{c}_1) > u(c_0, c_1)$. An allocation $(c_1^{-1}, (c^t, i_t, k_t)_{t\geq 0})$ which converges to a steady-state $(\bar{c}_0, \bar{c}_1, \bar{i}, \bar{k})$ is *long-run efficient*, if the steady-state $(\bar{c}_0, \bar{c}_1, \bar{i}, \bar{k})$ is steady-state optimal.

A steady-state allocation (c_0, c_1, i, k) can be steady-state inefficient even though the allocation defined by $c_1^{-1} = c_1, c^t = (c_0, c_1), i_t = i, k_t = k, t \geq 0$, is dynamically efficient. This occurs when the feasible steady-state allocations $(\tilde{c}_0, \tilde{c}_1, \tilde{i}, \tilde{k})$ which dominate (c_0, c_1, i, k) in the sense of Definition 3 are such that $\tilde{c}_0 > c_0, \tilde{c}_1 < c_1$. Viewed as allocations on $[0, \infty)$ such steady-state allocations do not lead to a Pareto improvement since the old agents who loose consumption at date 0 cannot be compensated in their youth at date -1, since the economy is assumed to "start" at date 0. More generally, when an allocation is dynamically efficient but long-run inefficient it is typically possible to change the path of consumption and investment so as to converge to a limiting steady-state in which the utility of the representative agent is higher: however, since the allocation is dynamically efficient such changes must decrease the utility of some agents in the early generations in order to improve the utility of an infinite number of future generations. Allocations which are dynamically efficient but long-run inefficient involve an inherent conflict of interest between the welfare of current and future generations: market structures which leads to equilibrium allocations which are both dynamically and long-run efficient, and avoid this conflict between long-run growth and current welfare, are thus of special interest.

A steady-state (c_0^*, c_1^*, i^*, k^*) is steady-state optimal if it is feasible and satisfies the first-order conditions

$$f'(k^*) - \beta = n, \qquad \frac{u_0'(c^*)}{u_1'(c^*)} - 1 = n \tag{16}$$

The capital labor k^* satisfying the first condition in (16) maximizes aggregate per-capita permanent consumption and is called the *Golden Rule* capital–labor ratio. The second condition requires that the distribution of consumption between young and old agents corresponds to the choice of the representative agent when faced with the interest rate $r(k^*) = n$ which supports the Golden Rule capital k^*.

The Golden Rule k^* is determined purely by technological (f, β) and demographic factors (n): the Diamond steady-state k_D defined by (15) depends in addition on agents' preferences (savings behavior). Thus for typical economies $k_D \neq k^*$, so that for most economies the Diamond steady-state is steady-state inefficient. When $k_D < k^*$, the interest rate $r_D = f'(k_D) - \beta$ at the Diamond steady-state exceeds the Golden Rule interest rate $r(k^*) = f'(k^*) - \beta = n$: the Diamond economy converges to a steady-state of *underaccumulation* characterized by a low level of capital, low output and a high interest rate. Under Assumption $\mathcal{S}$, $k_D < k^*$ is equivalent to $s(r(k^*), w(k^*)) < (1+n)k^*$, so that an alternative definition of *underaccumulation* is that the savings of the consumers at the prices $(r(k^*), w(k^*))$ at the Golden Rule are not sufficient to sustain the Golden Rule capital stock. To attain a steady-state with a higher level of capital and consumption, more investment than that undertaken in the Diamond equilibrium would be needed at some date. This is not feasible without a decrease in the welfare of some generation: in the case of underaccumulation, a Diamond equilibrium, though long-run inefficient, is dynamically efficient.

When $k_D > k^*$, $r_D < n$ and $s(r(k^*), w(k^*)) > (1+n)k^*$, so that the savings of the consumers at k^* can "buy" more capital than k^*: the Diamond economy converges to a steady-state of *overaccumulation* characterized by a high level of capital, a low interest rate $r_{\rm D}$ and high output level $y_{\rm D}$, much of which is absorbed by the need to maintain the capital stock rather than being used for consumption. In this case the Diamond steady-state, taken as an allocation on $[0, \infty)$ is dynamically inefficient since an improvement can be obtained by discarding some of the capital at the initial date. For any $k_0 > 0$, a Diamond equilibrium which converges to $k_{\rm D}$ is both dynamically and long-run inefficient.

Since we are adopting Assumptions $\mathcal{C}$ and $\mathcal{S}$, under which a Diamond equilibrium is unique, in the analysis that follows it will be convenient to refer to an economy $\mathcal{E}(u, F, \beta, n)$ as an *economy with underaccumulation* (resp. *overaccumulation*) if its characteristics (u, F, β, n) are such that $k_{\rm D} < k^*$ (resp. $k_{\rm D} > k^*$).

3.2. Positive discount equilibria

If the initial discount on equity is positive, $v_0 > 0$, then (ii) in Definition 1 implies $v_t > 0$ for all $t > 0$. Let us show that positive discount equilibria can be defined as solutions of a pair of difference equations in the per-unit capital and discount (k_t, v_t) with positive initial conditions (k_0, v_0), and some additional restrictions. As before equations (iv) and (v) of Definition 1—the firms' FOC—define the wage and interest rate as a function of the capital–labor ratio, while (i) defines the equity price, and (vii) gives the investment. A positive discount equilibrium trajectory $(k_t, v_t)_{t\geq 0}$ must thus satisfy for all $t \geq 0$

$$\begin{aligned}(1+n)k_{t+1} &= s(r(k_{t+1}), w(k_t)) + v_t \\ (1+n)v_{t+1} &= (1+r(k_{t+1}))v_t\end{aligned} \tag{E$_{\rm S}$}$$

$i_t > 0$, the inequality

$$0 \leq (1+n)v_{t+1} \leq (1-\beta)^2 k_t \tag{17}$$

to ensure that (iv) and (v) characterize the optimum behavior of firms, and the initial condition $(k_0, v_0) \gg 0$.

The first equation in $({\rm E_S})$ is the "savings = investment" equation (vi) of a stock market equilibrium. When there is a discount $v_t > 0$, the (per-capita) capital stock that young agents are able to acquire for use in the subsequent period $((1+n)k_{t+1})$, exceeds their savings because firms are sold on the equity market at a discount relative to their replacement cost. The discount v_t in essence acts like an additional "source of funds" that enables them to finance a higher level of capital accumulation than would be warranted by their savings in a Diamond equilibrium, where firms are sold for their replacement cost.

A steady-state solution (k, v) of $({\rm E_S})$ must satisfy

$$(1+n)k = s(r(k), w(k)) + v \tag{18}$$

$$(1+n)v = (1+r(k))v \tag{19}$$

and the inequality

$$0 \leq (1+n)v \leq (1-\beta)^2 k \tag{20}$$

(19) is equivalent to $(r(k) - n)v = 0$ and thus has two solutions, $v = 0$ and $r(k) = n$. When $v = 0$, (18) gives the Diamond steady-state $k_{\rm D}$, defined by (15) and (20) is satisfied.

$r(k) = n$ is equivalent to $k = k^*$, the Golden Rule capital–labor ratio, and (18) defines the associated discount

$$v^* = (1+n)k^* - s(r(k^*), w(k^*)) \tag{21}$$

As we have seen above, in the case of an economy with overaccumulation, $(1+n)k^* - s(r(k^*), w(k^*)) < 0$, so that (20) is not satisfied: for such economies the Golden Rule is not a steady-state stock market equilibrium. For an economy with underaccumulation, v^* defined by (21) is positive, but in order that the right hand inequality in (20) be satisfied, the characteristics (u, F, β, n) of the economy must be such that

$$s(r(k^*), w(k^*)) \geq (1+n)k^* - \frac{(1-\beta)^2}{1+n}k^* \tag{22}$$

in which case (k^*, v^*) defines a positive discount steady-state stock market equilibrium. Although the savings at the Golden Rule are not sufficient to cover the replacement cost of capital, they must not be too deficient in the sense made precise by (22). There is thus a limit to the extent to which the discount on equity can make up for the deficiency in savings needed to sustain the Golden Rule. This condition can also be written as

$$s(r(k^*), w(k^*)) \geq i^* + i^* \frac{1-\beta}{1+n}$$

where $i^* = (n+\beta)k^*$ is the (per-capita) investment needed to sustain the Golden Rule: thus, while the savings of the young may not be sufficient to cover the combined costs of new investment and installed capital at replacement value, they must be sufficient to cover current new investment and the depreciated investment of the previous period. From now on, for economies with underaccumulation, we restrict attention to those for which the Golden Rule is a steady-state stock market equilibrium satisfying (22).

Assumption $\mathcal{GR}$. For economies with underaccumulation the characteristics (u, F, β, n) are such that (22) is satisfied with strict inequality.

The system of equations (E_S) is related in an interesting way to the equations studied by Tirole (1985). He considered a Diamond economy in which the young, in addition to financing the capital used in the next period, could also use their savings to purchase an asset paying a zero dividend, which he called a "bubble". This leads to the system of equations

$$s_t = (1+n)k_{t+1} + b_t$$
$$(1+n)b_{t+1} = (1+r_{t+1})b_t$$

where b_t is the (per-capita) price of the bubble asset, and $b_t \geq 0$, since by free disposal the asset cannot have a negative price. This system of equations is the same as (E_S) with $b_t = -v_t$, but with $v_t \leq 0$. If formally when $v_t > 0$, the discount on equity plays the role of a "negative bubble" for Tirole's equations, in our market structure v_t is *not* a negative bubble. A security price is said to have a bubble component if the price differs from the present value of its future stream of dividends (the fundamental value): the difference between the price and the fundamental value is called the bubble (component). With free disposal of securities, all security prices must be non-negative, and as Tirole has shown, this implies that the bubble component can only be non-negative. When inequality (20) is satisfied, the (per-capita) equity price $q_t = (1-\beta)k_t - v_t$ is non-negative, so that, if there is a bubble in our model, it can only be non-negative. As we shall see in Section 4, when $v_t > 0$ the fundamental value of equity is not equal to $(1-\beta)k_{t+1}$, so that $-v_t$ is not the difference between the price of equity and its fundamental value. In short, in our model there is free disposal of securities and there are no negative bubbles.

Tirole's equations also hold for a Diamond economy in which a government incurs a debt b_0 with the young at date 0, and rolls it over indefinitely, borrowing from the young agents of generation t to reimburse the contemporaneous old agents. The case $b_t > 0$ (or $v_t < 0$) corresponds to government debt, so that formally the case obtained for $b_t < 0$ or ($v_t > 0$) corresponds to a government surplus. Most authors in macroeconomics have found it implausible to study a setting where the government runs a perpetual surplus used to finance investment in the private sector (see, for example, Azariadis, 1993): as a result the dynamics of the system (E_S) with $v_t > 0$ has not been studied in the macro literature.[4] Since as we shall see shortly, for economies with underaccumulation, the trajectories of the system (E_S) with $v_t > 0$ have good normative properties, it seems important to establish that this dynamics can be generated by a realistic market structure under plausible assumptions, without invoking government intervention and without violating any rationality assumption.

Under Assumption $\mathcal{C}$, (E_S) can be written as

$$\begin{aligned} k_{t+1} &= \psi(k_t, v_t) \\ v_{t+1} &= \frac{1-\beta+f'(\psi(k_t, v_t))}{1+n} v_t \end{aligned} \qquad (E'_S)$$

where $\psi : \mathbb{R}^2_t \to \mathbb{R}$ is an increasing, differentiable function. The phase diagram is determined by the curves $\mathcal{V}$ and $\mathcal{K}$ defined by

$$\begin{aligned} \mathcal{V} &= \{(k_t, v_t) \in \mathbb{R}^2_+ | v_{t+1} = v_t\} = \{(k_t, v_t) \in \mathbb{R}^2_+ | v_t = 0 \text{ or } \psi(k_t, v_t) = k^*\} \\ &= \{(k_t, v_t) \in \mathbb{R}^2_+ | v_t = 0 \text{ or } v_t = (1+n)k^* - s(r^*, w(k_t))\} \end{aligned}$$

and

$$\mathcal{K} = \{(k_t, v_t) \in \mathbb{R}^2_+ | k_{t+1} = k_t\} = \{(k_t, v_t) \in \mathbb{R}^2 | v_t = (1+n)k_t - s(r(k_t), w(k_t))\}$$

Under Assumptions $\mathcal{C}$ and $\mathcal{S}$, $\mathcal{V}$ is the union of the axis $v = 0$ and the graph of a decreasing function, while $\mathcal{K}$ is a U-shaped curve passing through the origin. The resulting phase diagrams which suggest—but are by themselves insufficient to prove—the stability properties

[4] Tirole has shown that, in the case of underaccumulation, there do not exist equilibria with $b_t > 0$, i.e. $v_t < 0$. Thus even if we relax the assumption that firms can be reproduced by installing the corresponding amount of capital, in economies with underaccumulation there do not exist equilibria with affine price expectations in which the price of a firm is always above the value of its capital (taken as the sum of past (depreciated) investments). For economies with overaccumulation such equilibria exist and as Tirole has shown, converge to the Diamond steady-state, except for the trajectory which coincides with the Golden Rule steady-state, which is saddle-point stable. It may be that the stability properties of the Diamond and Golden Rule steady-states for the dynamics with $b_t < 0$ (or $v_t > 0$) are a kind of Folk theorem ($b_t < 0$ is sometimes called "negative money"), but since we have not found a published reference, we state and prove these properties.

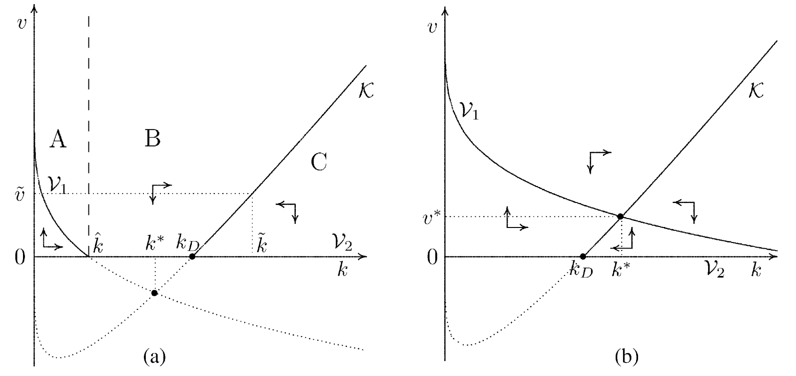

Fig. 1. Phase diagram for the stock market equilibrium equations ($E_S{}'$) in the cases of (a) overaccumulation and (b) underaccumulation.

of the steady-states, are shown in Fig. 1(a) for an economy with overaccumulation, and in Fig. 1(b) for an economy with underaccumulation.

As Fig. 1(a) suggests, the Diamond steady-state k_D is globally stable for an economy with overaccumulation.

Proposition 3. *Under Assumptions $\mathcal{C}$ and $\mathcal{S}$, if $k^* < k_D$ then any solution $(k_t, v_t)_{t\geq 0}$ of* (E_S) *with $k_0 > 0$, $v_0 \geq 0$ converges to* $(k_D, 0)$.

Proof. If $v_0 = 0$, then by Proposition 2, the trajectory converges to the Diamond steady-state. Suppose $v_0 > 0$. Consider the three regions A, B, C shown in Fig. 1(a). A is defined by $k \leq \hat{k}$, where $\hat{k}$ is such that $(1+n)k^* - s(n, w(\hat{k})) = 0$: since $(1+n)k^* - s(n, w(k^*)) < 0$, $0 < \hat{k} < k^*$. The definitions of B and C are clear from Fig. 1(a). Let us show that if $(k_0, v_0) \in$ A then the trajectory must leave the region A in a finite number of periods and enter B∪C. $v_t > 0$, $t \geq 0$ implies by induction that for $t \geq 1$, $k_t > k_t^D$, where $(k_t^D)_{t\geq 0}$ is the Diamond trajectory beginning at $(k_0, 0)$: $k_{t+1} = \psi(k_t, v_t) > \psi(k_t, 0) > \psi(k_t^D, 0) = k_{t+1}^D$. Since k_t^D converges to $k^D > \hat{k}$ the property follows. Let us show that if $(k_0, v_0) \in$ B∪C, the trajectory stays in B∪C. If $(k_t, v_t) \in$ B then $k_{t+1} > k_t$, so that $(k_{t+1}, v_{t+1}) \in$ B∪C. If $(k_t, v_t) \in$ C, then $k_t > k_D$, so that $k_{t+1} = \psi(k_t, v_t) > \psi(k_D, 0) = k_D$. Thus $(k_t, v_t) \in$ B∪C. When $(k_t, v_t) \in$ B∪C the sequence $(v_t)_{t\geq 0}$ is a decreasing sequence which is bounded below since $v_t > 0, \forall t \geq 0$: thus $v_t \to \bar{v}$. Either $\bar{v} > 0$ or $\bar{v} = 0$. Suppose $\bar{v} > 0$ then $k_t \to \bar{k}$ defined by $\bar{v} = (1/1+n)(1-\beta+f'(\psi(\bar{k}, \bar{v})))\bar{v} \Leftrightarrow \psi(\bar{k}, \bar{v}) = k^*$, so that $(\bar{k}, \bar{v})$ lies on the curve $\mathcal{K}$. Since there is no intersection of the $\mathcal{V}_1$ curve and the $\mathcal{K}$ curve in the non-negative orthant, it follows that $\bar{v} \in \mathcal{V}_2$. Thus $\bar{v} = 0$ and $\bar{k} = k_D$. □

A solution of (E_S) is an equilibrium trajectory for our economy if the inequalities $i_t > 0$, and $(1+n)v_{t+1} < (1-\beta)^2 k_t$ are satisfied at all dates. Let $\tilde{k}$, be the first value of the capital–labor ratio for which investment is zero under the Diamond dynamics, i.e. $\phi(\tilde{k}) = (1-\beta)\tilde{k}$. Then $\tilde{k} > k_D$, and $k < \tilde{k}$ implies $\phi(k) > (1-\beta)k$. Let $\tilde{v}$ be the discount such that $(\tilde{v}, \tilde{k}) \in \mathcal{K}$ (see Fig. 1(a)). Consider initial conditions (v_0, k_0) such that $k_0 < k_D$, $v_0 < \tilde{v}$, and $(v_0, k_0) \in$ B and $(1+n)v_0 \leq (1-\beta)^2 k_0$. Since, by the reasoning above $v_t \leq v_0$ and $k_t \geq k_0$, it follows that $(1+n)v_{t+1} \leq (1+n)v_0 \leq (1-\beta)^2 k_0 \leq (1-\beta)^2 k_t$, $t \geq 0$, so that the inequality (17) holds. Since $(v_0, k_0) \ll (\tilde{v}, \tilde{k})$, $\psi(v_0, k_0) < \psi(\tilde{v}, \tilde{k}) = \tilde{k}$. Thus $k_1 \leq \tilde{k}$, and since $v_1 \leq v_0$, $v_1 \leq \tilde{v}$. Thus $(v_1, k_1) < (\tilde{v}, \tilde{k})$, and by induction for all t, $(v_t, k_t) \ll (\tilde{v}, \tilde{k})$. But $\psi(v_t, k_t) \geq \psi(k_t, 0) > (1-\beta)k_t$ since $k_t < \tilde{k}$, so that investment is positive on the whole trajectory.

Thus in the case of overaccumulation it is easy to prove the existence of a stock market equilibrium trajectory and its convergence to the Diamond equilibrium. In this case the existence of a discount on equity does not improve the long-run efficiency of the equilibrium. This was to be expected since in the case of overaccumulation the propensity to save of the young agents is too high when compared to the productivity of capital. The discount on equity which is akin to an increase in savings can only make things worse. In the long run however the effect vanishes, since the discount on equity increases at a slower rate than the population and tends to disappear in per-capita terms, so that the equilibrium converges to the stable Diamond steady-state. A variety of methods have been proposed for absorbing the excess savings to restore convergence to the Golden Rule: social security, land as a third factor of production (McCallum and Bennett, 1987; Rhee, 1991) or unbacked debt (Pingle and Tesfatsion, 1998): each of these methods is applicable to our model.

For an economy with underaccumulation, the savings of the young are “scarce” and the discount on the equity prices acts like an additional source of funds, permitting increased investment. The phase diagram (Fig. 1(b)) suggests that the equilibrium trajectories converge to the Golden Rule steady-state. Global properties are more difficult to establish for economies with underaccumulation than for those with overaccumulation. Indeed even to prove the local stability of the Golden Rule (k^*, v^*), a stronger assumption is needed than that which assures the stability of the Diamond steady-state under the Diamond dynamics, namely, Assumptions $\mathcal{C}$ and $\mathcal{S}$.

Assumption $\mathcal{P}$. The production function f is such that $kf'(k)$ is an increasing function of k.

$\mathcal{P}$ is satisfied only if capital and labor are sufficiently substitutable: it requires that the marginal product of capital $f'(k)$ does not decrease too fast as the capital–labor ratio k increases, so that the amount of output (per-capita) going as payment to capital $(kf'(k))$ decreases. This property is satisfied for CES production functions $F(K, L)$ with elasticity of substitution greater than or equal to 1 (and hence for Cobb–Douglas production functions): thus for the class of CES functions $\mathcal{P}$ requires no additional restrictions over those needed to satisfy Assumption $\mathcal{S}$.

Proposition 4. *Under Assumptions $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{P}$, if $k^* > k_D$ then under the stock market equilibrium dynamics* (E_S), *the Golden Rule* (k^*, v^*) *is locally stable and the Diamond steady-state* $(k_D, 0)$ *is locally saddle-point stable.*

Proof. See Appendix A. □

Proposition 4 implies that for all economies satisfying Assumption $\mathcal{GR}$, a stock market equilibrium which converges to the Golden Rule exists for all initial conditions (k_0, v_0) in a neighborhood of the Golden Rule (k^*, v^*). In both the case of over- and underaccumulation, since there is a stable steady-state (k_D in overaccumulation, k^* in underaccumulation), stock market equilibria exist for a continuum of initial conditions (k_0, v_0), or equivalently (k_0, q_0), since $q_0 = (1-\beta)k_0 - v_0$.

Stock market equilibria of an economy with underaccumulation which converge to the Golden Rule are, by definition, long-run efficient: applying the Cass criterion shows that they are also dynamically efficient.

Proposition 5. *If f is strictly concave and u differentiably strictly quasi-concave,*[5] *a stock market equilibrium of an economy with underaccumulation which converges to the Golden Rule is both dynamically and long-run efficient.*

Proof. To prove dynamic efficiency note that, since the per-capita capital satisfies the first-order condition $f'(k_t) = r_t + \beta$ and agents maximize utility under the budget constraint $c_0^t + c_1^t/(1 + r_{t+1}) = w_t$, a stock market equilibrium can be viewed as an Arrow–Debreu equilibrium, with prices $p_0 = 1$, $p_t = \prod_{\tau=1}^{t} 1/(1+r_\tau)$, for an economy with the following characteristics: there is one good at each date, and each agent born at date t is the owner of a firm which lasts for two periods, $t-1$ and t, investing k_t units of the good at date $t-1$ to obtain $f(k_t)+(1-\beta)k_t$ units at date t. Each firm maximizes its profit $p_t(f(k_t)+(1-\beta)k_t)-p_{t-1}k_t$, or equivalently maximizes $p_t(f(k_t) + (1 - \beta)k_t - (1 + r_t)k_t)$, by choosing k_t such that $f'(k_t) = \beta + r_t$. The profit of each firm $p_t(f(k_t) - f'(k_t)k_t)$ goes to its owner, a young agent at date t. In this market structure, which is purely formal, i.e. does not claim to be realistic, young agents own firms which operate one period before they are born, and the labor income that agents receive in the standard model becomes a profit. An agent born at date t maximizes lifetime utility $u(c_0^t, c_1^t)$ subject to the constraint $p_t c_0^t + p_{t+1} c_1^t = p_t(f(k_t) - k_t f'(k_t))$. Market clearing at date $t \geq 0$ is given by

$$N_t c_0^t + N_{t-1} c_1^{t-1} + N_{t+1} k_{t+1} = N_t(f(k_t) + (1-\beta)k_t)$$

which is the same as the equation $c_0^t + c_1^t/(1 + n) + i_t = f(k_t)$ in the stock market model. Since an equilibrium trajectory $(c_t, k_t)_{t\geq 0}$ with $k_0 > 0$, $0 < c_{-1}^1 < (1+n)(f(k_0) + (1-\beta)k_0)$, which converges to the Golden Rule, lies in a compact subset of the positive orthant, the curvature condition needed to apply the efficiency condition developed by Cass (1972), Benveniste and Gale (1975), Balasko and Shell (1981) is satisfied. Thus the Arrow–Debreu allocation is Pareto optimal if

$$C = \sum_{t=1}^{\infty} \frac{1}{N_t p_t} = \sum_{t=1}^{\infty} \frac{\prod_{\tau=1}^{t}(1+r_\tau)}{N_0(1+n)^t} = \infty$$

If the equilibrium is also a stock market equilibrium converging to the Golden Rule, then there exists a sequence $v_t > 0$, $t \geq 0$ which converges to $v^* > 0$ such that $(1 + n)v_{t+1} = (1 + r_{t+1})v_t$, $t \geq 0$, so that $\prod_{\tau=1}^{t}(1 + r_\tau)v_0 = (1 + n)^t v_t$. Thus $C = \sum_{t=1}^{\infty} v_t$ and since $v_t \to v^* > 0$, $C = \infty$. □

[5] The condition insures that the indifference curves have positive Gaussian curvature (see Mas-Colell, 1985).

Note that the Arrow–Debreu equilibria constitute a two-dimensional family[6] of paths parametrized by (k_0, c_1^{-1}). The Diamond equilibria select the Arrow–Debreu equilibria with $c_1^{-1} = (1+n)(k_0 f'(k_0) + (1-\beta)k_0)$, while the stock market equilibria select the class of Arrow–Debreu equilibria with $c_1^{-1} = (1+n)(k_0 f'(k_0) + (1-\beta)k_0 - v_0)$ for appropriate restrictions on (k_0, v_0).

Assumption $\mathcal{GR}$ in essence imposes a restriction on how far the Diamond steady-state k_D is from the Golden Rule k^*: if k^* is too much greater than k_D then the funds in excess of the savings of the young needed to finance investment become too large to permit them to be covered by the discount on the equity prices. To get a feel for how the equilibrium behaves and to what extent these conditions are restrictive, let us consider a family of Cobb–Douglas economies.

Example. Let $\mathcal{E}(u, F, \beta, n)$ be a Cobb–Douglas economy:

$$u(c_0, c_1) = c_0^{1-\alpha} c_1^{\alpha}, \quad 0 < \alpha < 1, \qquad F(K, L) = AK^{\gamma} L^{1-\gamma}, \quad 0 < \gamma < 1$$

There are four parameters $(\alpha, \gamma, \beta, n)$ which characterize an economy: the parameter A is just a scale factor which does not matter for the analysis (for the graph in Fig. 2 we chose $A = 50$). α gives the propensity to save of the young ($s(r, w) = \alpha w$), γ determines the share of capital in output, $0 < \beta < 1$ is the depreciation rate of capital and n the population growth rate. Let us fix $\gamma = 0.25$ and $n = 0.35$ (which corresponds to an annual increase of population of about 1% for 30 years). The Golden Rule capital–labor ratio is $k^* = (A\gamma/(\beta + n))^{1/(1-\gamma)}$ and there is underaccumulation if

$$(1+n)k^* \geq A\alpha(1-\gamma)(k^*)^{\gamma} \Leftrightarrow (1+n) \geq \frac{\alpha(1-\gamma)}{\gamma}(\beta + n) \Leftrightarrow$$
$$\alpha \leq \frac{\gamma(1+n)}{(1-\gamma)(\beta+n)} \tag{23}$$

The Golden Rule k^* satisfies condition (22) if

$$A\alpha(1-\gamma)(k^*)^{\gamma} \geq (\beta+n)k^* + \frac{1-\beta}{1+n}(\beta+n)k^* \Leftrightarrow \alpha \geq \frac{\gamma}{1-\gamma} + \frac{(1-\beta)^2}{1+n}\frac{\gamma}{1-\gamma} \tag{24}$$

For the chosen parameters $(\gamma, n) = (0.25, 0.35)$, (23) and (24) give the admissible values of the parameters $(\alpha, \beta) \in (0, 1) \times (0, 1)$ for which the Golden Rule is a stock market equilibrium. Let $\ell(\beta) = \gamma/(1-\gamma)((1+n)/(\beta+n))$ denote the function in (23) defining

[6] To see this, note that, if we denote demand and supply functions by tilde, the equilibrium equations can be written as

$$\tilde{c}_0^0(p_0, p_1) + \frac{1}{1+n}c_1^{-1} + (1+n)\tilde{k}_1(p_0, p_1) = f(k_0) + (1-\beta)k_0$$
$$\tilde{c}_0^t(p_t, p_{t+1}) + \frac{1}{1+n}\tilde{c}_1^{t-1}(p_{t-1}, p_t) + (1+n)\tilde{k}_{t+1}(p_t, p_{t+1})$$
$$= f(\tilde{k}_t(p_{t-1}, p_t)) + (1-\beta)\tilde{k}_t(p_{t-1}, p_t), t \geq 1$$

.

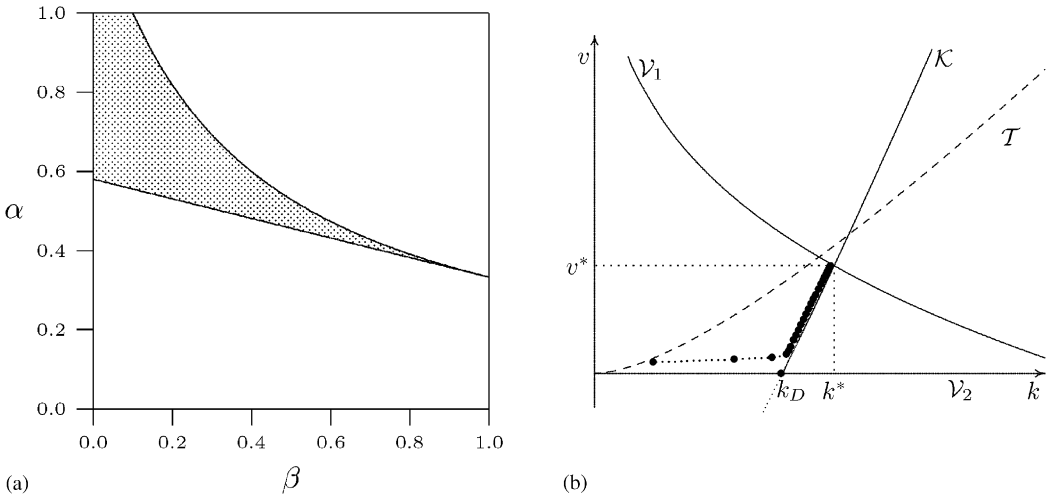

Fig. 2. Parameters for which the Cobb–Douglas economy is characterized by underaccumulation and satisfies $\mathcal{GR}$ (a) and convergence of a trajectory to the Golden Rule (b).

economies with "low" savings (underaccumulation) and let $p(\beta) = \gamma/(1-\gamma)((1+(1-\beta)/(1+n))$ denote the function in (24) defining economies satisfying condition $\mathcal{GR}$, then the admissible parameters (α, β) are given by the shaded region in Fig. 2(a). Thus, for example, if $\beta = 0.4$ (which corresponds to an annual depreciation rate of 1.7% for 30 years) then the interval of admissible α values is [0.48, 0.6]. Fig. 2(b) shows an equilibrium trajectories when $(\alpha, \beta) = (0.5, 0.4)$ beginning with a low level of (k_0, v_0). The region lying below the dashed curve $\mathcal{T}$ is satisfied defines the set of (k_t, v_t) pairs satisfying condition (17) which ensures that the young have the incentive to make positive investment in their firms. All initial conditions in the region below $\mathcal{T}$, above $\mathcal{K}$, and with $v_0 < v^*$ lead to stock market equilibria converging monotonically to the Golden Rule. While for this Cobb–Douglas economy the convergence is monotone, this may not be true generally since Proposition 5 does not preclude complex or negative eigenvalues at the Golden Rule.

3.3. *Initial conditions*

In a stock market equilibrium with affine price expectations, the equilibrium trajectory depends not only on the initial (per-capita) capital stock k_0 but also on the initial price of equity $q_0 = (1-\beta)k_0 - v_0$, or equivalently on the initial discount v_0. The discount v_0 should have the property that it justifies the price q_{-1} (or the discount v_{-1}) paid by the old agents of date 0 in their youth at date -1. However since the price q_{-1} is absent from the model when it is "started" at date 0, q_0 must be given exogenously: as is often the case in OLG models with asset markets the equilibrium is indeterminate. Note however that the equilibrium is determinate in the following sense: if the economy is cut at any moment of time t and agents are given the available information (k_t, q_t) from the markets, then the future course of capital accumulation and prices is determinate and can be anticipated by the agents.

In this section we have shown that the initial condition v_0 with which the economy begins is not important for the long-run behavior of economies with overaccumulation: $k_t \to k_{\mathrm{D}}$ for any initial condition $k_0 > 0$ and $v_0 \geq 0$. For economies with underaccumulation this is not the case, since $k_t \to k_{\mathrm{D}}$ if $v_0 = 0$ while $k_t \to k^*$ if $v_0 > 0$. Thus whether v_0 is zero or positive has important long-run consequences.

A fully consistent theory would need to explain the emergence of firms (how their capital stocks have evolved to give k_0), and when and how they have come to be priced on the equity market (how their price has evolved to give q_0). This is clearly out of the perview of a general equilibrium model with perfect competition and zero profits as studied in this paper, which can only apply to a situation in which all profit opportunities have been exploited and firms are "mature". Typically there is a phase in the creation of a firm in which an entrepreneur perceives some unexploited profit opportunities and sets up a firm to take advantage of them. This firm typically begins as a sole proprietorship or as a partnership. At this stage, before the firm becomes public and the stock market restores liquidity, installed capital is "illiquid"—it cannot be sold without a large loss because of the cost of adapting it it to some other uses—and such an imperfection may result in a discount in the equity price when the firm goes public (investors may have difficulty correctly assessing the potential of the firm and the entrepreneur may accept a discount for the benefit of diversifying his investment).

With this optic a positive discount v_0 may reflect the cost of nonshiftability of capital at an early stage when the firm first moved to the corporate form.

4. Financial valuation

The previous section analyzed the real equilibrium outcome of an economy in which capital once installed is a sunk cost, making it possible for the equity price of a firm to be less than its replacement cost. In this section we examine the equilibrium financial valuation of firms—in particular, the relation between the equity price of a firm and its fundamental value (the discounted sum of its future dividends).

In our model, as in all models in which the sequence of equilibrium interest rates is $(r_t)_{t\geq 0}$, the equity price[7] must satisfy the rate of return condition

$$Q_t = \frac{1}{1+r_{t+1}}(D_{t+1} + Q_{t+1}) \tag{25}$$

which by successive substitution gives

$$Q_t = \sum_{\tau=1}^{T} \frac{D_{t+\tau}}{(1+r_{t+1})\cdots(1+r_{t+\tau})} + \frac{Q_{t+T}}{(1+r_{t+1})\cdots(1+r_{t+\tau})}$$

Assuming that the limits exist (perhaps in the generalized sense of taking the values $+\infty$ or $-\infty$) the *fundamental value* of equity at date t is defined by

$$Q_t^f = \lim_{T\to\infty} \sum_{\tau=1}^{T} \frac{D_{t+\tau}}{(1+r_{t+1})\cdots(1+r_{t+\tau})}$$

and the *bubble component* as

$$Q_t^b = \lim_{T\to\infty} \frac{Q_{t+T}}{(1+r_{t+1})\cdots(1+r_{t+T})}$$

Free disposal of securities implies that the equity price is always non-negative, $Q_{t+T} \geq 0, \forall T \geq 0$, so that $Q_t^b \geq 0$. If $Q_t^b = 0$, the equity price is said to satisfy a *transversality condition*, and the price of equity is equal to its fundamental value $Q_t = Q_t^f$. If $Q_t^b > 0$, the equity price has a bubble component and its price exceeds its fundamental value, $Q_t > Q_t^f$.

The transversality condition holds in all economies with equilibria in which the asymptotic interest rate exceeds the rate of growth of the economy. This includes most models with infinite-lived agents who discount the future, the OLG model of Scheinkman (1983), Dechert and Yamamoto (1992), and the Diamond equilibria of economies with underaccumulation. For, if in such economies Q_t^b were positive, the price of equity Q_t would have to

[7] In what follows we assume that all firms follow, up to scale, the same investment and financial policy and omit the firm index.

grow asymptotically faster than the rate of growth of the economy and would at some date have to exceed the resources of the economy, which is not compatible with equilibrium.[8]

The transversality condition is often associated with efficiency. The reason is that, when the asymptotic interest rate exceeds the rate of growth, the Cass criterion for efficiency is satisfied. However it is known that the Cass criterion is weaker than the transversality condition. The analysis that follows will show that stock market equilibria with positive discount of a economy with underaccumulation are examples of equilibria which do not satisfy a transversality condition, but do satisfy the Cass criterion, as we saw in Proposition 5. The effect of the discount on equity is to lower interest rates relative to the Diamond equilibrium, so that in an economy with underaccumulation the interest rate converges to the rate of growth of the population n, rather than to the Diamond steady-state interest rate $r_{\rm D} > n$. As shown in Proposition 6, this downward shift in the equilibrium interest rates is sufficient to create a bubble component in the market value (equity plus debt) of firms.

So far we have restricted attention to the financial policy for firms which consists in financing investment at each date by a one-period debt which is fully reimbursed in the following period ($B_t = I_t, \forall t \geq 0$). Confining the analysis of equity prices to this case has two drawbacks. First, in economies with overaccumulation this financial policy leads to negative dividends, which is not realistic: for in such economies the low earnings of firms coupled with their high investment make it likely that they will either roll over their debt or issue new equity. Second, the proposition below which shows that the equity price has a bubble component may be criticized as being a result which depends on the particular financial policy chosen. To avoid these drawbacks, we note that for our model it can be shown that the Modigliani–Miller theorem holds:[9] this theorem asserts that "for financial policies which do not lead to bankruptcy: (i) the optimal investment policy of a firm and (ii) the market value of the firm (the sum of its equity and debt), are independent of its financial policy."

In our economy the *market value* of the firms at date t is given by

$$M_t = Q_t + B_t = (1-\beta)K_t - V_t + I_t = K_{t+1} - V_t \tag{26}$$

This can be evaluated by choosing the particular financial policy $B_t = I_t$ that we have considered above, and the timing in which firms are sold without debt and before their investment is made. Suppose we consider general debt policies $(B_t)_{t\geq 0}$ for the firms, with $B_t \geq 0, \forall t \geq 0$ but for simplicity assume that firms do not issue new equity. The following identity for the sources and uses of funds at date $t+1$ must then hold

$$F(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - (1+r_{t+1})B_t + B_{t+1} = I_{t+1} + D_{t+1} \tag{27}$$

The left side describes the sources of funds for the firm, internal (earnings) and external (borrowing), while the right side gives the two uses to which funds can be put: to pay for investment or to pay dividends to shareholders. In equilibrium, regardless of the firms' debt

[8] This is in essence the argument which eliminates the bubble component in the models of Scheinkman (1983), Magill and Quinzii (1996), Santos and Woodford (1997).

[9] The proof is given in the earlier version of the paper (Magill and Quinzii, 2000).

policy $(B_t)_{t\geq 0}$, the equity price $(Q_t)_{t\geq 0}$ must satisfy (25): adding B_t to both sides gives

$$Q_t + B_t = \frac{1}{1+r_{t+1}}(D_{t+1} + (1+r_{t+1})B_t + Q_{t+1})$$

which can be written as

$$M_t = \frac{1}{1+r_{t+1}}(\hat{D}_{t+1} + M_{t+1}) \tag{28}$$

where in view of (27)

$$\hat{D}_{t+1} = D_{t+1} + \Delta_{t+1} = F(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - I_{t+1} \tag{29}$$

D_{t+1} being the payment by the firms to to the shareholders and

$$\Delta_{t+1} = (1+r_{t+1})B_t - B_{t+1} \tag{30}$$

being the net payment to the debtholders. The right side of (29) is the real output which is available after compensating labor and deducting the part of output going to investment: this is the "real dividend" which is left to pay the "capital markets", i.e. the two claimants to the firms' income stream, the equity and debt holders. Integrating (28) gives

$$M_t = \sum_{\tau=1}^{T} \frac{\hat{D}_{t+\tau}}{(1+r_{t+1})\cdots(1+r_{t+\tau})} + \frac{M_{t+T}}{(1+r_{t+1})\cdots(1+r_{t+T})}$$

and by analogy with the definitions for equity, we define

$$M_t^f = \lim_{T\to\infty} \sum_{\tau=1}^{T} \frac{\hat{D}_{t+\tau}}{(1+r_{t+1})\cdots(1+r_{t+\tau})}, \quad M_t^b = \lim_{T\to\infty} \frac{M_{t+T}}{(1+r_{t+1})\cdots(1+r_{t+T})}$$

as the fundamental value and the bubble component of the firms' market value. Since $M_{t+T} = Q_{t+T} + B_{t+T}$, if $M_t^b = 0$ then both the present value of the equity price and of the debt tend to zero: there is no bubble on equity and firms do not accumulate debt at infinity. If $M_t^b > 0$, then either $Q_t^b > 0$ and there is a bubble on equity, or $\lim_{T\to\infty}(B_{t+T})/(1 + r_{t+1})\cdots(1+r_{t+T}) > 0$ and firms accumulate debt at infinity, or both. If the present value of debt at infinity is positive, we will say that there is a bubble on debt.

Proposition 6. *The market value of firms* $(M_t)_{t\geq 0}$ *in a stock market equilibrium with affine price expectations has the following properties*:

(i) *If* $(K_t, V_t)_{t\geq 0}$ *is an equilibrium trajectory of an economy with underaccumulation, then*
  (α) *if* $V_0 = 0$, *then* $M_t = M_t^f$ *and* $M_t^b = 0$;
  (β) *if* $V_0 > 0$, *and the trajectory converges to the Golden Rule, then* $M_t > M_t^f$ *and* $M_t^b > 0$.

(ii) *If* $(K_t, V_t)_{t\geq 0}$ *is an equilibrium trajectory of an economy with overaccumulation, then* $M_t^f = -\infty$ *and* $M_t^b = +\infty$

Proof. (i)(α): Since $V_t = 0, \forall t \geq 0$, by (26)

$$\frac{M_{t+T}}{(1+r_{t+1})\cdots(1+r_{t+T})} = \frac{N_t(1+n)^{T+1}k_{t+T+1}}{(1+r_{t+1})\cdots(1+r_{t+T})}$$

and since the equilibrium is a Diamond equilibrium, $k_{t+T+1} \to k_D, r_{t+T} \to r_D$ with $r_D > n$, and $\lim_{T\to\infty}(1+n)^{T+1}/(1+r_{t+1})\cdots(1+r_{t+T}) = 0$, so that $M_t = M_t^f$ and $M_t^b = 0$.

(i)(β): Since $V_t > 0, \forall t \geq 0$

$$\frac{M_{t+T}}{(1+r_{t+1})\cdots(1+r_{t+T})} = \frac{N_t(1+n)^T((1+n)k_{t+T+1} - v_{t+T})}{(1+r_{t+1})\cdots(1+r_{t+T})} = N_t\frac{v_t}{v_{t+T}}((1+n)k_{t+T+1} - v_{t+T})$$

and since the equilibrium is a stock market equilibrium converging to the Golden Rule, $k_{t+T+1} \to k^*, v_{t+T} \to v^*$ with $(1+n)k^* - v^* = s(n, w(k^*)) > 0$ and $M_t^b = N_t v_t (s(n, w(k^*)))/v^* > 0$.

(ii) If $V_0 = 0$, then the expression for M_{t+T} is the same as in (i)(α). Since $k_{t+T+1} \to k_D, r_{t+T} \to r_D$ with $r_D < n$, it follows that $\lim_{T\to\infty}(1+n)^T/(1+r_{t+1})\cdots(1+r_{t+T}) = +\infty$. When $V_0 > 0$, the expression for M_{t+T} is the same as in (i)(β) and since $k_{t+T+1} \to k_D, v_{t+T} \to 0$, it follows that $M_t^b = +\infty$. Since M_t is finite, it follows that $M_t^f = -\infty$, which can also be checked directly by evaluating M_t^f. □

The only case in which the market value of the firms concides with the fundamental value of their real dividends is in the Diamond equilibrium of an economy with underaccumulation. Since in this case there is no discount on equity, the market value also coincides with the replacement value of the capital embodied in the firms at the end of the period

$$M_t^f = M_t = K_{t+1}$$

In this case there is no bubble on equity and the equity price fits with the conventional measures: comparing the financial and the real side, the market value of the firms (equity + debt) will corrrespond with the “book value” or replacement cost, i.e. the accumulated value of investments once depreciation has been taken into account. Or, if analysts were to correctly forecast future interest rates and future dividends and evaluate the fundamental value of equity, their valuation would coincide with the observed value of equity.

In all other cases the conventional measures—replacement cost and fundamental value—will not provide exact estimates, they will only provide bounds since

$$M_t^f < M_t \leq K_{t+1}$$

with strict inequality on the right side when $V_t > 0$. Firms are cheap when compared to their book value, but expensive when compared to the stream of dividends that they generate.

With the financial policy considered in Sections 2 and 3, where firms finance investment by one period loans, there is no bubble on debt and thus it follows from Proposition 6 that, except in the case of underaccumulation and no discount, there is a positive bubble on equity.

When investment is financed by one-period loans the dividend on equity is

$$D_{t+1} = F(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - (1 + r_{t+1})I_t$$
$$D_{t+1} = N_t((1 + n)k_{t+1}f'(k_{t+1}) - (1 + r_{t+1})i_t)$$

which, after using the relations $f'(k_{t+1}) = \beta + r_{t+1}$ and $i_t = (1 + n)k_{t+1} - (1 - \beta)k_t$ can be written as

$$D_{t+1} = N_t(1 - \beta)((1 + r_{t+1})k_t - (1 + n)k_{t+1}) \tag{31}$$

Summing the present value of dividends given by (31) gives

$$\sum_{\tau=1}^{T} \frac{D_{t+\tau}}{(1 + r_{t+1}) \cdots (1 + r_{t+\tau})} = N_t(1 - \beta) \left(k_t - \frac{(1 + n)^T k_{t+T}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})} \right) \tag{32}$$

In the case of overaccumulation where the equilibrium converges to the Diamond equilibrium the limit when $T \to \infty$ of the sum in (32) is $-\infty$. In this case, the investment made by the firms is not sufficiently productive to permit paying back the debt and distributing positive dividends. If firms are to pay non-negative dividends because of limited liability rules, then they must either roll over their debt to infinity or use equity financing, or both. A more detailed analysis of this case can be found in Magill and Quinzii (2000).

In the case of underaccumulation with no discount (Diamond equilibrium), the limit of the sum in (32) is $N_t(1 - \beta)k_t = (1 - \beta)K_t = Q_t$, as predicted by Proposition 6, and the equity price is equal to the discounted value of dividends.

In the case of underaccumulation with a positive discount the recursive equations (E_S) imply $(1 + n)^T/(1 + r_{t+1}) \cdots (1 + r_{t+T}) = v_t/(v_{t+T})$, so that the fundamental value of equity is given by

$$Q_t^f = N_t(1 - \beta) \lim_{T \to \infty} \left(k_t - \frac{v_t}{v_{t+T}} k_{t+T} \right) = N_t(1 - \beta) \left(k_t - \frac{k^*}{v^*} v_t \right)$$

which can be written as

$$Q_t^f = (1 - \beta)K_t - (1 - \beta)\frac{k^*}{v^*}V_t \tag{33}$$

Since, by Assumption $\mathcal{GR}$, inequality (20) is satisfied at the Golden Rule, it follows that $v^* < (1 - \beta)^2/(1 + n)k^* < (1 - \beta)k^*$. By (33), the fundamental value of dividends is less than the equity price $Q_t = (1 - \beta)K_t - V_t$ and

$$Q_t^f < Q_t < (1 - \beta)K_t$$

The equity price is bounded above by the replacement cost and below by the fundamental value. As mentioned in Section 3, $(1 - \beta)K_t$ is not the fundamental value of equity, so that the discount $-V_t$ cannot be considered as a negative bubble attached to the fundamental value. The bubble component of equity is $Q_t^b = ((1 - \beta)k^*/v^* - 1)V_t > 0$.

The Golden Rule steady-state of an economy with underaccumulation satisfying Assumption $\mathcal{GR}$ is an example of an equilibrium with positive discount to which this analysis applies. It follows from (31) and (33) that at the Golden Rule, the dividends and the fundamental value of equity are zero: firms use all their earnings to finance the growth of capital—all the earnings are used to pay off the debt incurred in the previous period—and shareholders obtain their return solely by capital gains, i.e. by the increase in the price of equity that agents of the next generation will pay. In this case equity is a pure bubble (it gives zero dividends) and like money in the exchange model, has value purely by virtue of its role as a store of value.

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Appendix A

Proof of Proposition 1. If $V^j_{t+1} > (1-\beta)^2 K^j_t$ then, when $I^j_t \in [0\ \bar{I}]$, where $\bar{I}$ is defined by $(1-\beta)^2 K^j_t + (1-\beta)\bar{I} - V^j_{t+1} = 0$, $Q^j_{t+1}((1-\beta)K^j_{t+1}) = 0$ (where $K^j_{t+1} = (1-\beta)K^j_t + I^j_t$). On the other hand, if $I^j_t \geq \bar{I}$ then $Q^j_{t+1}((1-\beta)K^j_{t+1}) = (1-\beta)K^j_{t+1} - V^j_{t+1}$. Thus the objective function (3), to be maximized with respect to (I^j_t, L^j_{t+1}), that we will denote $\pi(I^j_t, L^j_{t+1})$, is equal to

$$\pi(I^j_t, L^j_{t+1}) = \begin{cases} -I^j_t + \dfrac{1}{1+r_{t+1}}[F((1-\beta)K^j_t + I^j_t, L^j_{t+1}) - w_{t+1}L^j_{t+1}] & \text{if } 0 \leq I^j_t \leq \bar{I} \\ -I^j_t + \dfrac{1}{1+r_{t+1}}[F((1-\beta)K^j_t + I^j_t, L^j_{t+1}) & \\ \quad -w_{t+1}L^j_{t+1} + (1-\beta)^2 K^j_t + (1-\beta)I^j_t - V^j_{t+1}] & \text{if } I^j_t \geq \bar{I} \end{cases} \tag{A.1}$$

When $I^j_t \geq \bar{I}$, substituting $I^j_t = K^j_{t+1} - (1-\beta)K^j_t$, this function can also be written as a function of capital and labor as

$$\pi(I^j_t, L^j_{t+1}) = \frac{1}{1+r_{t+1}}[F(K^j_{t+1}, L^j_{t+1}) - w_{t+1}L^j_{t+1} - (\beta + r_{t+1})K^j_{t+1} + (1-\beta)(1+r_{t+1})K^j_t - V^j_{t+1}] \tag{A.2}$$

Suppose that there exists a solution (I^{j*}_t, L^{j*}_{t+1}) to maximizing (A.1) with $I^{j*}_t > 0$. Suppose first that $I^{j*}_t \in (0\ \bar{I}]$. Then (I^{j*}_t, L^{j*}_{t+1}) must satisfy the first-order conditions for maximizing

(A.1) on $[0\ \bar{I}] \times \mathbb{R}_+$, where the constraint $I_t^j \geq 0$ is not binding. This implies that

$$F'_K(K_{t+1}^{j*}, L_{t+1}^{j*}) \geq 1 + r_{t+1}, \qquad F'_L(K_{t+1}^{j*}, L_{t+1}^{j*}) = w_{t+1} \tag{A.3}$$

with $K_{t+1}^{j*} = (1-\beta)K_t^j + I_t^{j*}$. Let f denote the production function per unit of labor, $f(k) = F(k,1)$. As is well known, by the linear homogeneity of F, $F'_K(K,L) = f'(K/L)$ and $F'_L(K,L) = f(K/L) - (K/L)f'(K/L)$.

Consider the capital–labor ratio $k_{t+1}^{j*} = K_{t+1}^{j*}/L_{t+1}^{j*}$ and large values of investment and labor such that $K_{t+1}^j > (1-\beta)K_t^j + \bar{I}$, $K_{t+1}^j/L_{t+1}^j = k_{t+1}^{j*}$. Then (A.2) evaluated at such pairs gives

$$\begin{aligned}\pi(I_t^j, L_{t+1}^j) &= \frac{L_{t+1}^j}{1+r_{t+1}}(f(k_{t+1}^{j*}) - w_{t+1} - (\beta + r_{t+1})k_{t+1}^{j*}) \\ &\quad + \frac{1}{1+r_{t+1}}((1-\beta)(1+r_{t+1})K_t^j - V_{t+1}^j)\end{aligned}$$

which, when (A.3) holds, can be made arbitrarily large since

$$f(k_{t+1}^{j*}) - w_{t+1} = F'_K(K_{t+1}^{j*}, L_{t+1}^{j*})k_{t+1}^{j*} \geq (1+r_{t+1})k_{t+1}^{j*} > (\beta + r_{t+1})k_{t+1}^{j*}$$

Thus $I_t^{j*} \leq \bar{I}$ cannot be a solution.

Suppose $I_t^{j*} \in (\bar{I}, \infty)$. Then $(K_{t+1}^{j*}, L_{t+1}^{j*})$ must be an interior solution to maximizing (A.2), so that the first-order conditions

$$F'_K(K_{t+1}^{j*}, L_{t+1}^{j*}) = \beta + r_{t+1}, \qquad F'_L(K_{t+1}^{j*}, L_{t+1}^{j*}) = w_{t+1} \tag{A.4}$$

must hold and the value of the objective function is $\pi(I_t^{j*}, L_{t+1}^{j*}) = 1/(1+r_{t+1})((1-\beta)(1+r_{t+1})K_t^j - V_{t+1}^j)$. Since $V_{t+1}^j > (1-\beta)^2 K_t^j$, $\pi(I_t^{j*}, L_{t+1}^{j*}) < 1/(1+r_{t+1})((1-\beta)(\beta + r_{t+1})K_t^j)$. Let us show that the shareholders would be made better off by not investing. If they do not invest the objective function will be greater or equal to $\pi(0, \tilde{L}_{t+1}^j)$, where $\tilde{L}_{t+1}^j$ is chosen, so that $((1-\beta)K_t^j)/\tilde{L}_{t+1}^j = k_{t+1}^{j*}$. When the FOC (A.4) hold

$$\pi(0, \tilde{L}_{t+1}^j) = \frac{\tilde{L}_{t+1}^j}{1+r_{t+1}}(f(k_{t+1}^{j*}) - w_{t+1}) = \frac{\tilde{L}_{t+1}^j}{1+r_{t+1}}(\beta + r_{t+1})k_{t+1}^{j*}$$

$$\begin{aligned}\pi(0, \tilde{L}_{t+1}^j) &= \frac{\tilde{L}_{t+1}^j}{1+r_{t+1}}(f(k_{t+1}^{j*}) - w_{t+1}) \\ &= \frac{1}{1+r_{t+1}}(1-\beta)(\beta + r_{t+1})K_t^j > \pi(I_t^{j*}, L_{t+1}^{j*})\end{aligned}$$

Thus the problem of maximizing (A.1) cannot have a solution such that $I_t^{j*} > 0$. □

Proof of Proposition 2. A Diamond steady-state is a solution of the equation $S(k)/k = 1+n$ and it is clear that assumption $\mathcal{S}$ implies that the equation has a unique positive solution

k_D. To prove global stability we show (i) ϕ is increasing; (ii) $\phi(k) > k$ if $0 < k < k_D$; and (iii) $\phi(k) < k$ if $k > k_D$.

(i) $s(r(k_{t+1}), w(k_t)) = (1+n)k_{t+1} \Leftrightarrow s(r(\phi(k_t)), w(k_t)) = (1+n)\phi(k_t) \Rightarrow s'_r r'(k_{t+1})$-$\phi'(k_t) + s'_w w'(k_t) = (1+n)\phi'(k_t) \Rightarrow [(1+n) - s'_r f''(k_{t+1})]\phi'(k_t) = -s'_w k_t f''(k_t) \Rightarrow$

$$\phi'(k_t) = \frac{-s'_w k_t f''(k_t)}{(1+n) - s'_r f''(k_{t+1})} > 0 \tag{A.5}$$

(ii) Suppose not, $\phi(k) \le k$; then $r(\phi(k)) \ge r(k)$ and $s(r(\phi(k)), w(k)) \ge s(r(k), w(k)) > (1+n)k \ge (1+n)\phi(k)$, where the first inequality follows from $s'_r \ge 0$ and the second from $S(k)/k > S(k_D)/k_D = (1+n)$ since $k < k_D$: but this contradicts (E_D), namely, $s(r(\phi(k)), w(k)) = (1+n)\phi(k)$.

(iii) Suppose not, $\phi(k) \ge k$; then $r(\phi(k)) \ge r(k)$ and $s(r(\phi(k)), w(k)) \le s(r(k), w(k)) < (1+n)k \le (1+n)\phi(k)$, contradicting ($E_D$).

To complete the proof, suppose $0 < k_0 > k_D$ (resp. $k_0 < k_D$) then (i)–(iii) imply that k_t is an increasing (decreasing) sequence which is bounded above (below) by k_D: thus $k_t \to k_D$ as $t \to \infty$. □

Proof of Proposition 4. The difference equation system ($E_S{}'$) can be written as

$$k_{t+1} = \psi(k_t, v_t)$$
$$v_{t+1} = h(k_t, v_t)$$

where ψ is defined implicitly by the equation

$$(1+n)\psi(k_t, v_t) - s(f'(\psi(k_t, v_t)) - \beta, w(k_t)) - v_t = 0$$

and $h(k_t, v_t) = g(\psi(k_t, v_t))v_t$ with $g(x) = (1 - \beta + f'(x))/(1+n)$. Thus the linearized system associated with ($E_S{}'$) around a steady-state $(\bar{k}, \bar{v})$, expressed in terms of the deviation variables $(\kappa_t, \nu_t) = (k_t - \bar{k}, v_t - \bar{v})$ is given by

$$\begin{bmatrix} \kappa_{t+1} \\ \nu_{t+1} \end{bmatrix} = \begin{bmatrix} \psi'_k(\bar{k}, \bar{v}) & \psi'_v(\bar{k}, \bar{v}) \\ h'_k(\bar{k}, \bar{v}) & h'_v(\bar{k}, \bar{v}) \end{bmatrix} \begin{bmatrix} \kappa_t \\ \nu_t \end{bmatrix} \tag{L_S}$$

where

$$\psi'_k(\bar{k}, \bar{v}) = \frac{-s'_w \bar{k} f''(\bar{k})}{1+n - s'_r f''(\bar{k})}, \qquad \psi'_v(\bar{k}, \bar{v}) = \frac{1}{1+n - s'_r f''(\bar{k})}$$

$$h'_k(\bar{k}, \bar{v}) = \frac{f''(\bar{k})}{1+n}\left(\frac{-s'_w \bar{k} f''(\bar{k})}{1+n - s'_r f''(\bar{k})}\right)\bar{v}, \quad h'_v(\bar{k}, \bar{v}) = g(\bar{k}) + \frac{f''(\bar{k})\bar{v}}{(1+n)(1+n - s'_r f''(\bar{k}))}$$

Let $\bar{M}$ denote the matrix of coefficients in (L_S) evaluated at $(\bar{k}, \bar{v})$, and let $p(\lambda) = \lambda^2 - tr(\bar{M})\lambda + \det \bar{M} = 0$ denote the associated characteristic polynomial. To show that the Golden Rule steady-state $(\bar{k}, \bar{v}) = (k^*, v^*)$ is locally stable we show that both roots of the characteristic polynomial lie inside the unit circle ($|\lambda_i| < 1, i = 1, 2$). Note that det $M^* = \psi'_k(k^*, v^*) = (-s'_w k^* f''(k^*))/(1+n-s'_r f''(k^*)) > 0$ by Assumption $\mathcal{C}$. Since there is underaccumulation, $k^* > k_D$ and by Assumption $\mathcal{S}$, $S'(k^*) < S(k_D)/k_D = 1+n$. Since

$S'(k^*) = s_r' f''(k^*) - s_w' k^* f''(k^*)$, this implies $0 < \det M^* < 1$. Since $\det M^* = \lambda_1\lambda_2$, if both roots are complex, they lie inside the unit circle. The condition $0 < \det M^* < 1$ implies that if both roots are real they lie in the unit interval $(-1, 1)$ if and only if $p(1) > 0$ and $p(-1) > 0$. Now

$$p(1) = 1 - trM^* + \det M^* = \frac{-f''(k^*)v^*}{(1+n)(1+n-s_r' f''(k^*))} > 0$$

since $v^* > 0$, $s_r' \geq 0$, $f''(k^*) < 0$ and

$$p(-1) = 1 + trM^* + \det M^* = 2 - \frac{2s_w' k^* f''(k^*)}{1+n-s_r' f''(k^*)} + \frac{f''(k^*)v^*}{(1+n)(1+n-s_r' f''(k^*))}$$

The first two terms are positive: to show $p(-1) > 0$ it suffices to show that the third term is bounded below by -1. Since $S(k^*) > 0$, $v^* < (1+n)k^*$ and since Assumption $\mathcal{P}$ implies $k^* f''(k^*) \geq -f'(k^*) = -(\beta + n)$, it follows that

$$\frac{f''(k^*)v^*}{(1+n)(1+n-s_r' f''(k^*))} > \frac{f''(k^*)k^*}{1+n-s_r' f''(k^*)} \geq \frac{-f'(k^*)}{1+n-s_r' f''(k^*)} > -\frac{\beta+n}{1+n} > -1$$

Thus both roots lie inside the unit circle and (k^*, v^*) is locally stable.

At the Diamond steady-state $(\bar{k}, \bar{v}) = (k_D, 0)$, $h_k'(k_D, 0) = 0$, so that M_D is triangular and $P(\lambda) = (\psi_k'(k_D, 0) - \lambda)(h_v'(k_D, 0) - \lambda)$. Since $0 < \psi_k'(k_D, 0) = (-s_w' k_D f''(k_D))/(1 + n - s_r' f''(k_D)) < 1$, where the latter inequality follows from $S'(k_D) < S(k_D)/k_D = 1 + n$, and $h_v'(k_D, 0) = g(k_D) = (1 + r(k_D))/(1+n) > 1$, it follows that the Diamond steady-state is locally saddle-point stable. □

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