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Which improves welfare more: A nominal or an indexed bond?★

Michael Magill[1] **and Martine Quinzii**[2]

[1] Department of Economics, University of Southern California, Los Angeles, CA 90089-0253, USA

[2] Department of Economics, University of California, Davis, CA 95616-8578, USA

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Summary. Economists have long argued that loan contracts should be indexed to remove the risks arising from fluctuations in the purchasing power of money: indexation however while eliminating one risk, substitutes another, arising from fluctuations in relative prices of goods. We present a theoretical framework which permits the relative merits of a nominal versus an indexed bond to be assessed in a general equilibrium setting.

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1 Introduction

Despite economists' long standing arguments in favor of systematic indexation of loan contracts to remove the risks associated with fluctuations in the purchasing power of money (Jevons [12], Marshall [15, 16], Fisher [6], Friedman [8]), surprisingly few loan contracts are indexed in most Western Economies. In the United States even thirty year corporate and government bonds are not indexed. The situation is however different in many Latin American countries where indexing is widely used as a way of coping with high and variable inflation rates. What seems difficult to explain is that it takes high variability in inflation rates before private sector agents shift from unindexed to indexed contracts. The object of this paper is to provide a theoretical framework for explaining this phenomenon.

The first formal analysis of the demand for nominal and indexed bonds in an equilibrium framework was given by Fischer [5]: he used the continuous-

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Correspondence to: M. Magill

time, Brownian-motion version of the one-good CAPM model, in which there are price level fluctuations and in which agents can trade a nominal bond, a perfectly indexed bond and an equity contract. As Modigliani [17] pointed out, since the indexed bond permits the riskless transfer of income and since the two-fund separation theorem holds, in Fischer's model there is no trade in the nominal bond in equilibrium: with perfect indexation, an indexed bond will always drive out the nominal bond. This result, while providing a formalization of the classical argument in favor of indexation, does not provide a model that explains why in practice so few indexed loans are traded.[1]

To move out of this *cul de sac* the analysis must be posed in the framework of a *multigood* model where the imperfections of indexation can be made explicit. To provide a coherent equilibrium framework, the model should in addition incorporate money, since fluctuations in the purchasing power of money are fundamentally of monetary origin; furthermore, to permit the relative benefits of nominal versus indexed bonds to be assessed, the financial markets on which agents trade should be incomplete. Thus the model we use is a variant of the two-period general equilibrium model with incomplete markets (GEI), in which the purchasing power of money depends on a (broadly defined) measure of the amount of money available for transactions, and on an index of real output. The model is sufficiently simple to permit the welfare of agents in equilibrium to be expressed explicitly as a function of the payoffs of the securities traded on the financial markets, and yet is sufficiently rich to capture some of the imperfections of indexation.

More precisely, we make assumptions on the characteristics of the economy, the agents' preferences and endowments and the security structure (described in section 2) which ensure that:

(i) the multigood model can be mapped into a purchasing power economy (essentially a one-good economy) in which there are well-defined (utility-based) indices of the purchasing power of money and aggregate output;

(ii) an efficient equilibrium is obtained if agents can trade a bond whose purchasing power payoff is constant;

(iii) if there is no such (real) riskless bond, but only a risky bond, then the loss in welfare depends on the distance (in the appropriate probability metric) of the financial market subspace from the riskless income stream;

(iv) if the future payoff of a bond could be indexed on the value of a state-dependent bundle of goods, then an indexed bond with a constant purchasing-power payoff could be obtained. However a bond indexed on the value of a state-independent reference bundle is imperfect, since its payoff fluctuates not only with the price level (its virtue), but also varies with changes in the relative prices of the goods (its defect).

[1] A step in this direction was made by Viard [19], using Fischer's model with constant relative risk aversion preferences: he argued that for some values of the parameters the welfare gains from introducing an indexed bond are small, once the nominal bond is traded.

We capture the imperfections of indexation by requiring that the reference bundle be non-contingent: this avoids relying too much on the specific structure of the model – after all when agents have preferences that are more heterogeneous than those that we consider, even a state-dependent reference bundle that leads to a perfect index does not exist – and it corresponds to the standard practice of indexation: in order that indexed contracts be enforceable, they must be indexed on officially computed price indices such as the Consumer Price Index (CPI) whose reference bundle, for reasons of practicality and credibility, is seldom changed.

The objective of the analysis is to compare two second-best situations, in which in addition to a given security structure, there is either a *nominal* bond which has the risks induced by fluctuations in the purchasing power of money or an *indexed* bond which has the risks induced by relative price fluctuations. In order to make such a comparison, we begin by studying how the welfare of agents in an equilibrium of the purchasing power economy is increased when a bond with a given payoff structure is added to an existing collection of securities (the equity contracts of firms). In general, adding a bond to an existing market structure has two effects. The first is the *direct* effect of increasing the span of the financial markets: this always increases the welfare of agents. The second is the *indirect* effect of changing spot and security prices: this can either increase or decrease agents' utilities. When the indirect effects are strong enough, they can more than offset the gains from increasing agents' trading opportunities (see Cass-Citanna [2] and Elul [4] for a complete local analysis of the combined effects). In this paper all indirect effects are absent by virtue of the specification of agents' preferences, so that introducing a bond always increases agents' utilities.

The welfare gain attributable to a bond is measured by the extent to which it reduces the riskiness of the financial market opportunity set: more precisely, the gain is measured by how much closer the financial market subspace moves to the riskless income stream. The welfare gain is expressed by a function, which we call the *statistical gains function*, since it depends on the statistical properties of the bond's real payoff, its standard deviation per unit of expectation and its vector of correlation coefficients with the existing securities.

A complete analysis of the properties of the gains function (Propositions 3 and 4) is the main mathematical contribution of the paper: this is a necessary preliminary for determining which type of bond (nominal or indexed) leads to higher welfare. It follows from the properties of this function that either a low variability of the bond's (real) income stream or a strong (positive or negative) correlation of its payoff with the payoffs of the other securities (or a combination of the two) permits a high proportion of the potential welfare gains to be captured: a low variability directly creates a security without much risk, while a high correlation permits a hedge portfolio of the bond and the underlying securities to reduce risk.

In the multigood economy, three groups of factors influence the real payoffs of the indexed and nominal bonds. The first are sectoral shocks which affect the relative output of the different sectors (goods) and hence the relative prices of the goods: these shocks determine the variability of the payoff of the indexed bond. The second are economy-wide shocks which affect aggregate output, and the third are monetary shocks which influence the "amount" of purchasing power: the ratio of these two magnitudes determines the purchasing power of money, which is the payoff of the nominal bond. In Proposition 5 it is shown that in an economy in which inflation and output are positively correlated and sectoral shocks lead to relative price fluctuations, there is a critical level of fluctuations in the purchasing power of money below (above) which the nominal (indexed) bond is preferred. Thus in the framework of this model, it is the existence of sectoral shocks, in conjunction with a relatively strong positive correlation between inflation and output, characteristic of the Phillips curve, which serve to explain the lack of indexation.

2 The economy

In this section we present a variant of the general equilibrium model with incomplete markets (GEI) which leads to a tractable study of the issue of indexation of nominal bonds. Consider a two-period ($t = 0, 1$) economy with $S \geq 2$ states of natures ($s = 1, \ldots, S$) at date 1; for convenience we include date 0 as state 0 and write $s = 0, 1, \ldots, S$. There are I agents; each agent i is characterized by an initial endowment consisting of a vector $\omega^i = (\omega^i_0, \omega^i_1, \ldots, \omega^i_S)$ of L goods in each state and a utility function $U^i : \mathbb{R}^{L(S+1)}_+ \longrightarrow \mathbb{R}$ reflecting his preferences for the goods across the states. Agents can trade on two types of markets. Goods can be bought and sold on spot markets, the vector of spot prices $p_s = (p_{s1}, \ldots, p_{sL})$ in state s being expressed in units of money. Let $p = (p_0, p_1, \ldots, p_S)$ denote the vector of spot prices. In addition agents can trade (at date 0) on a system of financial markets. To provide a convenient framework for analyzing the potential benefits of indexing a bond, we consider a family of $J + 1$ securities. Security zero, which is the bond that may or may not be indexed, has a date 0 price q_0 and a date 1 payoff stream

$$A = (A_1, \ldots, A_S)$$

The remaining J securities have prices $(q_1, \ldots, q_J)$ at date 0 and date 1 payoffs summarized by an $S \times J$ matrix

$$Y = \begin{bmatrix} Y^1_1 & \ldots & Y^J_1 \\ \vdots & & \vdots \\ Y^1_S & \ldots & Y^J_S \end{bmatrix}$$

the payoff of security j in state s being Y^j_s. Let

$$q = (q_0, q_1, \ldots, q_J), \quad [A \ Y]$$

denote the vector of prices of the $J + 1$ securities and their combined date 1 payoff matrix. The payoffs of the securities can be either *real* (dependent on the spot prices) or *nominal* (independent of the spot prices) and in both cases are denominated in units of money. When security zero is indexed (unindexed) its payoff is real (nominal). The payoffs on the remaining securities can be either real or nominal, but will be required to satisfy certain spanning conditions (Assumption **S**) which imply that some of these securities are real (in essence, that they be equity contracts). To simplify notation, we omit the explicit dependence of the securities' payoffs on the spot prices.

If $z^i = (z_0^i, z_1^i, \ldots, z_J^i) \in \mathbb{R}^{J+1}$ denotes the portfolio of the $J+1$ securities purchased by agent i and if $x^i = (x_0^i, x_1^i, \ldots, x_S^i) \in \mathbb{R}_+^{L(S+1)}$ denotes his consumption stream of the L goods, then the agent's budget set is given by

$$\mathbb{B}(p, q, \omega^i) = \left\{ x^i \in \mathbb{R}_+^{L(S+1)} \;\middle|\; \begin{array}{l} p_0(x_0^i - \omega_0^i) = -qz^i, \; z^i \in \mathbb{R}^{J+1} \\ p_s(x_s^i - \omega_s^i) = [A_s \; Y_s]z^i, \; s = 1, \ldots, S \end{array} \right\}$$

where $[A_s \; Y_s]$ denotes row s of the matrix $[A \; Y]$.

One of the interesting properties of the GEI model with nominal securities is that price levels affect the real equilibrium allocation. This result can either be interpreted as exhibiting the indeterminacy of equilibrium allocations when there are no forces determining price levels (Balasko-Cass [1], Geanakoplos-Mas-Colell [9]) or as exhibiting the fact that fluctuations in the purchasing power of money (*ppm*) induced by monetary policy have real effects (Magill-Quinzii [13]). In this paper we adopt the latter interpretation. The general idea is to draw on the logic of the quantity theory: agents use money for transactions and a combination of a private sector banking system and a monetary authority determines the quantity of money that is available for making transactions. If $p_s \sum_{i=1}^I x_s^i$ is the demand for money in state s and M_s is the quantity of money made available, then the price level in state s is determined by the monetary equation

$$p_s \sum_{i=1}^{I} x_s^i = M_s, \quad s = 0, 1, \ldots, S$$

For the sake of interpretation we suppose there is a monetary authority with some (in certain cases very little) control over $M = (M_0, M_1, \ldots, M_S)$ and we call M the *monetary policy*. If $U = (U^1, \ldots, U^I)$ and $\omega = (\omega^1, \ldots, \omega^I)$, then $\mathscr{E}(U, \omega, A, Y, M)$ denotes the economy with agents' characteristics (U, ω), financial structure (A, Y) and monetary policy M. The exogenously given random variables (ω, M) which describe the underlying real and monetary sides of the economy, can have a very general stochastic dependence. This permits a wide class of economies to be considered which can differ not only in the way in which monetary policy or shocks intervene, but also in the way money and output are correlated.

2.1 Definition: An *equilibrium* of the economy $\mathscr{E}(U, \omega, A, Y, M)$ is a pair of actions and prices $\big((\bar{x}, \bar{z}), (\bar{p}, \bar{q})\big) = \big((\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I), (\bar{p}, \bar{q})\big)$ such that

(i) $\bar{x}^i \in \arg\max \{U^i(x^i) \mid x^i \in \mathbb{B}(\bar{p}, \bar{q}, \omega^i)\}$ and $\bar{z}^i$ finances $\bar{x}^i$, $i = 1, \ldots, I$

(ii) $\sum_{i=1}^{I}(\bar{x}^i - \omega^i) = 0$

(iii) $\sum_{i=1}^{I}\bar{z}^i = 0$

(iv) $\bar{p}_s \sum_{i=1}^{I}\bar{x}_s^i = M_s, \quad s = 0, 1, \ldots, S.$

The abstract model presented above is capable of covering many different types of financial securities, in particular, two important classes of securities which are used to finance many activities in an economy – bonds and equity contracts. Equity contracts are readily included by adapting the abstract exchange economy to represent a production economy in which firms have fixed production plans. The initial ownership of the K firms in the economy is distributed among the I agents, δ_k^i of firm k being owned by agent i. Agent i then has initial resources in the abstract economy consisting of two components

$$\omega^i = \underset{\sim}{\omega}^i + \sum_{k=1}^{K}\delta_k^i y^k \tag{1}$$

where $\underset{\sim}{\omega}^i \in \mathbb{R}_+^{L(S+1)}$ is a proxy for the agent's labor income and y^k is the production plan of firm k. If the financial markets include a stock market on which the equity contract of each firm is traded, then there is a security with payoff in state s ($s = 1, \ldots, S$) given by

$$Y_s^k = p_s y_s^k, \quad k = 1, \ldots, K$$

If θ_k^i is the amount of equity k purchased by agent i (at date 0), then $z_k^i = \theta_k^i - \delta_k^i$ is the agent's net trade in the k^{th} equity contract. As a class of contracts, bonds are typically designed to be less risky than equity contracts: modulo the problem of default, a bond promises a stable nominal payoff across the states of nature, while equity contracts have payoffs which fluctuate directly with the contingencies that affect the performance of individual firms. However, the stable nominal payoff of a bond only translates into a stable real payoff if there are no fluctuations in the purchasing power of money. The fact that variations in *ppm* introduce risks into securities designed to be essentially riskfree has long been viewed by economists as introducing an inefficiency that should be avoided. Hence the idea that monetary policy should seek, as far as possible, to achieve a stable *ppm* or, if imperfections in the control of the monetary transmission mechanism or political factors make this unfeasible, that bonds should be indexed.

Our objective is to find a way of formalizing these ideas. We will not try to address the general problem of indexing a family of nominal securities. Rather, we shall focus on the benefits and costs of indexing the least risky nominal bond – namely the *default-free* bond. To do this, we need to give more specific structure to the characteristics of the economy – basically assumptions on agents' endowments and preferences and on the security structure which ensure that agents would really benefit from the presence of a bond with a riskless real purchasing power. We want to show that, in a multigood setting, indexing is not the universal panacea for neutralizing fluctuations in *ppm* that is often suggested: indexing inevitably introduces the risks of relative-price fluctuations, and in some cases these risks may exceed the risks arising from fluctuations in *ppm*. The first assumption places a restriction on agents' preferences which implies that spot prices are independent of the income distribution and are thus independent of agents' choices on the financial markets. This eliminates a feedback between the spot markets and the financial markets, and greatly simplifies the analysis of the model.

Assumption H: Agents have separable-homothetic utility functions of the form

$$U^i(x^i) = \lambda_0^i h(x_0^i) + \sum_{s=1}^{S} \gamma_s f^i\big(h(x_s^i)\big), \quad i = 1, \dots, I$$

where $\gamma_1, \dots, \gamma_S$ are strictly positive probabilities of the states, $\lambda_0^i > 0$, $h : \mathbb{R}_+^L \longrightarrow \mathbb{R}$, $f^i : \mathbb{R} \longrightarrow \mathbb{R}$, both h and f^i are increasing, strictly concave and differentiable and h is homogeneous of degree 1.

In an equilibrium, the maximum problem of each agent can be decomposed into two steps: the first is a choice of a portfolio (z^i) on the financial markets, the second is the choice of a vector of consumption (x^i) on the spot markets. The choice of a portfolio by agent i generates an income stream across the states

$$m_0^i = p_0 \omega_0^i - q z^i$$
$$m_s^i = p_s \omega_s^i + [A_s \ Y_s] z^i, \ s = 1, \dots, S$$

Given the income stream $m^i = (m_0^i, m_1^i, \dots, m_S^i)$, the separability of the utility function U^i implies that the agent's optimal choice of consumption is the solution of $S + 1$ separate choice problems

$$\max \ \{h(x_s^i) \mid p_s x_s^i = m_s^i, \ x_s^i \in \mathbb{R}_+^L\}, \quad s = 0, 1, \dots, S$$

At the optimal choice x_s^i (unique by strict concavity of h) the gradient $\nabla h(x_s^i)$ is collinear to the spot price vector p_s. By homogeneity of degree one of h, the gradient vectors $\nabla h(x_s^i)$ are collinear if and only if the consumption vectors are collinear. Thus at an equilibrium each agent's consumption x_s^i is some proportion of the aggregate endowment $w_s = \sum_{i=1}^{I} \omega_s^i$ in state s and the equilibrium vector of spot prices is proportional to the gradient of h at the aggregate endowment. Using the Euler identity $\nabla h(w_s) w_s = h(w_s)$ and writing the monetary equations as $p_s w_s = M_s$, $s = 0, 1, \dots, S$ leads to the *equilibrium spot prices*

$$\bar{p}_s = \frac{M_s}{h(w_s)} \nabla h(w_s), \quad s = 0, 1, \dots, S \tag{2}$$

which are independent of the financial choices of the agents. Since agent i's share of aggregate output must be his share (m_s^i / M_s) of aggregate expenditure $M_s = p_s w_s$, his equilibrium consumption vector can be deduced once the expenditure stream $m^i = (m_0^i, m_1^i, \dots, m_S^i)$ is known

$$x_s^i = \left(\frac{m_s^i}{M_s}\right) w_s, \; s = 0, 1, \dots, S$$

Substituting this expression into the utility function $U^i(x^i)$ in Assumption **H** and exploiting the homogeneity of degree 1 of h, gives the utility of agent i as a function of his expenditure stream m^i

$$\tilde{u}^i(m^i) = \lambda_0^i v_0 m_0^i + \sum_{s=1}^{S} \gamma_s f^i(v_s m_s^i) \tag{3}$$

where

$$v_s = \frac{\nabla h(w_s) w_s}{M_s} = \frac{h(w_s)}{M_s}, \; s = 0, 1, \dots, S \tag{4}$$

is a *utility index of the purchasing power of money*. The numerator in (4) is an ideal (utility based) *index of aggregate output* in state s. The aggregate output $w_{s\ell}$ of good ℓ in state s is weighted by its social (representative agent) marginal utility in state s, $\frac{\partial h(w_s)}{\partial w_{s\ell}}$, and the index[2] measures the representative agent's utility $h(w_s)$ at the total output w_s. The purchasing power v_s is the utility that can be obtained by optimally spending one unit of money in state s.

Purchasing power economy. Since agents' preferences over expenditure streams are expressed by (3), the analysis of the equilibrium problem for the economy $\mathscr{E}(U, \omega, A, Y, M)$ can be reduced to the analysis of the equilibrium of a finance economy in which all quantities (income and expenditure streams, security payoffs) are converted to real (i.e. purchasing power) values. To this end, define each agent's *real* income and expenditure stream $(i = 1, \dots, I)$

$$e_s^i = v_s p_s \omega_s^i, \quad \mu_s^i = v_s m_s^i, \quad s = 0, 1, \dots, S$$

and let

$$u^i(\mu^i) = \lambda_0^i \mu_0^i + \sum_{s=1}^{S} \gamma_s f^i(\mu_s^i)$$

denote the utility to agent i of the real expenditure stream $\mu^i \in \mathbb{R}_+^{S+1}$. If we define the *real* prices and payoff streams of the securities $(j = 0, 1, \dots, J)$

$$q_j' = v_0 q_j, \quad a_s = v_s A_s, \quad v_s^j = v_s Y_s^j, \quad s = 1, \dots, S$$

then the financial problem of agent i reduces to choosing a portfolio $z^i \in \mathbb{R}^{J+1}$ which maximizes u^i in the budget set

[2] If h is the Cobb-Douglas utility function then the index of output in state s is the geometric mean of the L components of aggregate output $(w_{s1}, \dots, w_{sL})$, the weight assigned to good ℓ being its coefficient in the Cobb Douglas function. The purchasing power of money v_s is then obtained by dividing the index of aggregate output by the money supply M_s.

$$\mathscr{B}(q', e^i) = \left\{ \mu^i \in \mathbb{R}_+^{S+1} \,\middle|\, \begin{array}{ll} \mu_0^i = e_0^i - q' z^i, & z^i \in \mathbb{R}^{J+1} \\ \mu_s^i = e_s^i + [a_s \ V_s]\, z^i, & s = 1, \ldots, S \end{array} \right\}$$

where $V = [v^1 \ldots v^J]$ is the matrix of real payoffs of securities $j = 1, \ldots, J$ and V_s is the row corresponding to state s. Let $e^i = (e_0^i, e_1^i, \ldots, e_S^i)$, $e = (e^1, \ldots, e^I)$ denote the real values of agents' endowments, $u = (u^1, \ldots, u^I)$ their utility functions for real income and $\mathbf{a} = (a_1, \ldots, a_S)$ the real payoff stream on the default-free bond, then we call $\mathscr{E}(u, e, \mathbf{a}, V)$ the *purchasing power economy* induced by the monetary economy $\mathscr{E}(U, \omega, A, Y, M)$.

The next assumption permits explicit calculations to be made of the welfare consequences of alternative real payoff streams $\mathbf{a}$ for the bond, depending on whether the nominal payoff $\boldsymbol{A}$ is indexed or unindexed; furthermore the welfare comparisons have a natural economic and geometric interpretation. The assumption requires that agents have mean-variance preferences – a convenient (if crude) first approximation for describing the way agents evaluate risks.

Assumption Q: For each agent the function $f^i : \mathbb{R} \longrightarrow \mathbb{R}$ in Assumption **H** is quadratic

$$f^i(\mu) = -\frac{1}{2}(\alpha^i - \mu)^2, \quad i = 1, \ldots, I$$

where $\alpha^i > h(w_s), \ s = 1, \ldots, \ S.$

Finally we include a spanning assumption on the security structure Y which ensures that in the purchasing power (pp) economy the riskless real income stream $\mathbb{1} = (1, \ldots, 1)$ becomes a reference income stream for measuring the losses due to fluctuations in ppm and the potential gains from indexation. For when the security structure Y is well-adapted to the agents' endowment risks $(\bar{p}_s \omega_s^i)_{s=1}^S$, then in the pp economy the most important missing security is the riskless real bond $\mathbb{1}$ and welfare losses or gains can be expressed in terms of the distance of the market subspace $\langle [\mathbf{a} \ V] \rangle$ from $\mathbb{1}$. We use the following notation: for any vector $x = (x_0, x_1, \ldots, x_s)$, $x_\mathbf{1} = (x_1, \ldots, x_S)$ denotes the vector of date 1 components.

Assumption S: For each agent $i = 1, \ldots, I$

$$(\bar{p}_1 \omega_1^i, \ldots, \bar{p}_S \omega_S^i) \in \langle Y \rangle \Longleftrightarrow e_\mathbf{1}^i \in \langle V \rangle$$

If the agents' endowments have the form given in (1), then the spanning assumption amounts to requiring that Y contains the equity contracts of the corporate firms and enough additional securities to permit agents to share their personal income risks $(\bar{p}_s \underset{\sim}{\omega}_s^i)_{s=1}^S$ – or equivalently, that their private sources of income (for example their wage income or their income from individually owned firms) are subject to the same shocks as the corporate sector. However we assume that the security structure is incomplete in that the subspace $\langle V \rangle$ of the pp economy does not contain $\mathbb{1}$ and has dimension less than $S - 1$ (there are no securities which provide direct insurance against monetary shocks and it would take more than one additional bond to complete the markets). For convenience we add two purely technical conditions: real payoff streams are non-redundant and have positive expected values.

Assumption I: (i) $\mathbb{1} \notin \langle V \rangle$ (ii) rank $V = J$ (iii) $J \leq S-2$ (iv) $E(v^j) > 0,\ j = 1, \ldots, J$.

Assumption **H** reduces the analysis of the multigood economy $\mathscr{E}(U, \omega, A, Y, M)$ to the analysis of the purchasing power economy $\mathscr{E}(u, e, \mathbf{a}, V)$. Under Assumptions **Q**, **S** and **I**, this *pp* economy satisfies the assumptions of the Capital Asset Pricing Model (CAPM), in which however, if $\mathbf{a}$ is risky (or more precisely if $\mathbb{1} \notin \langle \mathbf{a}, V \rangle$), the riskless transfer of income is not possible. If $\mathbf{a} = \mathbb{1}$ or $\mathbb{1} \in \langle \mathbf{a}, V \rangle$, then by a standard result, the equilibria of $\mathscr{E}(u, e, \mathbf{a}, V)$ are Pareto optimal; when $\mathbf{a}$ is risky there is a loss relative to the ideal situation $\mathbf{a} = \mathbb{1}$. If A is the default-free nominal bond, then its nominal payoff is $A^N = \mathbb{1}$ and its real payoff is just the purchasing power of money $\mathbf{a}^N = v = (v_1, \ldots, v_S)$: the greater the fluctuations in *ppm*, the greater the risks of $\mathbf{a}^N$. On the other hand if A is indexed on the value of a reference bundle of goods $b = (b_1, \ldots, b_L) \in \mathbb{R}^L$, then its nominal payoff stream is $A^R = (\bar{p}_1 b, \ldots, \bar{p}_S b)$, and in view of (2) and (4), its real payoff stream is $\mathbf{a}^R = (\nabla h(w_1) b, \ldots, \nabla h(w_S) b)$. While $\mathbf{a}^R$ is isolated from fluctuations in *ppm*, it does however vary with fluctuations in $\nabla h(w_s)$ i.e. those induced by underlying real shocks which affect the relative aggregate supplies of the goods. In order to explain the conditions under which the agents are better off using the nominal or the indexed bond, we need to understand how the welfare of the agents in an equilibrium depends on the characteristics of the income stream $\mathbf{a}$ – its variability and the way it covaries with the other securities in the economy summarized by V.

3 Welfare and the statistical characteristics of the bond

A geometric approach to the welfare analysis of equilibria of an economy in which agents have mean-variance preferences can be obtained using projections under the probability induced *inner product* on $\mathbb{R}^S$ defined by

$$[[x, y]] = \sum_{s=1}^{S} \gamma_s x_s y_s = E(xy) = E(x)E(y) + \text{cov}(x, y) \tag{1}$$

and its associated *norm*

$$\| x \|_\gamma = \left(\sum_{s=1}^{S} \gamma_s x_s^2 \right)^{\frac{1}{2}} = \left(E(x^2) \right)^{\frac{1}{2}} = \left((E(x))^2 + \text{ var } x \right)^{\frac{1}{2}} \tag{2}$$

Two vectors $x, y \in \mathbb{R}^S$ are said to be γ-*orthogonal* if $[[x, y]] = 0$. For a subspace $\mathscr{W} \subset \mathbb{R}^S$, let $\mathscr{W}^{\perp\!\!\perp}$ denote the γ-orthogonal complement, namely the subspace of vectors γ-orthogonal to all vectors in $\mathscr{W}$. Since $\mathbb{R}^S$ can be decomposed as a direct sum $\mathbb{R}^S = \mathscr{W} \oplus \mathscr{W}^{\perp\!\!\perp}$, any vector $x \in \mathbb{R}^S$ can be written uniquely

$$x = x^* + x', \quad x^* \in \mathscr{W}, \quad x' \in \mathscr{W}^{\perp\!\!\perp}$$

x^* (resp. x') is called the γ-*orthogonal* projection of x onto $\mathscr{W}$ (resp. onto $\mathscr{W}^{\perp\!\!\perp}$) and we write $x^* = \text{proj}_{\mathscr{W}} x$, $x' = \text{proj}_{\mathscr{W}^{\perp\!\!\perp}} x$. The γ-projection x^* onto $\mathscr{W}$ is the vector in the subspace $\mathscr{W}$ which lies closest to x in the γ-norm i.e. it solves the problem

$$x^* = \arg\min\{\| x - y \|_\gamma \mid y \in \mathscr{W}\} \tag{3}$$

If $\mathscr{W}$ is the subspace of $\mathbb{R}^S$ spanned by the k linearly independent columns of an $S \times k$ matrix W (i.e. $\mathscr{W} = \langle W \rangle$ and rank $W = k$) then the matrix which represents the γ-projection (in the standard basis) is

$$B_{\mathscr{W}} = W[W^T[\gamma]W]^{-1}W^T[\gamma] \tag{4}$$

where

$$[\gamma] = \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_S \end{bmatrix}$$

is the diagonal matrix of probabilities. The matrix $B_{\mathscr{W}}$ in (4) can be readily derived by solving the problem (3) and showing that $x^* = B_{\mathscr{W}} x$. Note that if $x \in \mathscr{W}$ then $B_{\mathscr{W}} x = x$.

If W is the payoff matrix of k securities in a one-good two-period economy $\mathscr{E}(u, e, W)$ in which agents' utility functions are linear-quadratic

$$u^i(x^i) = \lambda_0^i x_0^i - \frac{1}{2}\sum_{s=1}^{S} \gamma_s(\alpha^i - x_s^i)^2, \quad i = 1, \ldots, I \tag{5}$$

then the welfare of the agents at an equilibrium can be expressed as a function of the subspace $\mathscr{W} = \langle W \rangle$. The expression is simplified when the date 1 initial endowments of the agents lie in the market subspace i.e. when $e_{\mathbb{1}}^i \in \mathscr{W}$, $i = 1, \ldots, I$.

Proposition 1 (equilibrium welfare of agents): *Let $\mathscr{E}(u, e, W)$ be a one-good, two-period economy in which agents have linear-quadratic utility functions (5) and in which $e_{\mathbb{1}}^i \in \mathscr{W}$, $i = 1, \ldots, I$. Then the welfare of the agents at the equilibrium is given by*

$$\bar{u}_{\mathscr{W}}^i = \frac{1}{2}\lambda_0^i \left(\frac{\alpha^i}{\lambda_0^i} - \frac{\alpha}{\lambda_0}\right)^2 \| \text{proj}_{\mathscr{W}} \mathbb{1} \|_\gamma^2 + k^i, \quad i = 1, \ldots, I \tag{6}$$

where $\alpha = \sum_{i=1}^I \alpha^i$, $\lambda_0 = \sum_{i=1}^I \lambda_0^i$ and $(k^i)_{i=1}^I$ are constants depending on the characteristics $(\lambda_0^i, \alpha^i, e^i)_{i=1}^I$ of the economy.

Proof: Let $(\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I, \bar{q})$ denote the equilibrium and let $e_{\mathbf{1}} = \sum_{i=1}^I e_{\mathbf{1}}^i$ denote the date 1 aggregate endowment of the economy. A straightforward calculation (see Magill-Quinzii [14, Exercise 5, Chapter 3]) shows that the equilibrium security prices are given by

$$\bar{q} = \frac{1}{\lambda_0}(\alpha\mathbb{1} - e_{\mathbf{1}})^T[\gamma]W \tag{7}$$

the agents' portfolio vectors are

$$\bar{z}^i = \left[W^T[\gamma]W\right]^{-1}W[\gamma]\left(\left(\alpha^i - \frac{\lambda_0^i}{\lambda_0}\alpha\right)\mathbb{1} - \left(e_{\mathbf{1}}^i - \frac{\lambda_0^i}{\lambda_0}e_{\mathbf{1}}\right)\right)$$

and their equilibrium consumption streams are

$$\bar{x}_0^i = e_0^i - \frac{1}{\lambda_0}\left(\alpha^i - \frac{\lambda_0^i}{\lambda_0}\alpha\right)(\alpha\mathbb{1} - e_{\mathbf{1}})^T[\gamma]B_{\mathscr{W}}\mathbb{1} + \frac{1}{\lambda_0}(\alpha\mathbb{1} - e_{\mathbf{1}})^T[\gamma]\left(e_{\mathbf{1}}^i - \frac{\lambda_0^i}{\lambda_0}e_{\mathbf{1}}\right)$$

$$\bar{x}_{\mathbf{1}}^i = \frac{\lambda_0^i}{\lambda_0}e_{\mathbf{1}} + \left(\alpha^i - \frac{\lambda_0^i}{\lambda_0}\alpha\right)B_{\mathscr{W}}\mathbb{1}, \quad i = 1, \ldots, I$$

where we have used the equality $B_{\mathscr{W}}e_{\mathbf{1}}^i = e_{\mathbf{1}}^i$ implied by $e_{\mathbf{1}}^i \in \mathscr{W}$. Inserting the expression for $\bar{x}^i$ into the utility functions (5) leads to (6). □

Since there is a sufficiently rich structure of financial securities for agents to share their endowment risks, the maximum welfare is obtained when, in addition, the riskless transfer of income is possible ($\mathbb{1} \in \mathscr{W}$); in this case, $\|\text{proj}_{\mathscr{W}}\mathbb{1}\|_\gamma = \|\mathbb{1}\|_\gamma = 1$ and the equilibrium allocation is Pareto optimal, since the allocation is the same as if the markets were complete ($\mathscr{W} = \mathbb{R}^S$). When the riskless transfer of income is not possible ($\mathbb{1} \notin \mathscr{W}$), then $\|\text{proj}_{\mathscr{W}}\mathbb{1}\|_\gamma < 1$ and if agents do not have identical preferences ($\frac{\alpha^i}{\lambda_0^i} \neq \frac{\alpha}{\lambda_0}$ for some i), there is a loss of welfare. The smaller the γ-distance of the market subspace $\mathscr{W}$ from $\mathbb{1}$, the greater the norm $\|\text{proj}_{\mathscr{W}}\mathbb{1}\|_\gamma$ of the γ-projection of $\mathbb{1}$ onto $\mathscr{W}$, and the greater the welfare of the agents.

Since the vector $\text{proj}_{\mathscr{W}}\mathbb{1}$ plays an important role in the analysis that follows, it is useful to introduce the shorthand notation

$$\eta_{\mathscr{W}} = \text{proj}_{\mathscr{W}}\mathbb{1}$$

and to summarize its main properties.

Proposition 2 (properties of least risky security): *The γ-projection $\eta_{\mathscr{W}}$ of $\mathbb{1}$ onto $\mathscr{W}$ has the following properties*:

(i) *Under the γ-inner product on $\mathbb{R}^S$, $\eta_{\mathscr{W}}$ represents the expectation operator on $\mathscr{W}$*

$$E(\eta_{\mathscr{W}}y) = [[\eta_{\mathscr{W}}, y]] = E(y) \text{ for all } y \in \mathscr{W}$$

(ii) *$\eta_{\mathscr{W}}$ is the least risky income stream in the market subspace $\mathscr{W}$ in the following two senses:*

(a) *geometrically, it is the vector in $\mathscr{W}$ which lies closest to $\mathbb{1}$*

$$\eta_{\mathscr{W}} = \arg\ \min\left\{ \| \mathbb{1} - y \|_\gamma^2 \ \middle|\ y \in \mathscr{W} \right\}$$

(b) *statistically, it is the vector in* $\mathscr{W}$ *which has the minimum standard deviation per unit of expected return*

$$\frac{\eta_{\mathscr{W}}}{E(\eta_{\mathscr{W}})} = \arg\ \min\left\{ \sigma\left(\frac{y}{E(y)}\right) \middle| y \in \mathscr{W},\ E(y) \neq 0 \right\}$$

(iii) *the minima in (a) and (b) lead to two measures of the riskiness of the market subspace:*

(a)′ $1 - E(\eta_{\mathscr{W}}) = \|\mathbb{1} - \eta_{\mathscr{W}}\|_\gamma^2$

(b)′ $\dfrac{1}{E(\eta_{\mathscr{W}})} - 1 = \sigma^2\left(\dfrac{\eta_{\mathscr{W}}}{E(\eta_{\mathscr{W}})}\right)$

Proof: (i) Since $\mathbb{1} - \eta_{\mathscr{W}} \in \mathscr{W}^{\perp\perp}$, $E(y) = [[\mathbb{1}, y]] = [[\eta_{\mathscr{W}}, y]]$ for all $y \in \mathscr{W}$. (ii) (a) follows from (3). To prove (b), consider the problem: min $\{\text{var}\ (y)\ \mid y \in \mathscr{W},\ E(y) = 1\}$ and suppose that $\eta_{\mathscr{W}}/E(\eta_{\mathscr{W}})$ is not the solution. Then there exists $y' \in \mathscr{W}$ with $E(y') = 1$ and var $(y') <$ var $(\eta_{\mathscr{W}})/\ E(\eta_{\mathscr{W}})^2$. Let $\bar{y} = E(\eta_{\mathscr{W}})y'$ then $\bar{y}$ satisfies $E(\bar{y}) = E(\eta_{\mathscr{W}})$ and var $(\bar{y}) <$ var$(\eta_{\mathscr{W}}) \Longrightarrow E(\bar{y}^2) < E(\eta_{\mathscr{W}}^2)$. Then $\| \mathbb{1} - \bar{y} \|_\gamma^2 = 1 - 2E(\bar{y}) + E(\bar{y}^2)\ < 1 - 2E(\eta_{\mathscr{W}}) + E(\eta_{\mathscr{W}}^2)\ = \| \mathbb{1} - \eta_{\mathscr{W}} \|_\gamma^2$ contradicting the definition of $\eta_{\mathscr{W}}$. (iii) follows by noting that (i) implies $E(\eta_{\mathscr{W}}^2) = E(\eta_{\mathscr{W}})$. □

Welfare gains function. We want to apply Proposition 1 to a purchasing power economy $\mathscr{E}(u, e,\ \mathbf{a}, V)$, namely a one-good economy with payoff matrix

$$W = [\mathbf{a}\ V] = V_a \tag{8}$$

When **a** changes, it alters the market subspace

$$\mathscr{V}_a = \langle V_a \rangle$$

and our objective is to understand how the welfare of agents varies with the "characteristics" of the bond **a**. Since in (8), V is taken as fixed, a convenient way of analyzing how welfare depends on **a** is to make the comparison with the case where **a** is redundant ($\mathbf{a} \in \mathscr{V} = \langle V \rangle$). The utility of agent i at the equilibrium with market subspace $\mathscr{V}_a$ can be written as

$$\bar{u}^i_{\mathscr{V}_a} = (\bar{u}^i_{\mathscr{V}_a} - \bar{u}^i_{\mathscr{V}}) + \bar{u}^i_{\mathscr{V}}$$

where the first term $G^i_a = \bar{u}^i_{\mathscr{V}_a} - \bar{u}^i_{\mathscr{V}}$ can be interpreted as the utility gain to agent i of having the bond with characteristics **a**. By Proposition 1, this gain can be written as

$$G^i_a = c^i\left(\| \eta_{\mathscr{V}_a} \|_\gamma^2 - \| \eta_{\mathscr{V}} \|_\gamma^2\right)$$

where $c^i = \frac{1}{2}\lambda^i_0\left(\frac{\alpha^i}{\lambda^i_0} - \frac{\alpha}{\lambda_0}\right)^2$ is a non-negative coefficient which is positive for all "non-average" agents. Since the subspace $\mathscr{V}_a$ contains $\mathscr{V}$, $\| \eta_{\mathscr{V}_a} \|_\gamma^2 \geq \| \eta_{\mathscr{V}} \|_\gamma^2$, so that the gain G^i_a is non-negative for all agents and is strictly positive if $c^i > 0$ and $\mathbf{a} \notin \mathscr{V}$. We are thus led to study the function $G : \mathbb{R}^S \longrightarrow \mathbb{R}$ defined by

$$G(\mathbf{a}) = \| \eta_{\mathscr{V}_a} \|_\gamma^2 - \| \eta_{\mathscr{V}} \|_\gamma^2 \tag{9}$$

which we call the *welfare gains function*, since the utility gains to all agents are proportional to the function G: by Proposition 2, this function measures the reduction in the riskiness of the market subspace achieved by introducing the security **a**. This property of the model, that the utility gains of all agents are proportional to the common function G – in particular that all agents are made better off when a nonredundant bond **a** is added to an existing security structure V – requires some explanation.

In general, introducing a new security has two effects: the first – which we may call the *direct effect* – is to increase the span of the markets i.e. the trading opportunities available in the economy, and this tends to increase the welfare of the agents; the second – which we may call the *indirect effect* – is to change all prices, both spot and security prices, and this can either increase or decrease agents' utilities. Combining the two effects can lead to the apparently paradoxical result that introducing a new security decreases the welfare of all agents, as first shown in an example by Hart [10]. More recently Cass-Citanna [2] and Elul [4] have studied the case where all (and hence the indirect) effects are marginal and have shown that if the markets are sufficiently incomplete then, in a multigood economy, the combination of the two effects can lead to any possible local change in agents' utilities. In a one-good two-period model, since there are no spot prices, the security prices are the only equilibrium parameters that can change, and this reduces the range of possible changes in agents' utilities: it is not possible for all agents to loose from the introduction of a new security – typically some agents gain and some agents loose from the resulting change in the prices of the existing securities. In this paper all indirect price effects are canceled: there is no effect from spot prices because of Assumption **H**, and no effect from security prices because of the linear-quadratic form of the agents' utility functions, as can be seen from formula (7) for the equilibrium security prices.[3] Thus the analysis concentrates on the direct effect of changing the span of the markets and this effect is present in all economies. The analysis can thus be applied to an economy in which the price effects are sufficiently small, or it can be taken as the first half of a more complete study in which indirect effects are also explicitly taken into account.

The next step is to analyze the properties of the welfare gains function $G(\mathbf{a})$: we will show that the gain depends only on the statistical properties of the income stream $\mathbf{a} \in \mathbb{R}^S$ summarized by its mean, standard deviation and its correlation coefficients with the securities $v^1, \ldots, v^J$. Furthermore we will show that the gain can be described in a very complete way for any number

[3] Elul [3] shows that in any one-good two-period model it is possible to cancel the indirect price effects and to exhibit the benefits of the increase in spanning brought about by the introduction of a new security. In a general model only some carefully chosen securities (which exist generically) are such that their inclusion does not change the existing security prices (no indirect effect), while with linear-quadratic utility this holds for all new securities.

of securities J, and any number of states of nature S. The derivation of the properties of G as a function of the statistical attributes of $\mathbf{a}$ requires some calculations which are left to section 5. Here we summarize these properties and provide a simple geometric interpretation of the results for the case $(J = 1,\ S = 3)$.

Since $G(\mathbf{a})$ is derived by projecting $\mathbb{1}$ onto $\mathscr{V}_a$ and $\mathscr{V}$, it depends only on the directions of the vectors $(\mathbf{a}, v^1, \ldots, v^J)$ and not on their lengths. Thus all these vectors can be normalized and the most natural economic interpretation is obtained by normalizing each vector so that its expected value is one. This requires that each of these date 1 payoff streams have a non-zero expected value: this is assured for $v^1, \ldots, v^J$ by Assumption **I** (iv), and will be assured for the bond by restricting attention to bonds with positive expected values ($\mathbf{a} \in \mathbb{R}^S$ with $E(\mathbf{a}) > 0$). The following notation for normalized variables is convenient: for any random variable $x \in \mathbb{R}^S$ with $E(x) > 0$, the *normalized* variable with expectation 1 is denoted by

$$\widehat{x} = \frac{x}{E(x)}$$

If $\sigma(x)$ denotes the standard deviation of x, then $\sigma(\widehat{x}) = \frac{\sigma(x)}{E(x)}$ measures the standard deviation of the income stream x per unit of expected value: for brevity we write $\sigma_{\hat{x}} = \sigma(\widehat{x})$. Since the correlation coefficient $\rho(x, y) = \frac{\mathrm{cov}\,(x,y)}{\sigma(x)\sigma(y)}$ between a pair of vectors $x, y \in \mathbb{R}^S$ does not depend on their lengths, $\rho(\widehat{x}, \widehat{y}) = \rho(x, y)$: for brevity we write ρ_{xy}. Let

$$\rho_a = (\rho_{av^1}, \ldots, \rho_{av^J}), \quad \rho_V = \begin{bmatrix} \rho_{v^1v^1} & \cdots & \rho_{v^1v^J} \\ \vdots & & \vdots \\ \rho_{v^Jv^1} & \cdots & \rho_{v^Jv^J} \end{bmatrix}$$

denote the vector of correlation coefficients between the bond $\mathbf{a}$ and the securities $v^1, \ldots, v^J$ and the matrix of correlation coefficients between these securities, respectively.

The next proposition asserts that the gain $G(\mathbf{a})$ depends only on $(\sigma_{\hat{a}}, \rho_a)$ i.e. there exists a function $g : \mathbb{R} \times \mathbb{R}^J \longrightarrow \mathbb{R}$ such that $G(\mathbf{a}) = g(\sigma_{\hat{a}}, \rho_a)$. In order to deduce the properties of G from those of g it is necessary to determine the subset (domain) of $\mathbb{R} \times \mathbb{R}^J$ on which g coincides with G i.e. the values $(\sigma, \rho) \in \mathbb{R} \times \mathbb{R}^J$ which correspond to the standard deviation and vector of correlation coefficients of a normalized random variable $\widehat{\mathbf{a}} \in \mathbb{R}^S$.

Proposition 3 (existence of statistical gains function):

(i) *Let* $(\sigma, \rho) \in \mathbb{R} \times \mathbb{R}^J$*, then there exists a random variable* $\mathbf{a} \in \mathbb{R}^S$ *with* $E(\mathbf{a}) > 0$ *such that* $(\sigma_{\hat{a}}, \rho_a) = (\sigma, \rho)$ *if and only if either* $(\sigma, \rho) = (0, 0)$ *or* $\sigma > 0$ *and* ρ *belongs to the convex domain* $\mathscr{R}$ *defined by*

$$\mathscr{R} = \left\{ \rho \in \mathbb{R}^J \,\middle|\, [\rho_V - \rho\rho^T] \quad \text{is positive semi-definite} \right\} \tag{10}$$

(ii) *The boundary of* $\mathscr{R}$ *is* $\partial\mathscr{R} = \left\{\rho \in \mathscr{R} \middle| \det\left[\rho_{\mathscr{V}} - \rho\rho^T\right] = 0\right\}$. *If* **a** *is a random variable with* $\rho_a \in \partial\mathscr{R}$, *then there exists* $y \in \mathscr{V}$ *such that*

$$\rho(\mathbf{a}, y) = \pm 1 \Longleftrightarrow \hat{\mathbf{a}} - \mathbb{1} = \lambda(y - E(y)\mathbb{1}) \quad \textit{for some } \lambda \in \mathbb{R} \tag{11}$$

(iii) *There exists a function* $g : \mathbb{R} \times \mathbb{R}^J \longrightarrow \mathbb{R}$ *such that if* $(\sigma, \rho) \in \mathbb{R}_{++} \times \mathscr{R} \cup \{0, 0\}$ *then* $g(\sigma, \rho) = G(\mathbf{a})$ *for all* $\mathbf{a} \in \mathbb{R}^S$ $\big($*with* $E(\mathbf{a}) > 0\big)$ *such that* $(\sigma_{\hat{a}}, \rho_a) = (\sigma, \rho)$.

Proof: (See Section 5)

The next proposition describes the properties of the function g, which we call the *statistical gains function*, since it expresses the gain from a bond **a** as a function of its statistical properties $(\sigma_{\hat{a}}, \rho_a)$. Since the securities $(v^1, \ldots, v^J)$ are taken as fixed, the projection $\eta_{\mathscr{V}}$ of $\mathbb{1}$ onto $\mathscr{V}$ forms part of the data of the problem: to reflect this we let

$$\eta = \eta_{\mathscr{V}} = \text{proj}_{\mathscr{V}}\mathbb{1}$$

In Proposition 2 we introduced the two measures, $1 - E(\eta)$ and $\sigma_{\hat{\eta}}$, of the riskiness of $\mathscr{V}$ (i.e. the market subspace in the absence of **a**). Both play an important role in the next proposition. $1 - E(\eta)$ measures the *maximum gain* that can be attributed to any bond **a** since

$$\| \eta_{\mathscr{V}_a} \|_\gamma^2 - \| \eta \|_\gamma^2 \leq 1 - \| \eta \|_\gamma^2 = 1 - E(\eta^2) = 1 - E(\eta)$$

The maximum gain is attained when $\mathbb{1} \in \mathscr{V}_a$, which happens either if $\mathbf{a} = \mathbb{1}$, or if **a** is risky and $\mathbb{1}$ can be obtained by a combination of **a** and some vector in $\mathscr{V}$: by (11) this occurs when **a** is perfectly correlated with some vector $y \neq \mathbf{a}$ in $\mathscr{V}$.

For normalized bonds of a given variability σ, the minimum gain as a function of ρ depends on whether the bond is less or more variable than the (normalized) least risky income stream $\hat{\eta}$ in $\mathscr{V}$: when $\sigma < \sigma_{\hat{\eta}}$ the bond is less risky than any security in $\mathscr{V}$ and thus necessarily leads to a positive gain; when $\sigma \geq \sigma_{\hat{\eta}}$ then the bond will not contribute towards risk reduction if it does not permit the risks in $\mathscr{V}$ to be hedged.

All bonds $\mathbf{a} \in \mathbb{R}^S$ with the same vector of correlation coefficients $\rho_a = \rho$ with $v^1, \ldots, v^J$, have the same correlation coefficient $\rho(\mathbf{a}, \eta)$ with the least risky security η in $\mathscr{V}$, regardless of their variability: for η can be written as $\eta = \sum_{j=1}^J \lambda_j v^j$, so that

$$\rho(\mathbf{a}, \eta) = \frac{\text{cov}\,(\mathbf{a}, \eta)}{\sigma_a \sigma_\eta} = \frac{\sum\limits_{j=1}^J \lambda_j \rho_{av^j} \sigma_a \sigma_{v^j}}{\sigma_a \sigma_\eta} = \frac{\sum\limits_{j=1}^J \lambda_j \rho_{av^j} \sigma_{v^j}}{\sigma_\eta}$$

which is a linear function of $(\rho_{av^1}, \ldots, \rho_{av^J})$ which is independent of σ_a. As a result a coefficient of correlation $r \in [-1, 1]$ defines a subset of $\mathscr{R}$

$$\mathcal{R}_r = \{\rho \in \mathcal{R} |\ \rho(\mathbf{a}, \eta) = r \text{ for all } \mathbf{a} \in \mathbb{R}^S \ \text{ with } \rho_a = \rho\}$$

which is the intersection of $\mathcal{R}$ by a hyperplane in $\mathbb{R}^J$. The domain $\mathcal{R}$ is thus partitioned into two regions depending on the sign of the correlation coefficient between $\mathbf{a}$ and the least risky security η

$$\mathcal{R}^+ = \{\rho \in \mathcal{R} |\ \rho(\mathbf{a}, \eta) > 0 \text{ for all } \mathbf{a} \in \mathbb{R}^S \text{ with } \rho_a = \rho\}$$
$$\mathcal{R}^- = \{\rho \in \mathcal{R} |\ \rho(\mathbf{a}, \eta) \leq 0 \text{ for all } \mathbf{a} \in \mathbb{R}^S \text{ with } \rho_a = \rho\}$$

Proposition 4 (Properties of the Statistical Gains Function):

(A) *Properties of g as a function of ρ (for fixed $\sigma > 0$):*

(0) *For any $\sigma > 0$, $g(\sigma, \cdot)$ is a convex function on the interior of $\mathcal{R}$.*

(i) (Low variability): *if $0 < \sigma < \sigma_{\hat{\eta}}$, then the maximum of $g(\sigma, \cdot)$ is attained for all $\rho \in \partial\mathcal{R}$ and*

$$g(\sigma, \rho) = 1 - E(\eta) = 1 - \frac{1}{1 + \sigma_\eta^2} \quad \text{for all } \rho \in \partial\mathcal{R}$$

The minimum is attained for the unique vector $\rho^* = (\sigma/\sigma_{\hat{v}^1}, \ldots, \sigma/\sigma_{\hat{v}^J})$ and

$$g(\sigma, \rho^*) = \frac{1}{1 + \sigma^2} - \frac{1}{1 + \sigma_{\hat{\eta}}^2}$$

(ii) (High variability): *if $\sigma > \sigma_{\hat{\eta}}$, the minimum of $g(\sigma, \cdot)$ is attained for the vectors ρ which lie in the $J - 1$ dimensional subset $\mathcal{R}_{r_\sigma}$ with $r_\sigma = \sigma_{\hat{\eta}}/\sigma$ and $g(\sigma, \rho) = 0$ for all $\rho \in \mathcal{R}_{r_\sigma}$. The maximum of $g(\sigma, \cdot)$ is attained for all vectors $\rho \in \partial\mathcal{R}\backslash\mathcal{R}_{r_\sigma}$ and*

$$g(\sigma, \rho) = 1 - E(\eta) = 1 - \frac{1}{1 + \sigma_{\hat{\eta}}^2} \quad \text{for all } \rho \in \partial\mathcal{R}\backslash\mathcal{R}_{r_\sigma} \tag{12}$$

If $\dim \mathcal{V} \geq 2$, then $g(\sigma, \cdot)$ is discontinuous at the points $\partial\mathcal{R} \cap \mathcal{R}_{r_\sigma}$.

(iii) (Intermediate case): *if $\sigma = \sigma_{\hat{\eta}}$, the subset $\mathcal{R}_{r_\sigma} = \mathcal{R}_1$ on which $g(\sigma, \cdot)$ attains its minimum reduces to the point $\rho^* = (\sigma/\sigma_{\hat{v}^1}, \ldots, \sigma/\sigma_{\hat{v}^J}) \in \partial\mathcal{R}$ and $g(\sigma, \rho^*) = 0$. The maximum of $g(\sigma, \cdot)$ is attained for all vectors in $\partial\mathcal{R}\backslash\mathcal{R}_1$ and is given by (12). If $\dim \mathcal{V} \geq 2$, then g is discontinuous at ρ^*.*

(B) *Properties of g as a function of σ (for fixed $\rho \in \mathcal{R}\backslash\partial\mathcal{R}$).*

If $\rho \in \mathcal{R}^-$, then $g(\cdot, \rho)$ is strictly decreasing for all $\sigma > 0$; if $\rho \in \mathcal{R}^+$, then there exists a critical variability $\sigma^* = \sigma_{\hat{\eta}}/\rho(\mathbf{a}, \eta)$ such that $g(\cdot, \rho)$ is strictly decreasing for $\sigma \in (0, \sigma^*)$ and strictly increasing for $\sigma \in (\sigma^*, \infty)$. Thus $g(\cdot, \rho)$ is strictly decreasing for all $\sigma > 0$ if and only if $\rho \in \mathcal{R}^-$.

Proof: (See Section 5)

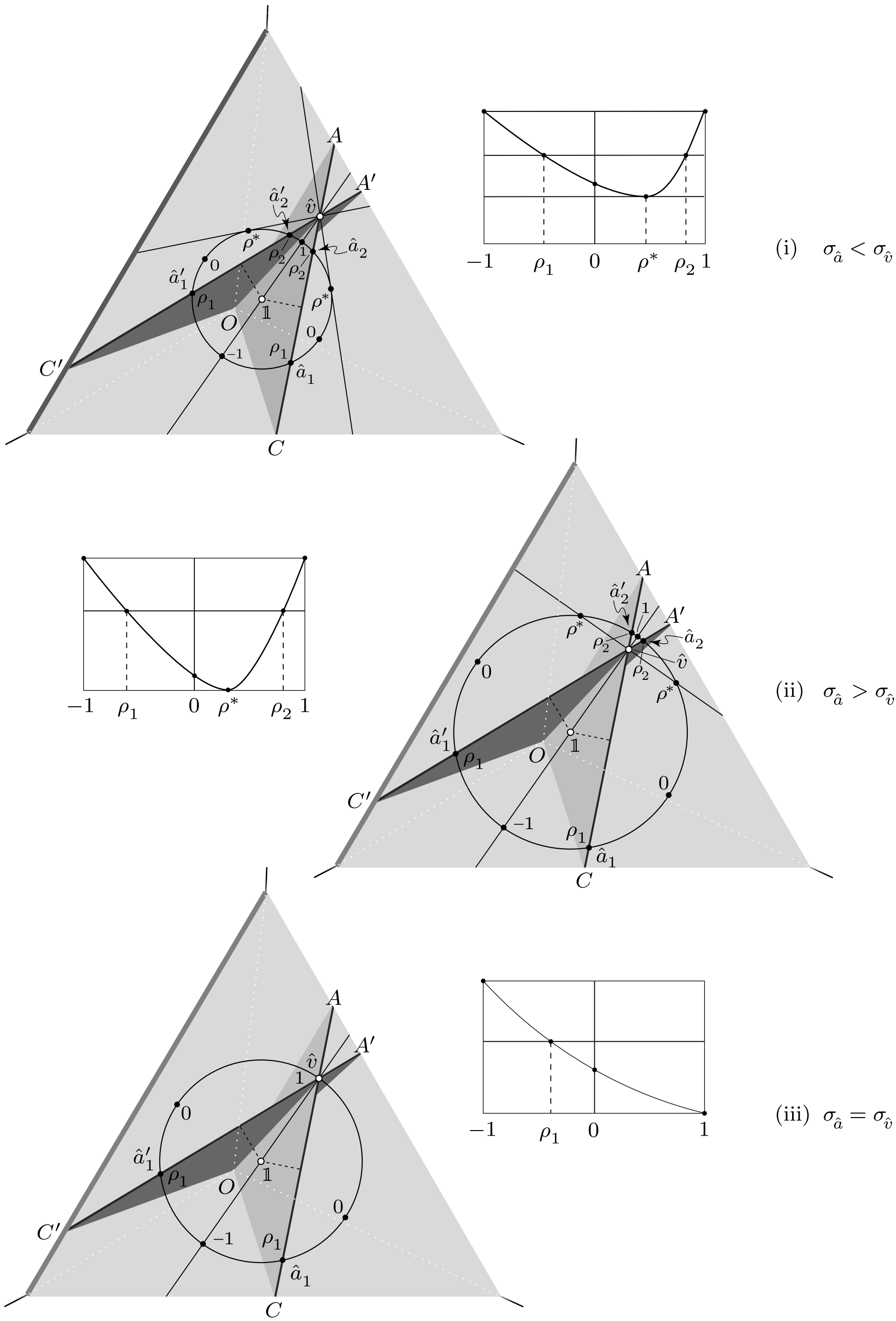

Figure 1. A family of normalized bonds with fixed standard deviation σ and different correlations ρ with v, and the graph of the gains function $g(\sigma, \cdot)$: (i) $\sigma < \sigma_{\hat{v}}$ (ii) $\sigma > \sigma_{\hat{v}}$ (iii) $\sigma = \sigma_{\hat{v}}$. In each case a pair of market subspaces OAC (OA′C′) is shown.

Geometric interpretation. An intuitive geometric interpretation of Propositions 3 and 4 can be given in the simplest case $J = 1$, $S = 3$ (recall that Assumption **I**(iii) requires $S \geq J + 2$). Let v denote the payoff on the single security, $\mathscr{V} =< v >$. Since the welfare gain only depends on the normalized income streams, it suffices to restrict attention to normalized bonds $\widehat{\mathbf{a}}$ with expectation 1 i.e. to vectors $\widehat{\mathbf{a}} = (\widehat{a}_1, \widehat{a}_2, \widehat{a}_3) \in \mathbb{R}^3$ such that $E(\widehat{\mathbf{a}}) = \gamma_1 \widehat{a}_1 + \gamma_2 \widehat{a}_2 + \gamma_3 \widehat{a}_3 = 1$. To simplify the geometry, let us assume that $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$ so that the normalized bonds with non-negative payoffs belong to the simplex (see Figure 1)

$$\mathscr{S} = \left\{ \widehat{\mathbf{a}} \in \mathbb{R}^3_+ \middle| \frac{1}{3}(\widehat{a}_1 + \widehat{a}_2 + \widehat{a}_3) = 1 \right\}$$

Furthermore with this assumption of equal probabilities the γ-inner product (1) coincides (up to the coefficient 1/3) with the Euclidean inner product.[4] Under the γ-norm (2), the standard deviation $\sigma(\widehat{\mathbf{a}})$ is the distance of a normalized bond $\widehat{\mathbf{a}} \in \mathscr{S}$ from the center $\mathbb{1} = (1, 1, 1)$ of the simplex, since $\sigma(\widehat{\mathbf{a}}) = \left[E(\widehat{\mathbf{a}} - E(\widehat{\mathbf{a}})\mathbb{1})^2 \right]^{1/2} = \| \widehat{\mathbf{a}} - \mathbb{1} \|_\gamma$. Thus all normalized bonds with the same standard deviation σ belong to a circle (in the simplex) of radius σ, centered at $\mathbb{1}$ (see Figure 1). Since $J = 1$, the subspace $\mathscr{V}$ is one dimensional and the least risky security η in $\mathscr{V}$ is collinear to the unique vector v generating $\mathscr{V}$: thus after normalization $\widehat{\eta} = \widehat{v}$. The intersection of the subspace $\mathscr{V}$ with the simplex is simply the vector $\widehat{v}$. When the bond $\widehat{\mathbf{a}}$ is introduced, the market subspace becomes $\mathscr{V}_a =< \widehat{v}, \widehat{\mathbf{a}} >$ which intersects the simplex along the line $(\widehat{v}, \widehat{\mathbf{a}})$: for example when the bond $\widehat{\mathbf{a}}_1$ is introduced, then in each of the cases (i)–(iii) in Figure 1, the market subspace $< \widehat{v}, \widehat{\mathbf{a}}_1 >$ is given by the plane OAC which intersects $\mathscr{S}$ along the line segment AC. *There is a gain in welfare from introducing the bond* $\widehat{\mathbf{a}}$ *if the line* $(\widehat{v}, \widehat{\mathbf{a}})$ *is closer to* $\mathbb{1}$ *than* $\widehat{v}$.[5]

Given $\widehat{v}$, the distance of the line $(\widehat{\mathbf{a}}, \widehat{v})$ from $\mathbb{1}$ depends only on the radius σ of the circle on which $\widehat{\mathbf{a}}$ lies and on the angle $\theta(\widehat{\mathbf{a}} - \mathbb{1}, \widehat{v} - \mathbb{1})$ between the vectors $\widehat{\mathbf{a}} - \mathbb{1}$ and $\widehat{v} - \mathbb{1}$ — or more precisely (by symmetry) on the cosine of this angle. This cosine is the correlation coefficient ρ_a between $\mathbf{a}$ and v, since

$$\cos\ \theta(\widehat{\mathbf{a}} - \mathbb{1}, \widehat{v} - \mathbb{1}) = \frac{[[\widehat{\mathbf{a}} - \mathbb{1}, \widehat{v} - \mathbb{1}]]}{\| \widehat{\mathbf{a}} - \mathbb{1} \|_\gamma \| \widehat{v} - \mathbb{1} \|_\gamma} = \frac{\text{cov}\ (\widehat{\mathbf{a}}, \widehat{v})}{\sigma(\widehat{a})\sigma(\widehat{v})} = \rho(\mathbf{a}, v) = \rho_a$$

Thus the gain from introducing a bond is a function of its standard deviation and of its correlation coefficient with $\widehat{v}$: $G(\mathbf{a}) = g(\sigma, \rho)$ for all $\mathbf{a}$ such that $(\sigma_{\hat{a}}, \rho_a) = (\sigma, \rho)$. For example in each of the cases (i)–(iii) in Figure 1, $\widehat{\mathbf{a}}_1$ and

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[4]Figures 1 (i)–(iii) can be used to represent the general case of unequal probabilities by appropriately changing units along the co-ordinate axes i.e. by changing from the standard basis $\{e_1, e_2, e_3\}$ to the basis $\{e'_1, e'_2, e'_3\}$ with $e'_s = \frac{1}{\sqrt{\gamma_s}} e_s,\ s = 1, 2, 3.$

[5] By Proposition 2 (ii)(b) the distance of the line $(\widehat{v}, \widehat{a})$ from $\mathbb{1}$ is equal to $\sigma(\widehat{\eta}_{\mathscr{V}_a})$. By (i) and (iii) of Proposition 2, $\sigma(\widehat{\eta}_{\mathscr{V}_a}) < \sigma(\widehat{\eta}_{\mathscr{V}}) \iff E(\eta_{\mathscr{V}_a}) > E(\eta_{\mathscr{V}}) \iff E(\eta^2_{\mathscr{V}_a}) > E(\eta^2_{\mathscr{V}}) \iff \| \eta_{\mathscr{V}_a} \|^2 > \| \eta_{\mathscr{V}} \|^2 \iff G(\mathbf{a}) > 0$. Thus if $\widehat{\eta}_{\mathscr{V}_a}$ is closer to $\mathbb{1}$ than $\widehat{\eta}_{\mathscr{V}}$ (i.e. when the riskiness of the market subspace is reduced) then the welfare is increased.

$\widehat{\mathbf{a}}_1'$ have the same standard deviation (lie on the same circle) and the same correlation coefficient ρ_1 with $\widehat{v}$ (since the angles satisfy $\theta(\widehat{\mathbf{a}}_1 - \mathbb{1}, \widehat{v} - \mathbb{1}) = -\theta(\widehat{\mathbf{a}}_1' - \mathbb{1}, \widehat{v} - \mathbb{1})$). They generate the same gain since the line segments AC and A′C′ lie at the same distance from $\mathbb{1}$.

To study the behavior of $g(\sigma, \rho)$ as a function of ρ, when σ is fixed, it is thus sufficient to study how the distance of the line $(\widehat{v}, \widehat{a})$ from $\mathbb{1}$ varies when $\widehat{a}$ "moves around" on a circle of radius σ. The qualitative behavior of g falls into three categories, corresponding to the cases (i), (ii) and (iii) in Proposition 4: they depend on whether $\widehat{v}$ lies outside the circle ($\sigma < \sigma_{\widehat{v}} = \sigma_{\widehat{\eta}}$), inside the circle ($\sigma_{\widehat{\eta}} < \sigma$) or on the circle ($\sigma_{\widehat{\eta}} = \sigma$) of radius σ. In each case the maximum gain is obtained when the line $(\widehat{v}, \widehat{a})$ passes through $\mathbb{1}$ (so that the distance from $\mathbb{1}$ is zero) and this occurs when the cosine of the angle $\theta(\widehat{a} - \mathbb{1}, \widehat{v} - \mathbb{1})$ is +1 or −1 (the correlation coefficient between $\widehat{a}$ and $\widehat{v}$ is +1 or −1). In economic terms this corresponds to a case where the bond provides a perfect (riskless) hedge against the risk $\widehat{v}$. What distinguishes the three cases (i)–(iii), is the geometric behavior of the market line $(\widehat{v}, \widehat{\mathbf{a}})$ at its maximum distance from $\mathbb{1}$. When $\widehat{v}$ lies outside the circle of radius σ, the maximum distance (and hence the minimum gain) is obtained for the lines $(\widehat{\mathbf{a}}^*, \widehat{v})$ and $(\widehat{\mathbf{a}}^{*\prime}, \widehat{v})$ (corresponding to the same ρ^* in Figure 1(i)) which are tangent to the circle. Since their distance from $\mathbb{1}$ is smaller than $\| \widehat{v} - \mathbb{1} \|_\gamma$, the minimum gain is strictly positive. When $\widehat{v}$ lies inside the circle of radius σ, the maximum distance is obtained when the market line $(\widehat{v}, \widehat{\mathbf{a}})$ is perpendicular to $\widehat{v} - \mathbb{1}$: its distance from $\mathbb{1}$ is exactly $\| \widehat{v} - \mathbb{1} \|_\gamma$ so that the minimum gain is zero. When $\widehat{v}$ lies on the circle of radius σ, the maximum distance is obtained for the degenerate case where $\widehat{\mathbf{a}} = \widehat{v}$ (the dimension of $\mathscr{V}_a$ collapses to 1 and $\mathscr{V}_a = \mathscr{V}$): once again the minimum gain is zero. The collapse in rank when $\widehat{\mathbf{a}} = \widehat{v}$ does not create a discontinuity in the gain function when $J = 1$, however it does when $J \geq 2$.

In all three cases the minimum gain always occurs when there is no "synergy" between the bond $\widehat{\mathbf{a}}$ and the security $\widehat{v}$ for reducing market risks: the projection of $\mathbb{1}$ onto the market line $(\widehat{\mathbf{a}}, \widehat{v})$ is $\widehat{\mathbf{a}}$ when $\sigma < \sigma_{\widehat{v}}$ and is $\widehat{v}$ when $\sigma \geq \sigma_{\widehat{v}}$. For all other values of ρ (i.e. $\rho \neq \rho^*$), a combination of $\widehat{\mathbf{a}}$ and $\widehat{v}$ creates the least risky security on the line $(\widehat{\mathbf{a}}, \widehat{v})$ and this security is less risky than either $\widehat{\mathbf{a}}$ of $\widehat{v}$ taken on their own. Placing the family of curves in the small diagrams in Figure 1 on a common graph (figure 2) shows how the welfare gains change when the variability of the bond is increased,[6] for a given ρ. For negative correlation $-1 < \rho \leq 0$, $g(\cdot, \rho)$ is a decreasing function for all va-

[6] The fact that even a bond with *very high variance* can essentially provide a riskless hedge provided the magnitude of its correlation with $\widehat{v}$ is sufficiently large, comes from neglecting the non-negativity constraints on the real value of consumption in the linear-quadratic *pp* economy. With non-negativity constraints and no possibility of default, the feasible hedging strategies using a bond with extremely variable payoffs are much more limited and the amount of trade on the bond must go to zero for extremely high variance: for a study of the consequences of these no-bankruptcy constraints see Neumeyer [18].

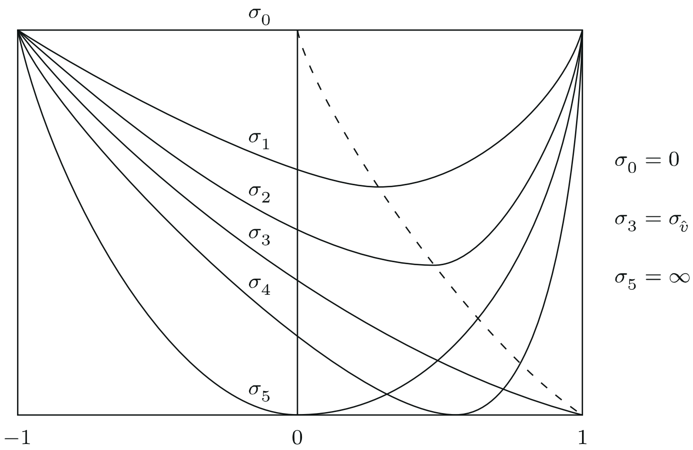

Figure 2. Welfare gains for a family of bonds of increasing variability.

lues of σ; for positive correlation $0 < \rho < 1$, $g(\cdot, \rho)$ is decreasing for $\sigma < \sigma_{\hat{v}}/\rho$ and increasing for $\sigma > \sigma_{\hat{v}}/\rho$.

Direct geometric arguments of the kind given above are no longer available in the multidimensional case $J \geq 2$, since by Assumption **I**, $S \geq J + 2$. What is remarkable is that by explicitly calculating the function $g(\sigma, \rho)$ and deriving analytically its properties, it is possible to show how these results extend to the multi-dimensional case $J \geq 2$. The explicit derivation of the function $g(\sigma, \rho)$ and the study of its properties is postponed to section 5. In the next section we use Proposition 4 to address the question posed at the beginning of this paper: under what circumstances does a nominal bond provide greater social welfare than an indexed bond, or conversely?

4 Nominal versus indexed bond

Let us first consider two extreme cases where the answer is clear cut, since one of the bonds is the "ideal" bond with constant real purchasing power payoff.

(a) *Conditions under which* $\mathbf{a}^N = \mathbb{1}$ *or* $\mathbf{a}^R = \mathbb{1}$

The purchasing power of a nominal bond paying one unit of money in every state is

$$\mathbf{a}^N = (v_s)_{s=1}^S = \left(\frac{h(w_s)}{M_s}\right)_{s=1}^S$$

The variations in the purchasing power of money v_s depend on how the money supply M_s varies with aggregate output, as measured by the index $h(w_s)$. In order for v_s to be constant across the states, the money supply M_s must be proportional to $h(w_s)$ or, in terms of growth rates, the rate of growth

m_s of the money supply must match the rate of growth g_s of real output so that (for some constant c)

$$\frac{v_s}{v_0} = \frac{h(w_s)/h(w_0)}{M_s/M_0} = \frac{1+g_s}{1+m_s} = c, \quad s = 1, \ldots, S$$

This condition would be satisfied in the idealized setting where a monetary authority (or a banking system) perfectly controls (adapts) the money supply to the fluctuations in real output $\big(h(w_s)\big)$. In this case, since the nominal bond is the ideal bond $\mathbf{a}^N = v = \mathbb{1}$, there is no role for an indexed bond.

If the bond is indexed on the value of a bundle of goods $b \in \mathbb{R}^L$, then it becomes a real bond whose purchasing power across the states

$$\mathbf{a}^R = (\nabla h(w_s) b)_{s=1}^S$$

is not influenced by fluctuations in the purchasing power of money. However if the relative prices of the goods $\big($proportional to$\nabla h(w_s)\big)$ vary across the states, then the purchasing power $\mathbf{a}^R$ fluctuates. In this model, in view of Assumption **H**, it would be possible to avoid these fluctuations by indexing on a state-dependent bundle $b_s = w_s/h(w_s)$ which is proportional to aggregate output in state s. Indexing on this ideal state-dependent bundle permits the creation of the riskless real income stream

$$\mathbf{a}^R = \left(\frac{\nabla h(w_s) w_s}{h(w_s)}\right)_{s=1}^S = \mathbb{1}$$

If indexation could create such a riskless real income stream, then agents would only use the indexed bond and the nominal bond would disappear.

In a more realistic model in which agents do not have identical preferences for goods within each state, no such ideal reference bundle – and hence no such ideal index – exists. We invoked Assumption **H** to simplify the analysis of equilibrium – by factoring out the influence of the income earned by agents on financial markets on the determination of spot prices – certainly not to suggest that there is an ideal index. To capture the inherent imperfections of indexation in spite of the simplifying Assumption **H**, we assume that *the reference bundle must be state independent*. This assumption also captures the fact that in practice an index is more credible if its computation does not involve the use of a state-dependent reference bundle, since the possibility of changing the bundle as the contingencies vary opens the door to manipulations to either understate or overstate inflation, depending on the interests of the parties involved.

Since neither of the extreme cases where the purchasing power of money is constant or there exists an ideal index is likely to be met in practice, it is instructive to identify the circumstances in which one of the two types of bond – nominal or indexed – has a relative advantage over the other. This may be done by analyzing which bond creates the greater social welfare, under the assumption that only one of the two bonds is traded.

(b) *Conditions under which* $\mathbf{a}^N$ *or* $\mathbf{a}^R$ *is socially preferred.*

We want to apply the analysis of section 3 to a purchasing power economy $\mathscr{E}(u, e, \mathbf{a}, V)$ where $\mathbf{a}$ denotes either the nominal or the indexed bond and V is the matrix of payoffs on the underlying risk-sharing securities, all payoffs being expressed in purchasing power. Consider first the simplest case where V consists of a single security ($J = 1$). Given Assumption **S** its payoff v must be

$$v = (\nabla h(w_s)w_s)_{s=1}^{S} = (h(w_s))_{s=1}^{S} \equiv h(w_{\mathbf{1}})$$

The projection η of $\mathbb{1}$ onto $\mathscr{V}$ must then be collinear to v so that $\widehat{\eta} = h(w_{\mathbf{1}})/E\big(h(w_{\mathbf{1}})\big)$. Thus $\sigma_{\widehat{\eta}}$ depends on the variability of aggregate output (measured with the aggregator h).

The risk characteristics of the real bond depend on the underlying real side of the economy. Since $\mathbf{a}^R = \nabla h(w_{\mathbf{1}})b$, the variability $\sigma_{\hat{a}^R}$ of the normalized indexed bond depends on the magnitude of the fluctuations in relative prices, which in turn depends on the extent to which supply-side shocks influence the relative quantities of the goods across the states. If the real shocks which affect the economy are primarily economy-wide, affecting all sectors (goods) in a similar fashion, then the fluctuations in output captured by $\sigma_{\hat{v}}$ will be greater than the fluctuations in relative prices summarized in $\sigma_{\hat{a}^R}$ (see Figure 3(a)). Conversely the case $\sigma_{\hat{a}^R} > \sigma_{\widehat{\eta}}$ arises when the real shocks are primarily sectoral, affecting sectors differentially while creating only small fluctuations in the level of output (see Figure 3(b)). Clearly the greater the relative price fluctuations the smaller the potential gains from an indexed bond. The correlation ρ_{a^R} depends on how the prices of the goods which are most heavily weighted in b covary with aggregate output: if the supply $w_{\ell s}$ of the goods ℓ, whose components b_ℓ in the index have a substantial weight, are positively (negatively) correlated with aggregate output $\big(h(w_s)\big)$, then ρ_{a^R} will be negative (positive). In view of Figure 2, when the correlation is relatively small, the potential gain is greater when the correlation is negative than when it is positive.

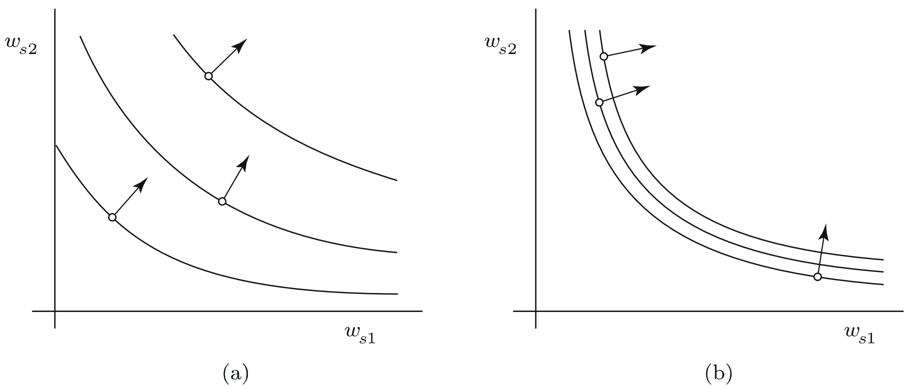

Figure 3. In **a**, economy-wide shocks are greater than sectoral shocks ($\sigma_{\hat{v}} > \sigma_{\hat{a}^R}$); in **b**, the reverse ($\sigma_{\hat{v}} < \sigma_{\hat{a}^R}$).

The risk characteristics of the nominal bond depend on the interaction between the real and the monetary sides of the economy. In the analysis that follows it is useful to distinguish two categories of economies depending on the role attributed to monetary policy:
(i) Economies in which a primary objective of monetary policy is to stabilize the purchasing power of money. Most developed countries are in this category with average annual inflation lying between 1 and 15% per annum and standard deviation of the same order of magnitude. Even in these economies, there is always some variability in the purchasing power of money due to imperfections in the control of the money supply process by the Central Bank or to the fact that monetary policy must also meet other objectives such as full employment. This is the category of economies in which the absence of indexed bonds has been somewhat of a puzzle to economists.
(ii) Economies in which the money supply is used to finance government expenditure. These are typically economies in which inflation is high and very variable, the variability in inflation being due to periodic attempts to drastically lower the rate of inflation. Many less developed countries are in this category, having mean and standard deviation of inflation per annum in excess of 200%. In these economies indexation is pervasive, although nominal bonds of short-term maturity continue to be traded.

The magnitude of $\sigma_{\hat{a}^N}$ is greater in economies of type (ii) than in those of type (i). As for the sign of ρ_{a^N}, in most economies – and especially in economies of type (i) – the statistical relation between inflation and output underlying the Phillips curve, suggests that the purchasing power of money is negatively correlated with aggregate output ($\rho_{a^N} < 0$). The fact that nominal bonds are typically used in economies of type (i), while indexation is pervasive in economies of type (ii), can then be explained by the following proposition which is a corollary of Proposition 4.

Proposition 5 (nominal versus indexed bond): *Given* $(\sigma_{\hat{a}^R}, \rho_{a^R})$ *which depend on the real side of the economy, with* $\rho_{a^R} \neq \pm 1$, *and given* ρ_{a^N} *satisfying* $-1 < \rho_{a^N} \leq 0$, *there exists* σ^* *such that if* $\sigma_{\hat{a}^N} < \sigma^*$, *then the nominal bond leads to greater social welfare and if* $\sigma_{\hat{a}^N} > \sigma^*$, *then the indexed bond leads to greater social welfare.*

Proof: Since $-1 < \rho_{a^N} \leq 0$, by Proposition 4B, the function $g(\cdot, \rho_{a^N})$ is strictly decreasing in σ. Thus if σ^* is defined by

$$g(\sigma^*, \rho_{a^N}) = g(\sigma_{\hat{a}^R}, \rho_{a^R}) \equiv \bar{g}$$

then $g(\sigma_{\hat{a}^N}, \rho_{a^N}) > \bar{g}$ if $\sigma_{\hat{a}^N} < \sigma^*$, and $g(\sigma_{\hat{a}^N}, \rho_{a^N}) < \bar{g}$ if $\sigma_{\hat{a}^N} > \sigma^*$. □

Thus in an economy which is subjected to real (sectoral) shocks there is always an interval $[0, \sigma^*)$ of fluctuations in the purchasing power of money on which the nominal bond is preferred. This interval is larger, the greater the relative price fluctuations $\sigma_{\hat{a}^R}$ and the more negative the correlation $\rho_{a^N v}$ between the purchasing power of money and aggregate output. The existence of sectoral shocks leading to relative price fluctuations and a relatively strong positive correlation between inflation and output may thus be two important elements which help to explain the lack of indexation in Western economies. With slight abuse of the model, the proposition can also be used to obtain insight into the bonds traded in economies with high and variable inflation. The response to high inflation is often to switch to nominal bonds of shorter maturities: such a switch reduces the uncertainty over the future *ppm* ($\sigma_{\hat{a}^N}$), while lags in collecting data and computing indices effectively add to the inherent uncertainty of the payoff on a short-term indexed bond ($\sigma_{\hat{a}^R}$). Indexation is typically introduced when certain agents (most notably the government) find it unpractical to shorten the maturity on bonds: thus in economies of type (ii), it is for medium to long-term bonds that indexation is typically used.

Proposition 5 extends in a relatively straightforward way to the case where there are many securities that generate the market subspace $\mathscr{V}$ $(J > 1)$. If neither the indexed nor the nominal bond is perfectly correlated with a marketed (real) income stream, and if $\rho(\mathbf{a}^N, \eta) \leq 0$, then by Proposition 4B, $g(\cdot, \rho_{a^N})$ is strictly decreasing in σ, so that there exists a σ^* with the properties stated in Proposition 5: if $\sigma_{\hat{a}^N} < \sigma^*$ then the nominal bond is preferred, while if $\sigma_{\hat{a}^N} > \sigma^*$ then the indexed bond gives greater social welfare. Note that if the least risky income stream η in $\mathscr{V}$ is positively correlated with aggregate output $h(w_1)$, then the condition $\rho(\mathbf{a}^N, \eta) \leq 0$ is likely to be satisfied. A qualitative analysis similar to that given for the single security case can then be made in the more realistic case $J > 1$ – many securities inevitably being required if the spanning Assumption **S** is to be a reasonable approximation.

(c) *When the restriction to trading only one of the two bonds is a reasonable assumption.*

The analysis in (b) was based on the assumption that only one of the two bonds is traded. We need to clarify the conditions under which this restriction is reasonable. For there can be circumstances when the correlations $\rho(\mathbf{a}^N, v^j)$, $\rho(\mathbf{a}^R, v^j)$ and $\rho(\mathbf{a}^R, \mathbf{a}^N)$ are such that agents would be much better off trading both the nominal and the indexed bond, so that restricting them to trading only one of the two securities gives an artificial result. The analysis in (b) leads to a result with explanatory power only if, when agents trade the preferred bond, augmenting their opportunity set by permitting trade in the other bond would not add much to their welfare. In such circumstances, even a small transaction cost would cancel the benefit of using the second-best bond.

To cover the two cases where the nominal (resp. indexed) bond is preferred, let $\mathbf{a}$ denote the preferred bond and let $\mathbf{a}'$ denote the second best bond. The market subspace when the preferred bond is used is $\mathscr{W} = \langle \mathscr{V}, \mathbf{a} \rangle$ and by Proposition 4, the maximum welfare gain from adding the second bond $\mathbf{a}'$ is $1 - E(\eta_{\mathscr{W}})$, where $\eta_{\mathscr{W}}$ is the least risky security in $\mathscr{W}$. There are two reasons why introducing the bond $\mathbf{a}'$ may add only a small welfare gain. First, the maximum potential gain $1 - E(\eta_{\mathscr{W}})$ from introducing any additional security may be small. Second, the characteristics of the bond $\mathbf{a}'$ may be such that only a small part of this maximum gain can be captured: since $\mathbf{a}$ is preferred to $\mathbf{a}'$, the least risky security $\eta_{\mathscr{W}}$ must be closer to $\mathbb{1}$ than $\mathbf{a}'$ i.e. $\sigma(\widehat{\mathbf{a}}') > \sigma(\widehat{\eta}_{\mathscr{W}})$, so that $\mathbf{a}'$ falls into the high variability category of Proposition 4, in which the gain may be zero.

In the case of economies of type (i), in which the nominal bond is preferred, a combination of these two reasons serves to explain why the indexed bond is not more widely used. First, if the nominal bond is negatively correlated with most of the securities $\mathscr{V}$ (the stocks), then diversification between the nominal bond and the stocks may permit risks to be significantly reduced, in which case $\sigma(\widehat{\eta}_{\mathscr{W}})$ is small. If $\sigma(\widehat{\mathbf{a}}^R)$ is relatively large and the correlation $\rho(\mathbf{a}^R, \widehat{\eta}_{\mathscr{W}})$ is positive, then the gains from introducing $\mathbf{a}^R$ may be close to the minimum, which is zero.

In the case of economies of type (ii) in which indexed bonds are preferred (at least for bonds with medium to long-term maturities) the nominal bond is simply too risky to be used. Given the general "noise" in the system, the high variability of $\widehat{\mathbf{a}}^N$ is not likely to be compensated by a high correlation with real variables. On the other hand, if neither of the bonds is traded, the assumption that only one can be traded is not restrictive: this will occur if both $\sigma_{\hat{a}^N}$ and $\sigma_{\hat{a}^R}$ are very large and the correlations $\rho_{a^N v}, \rho_{a^R v}$ and $\rho(\mathbf{a}^N, \mathbf{a}^R)$ are weak. For then neither of the bonds, nor a combination of the two, would add significant welfare gains. This will be the case for economies for which the random variables w_s, $\nabla h(w_s)$ and $h(w_s)/M_s$ are highly variable and uncorrelated: such a situation may closely represent economies with high and variable inflation, since the stylized facts (see Heymann and Leijonhufvud [11]) indicate that high variability of inflation is typically associated with high variability of relative prices, and a tendency for trade on both nominal and indexed bonds with medium to long-term maturities to disappear.

5 Proof of properties of the statistical gains function

In this section we prove Propositions 3 and 4. The order of the proof will not exactly follow the statements of these propositions. It is convenient to begin by calculating the statistical gains function, namely the function $g(\sigma_{\hat{a}}, \rho_a)$ which expresses the welfare gain $G(\mathbf{a})$ from a bond $\mathbf{a}$ as a function of its normalized standard deviation and its vector of correlation coefficients with the underlying securities $v^1, \dots, v^J$ ((iii) of Proposition 3). We then exhibit the domain on which the function $g(\sigma, \rho)$ expresses the welfare gain of some random income stream $\mathbf{a} \in \mathbb{R}^S$ ((i) and (ii) of Proposition 3). Finally we establish the properties of g as a function of ρ and σ (A and B of Proposition 4).

Some matrix notation simplifies the calculation of g. Since the purchasing power payoffs on the securities can be normalized to have unit expectation, we let

$$\widehat{V} = \begin{bmatrix} \widehat{v}_1^1 & \dots & \widehat{v}_1^J \\ \vdots & & \vdots \\ \widehat{v}_S^1 & \dots & \widehat{v}_S^J \end{bmatrix}, \quad \widehat{v}^j = \frac{v^j}{E(v^j)}$$

denote the matrix of normalized payoffs. The $J \times J$ diagonal matrix of standard deviations of these J normalized payoffs is denoted by

$$\sigma_{\hat{V}} = \begin{bmatrix} \sigma_{\hat{v}^1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{\hat{v}^J} \end{bmatrix}$$

and $\rho_V = [\rho_{v^i v^j}]_{i,j=1,\dots,J}$ denotes their $J \times J$ matrix of correlation coefficients. For some intermediate calculations, it is convenient to introduce the following measure of stochastic dependence, defined for non-centered random variables: if $x, y \in \mathbb{R}^S,\ E(x) \neq 0,\ E(y) \neq 0$ define

$$k(x,y) = E(\widehat{xy}) = \frac{E(xy)}{E(x)E(y)}$$

Since $k(x,y) = 1 + \rho(x,y)\sigma(\widehat{x})\sigma(\widehat{y}),\ k(x,y)$ is greater (less) than 1 for positively (negatively) correlated random variables. This measure of stochastic dependence appears naturally in the projection formulae. Thus we define

$$k_a = (k_{a1}, \dots, k_{aJ}) = \big(k(\mathbf{a}, v^1), \dots, k(\mathbf{a}, v^J)\big)$$

$$K = \begin{bmatrix} k(v^1, v^1) & \dots & k(v^1, v^J) \\ \vdots & & \vdots \\ k(v^J, v^1) & \dots & k(v^J, v^J) \end{bmatrix} = \widehat{V}^T [\gamma] \widehat{V}$$

Computation of the function g. Recall that the gain function $G : \mathbb{R}^S \longrightarrow \mathbb{R}$ is defined by $G(\mathbf{a}) = \| \eta_{\mathscr{V}_a} \|_\gamma^2 - \| \eta \|_\gamma^2$. Not surprisingly, the reduction in the distance from $\mathbb{1}$ (or the increase of the length of the projection) achieved by changing the market subspace from $\mathscr{V}$ to $\mathscr{V}_a = \langle \mathscr{V}, \mathbf{a} \rangle$ depends only on the innovation component of $\mathbf{a}$ relative to the subspace $\mathscr{V}$. Let

$$\mathbf{a} = \mathbf{a}^* + \mathbf{a}', \quad \mathbf{a}^* \in \mathscr{V}, \quad \mathbf{a}' \in \mathscr{V}^{\perp\!\!\perp}$$

denote the decomposition of $\mathbf{a}$ into its component $\mathbf{a}^*$ on $\mathscr{V}$ and its *innovation component* $\mathbf{a}' \in \mathscr{V}^{\perp\!\!\perp}$ and let $\eta_{\mathbf{a}'} = \mathrm{proj}_{\langle \mathbf{a}' \rangle} \mathbb{1}$ denote the projection of $\mathbb{1}$ onto the one-dimensional subspace generated by $\mathbf{a}'$.

Lemma 1: *The welfare gain* $G(\mathbf{a})$ *from introducing a bond* $\mathbf{a} \in \mathbb{R}^S$ *is given by* $G(\mathbf{a}) = \| \eta_{\mathbf{a}'} \|_\gamma^2$.

Proof: The decomposition of $\mathbb{1}$ onto $\mathscr{V}_a$ and its orthogonal complement $\mathscr{V}_a^{\perp\!\!\perp}$ gives

$$\mathbb{1} = \eta_{\mathscr{V}_a} + \mathbb{1}', \quad \eta_{\mathscr{V}_a} \in \mathscr{V}_a, \quad \mathbb{1}' \in \mathscr{V}_a^{\perp\!\!\perp}$$

Since $\mathscr{V}_a = \mathscr{V} \oplus \langle \mathbf{a}' \rangle,\ \eta_{\mathscr{V}_a}$ can in turn be decomposed into

$$\eta_{\mathscr{V}_a} = u + v, \quad u \in \mathscr{V}, \quad v \in \langle \mathbf{a}' \rangle$$

so that $\mathbb{1} = u + v + \mathbb{1}'$. Since $v \in \langle \mathbf{a}' \rangle \subset \mathscr{V}^{\perp\!\perp}$ and $\mathbb{1}' \in \mathscr{V}_a^{\perp\!\perp} \subset \mathscr{V}^{\perp\!\perp}$, by uniqueness of the orthogonal decomposition $u = \eta_{\mathscr{V}}$. Since $u \in \mathscr{V} \subset \langle \mathbf{a}' \rangle^{\perp\!\perp}$ and $\mathbb{1}' \in \mathscr{V}_a^{\perp\!\perp} \subset \langle \mathbf{a}' \rangle^{\perp\!\perp}$, $v = \eta_{\mathbf{a}'}$. Thus $\eta_{\mathscr{V}_a} = \eta_{\mathscr{V}} + \eta_{\mathbf{a}'}$ and by Pythagoras theorem $G(a) = \| \eta_{\mathscr{V}_a} \|_\gamma^2 - \| \eta_{\mathscr{V}} \|_\gamma^2 = \| \eta_{\mathbf{a}'} \|_\gamma^2$. □

Lemma 2: *The welfare gains function* $G : \mathbb{R}^S \longrightarrow \mathbb{R}$ *can be expressed as a function* $\widetilde{g} : \mathbb{R}_+ \times \mathbb{R}^J \longrightarrow \mathbb{R}$ *of the normalized variables* $\big(E(\widehat{\mathbf{a}}^2), k_a\big)$ *for all* $\mathbf{a} \in \mathbb{R}^S$ *such that* $E(\mathbf{a}) \neq 0$

$$G(\mathbf{a}) = \widetilde{g}\big(E(\widehat{\mathbf{a}}^2), k_a\big) = \begin{cases} \dfrac{(1 - \mathbb{1}^T K^{-1} k_a)^2}{E(\hat{a}^2) - k_a^T K^{-1} k_a}, & \text{if } \mathbf{a} \notin \langle \mathscr{V} \rangle \\ 0, & \text{if } \mathbf{a} \in \langle \mathscr{V} \rangle \end{cases} \tag{1}$$

Proof: By formula (4) of section 3 for the projection matrix $B_{\mathscr{W}}$ with $\mathscr{W} = \langle \mathbf{a}' \rangle$,

$$\eta_{\mathbf{a}'} = \mathbf{a}' [\mathbf{a}'^T [\gamma] \mathbf{a}']^{-1} \mathbf{a}'^T [\gamma] \mathbb{1} = \frac{E(\mathbf{a}')}{E(\mathbf{a}'^2)} \mathbf{a}'$$

so that

$$G(\mathbf{a}) = \| \ \eta_{\mathbf{a}'} \ \|_\gamma^2 = \frac{E(\mathbf{a}')^2}{E(\mathbf{a}'^2)} = \frac{(\mathbb{1}^T [\gamma] \mathbf{a}')^2}{\mathbf{a}'^T [\gamma] \mathbf{a}'}$$

Since $\mathbf{a}' = \mathbf{a} - B_{\mathscr{V}} \mathbf{a}$,

$$G(\mathbf{a}) = \frac{\big(E(\mathbf{a}) - \mathbb{1}^T [\gamma] B_{\mathscr{V}} \mathbf{a}\big)^2}{(\mathbf{a} - B_{\mathscr{V}} \mathbf{a})^T [\gamma] (\mathbf{a} - B_{\mathscr{V}} \mathbf{a})} = \frac{(1 - \mathbb{1}^T [\gamma] B_{\mathscr{V}} \widehat{\mathbf{a}})^2}{E(\widehat{\mathbf{a}}^2) - \widehat{\mathbf{a}}^T [\gamma] B_{\mathscr{V}} \widehat{\mathbf{a}}}$$

where the second equality is obtained by dividing the numerator and denominator by $E(\mathbf{a})^2$ and exploiting the orthogonality of $\mathbf{a}$ and $\mathbf{a}'$: $\mathbf{a}^T [\gamma] (\mathbf{a} - B_{\mathscr{V}} \mathbf{a}) = 0$. Since the γ-projection onto $\langle \mathscr{V} \rangle$ is not affected by the length of the vectors which span the subspace $\langle \mathscr{V} \rangle$, the γ-projection matrix can be written as

$$B_{\mathscr{V}} = \widehat{V} [\widehat{V}^T [\gamma] \widehat{V}]^{-1} \widehat{V}^T [\gamma] = \widehat{V} K^{-1} \widehat{V}^T [\gamma]$$

Using the relations $\mathbb{1}^T [\gamma] \widehat{V} = \mathbb{1}^T$ and $\widehat{V}^T [\gamma] \widehat{\mathbf{a}} = k_a$ leads to formula (1).

Since the variables $\big(E(\widehat{\mathbf{a}}^2), k_a\big)$ can be expressed as functions of (σ_a, ρ_a),

$$E(\widehat{\mathbf{a}}^2) = 1 + \sigma_{\hat{a}}^2, \quad k_a = \mathbb{1} + \sigma_{\hat{a}} [\sigma_{\widehat{V}}] \rho_a \tag{2}$$

substituting the expressions in (2) into equation (1), leads to a function $g(\sigma_{\hat{a}}, \rho_a)$ satisfying

$$G(\mathbf{a}) = g(\sigma_{\hat{a}}, \rho_a) = \widetilde{g}\big(1 + \sigma_{\hat{a}}^2 \ , \ \mathbb{1} + \sigma_{\hat{a}} [\sigma_{\widehat{V}}] \rho_a\big)$$

which proves (iii) of Proposition 3. The exact formula for g is cumbersome and it is always more convenient to make calculations using the function $\widetilde{g}$.

Consider therefore the functions $\widetilde{g} : \mathbb{R} \times \mathbb{R}^J \longrightarrow \mathbb{R}$ defined by

$$\widetilde{g}(m,k) = \begin{cases} \dfrac{(1-\mathbb{1}^T K^{-1}k)^2}{m - k^T K^{-1} k}, & \text{if } m \neq k^T K^{-1} k \\ 0, & \text{if } m = k^T K^{-1} k \end{cases} \tag{3}$$

and $g : \mathbb{R} \times \mathbb{R}^J \longrightarrow \mathbb{R}$ defined by

$$g(\sigma,\rho) = \widetilde{g}\big(1+\sigma^2,\ \mathbb{1} + \sigma[\sigma_{\hat{V}}]\rho\big) \tag{4}$$

When the variables (σ, ρ) correspond to the standard deviation and vector of correlation coefficients of a normalized random variable $\widehat{\mathbf{a}} \in \mathbb{R}^S$, then $g(\sigma,\rho)$ is the welfare gain attributable to the bond **a**. Thus the properties of g need to be studied only for these relevant values of (σ,ρ) which we now characterize.

Relevant domain of g. We begin by proving the sufficiency part of Proposition 3(i).

Lemma 3: *If* $\mathbf{a} \in \mathbb{R}^S$, *then either* $(\sigma_a, \rho_a) = (0,0)$ *or* $\sigma_a > 0$ *and* ρ_a *is such that* $[\rho_V - \rho_a\rho_a^T]$ *is positive semi-definite. Furthermore if* $E(\mathbf{a}) \neq 0$ *and* $\sigma_a > 0$ *then the following properties are equivalent*:

(i) det $[\rho_V - \rho_a\rho_a^T] = 0$
(ii) *there exists* $y \in \mathscr{V}$ *such that* $\rho(\mathbf{a}, y) = \pm 1$
(iii) *there exist* $y \in \mathscr{V}$ *and* $\lambda \in \mathbb{R}$ *such that* $\widehat{\mathbf{a}} - \mathbb{1} = \lambda\big(y - E(y)\mathbb{1}\big)$.

Note that if $\mathbf{a} \in \mathscr{V}$ then (iii) implies that $\mathbb{1} \in \mathscr{V}_a$

Proof: If $\mathbf{a} \in \mathbb{R}^S$, then $-1 \leq \rho(\mathbf{a}, y) \leq 1$ for all $y \in \mathbb{R}^S$ and in particular for all $y \in \mathscr{V}$. If $\sigma(\mathbf{a}) = 0$ and $E(\mathbf{a}) \neq 0$ then $\mathbf{a} = \lambda\mathbb{1}$ and $\rho_a = 0$. If $\sigma(\mathbf{a}) > 0$, then $-1 \leq \rho(a,y) \leq 1,\ \forall\, y \in \mathscr{V}$ is equivalent to

$$\left(\sum_{j=1}^{J} \widetilde{\lambda}_j \rho_{av^j} \sigma_a \sigma_{v^j}\right)^2 \leq \sigma_a^2 \left(\sum_{i=1}^{J}\sum_{j=1}^{J} \widetilde{\lambda}_i \widetilde{\lambda}_j \rho_{v^i v^j} \sigma_{v^i} \sigma_{v^j}\right), \quad \forall\, \widetilde{\lambda} \in \mathbb{R}^J \tag{5}$$

Letting $\lambda_j = \widetilde{\lambda}_j \sigma_{v^j}$, (5) is equivalent to $\lambda^T \rho_a \rho_a^T \lambda \leq \lambda^T \rho_V \lambda$ for all $\lambda \in \mathbb{R}^J$ or $[\rho_V - \rho_a\rho_a^T]$ positive semi-definite. There exists $y \in \mathscr{V}$ such that $\rho(\mathbf{a}, y) = \pm 1$ if and only if there exists $\widetilde{\lambda} \in \mathbb{R}^J$ such that (5) holds with equality, or if and only if there exists $\lambda \in \mathbb{R}^J$ such that $\lambda^T[\rho_v - \rho_a\rho_a^T]\lambda = 0 \Longleftrightarrow$ det $[\rho_V - \rho_a\rho_a^T] = 0$. Thus (i) is equivalent to (ii). On the other hand (ii) is equivalent to

$$[[\mathbf{a} - E(\mathbf{a})\mathbb{1}, y - E(y)\mathbb{1}]]^2 = \| \mathbf{a} - E(\mathbf{a})\mathbb{1} \|_\gamma^2 \| y - E(y)\mathbb{1} \|_\gamma^2 \neq 0 \text{ for some } y \in \mathscr{V}$$

If $E(\mathbf{a}) \neq 0$, dividing by $\big(E(\mathbf{a})\big)^2$ gives

$$[[\widehat{\mathbf{a}} - \mathbb{1}, y - E(y)\mathbb{1}]]^2 = \| \widehat{\mathbf{a}} - \mathbb{1} \|_\gamma^2 \| y - E(y)\mathbb{1} \|_\gamma^2 \neq 0 \text{ for some } y \in \mathscr{V}$$

By the Cauchy-Schwartz inequality this occurs if and only if $\widehat{\mathbf{a}} - \mathbb{1}$ and $y - E(y)\mathbb{1}$, which are non-zero, are linearly dependent, which gives (iii). (iii) can be written as $\big(1-\lambda E(y)\big)\mathbb{1} = \mathbf{a} - \lambda y$ for some $\lambda \in \mathbb{R}$. If $\mathbf{a} \notin \mathscr{V}$ then $1-\lambda E(y) \neq 0$ and $\mathbb{1} \in \mathscr{V}_a$. □

The next lemma proves that the restriction $\sigma > 0$ and $[\rho_V - \rho\rho^T]$ positive semi-definite, completely characterizes the (σ, ρ) which correspond to the standard deviation and vector of correlation coefficients of non-constant random variables in $\mathbb{R}^S$.

Lemma 4: *Let* $\mathscr{R} = \{\rho \in \mathbb{R}^J | [\rho_V - \rho\rho^T]$ is positive semi-definite$\}$

(i) $\mathscr{R}$ is a convex subset of $\mathbb{R}^J$
(ii) $\partial\mathscr{R} = \{\rho \in \mathbb{R}^J | \det[\rho_V - \rho\rho^T] = 0\}$
(iii) *If* $(\sigma, \rho) \in (0,0) \cup \mathbb{R}_{++} \times \mathscr{R}$, *then there exists* $\mathbf{a} \in \mathbb{R}^S$ *with* $E(\mathbf{a}) \neq 0$ *such that* $(\sigma_a, \rho_a) = (\sigma, \rho)$.

Proof: The proof of (i) and (ii) is straightforward and is left to the reader. Proving (iii) is equivalent to showing that if $\sigma > 0$ and $\rho \in \mathscr{R}$ then the following system of equations has a solution:

Find $\mathbf{a} \in \mathbb{R}^S$ *such that*

$$(\mathrm{A}) \quad \begin{cases} \sum_{s=1}^{S} \gamma_s\big(\mathbf{a}_s - E(\mathbf{a})\big)\big(v_s^j - E(v^j)\big) = \rho_j\sigma\sigma_{v^j}, \ j = 1,\ldots,J \\ \sum_{s=1}^{S} \gamma_s\mathbf{a}_s = E(\mathbf{a}) \\ \sum_{s=1}^{S} \gamma_s\big(\mathbf{a}_s - E(\mathbf{a})\big)^2 = \sigma^2 \end{cases}$$

In terms of the standardized variables

$$x_s = \frac{\mathbf{a}_s - E(\mathbf{a})}{\sigma}, \quad c_s^j = \frac{v_s^j - E(v^j)}{\sigma_{v^j}}, \quad s = 1,\ldots,J, \quad j = 1,\ldots,J$$

the problem (A) is equivalent to:
Find $x \in \mathbb{R}^S$ *such that*

$$(\mathrm{A}') \quad \begin{cases} \sum_{s=1}^{S} \gamma_s x_s c_s^j = \rho_j, \ j = 1,\ldots,j \\ \sum_{s=1}^{S} \gamma_x x_s = 0 \\ \sum_{s=1}^{S} \gamma_s x_s^2 = 1 \end{cases} \iff \begin{cases} \widehat{C}[\gamma]x = \begin{bmatrix} \rho \\ 0 \end{bmatrix} \\ x^T[\gamma]x = 1 \end{cases}$$

$$\text{with} \qquad \widehat{C} = \begin{bmatrix} C \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} c_1^1 & \ldots & c_S^1 \\ \vdots & & \vdots \\ c_1^J & \ldots & c_S^J \\ 1 & \ldots & 1 \end{bmatrix}$$

Since rank $\widehat{C} \leq J+1 < S$, the problem (A′) has a solution if and only if the minimum value function

$$(\mathscr{P}) \qquad h(\rho) = \min \left\{ x^T[\gamma]x \,\middle|\, \widehat{C}[\gamma]x = \begin{bmatrix} \rho \\ 0 \end{bmatrix},\ x \in \mathbb{R}^S \right\}$$

satisfies $h(\rho) \leq 1$. For if x^* gives the minimum of this problem then, for all solutions $y \in \mathbb{R}^S$ of the homogeneous equations $\widehat{C}[\gamma]y = 0,\ x = x^* + \lambda y$ satisfies $\widehat{C}[\gamma]x = \begin{bmatrix} \rho \\ 0 \end{bmatrix}$ and an appropriate choice of λ leads to $x^T[\gamma]x = 1$. The solution of the problem $(\mathscr{P})$ is given by $x^* = C\rho_V^{-1}\rho$ where $\rho_V = C[\gamma]C^T$ is the symmetric positive definite matrix of correlation coefficients of the vectors $v^1, \ldots, v^J$, and $h(\rho) = x^{*T}[\gamma]x^* = \rho^T\rho_V^{-1}\rho$. If $[\rho_V - \rho\rho^T]$ is positive semi-definite, then for $\xi = \rho_V^{-1}\rho,\ \xi^T[\rho_V - \rho\rho^T]\xi \geq 0$ which implies $\rho^T\rho_V^{-1}\rho - (\rho^T\rho_V^{-1}\rho)^2 \geq 0$ and since $\rho^T\rho_V^{-1}\rho > 0,\ h(\rho) \leq 1$.

Note that for any $(\sigma, \rho) \in \mathbb{R}_{++} \times \mathscr{R}$, the expected value of the random variables $\mathbf{a} \in \mathbb{R}^S$ such that $(\sigma_a, \rho_a) = (\sigma, \rho)$ is arbitrary: if x is a solution to (A'), then for any $\lambda \in \mathbb{R}, \mathbf{a} = \sigma x + \lambda\mathbb{1}$ is a solution to (A). □

Lemmas 1–4 complete the proof of Proposition 3. It remains to establish the properties of the statistical gains function g on the domain $\mathbb{R}_{++} \times \mathscr{R}$.

Properties of the function g. The function $g(\sigma, \rho)$ defined by (4) is obtained from the function $\widetilde{g}(m, k)$ defined by (3), via the change of variable

$$m = 1 + \sigma^2,\ k = \mathbb{1} + \sigma[\sigma_V]\rho \tag{6}$$

While the variables (σ, ρ) have a more natural economic interpretation, the variables (m, k) are better adapted to analyzing properties derived from projection formulae: the properties of $g(\sigma, \rho)$ will thus be derived from the properties of the function $\widetilde{g}(m, k)$.

The function $\widetilde{g}(m, k)$ is rational function which we write as

$$\widetilde{g}(m, k) = \begin{cases} \dfrac{N(k)}{Q(m, k)}, & \text{if } Q(m, k) \neq 0 \\ 0, & \text{if } Q(m, k) = 0 \end{cases}$$

The relevant domain for $\widetilde{g}$ is the image of $\mathbb{R}_{++} \times \mathscr{R}$ under the change of variable (6). It is convenient to begin by studying when the denominator $Q(m, k)$ vanishes.

Lemma 5: *If* $(\sigma, \rho) \in \mathbb{R}_{++} \times \mathscr{R}$ *and* (m, k) *is defined by* (6), *then*

(i) $Q(m, k) \geq 0$
(ii) $Q(m, k) = 0 \Longleftrightarrow$ *every* $\mathbf{a} \in \mathbb{R}^S$ *such that* $(\sigma_a, \rho_a) = (\sigma, \rho)$ *satisfies* $\mathbf{a} \in \mathscr{V}$
(iii) $Q(m, k) = 0 \Longrightarrow \rho \in \partial\mathscr{R}$ *and* $\sigma \geq \sigma_{\hat{\eta}}$.

Proof: Let $\mathbf{a} \in \mathbb{R}^S$ be such that $(\sigma_{\hat{a}}, \rho_a) = (\sigma, \rho)$ and let (m, k) be deduced from (σ, ρ) by (6), then $\widetilde{g}(m, k) = \left(E(\mathbf{a}')\right)^2 / E(\mathbf{a}'^2)$ where $\mathbf{a}'$ is the innovation component of $\mathbf{a}$ relative to $\mathscr{V}$. Thus $Q(m, k) = E(\mathbf{a}'^2) \geq 0$ and $Q(m, k) = 0$ if and only if $\mathbf{a}' = 0 \Longleftrightarrow \mathbf{a} \in \mathscr{V}$, which proves (i) and (ii). If $\mathbf{a} \in \mathscr{V}$, then there exists $y \in \mathscr{V}$ $(y = \mathbf{a})$ such that $\rho(\mathbf{a}, y) = 1$, and by Lemmas 3 and 4, $\rho \in \partial\mathscr{R}$. Moreover in this case $\sigma = \sigma_{\hat{a}} \geq \sigma_{\hat{\eta}}$, since $\sigma_{\hat{a}} < \sigma_{\hat{\eta}}$ would contradict the minimum risk property of $\hat{\eta}$ in Proposition 2 (ii) b. □

Lemma 6: *For all* $\sigma \in \mathbb{R}_{++}$, $g(\sigma, \cdot)$ *is a convex function on* int $\mathcal{R}$.

Proof: Given the linearity of the change of variable (6), it suffices to prove that $k \mapsto \widetilde{g}(m, k)$ is a convex function of k on the domain $Q(m, k) > 0$. The matrix of second derivatives of $\widetilde{g}$ with respect to k is given by

$$\begin{aligned} D^2_{kk}\widetilde{g}(m,k) = & \frac{D^2 N(k)}{Q(m,k)} + N(k) D^2_{kk}\left(\frac{1}{Q(m,k)}\right) \\ & + \nabla N(k) \nabla_k^T \left(\frac{1}{Q(m,k)}\right) + \nabla_k \left(\frac{1}{Q(m,k)}\right) \nabla^T N(k) \end{aligned} \tag{7}$$

where ∇ (resp. ∇^T) denotes the gradient (resp. transpose of the gradient) and where

$$\begin{aligned} \nabla N(k) &= -2(1 - \mathbb{1}^T K^{-1} k) K^{-1} \mathbb{1} \\ D^2 N(k) &= 2K^{-1} \mathbb{1} \mathbb{1}^T K^{-1} \\ \nabla_k \left(\frac{1}{Q(m,k)}\right) &= \frac{2K^{-1} k}{Q^2(m,k)} \\ D^2_{kk}\left(\frac{1}{Q(m,k)}\right) &= \frac{2K^{-1}}{Q^2(m,k)} + \frac{8K^{-1} k k^T K^{-1}}{Q^3(m,k)} \end{aligned}$$

Inserting these expressions into (7) leads to

$$\begin{aligned} x^T [D^2_{kk}\widetilde{g}\,] x = & \frac{2}{Q}\left(x^T K^{-1} \mathbb{1} - \frac{2(1 - \mathbb{1}^T K^{-1} k) x^T K^{-1} k}{Q}\right)^2 \\ & + \frac{2x^T K^{-1} x (1 - \mathbb{1}^T K^{-1} k)^2}{Q^2} \end{aligned}$$

which is non-negative for all $x \in \mathbb{R}^J$, since K^{-1} is positive definite and $Q > 0$. □

We now study the minima of the function $g(\sigma, \cdot)$. Since g is a convex function on int $\mathcal{R}$, the values of $\rho \in$ int $\mathcal{R}$ for which g attains a minimum are the solutions of the first order condition $\nabla_\rho g(\sigma, \rho) = 0$. Since

$$\nabla_\rho g(\sigma, \rho) = \sigma [\sigma_{\hat{V}}] \nabla_k \widetilde{g}$$

and since $[\sigma_{\hat{V}}]$ is invertible, these values of ρ correspond to the values of k such that $\nabla_k \widetilde{g}(m, k) = 0$ (with $m = 1 + \sigma^2$). Define the functions $H : \mathbb{R}^J \longrightarrow \mathbb{R}$ and $F : \mathbb{R}^J \longrightarrow \mathbb{R}^J$

$$H(k) = 1 - \mathbb{1}^T K^{-1} k, \quad F(m,k) = (k^T K^{-1} k - m)\mathbb{1} + (1 - \mathbb{1}^T K^{-1} k) k \tag{8}$$

noting that the numerator of $\widetilde{g}$ satisfies $N(k) = \big(H(k)\big)^2$. Then

$$\nabla_k \widetilde{g}(m,k) = \frac{2H(k) K^{-1} F(m,k)}{Q^2(m,k)}$$

Since K^{-1} is invertible, $\nabla_k \widetilde{g}(m,k) = 0$ if and only if

$$\text{either (i) } H(k) = 0$$

$$\text{or (ii) } F(m,k) = 0$$

The next two lemmas locate the zeros of H and F respectively. For fixed $m = 1 + \sigma^2$, the zeros of H define a hyperplane $\mathscr{H}_\sigma$ in $\mathbb{R}^J$

$$\mathscr{H}_\sigma = \left\{\rho \in \mathbb{R}^J \middle| H(\mathbb{1} + \sigma[\sigma_{\widehat{V}}]\rho) = 0\right\}$$

Lemma 7: (α) *Let* $\rho \in \mathscr{H}_\sigma \cap \mathscr{R}$, *then* (i) $g(\sigma,\rho) = 0$ *and* (ii) $\mathbf{a} \in \mathbb{R}^S$ *is such that* $(\sigma_{\hat{a}}, \rho_a) = (\sigma, \rho)$ *if and only if*

$$\rho(\mathbf{a},\eta) = \frac{\sigma_{\hat{\eta}}}{\sigma} \Longleftrightarrow [\![\hat{\mathbf{a}},\, \mathbb{1} - \eta]\!] = 0 \Longleftrightarrow \eta_{\mathscr{V}_a} = \eta$$

(β) *If* $\sigma < \sigma_{\hat{\eta}}$, *then* $\mathscr{H}_\sigma$ *does not intersect* $\mathscr{R}$.
(γ) *If* $\sigma = \sigma_{\hat{\eta}}$, *then* $\mathscr{H}_\sigma$ *is tangent to* $\mathscr{R}$ *at the unique point* $\rho^* = \sigma_{\hat{\eta}}[\sigma_{\widehat{V}}]^{-1}\mathbb{1} \in \partial\mathscr{R}$.
(δ) *If* $\sigma > \sigma_{\hat{\eta}}$, *then* $\mathscr{H}_\sigma$ *intersects* $\mathscr{R}$ *and the relative interior of* $\mathscr{H}_\sigma \cap$ int $\mathscr{R}$ *is an open subset* of *dimension* $J-1$.

Proof: (α) (i) If $\rho \in \mathscr{H}_\sigma \cap \mathscr{R}$ then $g(\sigma,\rho) = \widetilde{g}(m,k) = \frac{(H(k))^2}{Q(m,k)} = 0$. (ii) Note that $E(\widehat{v}^j) = 1, j = 1 \ldots, J$ implies $\widehat{V}^T[\gamma]\mathbb{1} = \mathbb{1}$. Thus η, which is the projection of $\mathbb{1}$ onto $\langle V \rangle$, is given by

$$\eta = \widehat{V}\left[\widehat{V}^T[\gamma]\widehat{V}\right]^{-1}\widehat{V}^T[\gamma]\mathbb{1} = \widehat{V}K^{-1}\mathbb{1} \tag{9}$$

so that

$$E(\eta) = \mathbb{1}^T[\gamma]\widehat{V}K^{-1}\mathbb{1} = \mathbb{1}^T K^{-1}\mathbb{1} \tag{10}$$

Thus if $\mathbf{a} \in \mathbb{R}^S$, since $k_a = \left(E(\widehat{v}^1, \widehat{\mathbf{a}}), \ldots, E(\widehat{v}^J, \widehat{\mathbf{a}})\right) = \widehat{V}[\gamma]\widehat{\mathbf{a}}$

$$E(\widehat{\mathbf{a}}\eta) = \mathbb{1}^T K^{-1}\widehat{V}^T[\gamma]\widehat{\mathbf{a}} = \mathbb{1}^T K^{-1} k_a \tag{11}$$

(11) and the definition of H in (8) imply

$$H(k) = 0 \Longleftrightarrow 1 - E(\widehat{\mathbf{a}}\eta) = 0 \Longleftrightarrow [\![\widehat{\mathbf{a}}, \mathbb{1} - \eta]\!] = 0$$

Thus $\mathbb{1} - \eta$ is orthogonal to $\widehat{\mathbf{a}}$. Since by definition $\mathbb{1} - \eta$ is orthogonal to $\mathscr{V}, \mathbb{1} - \eta \in \mathscr{V}_a^{\perp\!\!\perp}$ which implies that η is the projection of $\mathbb{1}$ onto $\mathscr{V}_a^{\perp\!\!\perp}$ i.e. $\eta = \eta_{\mathscr{V}_a}$. Furthermore $1 - E(\widehat{\mathbf{a}}\eta) = 0 \Longleftrightarrow 1 - E(\eta) - \rho(\mathbf{a},\eta)\sigma\sigma_\eta = 0$ and dividing by $E(\eta)$ this is equivalent to

$$\frac{1}{E(\eta)} - 1 - \rho(\mathbf{a},\eta)\sigma\sigma_{\hat{\eta}} = 0 \Longleftrightarrow \rho(\mathbf{a},\eta) = \frac{\sigma_{\hat{\eta}}}{\sigma}$$

where the last step is derived from the equality $\sigma_{\hat{\eta}}^2 = \frac{1}{E(\eta)} - 1$ proved in Proposition 2 (iii). (β) By $(\alpha), \rho \in \mathscr{H}_\sigma \cap \mathscr{R}$ implies $\rho(\mathbf{a},\eta) = \sigma_{\hat{\eta}}/\sigma$ which is impossible if $\sigma < \sigma_{\hat{\eta}}$: thus $\mathscr{H}_\sigma \cap \mathscr{R} = \emptyset$. (γ) If $\mathbf{a} \in \mathbb{R}^S$ is such that

$(\sigma_{\hat{a}}, \rho_a) = (\sigma, \rho)$ with $\sigma = \sigma_{\hat{\eta}}$ and $\rho \in \mathscr{H}_\sigma \cap \mathscr{R}$ then by (α), $\rho(\mathbf{a}, \eta) = 1$ and by Lemma 3, $\widehat{\mathbf{a}} - \mathbb{1} = \lambda\big(\eta - E(\eta)\mathbb{1}\big) = \lambda'(\widehat{\eta} - \mathbb{1})$ with $\lambda' > 0$ since the correlation is positive. $\sigma_{\hat{a}} = \sigma_{\hat{\eta}} \iff \| \,\hat{\mathbf{a}} - \mathbb{1}\, \|_\gamma = \|\, \widehat{\eta} - \mathbb{1}\, \|_\gamma$ which implies $\widehat{\mathbf{a}} = \widehat{\eta}$ so that $\rho_a = \rho_\eta$, and by Lemma 5, $\rho_\eta \in \partial\mathscr{R}$. ρ_η is readily computed, since $k_\eta = \widehat{V}^T[\gamma]\widehat{\eta} = \frac{1}{E(\eta)}\widehat{V}^T[\gamma]\widehat{V}K^{-1}\mathbb{1} = \frac{1}{E(\eta)}KK^{-1}\mathbb{1} = \frac{\mathbb{1}}{E(\eta)}$ so that solving from $\mathbb{1} + \sigma_{\hat{\eta}}[\sigma_{\hat{V}}]\rho_\eta = k_{\hat{\eta}}$ gives

$$\rho^* = \rho_\eta = \frac{1}{\sigma_{\hat{\eta}}}[\sigma_{\hat{V}}]^{-1}\left(\frac{1}{E(\eta)} - 1\right)\mathbb{1} = \sigma_{\hat{\eta}}[\sigma_{\hat{V}}]^{-1}\mathbb{1}$$

(δ) Since $\mathscr{H}_\sigma$ is a hyperplane in $\mathbb{R}^J$, it suffices to show that $\mathscr{H}_\sigma \cap$ int $\mathscr{R} \neq \emptyset$. Consider $k^* = \frac{\mathbb{1}}{E(\eta)}$. By (10), $H(k^*) = 0$. Let us prove that ρ^* such that $k^* = \mathbb{1} + \sigma[\sigma_{\hat{V}}]\rho^*$ namely $\rho^* = \frac{\sigma^2_{\hat{\eta}}}{\sigma}[\sigma_{\hat{V}}]^{-1}\mathbb{1}$ lies in the interior of $\mathscr{R}$. For any $\lambda \in \mathbb{R}^J$, consider the vector $y = \sum_{j=1}^J \lambda_j \widehat{v}^j$ with co-ordinates λ on the normalized basis of $\mathscr{V}$. Then

$$\lambda^T[\sigma_{\hat{V}}][\rho_V - \rho^*\rho^{*T}][\sigma_{\hat{V}}]\lambda = \lambda^T[\sigma_{\hat{V}}]\rho_V[\sigma_{\hat{V}}]\lambda - \frac{\sigma^4_{\hat{\eta}}}{\sigma^2}\lambda^T\mathbb{1}\mathbb{1}^T\lambda$$
$$= \sigma_y^2 - \frac{\sigma^4_{\hat{\eta}}}{\sigma^2}\big(E(y)\big)^2 = \left(\sigma^2_{\hat{y}} - \frac{\sigma^4_{\hat{\eta}}}{\sigma^2}\right)\big(E(y)\big)^2$$

Since $\sigma > \sigma_{\hat{\eta}}$ and since $\hat{\eta}$ is the minimum risk income stream in $\mathscr{V}$, $\sigma_{\hat{y}} \geq \sigma_{\hat{\eta}}$, so that the expression is always strictly positive, implying that $\rho^* \in$ int $\mathscr{R}$. □

For fixed $m = 1 + \sigma^2$, the zeros of F define the subset of $\mathbb{R}^J$

$$\mathscr{F}_\sigma = \left\{\rho \in \mathbb{R}^J \middle| F(1+\sigma^2, \mathbb{1} + \sigma[\sigma_{\hat{V}}]\rho) = 0\right\}$$

Lemma 8: (α) *If* $\sigma < \sigma_{\hat{\eta}}$, *then* $\mathscr{F}_\sigma \cap \mathscr{R} = \{\rho^*\}$ *where* $\rho^* = \sigma[\sigma_{\hat{V}}]^{-1}\mathbb{1} = \left(\frac{\sigma}{\sigma_{\hat{v}^1}}, \ldots, \frac{\sigma}{\sigma_{\hat{v}^J}}\right) \in$ int $\mathscr{R}$, *and*

$$g(\sigma, \rho^*) = \frac{1}{1+\sigma^2} - \frac{1}{1+\sigma^2_{\hat{\eta}}} \tag{12}$$

(β) *If* $\sigma \geq \sigma_{\hat{\eta}}$, *then* $\mathscr{F}_\sigma \cap \mathscr{R} \subset \mathscr{H}_\sigma \cap \partial\mathscr{R}$.

Proof: By (8), $F(m,k) = -Q(m,k)\mathbb{1} + H(k)k$ so that F is a linear combination of the vectors $\{\mathbb{1}, k\}$. Either k is collinear to $\mathbb{1}$ or these vectors are linearly independent. In the first case $F = 0$ only if $k = m\mathbb{1}$ and this corresponds to a value ρ^* such that

$$\mathbb{1} + \sigma[\sigma_{\hat{V}}]\rho^* = (1+\sigma^2)\mathbb{1} \iff \rho^* = \sigma[\sigma_{\hat{V}}]^{-1}\mathbb{1} \tag{13}$$

$\rho^* \in \mathscr{R}$ if for all $\lambda \in \mathbb{R}^J$

$$\lambda^T[\sigma_{\hat{V}}][\rho_V - \rho^*\rho^{*T}][\sigma_{\hat{V}}]\lambda \;\geq\; 0$$

which is equivalent $\big($see proof of Lemma 7$(\delta)\big)$ to $\sigma^2_{\hat{y}} - \sigma^2 \;\geq\; 0$ for all $y \in \mathscr{V}$, or to $\sigma \leq \sigma_{\hat{\eta}}$. If $\sigma < \sigma_{\hat{\eta}}$ then the inequality is strict so that $\rho^* \in$ int $\mathscr{R}$. Since $k^* = m\mathbb{1}$, it follows from (10) that

$$
\begin{aligned}
g(\sigma, \rho^*) = \widetilde{g}(m, k^*) &= \frac{(1 - m\mathbb{1}^T K^{-1}\mathbb{1})^2}{m - m^2\mathbb{1}^T K^{-1}\mathbb{1}} = \frac{\big(1 - mE(\eta)\big)^2}{m\big(1 - mE(\eta)\big)} \\
&= \frac{1}{m} - E(\eta) = \frac{1}{1+\sigma^2} - \frac{1}{1+\sigma_{\hat{\eta}}^2}
\end{aligned}
\tag{14}
$$

If $\sigma = \sigma_{\hat{\eta}}$, then ρ^* is given by (13) with $\sigma = \sigma_{\hat{\eta}}$ and thus coincides with the point in $\mathscr{H}_\sigma \cap \partial\mathscr{R}$ given by Lemma 7(γ) and $g(\sigma, \rho^*) = 0$.

If the vectors $\{k, \mathbb{1}\}$ are linearly independent, then $F = 0$ if and only if $Q = 0$ and $H = 0$: by Lemma 5, the former implies $\rho \in \partial\mathscr{R}$ and $\sigma \geq \sigma_{\hat{\eta}}$ and the latter implies $\rho \in \mathscr{H}_\sigma$. Thus if $\sigma \geq \sigma_{\hat{\eta}}, \mathscr{F}_\sigma \cap \mathscr{R} \subset \mathscr{H}_\sigma \cap \partial\mathscr{R}$. □

Since by Lemma 6, $g(\sigma, \cdot)$ is convex on int $\mathscr{R}$, it follows from Lemmas 7 and 8 that if $\sigma \leq \sigma_{\hat{\eta}}$ then $g(\sigma, \cdot)$ attains its minimum at the unique point ρ^* given by (13), and $g(\sigma, \rho^*)$ is given by (14). If $\sigma > \sigma_{\hat{\eta}}$ then $g(\sigma, \cdot)$ attains its minimum for all points ρ on the intersection of the hyperplane $\mathscr{H}_\sigma$ with $\mathscr{R}$ and $g(\sigma, \rho) = 0$ for all such points. By Lemma 7(α), $\mathscr{H}_\sigma \cap \mathscr{R}$ coincides with the set $\mathscr{R}_{r_\sigma}$ with $r_\sigma = \sigma_{\hat{\eta}}/\sigma$ consisting of the vectors $\rho \in \mathscr{R}$ such that $\rho(a, \eta) = \sigma_{\hat{\eta}}/\sigma$ for all $\mathbf{a} \in \mathbb{R}^S$ with $\rho_a = \rho$.

The next lemma locates the values of ρ for which $g(\sigma, \cdot)$ attains its maximum on $\mathscr{R}$: this consists of all the boundary points of $\mathscr{R}$ which do not lie on the hyperplane $\mathscr{H}_\sigma$. Since g is zero on $\mathscr{H}_\sigma$, it follows that g has a discontinuity at the boundary points which lie on $\mathscr{H}_\sigma$, when $J \geq 2$.

Lemma 9: (α) $g(\sigma, \cdot)$ *attains its maximum on* $\mathscr{R}$ *for all* $\rho \in \partial\mathscr{R} \backslash \mathscr{H}_\sigma$ *and* $g(\sigma, \rho) = 1 - E(\eta), \quad \forall\, \rho \in \partial\mathscr{R} \backslash \mathscr{H}_\sigma$.

(β) *If* $\sigma < \sigma_{\hat{\eta}}$, *then* $g(\sigma, \cdot)$ *is continuous on* $\mathscr{R}$. *If* $\sigma \geq \sigma_{\hat{\eta}}$ *and* $J \geq 2$, *then* $g(\sigma, \cdot)$ *has a discontinuity at* $\rho \in \partial\mathscr{R} \cap \mathscr{H}_\sigma$ *and* $g(\sigma, \rho) = 0, \forall\, \rho \in \partial\mathscr{R} \cap \mathscr{H}_\sigma$.

Proof: (α) Since $g(\sigma, \rho) = G(\mathbf{a}) = \| \eta_{\mathscr{V}_a} \|_\gamma^2 - \| \eta_{\mathscr{V}} \|_\gamma^2$, for all $\mathbf{a} \in \mathbb{R}^S$ such that $(\sigma_{\hat{a}}, \rho_a) = (\sigma, \rho), g$ attains its maximum when $\eta_{\mathscr{V}_a} = \mathbb{1} \Longleftrightarrow \mathbb{1} \in \mathscr{V}_a$. By Lemmas 3 and 4, this occurs when $\rho \in \partial\mathscr{R}$ and $\mathbf{a} \notin \mathscr{V}$. Since $\mathbf{a} \in \mathscr{V}$ is equivalent to $E(\mathbf{a}') = 0$ where $\mathbf{a}'$ is the innovation component of $\mathbf{a}$, and since (see proof of Lemma 2) $E(\mathbf{a}') = H(k_a)$, the maximum of $g(\sigma, \cdot)$ is attained for $\rho \in \partial\mathscr{R} \backslash \mathscr{H}_\sigma$.

(β) Since g is a rational function it can be discontinuous only at the points where the denominator is zero. When $\sigma < \sigma_{\hat{\eta}}$, by Lemma 5, $Q > 0$, so that $g(\sigma, \cdot)$ is continuous on $\mathscr{R}$. When $\sigma \geq \sigma_{\hat{\eta}}$, $Q = 0$ when $\rho \in \partial\mathscr{R} \cap \mathscr{H}_\sigma$ and $g(\sigma, \cdot)$ has a potential discontinuity at such points. Since $\mathscr{R}$ is a manifold with boundary of dimension J, its boundary $\partial\mathscr{R}$ is a manifold of dimension $J - 1$. When $J - 1 = 0$, $\partial\mathscr{R}$ consists of isolated points and we saw in section 3 that $g(\sigma, \cdot)$ is not discontinuous at $\rho \in \partial\mathscr{R} \cap \mathscr{H}_\sigma$. For $J \geq 2$, when ρ moves in $\partial\mathscr{R}$, which is now of dimension $J - 1 \geq 1$, $g(\sigma, \cdot)$ has the value $1 - E(\eta)$ when $\rho \notin \mathscr{H}_\sigma$ and 0 when $\rho \in \mathscr{H}_\sigma$. Thus there is a discontinuity which arises from the drop in dimension of $\mathscr{V}_a$ which looses one dimension when $\mathbf{a}$ goes from being outside $\mathscr{V}$ (in which case it contributes a great deal) to being inside $\mathscr{V}$ (in which case it contributes nothing). □

Since $\mathscr{R}_{r_\sigma} = \mathscr{H}_\sigma \cap \mathscr{R}$, this completes the proof of part A of Proposition 4. It remains to study the properties of g as a function of σ. In section 3 it was shown that the correlation coefficient $\rho(a, \eta)$ with the least risky security η is the same for all $a \in \mathbb{R}^S$ with the same vector of correlation coefficients ρ_a. The expression for $\rho(a, \eta)$ as a function of ρ_a is

$$\rho(a, \eta) = \frac{\text{cov}\ (\hat{a}, \eta)}{\sigma_{\hat{a}}\sigma_\eta} = \frac{E(\hat{a}\eta) - E(\eta)}{\sigma_{\hat{a}}\sigma_\eta} = \frac{\mathbb{1}^T K^{-1} k_a - \mathbb{1}^T K^{-1}\mathbb{1}}{\sigma_{\hat{a}}\sigma_\eta}$$

Substituting the expression for k_a in (2) gives

$$\rho(a, \eta) = \frac{\mathbb{1}^T K^{-1}[\sigma_{\hat{V}}]\rho_a}{\sigma_\eta} \tag{15}$$

Thus

$$\rho \in \mathscr{R}^+ \text{ (resp.} \mathscr{R}^-) \Longleftrightarrow \mathbb{1}^T K^{-1}[\sigma_{\hat{V}}]\rho > 0 \text{ (resp. } \leq 0) \tag{16}$$

The behavior of g as a function of σ depends on whether ρ lies in $\mathscr{R}^+$ or $\mathscr{R}^-$.

Lemma 10: *Consider any* $\rho \in$ int $\mathscr{R}$.

(α) *If* $\rho \in \mathscr{R}^-$, *then* $g(\cdot, \rho)$ *is strictly decreasing for all* $\sigma > 0$.
(β) *If* $\rho \in \mathscr{R}^+$, *then there exists* $\sigma^* = \sigma_{\hat{\eta}}/\rho(a, \eta)$ *such that* $g(\cdot, \rho)$ *is strictly decreasing for* $\sigma \in (0, \sigma^*)$ *and strictly increasing for* $\sigma \in (\sigma^*, \infty)$.

Proof: $\dfrac{\partial g(\sigma, \rho)}{\partial \sigma} = 2\sigma \dfrac{\partial \tilde{g}(m, k)}{\partial m} + \rho^T[\sigma_{\hat{V}}]\nabla_k \tilde{g}(m, k)$ where

$$(m, k) = \left(1 + \sigma^2,\ \mathbb{1} + \sigma[\sigma_{\hat{V}}]\rho\right) \tag{17}$$

Define

$$L(\sigma, \rho) = \sigma\rho^T[\sigma_{\hat{V}}]K^{-1}F(m, k) - \sigma^2 H(k)$$

with (m, k) given by (17). Then

$$\frac{\partial g}{\partial \sigma} = \frac{2HL}{\sigma Q^2}$$

Let us show that $L(\sigma, \rho) < 0,\ \forall\ (\sigma, \rho) \in \mathbb{R}_{++} \times$ int $\mathscr{R}$ so that

$$\text{sgn}\left(\frac{\partial g}{\partial \sigma}\right) = -\text{sgn}(H)$$

L can be written as

$$L(\sigma, \rho) = (1 - m)H(k) + (k - \mathbb{1})^T K^{-1}(-Q(m, k)\mathbb{1} + H(k)k)$$

with (m, k) given by (17), which by appropriately regrouping terms gives

$$L = Q(g - (1 - E(\eta)) < 0$$

where $L < 0$ follows from $Q > 0$ and $g < 1 - E(\eta)$ for $\sigma > 0$ and $\rho \in$ int $\mathscr{R}$. Thus if $H > 0$ (resp. < 0) then $g(\cdot, \rho)$ is strictly decreasing (resp. increasing). The expression for H as a function of (σ, ρ) is

$$H(\sigma,\rho) = 1 - \mathbb{1}^T K^{-1}\big(\mathbb{1} + \sigma[\sigma_{\hat{V}}]\rho\big) = 1 - E(\eta) - \sigma \mathbb{1}^T K^{-1}[\sigma_{\hat{V}}]\rho$$

which by (15) can be written as

$$H(\sigma,\rho) = 1 - E(\eta) - \sigma\sigma_\eta \rho(a,\eta)$$

Thus if $\rho(a,\eta) \le 0$ then $H(\sigma,\rho) > 0$ for all $\sigma > 0$, which proves (α). If $\rho(a,\eta) > 0$, define

$$\sigma^* = \frac{1 - E(\eta)}{\sigma_\eta \rho(a,\eta)} = \frac{1}{\sigma_{\hat{\eta}}\rho(a,\eta)} \frac{1 - E(\eta)}{E(\eta)} = \frac{\sigma_{\hat{\eta}}}{\rho(a,\eta)}$$

If $\sigma \in (0,\sigma^*)$ then $H(\sigma,\rho) > 0$ and if $\sigma \in (\sigma^*,\infty)$ then $H(\sigma,\rho) < 0$, which proves (β).

This completes the proof of Proposition 4. □

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